University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2Q - Groups, Symmetry and Fractals

Candidates must not attempt more than THREE questions.

1. (i) Given that

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

describe the geometric effect of the isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ represented by each of the Seitz symbols $(A \mid \mathbf{0})$ and $(B \mid \mathbf{t})$. **7**

Determine the Seitz symbol of the composition $(A \mid \mathbf{0})(B \mid \mathbf{t})$ and describe briefly the type of isometry it represents. **5**

(ii) Show that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two parallel lines in the plane, the composition $\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}$ of the reflections in these lines is a translation $\mathrm{Trans}_{\mathbf{w}}$ for some vector $\mathbf{w}$. **6**

Determine the translation vector $\mathbf{w}$ when the lines are

$$\mathcal{L}_1 = \{(x, y) : x - 2y = 0\}, \quad \mathcal{L}_2 = \{(x, y) : x - 2y = 1\}.$$ **2**

2. (i) In the symmetric group S_6,

   (a) evaluate the product $\alpha = (1\ 6\ 5)(2\ 3\ 6)(3\ 4\ 6)$, expressing the answer in disjoint cycle form; **3**

   (b) determine the disjoint cycle decomposition of

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 5 & 2 & 3 & 4 \end{pmatrix};$$ **3**

   (c) determine $\operatorname{sgn} \alpha\beta$, the sign of the permutation $\alpha\beta$. **3**

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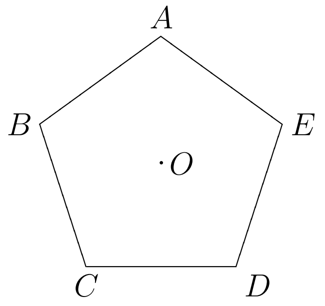
(ii) Let Γ be the group of symmetries of a regular pentagon, centred at the origin O and with vertices A, B, C, D, E.



By identifying Γ with a group of permutations of the vertices, describe the elements of Γ both geometrically and using permutation notation. **7**

Let Φ denote reflection in the line OA and let Θ denote rotation through $2\pi/5$ in the anti-clockwise direction about O. Determine the composition $\Theta \circ \Phi$ and describe its effect geometrically. **4**

3. (i) Define the Euclidean group $(\mathrm{Euc}(2), \circ)$. **2**

Show that the subset $\mathrm{O}(2) \subseteq \mathrm{Euc}(2)$ consisting of isometries that fix the origin is a subgroup of the Euclidean group $(\mathrm{Euc}(2), \circ)$. **3**

Explain briefly why the subset $\mathrm{SO}(2) \subseteq \mathrm{O}(2)$ consisting of rotations is also a subgroup. **3**

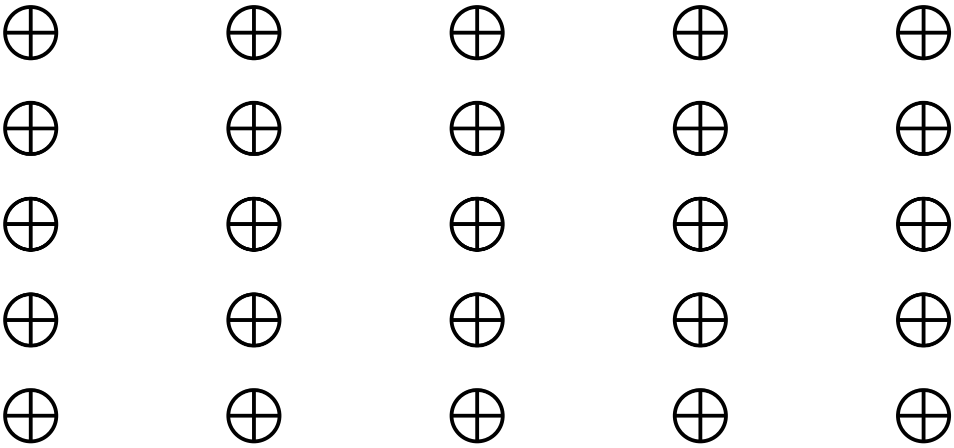
(ii) Suppose that $F \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is an isometry with Seitz symbol $(A \mid \mathbf{t})$. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $s \in \mathbb{R}$, show that

$$F(s\mathbf{x} + (1-s)\mathbf{y}) = sF(\mathbf{x}) + (1-s)F(\mathbf{y}).$$ **5**

