//euler/texmf/dvips/unigla600.bmp

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2Q - Groups, Symmetry and Fractals

Candidates must not attempt more than THREE questions.

1. (i) Given that

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

describe the geometric effect of the isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ represented by each of the Seitz symbols $(A \mid \mathbf{0})$ and $(B \mid \mathbf{t})$. **7**

Determine the Seitz symbol of the composition $(A \mid \mathbf{0})(B \mid \mathbf{t})$. **2**

(ii) Show that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two parallel lines in the plane, the composition $\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}$ of the reflections in these lines is a translation $\mathrm{Trans}_{\mathbf{w}}$ for some vector $\mathbf{w}$. **6**

Determine the translation vector $\mathbf{w}$ when the lines are

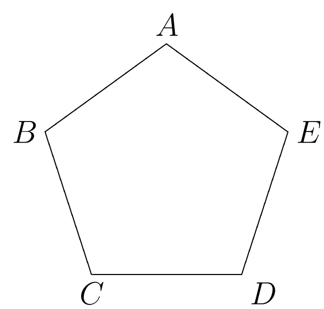
$$\mathcal{L}_1 = \{(x, y) : x + 2y = 0\}, \quad \mathcal{L}_2 = \{(x, y) : x + 2y = 1\}.$$ **2**

2. (i) In the symmetric group S_6, evaluate the product $(1\,6\,4)(1\,4\,5\,2)(2\,5)$. **3**

(ii) Determine the disjoint cycle decomposition of the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{pmatrix}.$$ **3**

(iii) Let Γ be the group of symmetries of the regular pentagon, centred at the origin O and with vertices A, B, C, D, E.



By identifying Γ with a group of permutations of the vertices, describe the elements of Γ both geometrically and using permutation notation. **7**

Let Φ denote reflection in the line OA and Θ rotation through $2\pi/5$ in the anticlockwise direction. Determine the composition $\Phi \circ \Theta$ and describe its effect geometrically. **4**

3. (i) Define the Euclidean group $(\mathrm{Euc}(2), \circ)$ of the plane. **2**

Show that the subset $\mathrm{Trans}(2) \subseteq \mathrm{Euc}(2)$ of translations forms a subgroup of $\mathrm{Euc}(2)$. **3**

(ii) Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a *finite* subgroup. Show that there is a point in the plane fixed by every element of Γ. **7**

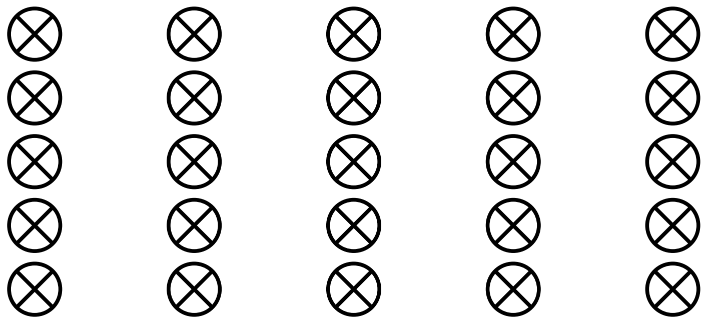
You may use, without proof, the following result:
For $t_1, \ldots, t_k \in \mathbb{R}$ satisfying $t_1 + \cdots + t_k = 1$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^2$, an isometry $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ satisfies

$$F(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = t_1F(\mathbf{v}_1) + \cdots + t_kF(\mathbf{v}_k).$$

(iii) Define what it means for two isometries of the plane to be *similar*. **2**

Show that the rotations through an angle θ about two different points P and Q in the plane are similar. **3**

4. (i) In a wallpaper pattern $\mathbb{W}$, part of which is shown below, the centres of the $\otimes$ symbols are located at positions $(5m, 2n)$ for values $m, n \in \mathbb{Z}$.



(a) Describe the group $\mathrm{Trans}(2)_{\mathbb{W}}$ of *translational symmetries* of $\mathbb{W}$, giving the answer in terms of a pair of generating vectors $\mathbf{u}$ and $\mathbf{v}$. **3**

(b) Determine the group of symmetries of $\mathbb{W}$ which fix the origin. **4**

(c) Describe the elements of the symmetry subgroup $\mathrm{Euc}(2)_{\mathbb{W}} \leqslant \mathrm{Euc}(2)$. **3**

(ii) Explain why the matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin. **2**

Determine the axis and angle of rotation for A. **5**

END]

Tuesday, 29*th May*, 2001 9.30 a.m. *to* 10.45 p.m.

