Tuesday, 28th May, 2002 *4 p.m. to 5.30 p.m.*

University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2Q - Groups, Symmetry and Fractals

Candidates must not attempt more than THREE questions.

1. (i) Given that for some $\alpha, \beta \in \mathbb{R}$,

$$S = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}, \quad T = \begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix},$$

show that the Seitz symbol $(S \mid \mathbf{0})$ represents the reflection in the line

$$\mathcal{L}_\alpha = \{(x, y) : x \sin \alpha - y \cos \alpha = 0\}.$$

5

Determine the Seitz symbol $(S \mid \mathbf{0})(T \mid \mathbf{0})$ and describe the isometry it represents.

3

(ii) List the four types of isometries of the plane. **2**

For the matrices

$$A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and any vector $\mathbf{t}$, state what kind of isometries are represented by each of the Seitz symbols $(A \mid \mathbf{t})$ and $(B \mid \mathbf{0})$. **3**

If $\mathbf{t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, determine the Seitz symbol of the composition $(B \mid \mathbf{0})(A \mid \mathbf{t})$ and describe the geometric effect of the isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ it represents. **7**

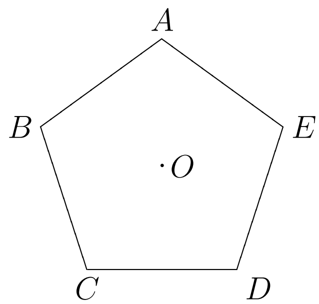
2. (i) In the symmetric group S_6,

   (a) evaluate the product $(2\ 6)(2\ 4\ 6)(1\ 3\ 6\ 5)$, expressing the answer in disjoint cycle form; **3**

   (b) determine the disjoint cycle decomposition of

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 6 & 1 & 4 \end{pmatrix}.$$ **3**

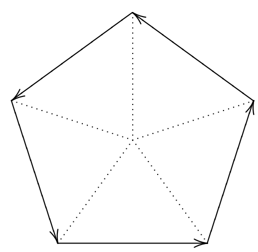
(ii) Let Γ be the group of symmetries of a regular pentagon, centred at the origin O and with vertices A, B, C, D, E.



By identifying Γ with a group of permutations of the vertices, describe the elements of Γ both geometrically and using permutation notation. **7**

Let Φ denote reflection in the line OA and Θ rotation through $2\pi/5$ in the anti-clockwise direction about O. Determine the composition $\Phi \circ \Theta$ and describe its effect geometrically. **4**

(iii) Explain how the symmetry group of the following regular pentagonal figure can be determined from that of part (ii). **3**



3. (i) Define the Euclidean group $(\mathrm{Euc}(2), \circ)$ of the plane. **2**

Show that the subset $\mathrm{O}(2) \subseteq \mathrm{Euc}(2)$ consisting of isometries that fix the origin is a subgroup of the Euclidean group $(\mathrm{Euc}(2), \circ)$. **5**

(ii) Let $\Gamma \leqslant \mathrm{Euc}(2)$ be a *finite* subgroup. Show that there is a point in the plane fixed by every element of Γ. **7**

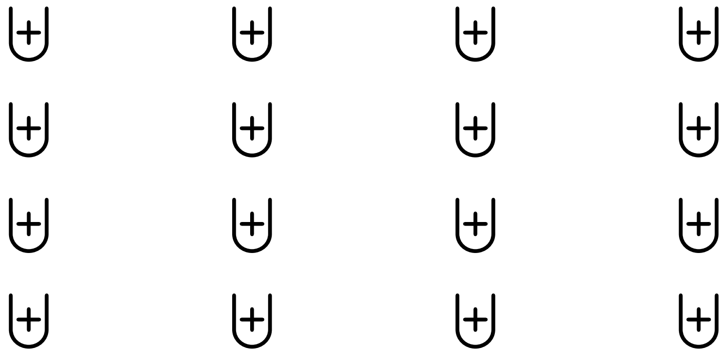
You may use, without proof, the following result:
For $t_1, \ldots, t_k \in \mathbb{R}$ *satisfying* $t_1 + \cdots + t_k = 1$ *and* $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^2$, *an isometry* $F \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *satisfies*

$$F(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = t_1 F(\mathbf{v}_1) + \cdots + t_k F(\mathbf{v}_k).$$