(iii) State what it means for two isometries of the plane to be *similar.* **2**

Show that the reflections in any two *non-parallel* lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are similar. **5**

4. (i) In a wallpaper pattern $\mathbb{W}$, part of which is shown below, the centres of the $\bigoplus$ symbols are located at positions $(7m, 3n)$ for values $m, n \in \mathbb{Z}$.



(a) Describe the group $\mathrm{Trans}(2)_{\mathbb{W}}$ of *translational symmetries* of $\mathbb{W}$, giving the answer in terms of a pair of generating vectors $\mathbf{u}$ and $\mathbf{v}$. **3**

(b) Determine the group of symmetries of $\mathbb{W}$ which fix the origin. **4**

(c) Describe the elements of the symmetry subgroup $\mathrm{Euc}(2)_{\mathbb{W}} \leqslant \mathrm{Euc}(2)$. **3**

(d) Explain why there are infinitely many glide reflections in $\mathrm{Euc}(2)_{\mathbb{W}}$. **3**

(ii) Explain why the matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin. **2**

Determine the axis and angle of rotation for A. **5**

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Wednesday, 4*th June*, 2003 2.30 *p.m. to* 4.00 *p.m.*

2Q Degree exam 2002–3 – Solutions

1. (i) $(A \mid \mathbf{0})$ represents reflection in the y-axis. **2**

Since

$$B = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix},$$

this represents a rotation about the origin through $\pi/4$. So $(B \mid \mathbf{t})$ represents a rotation through $\pi/4$ about the point with position vector

$$\begin{aligned} \mathbf{c} &= (I - B)^{-1}\mathbf{t} \\ &= \begin{bmatrix} 1 - 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1 - 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} - 1 & 1 \\ -1 & \sqrt{2} - 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{\sqrt{2}}{2(2 - \sqrt{2})} \begin{bmatrix} \sqrt{2} - 1 & -1 \\ 1 & \sqrt{2} - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \frac{(\sqrt{2} + 1)}{2} \begin{bmatrix} \sqrt{2} - 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ (\sqrt{2} + 1)/2 \end{bmatrix}. \end{aligned}$$

Hence the centre of rotation is $\mathbf{c} = (1/2, (\sqrt{2} + 1)/2)$. **5**

We have

$$(A \mid \mathbf{0})(B \mid \mathbf{t}) = (AB \mid A\mathbf{t}).$$

Here we have

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos 3\pi/4 & \sin 3\pi/4 \\ \sin 3\pi/4 & -\cos 3\pi/4 \end{bmatrix}, \\ A\mathbf{t} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \end{aligned}$$

giving the value of the required Seitz symbol. **3**

This represents a glide reflection in a line parallel to the line through the origin and making the angle $3\pi/8$ with the x-axis. **2**

(ii) Let $\mathbf{p}$ be the position vector of any point on $\mathcal{L}_1$ and $\mathbf{q}$ be the position vector of the unique point on $\mathcal{L}_2$ for which the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$ is perpendicular to the two lines. This vector is independent of the choice of $\mathbf{p}$.