2Q Degree exam 2000–1 – Solutions

1. (i) $(A \mid \mathbf{0})$ represents reflection in the y-axis. **2**

Since

$$B = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix},$$

this represents a rotation about the origin through $-\pi/4$. So $(B \mid \mathbf{t})$ also represents a rotation through $-\pi/4$ about the point with position vector given by

$$\begin{aligned} \mathbf{c} &= (I - B)^{-1}\mathbf{t} \\ &= \begin{bmatrix} 1 - 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 - 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} - 1 & -1 \\ 1 & \sqrt{2} - 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{\sqrt{2}}{2(2 - \sqrt{2})} \begin{bmatrix} \sqrt{2} - 1 & 1 \\ -1 & \sqrt{2} - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{(\sqrt{2} + 1)}{2} \begin{bmatrix} 1 \\ \sqrt{2} - 1 \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{2} + 1)/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

Hence the centre of rotation is $\mathbf{c} = ((\sqrt{2} + 1)/2, 1/2)$. **5**

We have

$$(A \mid \mathbf{0})(B \mid \mathbf{t}) = (AB \mid A\mathbf{t}).$$

Now

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \\ A\mathbf{t} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \end{aligned}$$

giving the value of the required Seitz symbol. **2**

(ii) Let $\mathbf{p}$ be the position vector of any point on $\mathcal{L}_1$ and $\mathbf{q}$ be the position vector of the unique point on $\mathcal{L}_2$ for which the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$ is perpendicular to the two lines. This vector is independent of the choice of $\mathbf{p}$.

Now let $\mathbf{x}$ be the position vector of any point. Choosing $\mathbf{p}$ so that $\mathbf{x} = \mathbf{p} + s\mathbf{v}$ for some $s \in \mathbb{R}$, we have

$$\mathbf{x}' = \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{p} - s\mathbf{v}.$$

Then the corresponding $\mathbf{q}$ satisfies $\mathbf{x}' = \mathbf{q} + t\mathbf{v}$ for some $t \in \mathbb{R}$, hence

$$\begin{aligned}\mathbf{x}'' = \mathrm{Refl}_{\mathcal{L}_2}(\mathbf{x}') &= \mathbf{q} - t\mathbf{v} \\ &= \mathbf{q} - (\mathbf{x}' - \mathbf{q}) \\ &= 2\mathbf{q} - (\mathbf{p} - s\mathbf{v}) \\ &= 2\mathbf{q} - \mathbf{p} + (\mathbf{x} - \mathbf{p}) \\ &= 2(\mathbf{q} - \mathbf{p}) + \mathbf{x} = 2\mathbf{v} + \mathbf{x}.\end{aligned}$$

Thus setting $\mathbf{w} = 2\mathbf{v}$ we have

$$\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{x} + \mathbf{w}.$$ **6**

Taking $\mathbf{p} = (0,0)$ the vector $(1,2)$ is normal to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the corresponding point $\mathbf{q} = (u, 2u)$ as above satisfies $u + 2(2u) = 1$, giving $u = 1/5$ and so $\mathbf{q} = (1/5, 2/5)$. Then $\mathbf{v} = (1/5, 2/5)$, and so $\mathbf{w} = (2/5, 4/5)$. **2**

2. (i) We have

$$(1\ 6\ 4)(1\ 4\ 5\ 2)(2\ 5) = (1)(2)(3)(4\ 5\ 6) = (4\ 5\ 6).$$ **3**

(ii) σ has the cycles

$$1 \longrightarrow 5 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1, \quad 4 \longrightarrow 6 \longrightarrow 4,$$

so $\sigma = (1\ 5\ 2\ 3)(4\ 6)$. **3**

(iii) There are 10 symmetries in all. 5 are rotations in anti-clockwise direction:

$$\begin{aligned}(A\ B\ C\ D\ E) &= \text{rotation through } 2\pi/5, \\ (A\ C\ E\ B\ D) &= \text{rotation through } 4\pi/5, \\ (A\ D\ B\ E\ C) &= \text{rotation through } 6\pi/5 = \text{rotation through } -4\pi/5, \\ (A\ E\ D\ C\ B) &= \text{rotation through } 8\pi/5 = \text{rotation through } -2\pi/5, \\ \iota &= \text{rotation through } 0 = \text{the identity}.\end{aligned}$$