(iii) Define what is meant by describing two isometries of the plane to be *similar*. **2**

Show that the reflections in any two *non-parallel* lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are similar. **4**

4. (i) In a wallpaper pattern $\mathcal{W}$, part of which is shown below, the centres of the $\uplus$ symbols are located at positions $(7m, 3n)$ for values $m, n \in \mathbb{Z}$.



(a) Describe the group $\mathrm{Trans}(2)_{\mathcal{W}}$ of *translational symmetries* of $\mathcal{W}$, giving the answer in terms of a pair of generating vectors $\mathbf{u}$ and $\mathbf{v}$. **3**

(b) Determine the group of symmetries of $\mathcal{W}$ which fix the origin. **4**

(c) Describe the elements of the symmetry subgroup $\mathrm{Euc}(2)_{\mathcal{W}} \leqslant \mathrm{Euc}(2)$. **3**

(d) If $\uplus$ was replaced by $\otimes$, what extra symmetries fixing the origin would there be? **2**

(ii) Explain why the matrix

$$R = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix},$$

corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin. **3**

Determine the axis and angle of rotation for R. **5**

END]

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2Q Degree exam 2001–2 – Solutions

1. (i) We have

$$S\begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix} = \begin{bmatrix}\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha\\ \sin 2\alpha\cos\alpha - \cos 2\alpha\sin\alpha\end{bmatrix} = \begin{bmatrix}\cos(2\alpha-\alpha)\\ \sin(2\alpha-\alpha)\end{bmatrix} = \begin{bmatrix}\cos\alpha\\ \sin\alpha\end{bmatrix},$$
$$S\begin{bmatrix}\sin\alpha\\ -\cos\alpha\end{bmatrix} = \begin{bmatrix}\cos 2\alpha\sin\alpha - \sin 2\alpha\cos\alpha\\ \sin 2\alpha\sin\alpha + \cos 2\alpha\sin\alpha\end{bmatrix} = \begin{bmatrix}-\sin(2\alpha-\alpha)\\ \cos(2\alpha-\alpha)\end{bmatrix} = -\begin{bmatrix}\sin\alpha\\ -\cos\alpha\end{bmatrix}.$$

This shows that $(S \mid \mathbf{0})$ has the effect

$$(S \mid \mathbf{0})\mathbf{u} = \mathbf{u}, \quad (S \mid \mathbf{0})\mathbf{v} = -\mathbf{v}.$$

on the perpendicular unit vectors $\mathbf{u} = (\cos\alpha, \sin\alpha)$ and $\mathbf{v} = (\sin\alpha, -\cos\alpha)$. Since $\mathbf{u}$ is parallel to the line $\mathcal{L}_\alpha$ and $\mathbf{v}$ is perpendicular to it, this isometry is the reflection in $\mathcal{L}_\alpha$. (Similarly, $(T \mid \mathbf{0})$ represents reflection in the line $\mathcal{L}_\beta$.) **5**

We have $(S \mid \mathbf{0})(T \mid \mathbf{0}) = (ST \mid \mathbf{0})$ with

$$\begin{aligned} ST &= \begin{bmatrix}\cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta & \cos 2\alpha\sin 2\beta - \sin 2\alpha\cos 2\beta\\ \sin 2\alpha\cos 2\beta - \cos 2\alpha\sin 2\beta & \cos 2\alpha\cos 2\beta + \sin 2\alpha\sin 2\beta\end{bmatrix}\\ &= \begin{bmatrix}\cos 2(\alpha-\beta) & -\sin 2(\alpha-\beta)\\ \sin 2(\alpha-\beta) & \cos 2(\alpha-\beta)\end{bmatrix}\end{aligned}$$