Now let $\mathbf{x}$ be the position vector of any point. Choosing $\mathbf{p}$ so that $\mathbf{x} = \mathbf{p} + s\mathbf{v}$ for some $s \in \mathbb{R}$, we have

$$\mathbf{x}' = \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{p} - s\mathbf{v}.$$

Then the corresponding $\mathbf{q}$ satisfies $\mathbf{x}' = \mathbf{q} + t\mathbf{v}$ for some $t \in \mathbb{R}$, hence

$$\begin{aligned}\mathbf{x}'' = \mathrm{Refl}_{\mathcal{L}_2}(\mathbf{x}') &= \mathbf{q} - t\mathbf{v} \\ &= \mathbf{q} - (\mathbf{x}' - \mathbf{q}) \\ &= 2\mathbf{q} - (\mathbf{p} - s\mathbf{v}) \\ &= 2\mathbf{q} - \mathbf{p} + (\mathbf{x} - \mathbf{p}) \\ &= 2(\mathbf{q} - \mathbf{p}) + \mathbf{x} = 2\mathbf{v} + \mathbf{x}.\end{aligned}$$

Thus setting $\mathbf{w} = 2\mathbf{v}$ we have

$$\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{x} + \mathbf{w}.$$ **6**

Taking $\mathbf{p} = (0,0)$, the vector $(1,-2)$ is normal to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the corresponding point $\mathbf{q} = (u,-2u)$ as above satisfies $u - 2(-2u) = 1$, giving $u = 1/5$ and so $\mathbf{q} = (1/5,-2/5)$. Then $\mathbf{v} = (1/5,-2/5)$, and so $\mathbf{w} = (2/5,-4/5)$. **2**

2. (i) (a) We have $\alpha = (1\ 6\ 5)(2\ 3\ 4)$. **3**

(b) β has the cycles

$$1 \longrightarrow 6 \longrightarrow 4 \longrightarrow 2 \longrightarrow 1, \quad 3 \longrightarrow 5 \longrightarrow 3,$$

so $\beta = (1\ 6\ 4\ 2)(3\ 5)$. **3**

(c) Since $\mathrm{sgn}\,\sigma\tau = \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau$, we have $\mathrm{sgn}\,\alpha\beta = \mathrm{sgn}\,\alpha\,\mathrm{sgn}\,\beta$ where

$$\mathrm{sgn}\,\alpha = \mathrm{sgn}(1\ 6\ 5)\,\mathrm{sgn}(2\ 3\ 4) = 1, \mathrm{sgn}\,\beta = \mathrm{sgn}(1\ 6\ 4\ 2)\,\mathrm{sgn}(3\ 5) = (-1)(-1) = 1.$$

Hence $\mathrm{sgn}\,\alpha\beta = 1$. **3**

(ii) There are 10 symmetries in all. 5 are rotations in the anti-clockwise direction:

$$\begin{aligned}(A\ B\ C\ D\ E) &= \text{rotation through } 2\pi/5, \\ (A\ C\ E\ B\ D) &= \text{rotation through } 4\pi/5, \\ (A\ D\ B\ E\ C) &= \text{rotation through } 6\pi/5 = \text{rotation through } -4\pi/5, \\ (A\ E\ D\ C\ B) &= \text{rotation through } 8\pi/5 = \text{rotation through } -2\pi/5, \\ \iota &= \text{rotation through } 0 = \text{the identity}.\end{aligned}$$

Five are reflections in lines through O and a vertex:

$$\begin{aligned}(B\ E)(C\ D) &= \text{reflection in } OA, \\ (A\ C)(D\ E) &= \text{reflection in } OB, \\ (A\ E)(B\ D) &= \text{reflection in } OC, \\ (A\ B)(C\ E) &= \text{reflection in } OD, \\ (A\ D)(B\ C) &= \text{reflection in } OE.\end{aligned}$$

7

We have

$$\Phi = (B\ E)(C\ D), \quad \Theta = (A\ B\ C\ D\ E).$$

Then

$$\Theta \circ \Phi = (A\ B\ C\ D\ E)(B\ E)(C\ D) = (A\ B)(C\ E)(D) = (A\ B)(C\ E),$$

which is reflection in OD. **4**

3. (i) The Euclidean group $(\mathrm{Euc}(2), \circ)$ consists of all isometries $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ under composition of functions. **2**