Five are reflections in lines through O and a vertex:

$$\begin{aligned}(B\ E)(C\ D) &= \text{reflection in } OA, \\ (A\ C)(D\ E) &= \text{reflection in } OB, \\ (A\ E)(B\ D) &= \text{reflection in } OC, \\ (A\ B)(C\ E) &= \text{reflection in } OD, \\ (A\ D)(B\ C) &= \text{reflection in } OE.\end{aligned}$$

7

We have

$$\Phi = (B\ E)(C\ D), \quad \Theta = (A\ B\ C\ D\ E).$$

Then

$$\Phi \circ \Theta = (B\ E)(C\ D)(A\ B\ C\ D\ E) = (A\ E)(B\ D)(C) = (A\ E)(B\ D),$$

which is reflection in OC. **4**

3. (i) Euc(2) consists of all isometries (distance preserving maps) $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ under composition of functions $\circ$ with the identity function as its identity ι and inversion of functions defining inverses. **2**

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, then

$$\mathrm{Trans}_{\mathbf{u}} \circ \mathrm{Trans}_{\mathbf{v}} = \mathrm{Trans}_{\mathbf{u}+\mathbf{v}},$$

so Trans(2) is closed under $\circ$. Translation by $\mathbf{0}$ $\mathrm{Trans}_{\mathbf{0}}$ agrees with the identity. Finally,

$$\mathrm{Trans}_{\mathbf{u}}^{-1} = \mathrm{Trans}_{-\mathbf{u}},$$

so Trans(2) is closed under taking inverses. Hence Trans(2) $\leqslant$ Euc(2). **3**

(ii) Let the distinct elements of Γ be $F_1, \ldots, F_n$, where $n = |\Gamma|$, the order of Γ. Let $\mathbf{p} \in \mathbb{R}^2$ be the position vector of any point. Define

$$\mathbf{p}_0 = \frac{1}{n} F_1(\mathbf{p}) + \cdots + \frac{1}{n} F_n(\mathbf{p}).$$

For any $k = 1, \ldots, n$, using the quoted result we have

$$F_k(\mathbf{p}_0) = \frac{1}{n} F_k F_1(\mathbf{p}) + \cdots + \frac{1}{n} F_k F_n(\mathbf{p}).$$

Now if $F_k F_i = F_k F_j$, then $F_k^{-1} F_k F_i = F_k^{-1} F_k F_j$ and so $F_i = F_j$. Also, every F_r can be written as $F_r = F_k^{-1}(F_k F_r)$ where $F_k F_r \in \Gamma$ has the form $F_k F_r = F_s$ for some s. So in this expression for $F_k(\mathbf{p}_0)$, the terms are the same as those in the formula for $\mathbf{p}_0$ apart from the order in which they appear. This shows that $F_k(\mathbf{p}_0) = \mathbf{p}_0$. **7**

(iii) $F_1, F_2 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are *similar* if there is a similarity transformation $H \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$F_2 = H \circ F_1 \circ H^{-1}.$$

Here H is a similarity transformation if it is obtained by composing an isometry with a *scaling* or *dilation* from a point and so has Sietz symbol of form $(\delta A \mid \mathbf{s})$, with scaling factor $\delta > 0$, A orthogonal and $\mathbf{s} \in \mathbb{R}^2$. **2**

Let the position vectors of these points be $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{t} = \mathbf{q} - \mathbf{p}$, then setting $H = \mathrm{Trans}_{\mathbf{t}}$, we see that

$$F = H \circ \mathrm{Rot}_{P,\theta} \circ H^{-1}$$

is an isometry and is a rotation since composing a rotation with a translation (in either order) gives a rotation. Then