This represents the clockwise rotation through $2(\alpha-\beta)$ about the origin. **3**

(ii) *Translation, rotation, reflection, glide reflection.* **2**

Since

$$A = \begin{bmatrix}\cos(\pi/4) & \sin(\pi/4)\\ \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix} = \begin{bmatrix}\cos(2\pi/8) & \sin(2\pi/8)\\ \sin(2\pi/8) & -\cos(2\pi/8)\end{bmatrix}, \quad \det A = -1,$$

$(A \mid \mathbf{0})$ represents reflection in the line

$$\mathcal{L}_\alpha = \{(x,y) : \sin(\pi/8)\,x - \cos(\pi/8)\,y = 0\}.$$

So $(A \mid \mathbf{t})$ is a glide reflection consisting of reflection in a line parallel to $\mathcal{L}_\alpha$ followed by a translation parallel to it. Since

$$B = \begin{bmatrix}\cos 2(-\pi/4) & \sin 2(-\pi/4)\\ \sin 2(-\pi/4) & -\cos 2(-\pi/4)\end{bmatrix}, \quad \det B = -1,$$

$(B \mid \mathbf{0})$ represents reflection in the line

$$\begin{aligned}\{(x,y) : \sin(-\pi/4)\,x - \cos(-\pi/4)\,y = 0\} &= \{(x,y) : (-1/\sqrt{2})x - (1/\sqrt{2})\,y = 0\}\\ &= \{(x,y) : x + y = 0\}.\end{aligned}$$

3

We have $(B \mid \mathbf{0})(A \mid \mathbf{t}) = (BA \mid B\mathbf{t})$ with

$$BA = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad \det(BA) = 1, \quad B\mathbf{t} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

So this Seitz symbol represents a rotation through the angle $-\dfrac{\pi}{2} - \dfrac{\pi}{4} = -\dfrac{3\pi}{4}$ about the point with position vector

$$\begin{aligned}
(I - BA)^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1 + 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1 + 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \frac{1}{(1 + 1/\sqrt{2})^2 + 1/2} \begin{bmatrix} 1 + 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 + 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
&= \frac{1}{1 + 1/2 + \sqrt{2} + 1/2} \begin{bmatrix} -1/\sqrt{2} \\ -1 - 1/\sqrt{2} \end{bmatrix} \\
&= \frac{1}{2 + \sqrt{2}} \begin{bmatrix} -1/\sqrt{2} \\ -1 - 1/\sqrt{2} \end{bmatrix} = \frac{\sqrt{2} - 1}{2} \begin{bmatrix} -1 \\ -\sqrt{2} - 1 \end{bmatrix} = \begin{bmatrix} (1 - \sqrt{2})/2 \\ -1/2 \end{bmatrix}.
\end{aligned}$$

7

2. (i) (a) We have $(2\ 6)(2\ 4\ 6)(1\ 3\ 6\ 5) = (1\ 3\ 6\ 5)(2\ 4)$. **3**

(b) σ has the cycles

$$1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 1, \quad 2 \longrightarrow 2, \quad 4 \longrightarrow 6 \longrightarrow 4,$$

so $\sigma = (1\ 3\ 5)(2)(4\ 6) = (1\ 3\ 5)(4\ 6)$. **3**

(ii) There are 10 symmetries in all. 5 are rotations in the anti-clockwise direction:

$$\begin{aligned}
(A\ B\ C\ D\ E) &= \text{rotation through } 2\pi/5, \\
(A\ C\ E\ B\ D) &= \text{rotation through } 4\pi/5, \\
(A\ D\ B\ E\ C) &= \text{rotation through } 6\pi/5 = \text{rotation through } -4\pi/5, \\
(A\ E\ D\ C\ B) &= \text{rotation through } 8\pi/5 = \text{rotation through } -2\pi/5, \\
\iota &= \text{rotation through } 0 = \text{the identity}.
\end{aligned}$$