For $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{O}(2)$ we have

$$(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0})$$

and

$$(AB)^T(AB) = (B^T A^T)(AB) = B^T(A^T A)B = B^T I_2 B = B^T B = I_2.$$

So $(A \mid \mathbf{0})(B \mid \mathbf{0}) \in \mathrm{O}(2)$. Also, $(I_2 \mid \mathbf{0}) \in \mathrm{O}(2)$ and

$$(A \mid \mathbf{0})^{-1} = (A^{-1} \mid \mathbf{0}) \in \mathrm{O}(2)$$

since $A^{-1} = A^T$ and

$$(A^T)^T(A^T) = AA^T = AA^{-1} = I_2,$$

hence A^{-1} is orthogonal. **3**

$(A \mid \mathbf{0}) \in \mathrm{O}(2)$ represents a rotation if and only if $\det A = 1$. If $(A \mid \mathbf{0}), (B \mid \mathbf{0}) \in \mathrm{SO}(2)$, then

$$(A \mid \mathbf{0})(B \mid \mathbf{0}) = (AB \mid \mathbf{0}) \in \mathrm{SO}(2)$$

since $\det(AB) = \det A \det B = 1$. Also, $\det I = 1$. Finally,

$$\det A^{-1} = \det A^T = \det A = 1.$$

Hence $\mathrm{SO}(2)$ is also a subgroup. **3**

(ii) We have

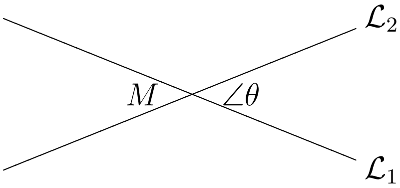
$$\begin{aligned}(A \mid \mathbf{t})(s\mathbf{x} + (1-s)\mathbf{y}) &= A(s\mathbf{x} + (1-s)\mathbf{y}) + \mathbf{t} \\ &= A(s\mathbf{x}) + A((1-s)\mathbf{y}) + \mathbf{t} \\ &= sA\mathbf{x} + A(1-s)\mathbf{y} + s\mathbf{t} + (1-s)\mathbf{t} \\ &= s(A \mid \mathbf{t})\mathbf{x} + (1-s)(A \mid \mathbf{t})\mathbf{y},\end{aligned}$$

giving the desired result. **5**

(iii) Two isometries of the plane F, G are *similar* if there is a similarity transformation $H = (\delta A \mid \mathbf{t})$ with $\delta > 0$, A orthogonal and $\mathbf{t} \in \mathbb{R}^2$, for which

$$G = H_* F = H \circ F \circ H^{-1}. \qquad \mathbf{2}$$

If the lines are not parallel they meet at a point M say. Then there is a rotation about M, $\mathrm{Rot}_{M,\theta}$ say, which sends $\mathcal{L}_1$ into $\mathcal{L}_2$.



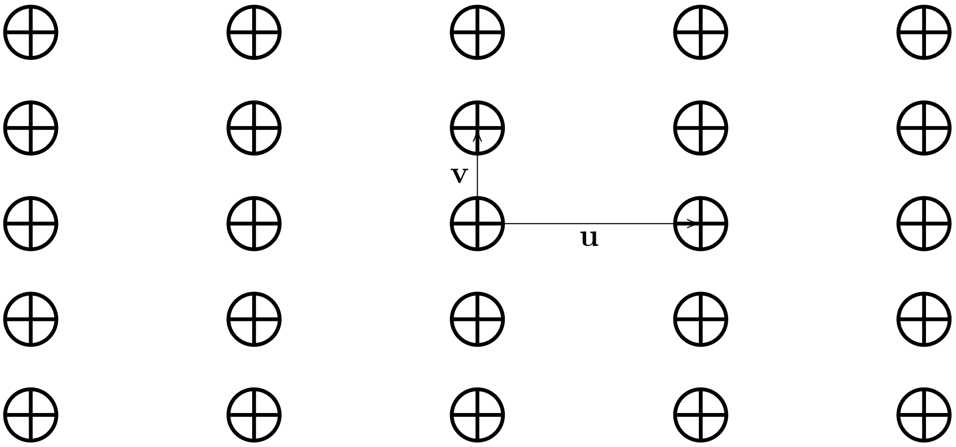
Then

$$\mathrm{Refl}_{\mathcal{L}_2} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Rot}_{M,-\theta} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ {\mathrm{Rot}_{M,\theta}}^{-1}. \qquad \mathbf{5}$$

