Friday, 5*th September*, 2003 2.30*pm to* 4*pm*

University of Glasgow

EXAMINATION FOR THE DEGREES OF
M.A. AND B.Sc.

Mathematics 2Q - Groups, Symmetry and Fractals

Candidates must not attempt more than THREE questions.

1. (i) List the four types of isometries of the plane and for each them describe the form of the Seitz symbol of such an isometry. **4**

(ii) Let

$$A = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ -1/2 & -\sqrt{3}/2 \end{bmatrix}, \quad \mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Determine the geometric effect of the isometry $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ represented by the Seitz symbol $(A \mid \mathbf{t})$. **8**

(iii) Show that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two parallel lines in the plane, the composition $\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}$ of the reflections in these lines is a translation $\mathrm{Trans}_{\mathbf{w}}$ for some vector $\mathbf{w}$. **6**

Determine the translation vector $\mathbf{w}$ when the lines are

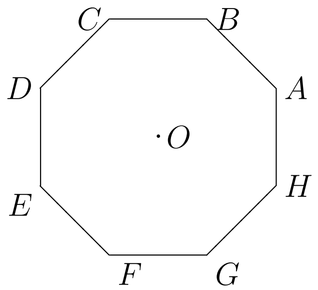
$$\mathcal{L}_1 = \{(x, y) : x + 2y = 0\}, \quad \mathcal{L}_2 = \{(x, y) : x + 2y = 5\}.$$ **2**

2. (i) Working in the symmetric group S_7, evaluate the following product of permutations, expressing the answer as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 3 & 7 & 1 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 1 & 6 & 5 & 7 & 2 \end{pmatrix}.$$ **6**

What is the sign of this permutation? **2**

(ii) Let Γ be the group of symmetries of the regular octagon whose vertices are A, B, C, D, E, F, G, H and which is centred at the origin O.



By identifying Γ with a group of permutations of the vertices, describe the elements of Γ which are rotations both geometrically and by using permutation notation. **8**

Write down the permutation corresponding to reflection in the line AE. By composing permutations, determine the effect of anti-clockwise rotation about O through $\pi/2$ followed by reflection in the line AE. **4**

3. (i) Define the Euclidean group $(\mathrm{Euc}(2), \circ)$ of the plane. **2**

Show that the subset $\mathrm{Trans}(2) \subseteq \mathrm{Euc}(2)$ of translations is a subgroup of $\mathrm{Euc}(2)$. **3**

(ii) Define a *dilation of the plane from a point* and explain what it means for two isometries of the plane to be *similar*. **3**

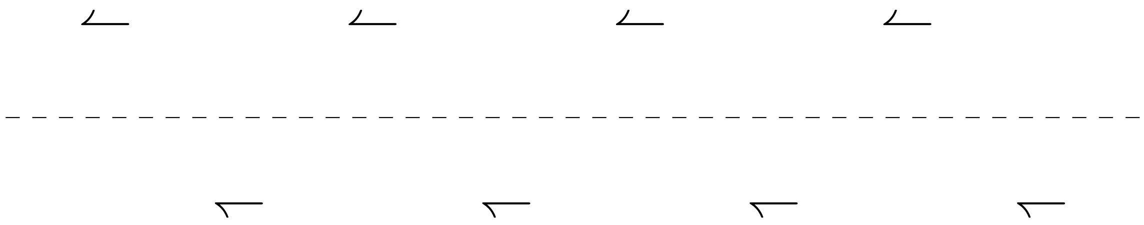
(a) Show that the rotations through the angle θ in the anti-clockwise direction about two points P and Q are similar. **7**

(b) Given two equilateral triangles Δ_1 and Δ_2 in the plane, explain how to construct a similarity transformation that transforms Δ_1 onto Δ_2. **5**

4. (i) For the frieze pattern $\mathcal{F}$, part of which is shown in the diagram below, indicate a generator of the group of translational symmetries and a glide reflection symmetry. Describe the elements of the symmetry subgroup $\mathrm{Euc}(2)_{\mathcal{F}} \leqslant \mathrm{Euc}(2)$ in terms of a suitable generator. **6**

Are there any points in the plane fixed by *all* the elements of $\mathrm{Euc}(2)_{\mathcal{F}}$? **2**

If the symbols $\leftharpoonup$ and $\leftharpoondown$ were replaced with $\leftharpoonup\!\!\!\!\rightharpoondown$ and $\leftharpoondown\!\!\!\!\rightharpoonup$ respectively, what extra symmetries would be introduced? **3**