Five are reflections in lines through O and a vertex:

$$\begin{aligned}
(B\ E)(C\ D) &= \text{reflection in } OA, \\
(A\ C)(D\ E) &= \text{reflection in } OB, \\
(A\ E)(B\ D) &= \text{reflection in } OC, \\
(A\ B)(C\ E) &= \text{reflection in } OD, \\
(A\ D)(B\ C) &= \text{reflection in } OE.
\end{aligned}$$

7

We have

$$\Phi = (B\ E)(C\ D), \quad \Theta = (A\ B\ C\ D\ E).$$

Then

$$\Phi \circ \Theta = (B\ E)(C\ D)(A\ B\ C\ D\ E) = (A\ E)(B\ D)(C) = (A\ E)(B\ D),$$

which is reflection in OC. **4**

(iii) This symmetry group is essentially the subgroup of Γ consisting of symmetries which preserve the directions of the arrows, *i.e.*, the 5 rotations. **3**

3. (i) Euc(2) consists of all isometries (distance preserving maps) $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ under composition of functions $\circ$ with the identity function as its identity ι and inversion of functions defining inverses. **2**

For $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{O}(2)$ we have

$$(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0})$$

and

$$(AB)^T(AB) = (B^T A^T)(AB) = B^T(A^T A)B = B^T I_2 B = B^T B = I_2.$$

So $(A \mid \mathbf{0})(B \mid \mathbf{0}) \in \mathrm{O}(2)$. Also, $(I_2 \mid \mathbf{0}) \in \mathrm{O}(2)$ and

$$(A \mid \mathbf{0})^{-1} = (A^{-1} \mid \mathbf{0}) \in \mathrm{O}(2)$$

since $A^{-1} = A^T$ and

$$(A^T)^T(A^T) = AA^T = AA^{-1} = I_2,$$

hence A^{-1} is orthogonal. **5**

(ii) Let the distinct elements of Γ be $F_1, \ldots, F_n$, where $n = |\Gamma|$ is the order of Γ. Let $\mathbf{p} \in \mathbb{R}^2$ be the position vector of any point. Define

$$\mathbf{p}_0 = \frac{1}{n}F_1(\mathbf{p}) + \cdots + \frac{1}{n}F_n(\mathbf{p}).$$

For any $k = 1, \ldots, n$, using the quoted result we have

$$F_k(\mathbf{p}_0) = \frac{1}{n}F_kF_1(\mathbf{p}) + \cdots + \frac{1}{n}F_kF_n(\mathbf{p}).$$

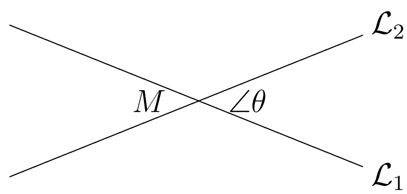
Now if $F_kF_i = F_kF_j$, then $F_k^{-1}F_kF_i = F_k^{-1}F_kF_j$ and so $F_i = F_j$. Also, every F_r can be written as $F_r = F_k(F_k^{-1}F_r)$ where $F_k^{-1}F_r \in \Gamma$ has the form $F_k^{-1}F_r = F_s$ for some s and therefore $F_r = F_kF_s$. So in the above expression for $F_k(\mathbf{p}_0)$, the terms are the same as those in the formula for $\mathbf{p}_0$ apart from the order in which they appear. This shows that $F_k(\mathbf{p}_0) = \mathbf{p}_0$. **7**

(iii) $F_1, F_2 \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are *similar* if there is a similarity transformation $H \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that

$$F_2 = H \circ F_1 \circ H^{-1}.$$

Here H is a similarity transformation if it is obtained by composing an isometry with a *scaling* or *dilation* from a point and so has Sietz symbol of form $(\delta A \mid \mathbf{s})$, with scaling factor $\delta > 0$, A orthogonal and $\mathbf{s} \in \mathbb{R}^2$. **2**

Since the lines are not parallel, they meet at some point M.