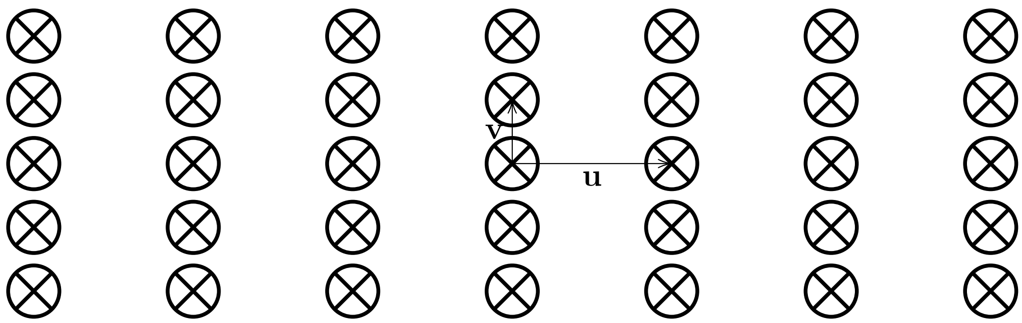
$$\begin{aligned} F(\mathbf{q}) &= H \circ \mathrm{Rot}_{P,\theta}(\mathbf{q} - (\mathbf{q} - \mathbf{p})) \\ &= H(\mathrm{Rot}_{P,\theta}(\mathbf{p})) \\ &= H(\mathbf{p}) = \mathbf{p} + (\mathbf{q} - \mathbf{p}) = \mathbf{q}, \end{aligned}$$

showing that F fixes Q. Hence F is rotation about Q through θ. **3**

4. (i) (a) The vectors $\mathbf{u} = (5, 0)$ and $\mathbf{v} = (0, 2)$ generate the lattice of centres L since $(5m, 2n) = m\mathbf{u} + n\mathbf{v}$. So the translation subgroup is

$$\begin{aligned}\mathrm{Trans}(2)_{\mathbb{W}} &= \{\mathrm{Trans}_{m\mathbf{u}+n\mathbf{v}} : m, n \in \mathbb{Z}\} \\ &= \{(\mathrm{Trans}_{\mathbf{u}})^m (\mathrm{Trans}_{\mathbf{v}})^n : m, n \in \mathbb{Z}\},\end{aligned}$$

and it is generated by translations by the vectors $\mathbf{u}$ and $\mathbf{v}$. **3**



(b) There are reflections in the x and y-axes, R_x, R_y. These compose to give a rotation through half a turn, $R_x \circ R_y = R_y \circ R_x$. There are no other rotational symmetries about O since the lattice L is rectangular and not square. Thus

$$\mathrm{Euc}(2)_{\mathbb{W},O} = \{\mathrm{Id}, R_x, R_y, R_x \circ R_y\}.$$ **4**

(c) Every element is obtained by composing a translational symmetry with a symmetry fixing O. Hence

$$\begin{aligned}\mathrm{Euc}(2)_{\mathbb{W}} = \mathrm{Trans}(2)_{\mathbb{W}} &\cup \{R_x \circ T : T \in \mathrm{Trans}(2)_{\mathbb{W}}\} \\ &\cup \{R_y \circ T : T \in \mathrm{Trans}(2)_{\mathbb{W}}\} \cup \{R_x \circ R_y \circ T : T \in \mathrm{Trans}(2)_{\mathbb{W}}\}.\end{aligned}$$

3

(ii) The matrix

$$\begin{aligned}A &= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos\pi/4 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & 0 & \sin(-\pi/4) \\ 0 & 1 & 0 \\ -\sin(-\pi/4) & 0 & \cos(-\pi/4) \end{bmatrix}\end{aligned}$$

is orthogonal and has determinant

$$\det A = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = \frac{1}{2} - \frac{-1}{2} = 1$$

so it corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin. **2**

The vector $\mathbf{e}_2 = (0, 1, 0)$ is fixed by A. Furthermore,

$$A\mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}(-\mathbf{e}_3) = \cos(-\pi/4)\mathbf{e}_1 + \sin(-\pi/4)\mathbf{e}_3,$$
$$A(-\mathbf{e}_3) = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}(-\mathbf{e}_3) = -\sin(-\pi/4)\mathbf{e}_1 + \cos(-\pi/4)(-\mathbf{e}_3).$$

So A represents a rotation through $(-\pi/4)$ about the y-axis, where the positive sense involves turning the xz-plane about the y-axis by rotating the x-axis towards the $(-z)$-axis. This is the same as rotating through $\pi/4$ about the y-axis, turning the xz-plane about the y-axis by rotating the x-axis towards the z-axis, so according to the motion of a left hand screwdriver along the y-axis. **5**

END]