(ii) Show that the matrix

$$R = \begin{bmatrix} \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & \sqrt{3}/2 \end{bmatrix},$$

corresponds to a rotation of $\mathbb{R}^3$ about a line through the origin and determine its axis and angle of rotation. **9**

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2Q Resit exam 2003 – Solutions

1. (i) Translation: $(I \mid \mathbf{t})$.
Rotation: $(A \mid \mathbf{t})$ where $A \neq I$ is orthogonal and $\det A = 1$.
Reflection or glide reflection: $(A \mid \mathbf{t})$ where A is orthogonal and $\det A = -1$. **4**

(ii) We have

$$A = \begin{bmatrix} \cos(7\pi/6) & -\sin(7\pi/6) \\ \sin(7\pi/6) & \cos(7\pi/6) \end{bmatrix}, \quad \det A = \cos^2(7\pi/6) + \sin^2(7\pi/6) = 1,$$

so $(A \mid \mathbf{t})$ represents an anti-clockwise rotation through $7\pi/6$. The centre of rotation is the point with position vector **3**

$$\begin{aligned}
\mathbf{c} = (I-A)^{-1}\mathbf{t} &= \begin{bmatrix} 1+\sqrt{3}/2 & -1/2 \\ 1/2 & 1+\sqrt{3}/2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2+\sqrt{3} & -1 \\ 1 & 2+\sqrt{3} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{2}{(2+\sqrt{3})^2+1} \begin{bmatrix} 2+\sqrt{3} & 1 \\ -1 & 2+\sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{2}{4+3+4\sqrt{3}+1} \begin{bmatrix} 1 \\ 2+\sqrt{3} \end{bmatrix} = \frac{1}{2(2+\sqrt{3})} \begin{bmatrix} 1 \\ 2+\sqrt{3} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1}{2(2+\sqrt{3})} \\ 1/2 \end{bmatrix} = \begin{bmatrix} (2-\sqrt{3})/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1-\sqrt{3}/2 \\ 1/2 \end{bmatrix}.
\end{aligned}$$

So the centre of rotation is at $\mathbf{c} = (1-\sqrt{3}/2, 1/2)$. **5**

(iii) Let $\mathbf{p}$ be the position vector of any point on $\mathcal{L}_1$ and $\mathbf{q}$ be the position vector of the unique point on $\mathcal{L}_2$ for which the vector $\mathbf{v} = \mathbf{q} - \mathbf{p}$ is perpendicular to the two lines. This vector is independent of the choice of $\mathbf{p}$.

Now let $\mathbf{x}$ be the position vector of any point. Choosing $\mathbf{p}$ so that $\mathbf{x} = \mathbf{p} + s\mathbf{v}$ for some $s \in \mathbb{R}$, we have

$$\mathbf{x}' = \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{p} - s\mathbf{v}.$$

Then the corresponding $\mathbf{q}$ satisfies $\mathbf{x}' = \mathbf{q} + t\mathbf{v}$ for some $t \in \mathbb{R}$, hence

$$\begin{aligned}
\mathbf{x}'' = \mathrm{Refl}_{\mathcal{L}_2}(\mathbf{x}') &= \mathbf{q} - t\mathbf{v} \\
&= \mathbf{q} - (\mathbf{x}' - \mathbf{q}) \\
&= 2\mathbf{q} - (\mathbf{p} - s\mathbf{v}) \\
&= 2\mathbf{q} - \mathbf{p} + (\mathbf{x} - \mathbf{p}) \\
&= 2(\mathbf{q} - \mathbf{p}) + \mathbf{x} = 2\mathbf{v} + \mathbf{x}.
\end{aligned}$$

Thus setting $\mathbf{w} = 2\mathbf{v}$ we have

$$\mathrm{Refl}_{\mathcal{L}_2} \circ \mathrm{Refl}_{\mathcal{L}_1}(\mathbf{x}) = \mathbf{x} + \mathbf{w}.$$ **6**

Taking $\mathbf{p} = (0,0)$ the vector $(1,2)$ is normal to $\mathcal{L}_1$ and $\mathcal{L}_2$. Then the corresponding point $\mathbf{q} = (u, 2u)$ as above satisfies $u + 2(2u) = 5$, giving $u = 1$ and so $\mathbf{q} = (1,2)$. Then $\mathbf{v} = \mathbf{q} = (1,2)$, and so $\mathbf{w} = (2,4)$. **2**