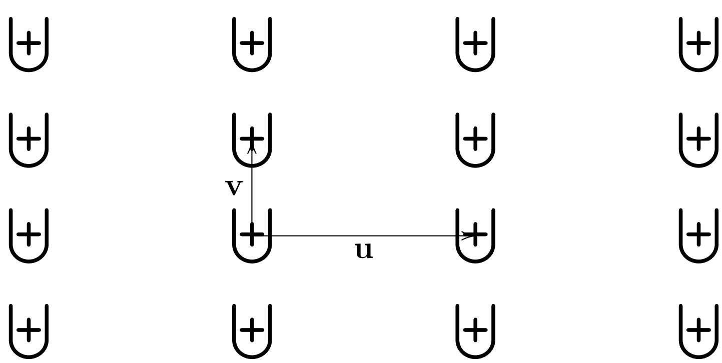
There is then a rotation about M, $\mathrm{Rot}_{M,\theta}$ say, which maps $\mathcal{L}_1$ into $\mathcal{L}_2$. Then

$$\mathrm{Refl}_{\mathcal{L}_2} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Rot}_{M,-\theta} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ {\mathrm{Rot}_{M,\theta}}^{-1}.$$ **4**

4. (i) (a) The vectors $\mathbf{u} = (7, 0)$ and $\mathbf{v} = (0, 3)$ generate the lattice of centres L since $(7m, 3n) = m\mathbf{u} + n\mathbf{v}$. So the translation subgroup is

$$\begin{aligned} \mathrm{Trans}(2)_{\mathcal{W}} &= \{\mathrm{Trans}_{m\mathbf{u}+n\mathbf{v}} : m, n \in \mathbb{Z}\} \\ &= \{(\mathrm{Trans}_{\mathbf{u}})^m (\mathrm{Trans}_{\mathbf{v}})^n : m, n \in \mathbb{Z}\} \end{aligned}$$

and it is generated by translations by the vectors $\mathbf{u}$ and $\mathbf{v}$. **3**



(b) There is a reflection in the y-axis, R_y but not in the x-axis. Since L is rectangular and not square, there are no other rotational symmetries about O since the lattice. Thus

$$\mathrm{Euc}(2)_{\mathcal{W},O} = \{\mathrm{Id}, R_y\}.$$ **4**

(c) Every element is obtained by composing a symmetry fixing O with a translational symmetry. Hence

$$\mathrm{Euc}(2)_{\mathcal{W}} = \mathrm{Trans}(2)_{\mathcal{W}} \cup \{T \circ R_y : T \in \mathrm{Trans}(2)_{\mathcal{W}}\}.$$ **3**

(d) There would be the reflection in the x-axis as well as the composition of the reflections in the x and y-axes which is equal to $-\,\mathrm{Id}$. **2**

(ii) Expanding along the middle row we obtain

$$\det R = \begin{vmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{vmatrix} = 1,$$

while R is orthogonal since

$$R^T R = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & 1/2 \end{bmatrix} = I_3.$$

Hence R represents a rotation about an axis through the origin. **3**

As $R\mathbf{e}_2 = \mathbf{e}_2$, the axis of rotation is the y-axis. Now let

$$\mathbf{w} = \mathbf{e}_2, \quad \mathbf{u} = \mathbf{e}_1, \quad \mathbf{v} = \mathbf{w} \times \mathbf{u} = \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3.$$

So $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a right handed orthonormal system. We have

$$R\mathbf{u} = (1/2)\mathbf{u} + (\sqrt{3}/2)\mathbf{v} = \cos(\pi/3)\mathbf{u} + \sin(\pi/3)\mathbf{v},$$
$$R\mathbf{v} = (-\sqrt{3}/2)\mathbf{u} + (1/2)\mathbf{v} = -\sin(\pi/3)\mathbf{u} + \cos(\pi/3)\mathbf{v},$$

so the angle of rotation is $\pi/3$ and the sense is the same as the turning of a right handed screw driver pointing along the positive y-axis. **5**

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