4. (i) (a) Since $(7m, 3n) = m\mathbf{u} + n\mathbf{v}$, the vectors $\mathbf{u} = (7,0)$ and $\mathbf{v} = (0,3)$ generate the lattice of centres L. So the translation subgroup is

$$\mathrm{Trans}(2)_{\mathbb{W}} = \{\mathrm{Trans}_{m\mathbf{u}+n\mathbf{v}} : m,n \in \mathbb{Z}\} = \{(\mathrm{Trans}_{\mathbf{u}})^m (\mathrm{Trans}_{\mathbf{v}})^n : m,n \in \mathbb{Z}\}$$

and it is generated by translations by the two vectors $\mathbf{u}$ and $\mathbf{v}$. **3**



(b) There are reflections in the x and y-axes, R_x, R_y. These compose to give a rotation through half a turn, $R_x \circ R_y = R_y \circ R_x$. There are no other rotational symmetries about O since the lattice L is rectangular and not square. Thus

$$\mathrm{Euc}(2)_{\mathbb{W},O} = \{\mathrm{Id}, R_x, R_y, R_x \circ R_y\}. \qquad \mathbf{4}$$

(c) Every element is obtained by composing a symmetry that fixes O with a translational symmetry. Hence

$$\begin{aligned}\mathrm{Euc}(2)_{\mathbb{W}} = \mathrm{Trans}(2)_{\mathbb{W}} &\cup \{T \circ R_x : T \in \mathrm{Trans}(2)_{\mathbb{W}}\} \\ &\cup \{T \circ R_y : T \in \mathrm{Trans}(2)_{\mathbb{W}}\} \cup \{T \circ R_x \circ R_y : T \in \mathrm{Trans}(2)_{\mathbb{W}}\}.\end{aligned}$$

3

(d) Each symmetry of the form $\mathrm{Trans}_{n\mathbf{v}} \circ R_y$ with $0 \neq n \in \mathbb{Z}$ is a glide reflection in a line parallel to the y-axis, while $\mathrm{Trans}_{n\mathbf{u}} \circ R_x$ is a glide reflection in a line parallel to the x-axis. **3**

(ii) The matrix

$$A = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \cos\pi/4 & 0 & -\sin\pi/4 \\ 0 & 1 & 0 \\ \sin\pi/4 & 0 & \cos\pi/4 \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & 0 & \sin(-\pi/4) \\ 0 & 1 & 0 \\ -\sin(-\pi/4) & 0 & \cos(-\pi/4) \end{bmatrix}$$

is orthogonal and has determinant

$$\det A = \begin{vmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = \frac{1}{2} - \frac{-1}{2} = 1$$

so it corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin. **2**

The vector $\mathbf{e}_2 = (0,1,0)$ is fixed by A. Furthermore,

$$\begin{aligned} A\mathbf{e}_1 &= \frac{1}{\sqrt{2}}\mathbf{e}_1 - \frac{1}{\sqrt{2}}(-\mathbf{e}_3) = \cos(-\pi/4)\mathbf{e}_1 + \sin(-\pi/4)(-\mathbf{e}_3), \\ A(-\mathbf{e}_3) &= \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}(-\mathbf{e}_3) = -\sin(-\pi/4)\mathbf{e}_1 + \cos(-\pi/4)(-\mathbf{e}_3). \end{aligned}$$

So A represents a rotation through $(-\pi/4)$ about the y-axis, where the positive sense involves turning the xz-plane about the y-axis by rotating the x-axis towards the $(-z)$-axis. This is the same as rotating through $\pi/4$ about the y-axis, turning the xz-plane about the y-axis by rotating the x-axis towards the z-axis, so according to the motion of a left hand screwdriver along the y-axis. **5**

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