2. (i) We have

$$\begin{pmatrix}1&2&3&4&5&6&7\\5&4&3&7&1&2&6\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6&7\\3&4&1&6&5&7&2\end{pmatrix}=\begin{pmatrix}1&2&3&4&5&6&7\\3&7&5&2&1&6&4\end{pmatrix}$$
$$=(1\ 3\ 5)(2\ 7\ 4)(6)=(1\ 3\ 5)(2\ 7\ 4).$$

The sign is $(-1)^2(-1)^2=(-1)^4=1$. **6,2**

(ii) There are eight rotations taken anti-clockwise:

$$\begin{aligned}\text{through } 0 &= \text{the identity} = \iota,\\ \text{through } 2\pi/8 = \pi/4 &= (A\,B\,C\,D\,E\,F\,G\,H),\\ \text{through } 4\pi/8 = \pi/2 &= (A\,C\,E\,G)(B\,D\,F\,H),\\ \text{through } 6\pi/8 = 3\pi/4 &= (A\,D\,G\,B\,E\,H\,C\,F),\\ \text{through } 8\pi/8 = \pi &= (A\,E)(B\,F)(C\,G)(D\,H),\\ \text{through } 10\pi/8 = 5\pi/4 &= (A\,F\,C\,H\,E\,B\,G\,D),\\ \text{through } 12\pi/8 = 3\pi/2 &= (A\,G\,E\,C)(B\,H\,F\,D),\\ \text{through } 14\pi/8 = 7\pi/4 &= (A\,H\,G\,F\,E\,D\,C\,B).\end{aligned}$$

Reflection in AE corresponds to the permutation $(B\,H)(C\,G)(D\,F)$. **8**

The symmetries being composed are $(A\,C\,E\,G)(B\,D\,F\,H)$ and $(B\,H)(C\,G)(D\,F)$ and we have

$$(B\,H)(C\,G)(D\,F)(A\,C\,E\,G)(B\,D\,F\,H)=(A\,G)(B\,F)(C\,E),$$

which is corresponds to reflection in the line HD. **4**

3. (i) Euc(2) consists of all isometries (distance preserving maps) $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ under composition of functions $\circ$ with the identity function as its identity ι and inversion of functions defining inverses. **2**

Let $\mathbf{u},\mathbf{v}\in\mathbb{R}^2$, then

$$\mathrm{Trans}_{\mathbf{u}}\circ\mathrm{Trans}_{\mathbf{v}}=\mathrm{Trans}_{\mathbf{u}+\mathbf{v}},$$

so Trans(2) is closed under $\circ$. Translation by $\mathbf{0}$ $\mathrm{Trans}_{\mathbf{0}}$ agrees with the identity. Finally,

$$\mathrm{Trans}_{\mathbf{u}}^{-1}=\mathrm{Trans}_{-\mathbf{u}},$$

so Trans(2) is closed under taking inverses. Hence Trans(2) $\leqslant$ Euc(2). **3**

(ii) A *dilation of the plane* is a function $H\colon\mathbb{R}^2\longrightarrow\mathbb{R}^2$ which has the form

$$H(\mathbf{x})=\delta(\mathbf{x}-\mathbf{c})+\mathbf{c}=\delta\mathbf{x}+(1-\delta)\mathbf{c},$$

where $\delta>0$ is the *dilation factor* and $\mathbf{c}$ is the position vector of the *centre C of the dilation.* **1**

Two isometries of the plane F,G are *similar* if there is a similarity transformation $H=(\delta A\mid\mathbf{t})$ with $\delta>0$, A orthogonal and $\mathbf{t}\in\mathbb{R}^2$, for which

$$G=H_*F=H\circ F\circ H^{-1}.$$ **2**

(a) Let $\mathbf{t} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$. Then

$$\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{\mathbf{t}}^{-1} = \mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}}$$

is another rotation through θ since its Seitz symbol is

$$(I \mid \mathbf{t})(R \mid \mathbf{u})(I \mid -\mathbf{t}) = (I \mid \mathbf{t})(R \mid \mathbf{u} - R\mathbf{t}) = (R \mid \mathbf{u} - R\mathbf{t} + \mathbf{t}),$$

where $\mathrm{Rot}_{P,\theta} = (R \mid \mathbf{u})$. Since P is the centre of rotation, $\mathrm{Rot}_{P,\theta}(\mathbf{p}) = \mathbf{p}$. Now applying this isometry to the position vector of Q we obtain

$$\begin{aligned}\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}}(\mathbf{q}) &= \mathbf{t} + \mathrm{Rot}_{P,\theta}(\mathbf{q} - \mathbf{t}) \\ &= (\mathbf{q} - \mathbf{p}) + \mathrm{Rot}_{P,\theta}(\mathbf{q} - (\mathbf{q} - \mathbf{p})) \\ &= (\mathbf{q} - \mathbf{p}) + \mathrm{Rot}_{P,\theta}(\mathbf{p}) \\ &= (\mathbf{q} - \mathbf{p}) + \mathbf{p} = \mathbf{q},\end{aligned}$$

hence $\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{P,\theta} \circ \mathrm{Trans}_{-\mathbf{t}}$ fixes Q which must therefore be its centre of rotation. Hence we have $\mathrm{Trans}_{\mathbf{t}*} \mathrm{Rot}_{P,\theta} = \mathrm{Rot}_{Q,\theta}$. **7**

(b) First let T be the translation (possibly the identity) that maps the midpoint M_1 of Δ_1 to the midpoint M_2 of Δ_2. Next let D be the dilation with centre M_2 an scaling factor

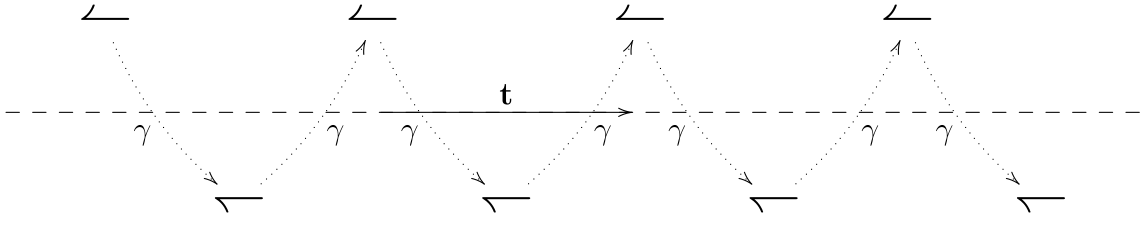
$$\delta = \frac{\text{length of side of } \Delta_2}{\text{length of side of } \Delta_1}.$$

Finally, let R be a rotation about M_2 that moves the vertices of $D \circ T\Delta_1$ (the image of Δ_1 under $D \circ T$) onto vertices of Δ_2. Then $R \circ D \circ T$ is a suitable similarity transformation with $R \circ D \circ T\Delta_1 = \Delta_2$. **5**

4. (i) There is a translation by $\mathbf{t}$ parallel to the dashed line and shifting the whole pattern one step the right. There is also a glide reflection γ whose effect is indicated below, with square equal to $\gamma \circ \gamma = \mathrm{Trans}_{\mathbf{t}}$. There are no rotations or reflections in lines perpendicular to the dashed line. The symmetry group of $\mathcal{F}$ is infinite cyclic with generator γ,

$$\mathrm{Euc}(2)_{\mathcal{F}} = \langle\gamma\rangle = \{\gamma^n : n \in \mathbb{Z}\}.$$

and consists of translations γ^n (n even) and glide reflections γ^n (n odd). **6**



There are no points simultaneously fixed by all the elements of this symmetry group since non-trivial translations and glide reflections have no fixed points. **2**

There would be reflections in the vertical lines through the midpoints of the $\leftharpoonup\!\!\!\rightharpoonup$ and $\leftharpoondown\!\!\!\rightharpoondown$ and half turn rotations about the points on the dashed line midway between these vertical lines. **3**

(ii) By expanding along the middle row we obtain

$$\det R = \begin{vmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{vmatrix} = \frac{(3+1)}{4} = 1,$$

so R represents a rotation about an axis through the origin. Since $R\mathbf{e}_2 = \mathbf{e}_2$, the axis of rotation is the y-axis. **4**

Now let

$$\mathbf{w} = \mathbf{e}_2, \quad \mathbf{u} = \mathbf{e}_3, \quad \mathbf{v} = \mathbf{w} \times \mathbf{u} = \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1.$$

So $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is a right handed orthonormal system. We have

$$\begin{aligned} R\mathbf{u} &= (\sqrt{3}/2)\mathbf{u} + (-1/2)\mathbf{v} = \cos(11\pi/6)\mathbf{u} + \sin(11\pi/6)\mathbf{v}, \\ R\mathbf{v} &= (1/2)\mathbf{u} + (\sqrt{3}/2)\mathbf{v} = -\sin(11\pi/6)\mathbf{u} + \cos(11\pi/6)\mathbf{v}, \end{aligned}$$

so the angle of rotation is $11\pi/6$ and the sense is that of a right handed screw driver pointing along the positive y-axis. **5**

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