1. Cohomology

Let $X$ be a space and $S^*(X)$ the graded abelian group of singular chains on $X$, so $S_n(X)$ is the free abelian group generated by the singular $n$-simplices $\Delta^n \to X$. With the usual boundary homomorphism $d$, $(S^*(X), d)$ is a chain complex and

$$H^*_s(X) = H^*_s(S^*(X), d).$$

Dually, for any abelian group $M$ we can from a cochain complex $(S^*(X; M), \delta)$, where

$$S^n(X; M) = \text{Hom}_{\mathbb{Z}}(S_n(X), M), \quad \delta = d^*.$$

The cohomology of $X$ with coefficients in $M$ is then the cohomology

$$H^*(X; M) = H^*_s(S^*(X; M), d).$$

This is a covariant homotopy invariant functor in $X$. For each $X$, $H^*(X; M)$ is also a covariant functor in $M$ and given a short exact sequence of abelian groups

$$0 \to L \to M \to N \to 0$$

there is an induced long exact sequence

$$\cdots \to H^{n-1}(X; N) \to H^n(X; L) \to H^n(X; M) \to H^n(X; N) \to H^{n+1}(X; L) \to \cdots$$

which is natural with respect to such short exact sequences.

2. External products and the Eilenberg-Zilber Theorem

Now suppose that we have two spaces $X, Y$ and want to calculate $H_*(X \times Y)$. As an abelian group, $S_*(X \times Y)$ has a basis consisting of the singular simplices $\Delta^n \to X \times Y$, but each of these is equivalent to a pair of singular simplices $\Delta^n \to X$ and $\Delta^n \to Y$. On the other hand we can also consider the tensor product $S_*(X) \otimes S_*(Y)$ where

$$[S_*(X) \otimes S_*(Y)]_n = \bigoplus_k S_k(X) \otimes S_{n-k}(Y),$$

and a suitable boundary homomorphism is defined. This is much bigger than $S_*(X \times Y)$ since it has a basis consisting of elements of the form $\alpha \otimes \beta$, where $\alpha: \Delta^r \to X$ and $\beta: \Delta^s \to Y$ are singular simplices.

**Theorem 2.1** (Eilenberg-Zilber Theorem). There are inverse chain homotopy equivalences

$$S_*(X) \otimes S_*(Y) \xrightarrow{\Phi} S_*(X \times Y)$$

Moreover these are unique up to chain homotopy and natural in $X$ and $Y$. 
Given this, we can now form an external multiplication of $S_*(X)$ and $S_*(Y)$ into $S_*(X \times Y)$. Dually, given any commutative ring $R$ with multiplication $\mu: R \otimes R \rightarrow R$, we can pair $\alpha \in S^p(X; R)$ and $\beta \in S^q(Y; R)$ to
\[ \mu \circ \alpha \otimes \beta \in \text{Hom}_\mathbb{Z}(S_p(X) \otimes S_q(Y), R). \]
Extending up to a graded homomorphism we obtain an external product
\[ S^*(X; R) \otimes S^*(Y; R) \rightarrow \text{Hom}_\mathbb{Z}(S_*(X) \otimes S_*(Y), R). \]
Then we can precompose this with $\Psi$ to form the external product
\[ \sqcup: S^*(X; R) \otimes S^*(Y; R) \rightarrow \text{Hom}_\mathbb{Z}(S_*(X \times Y), R). \]
These are both cochain maps.

Now specialising to the case $Y = X$ we can use the diagonal $\Delta: X \rightarrow X \times X$ and its induced map $\Delta^*$ to form the cup product
\[ \bigcup = \sqcup \circ \Delta^*: S^*(X; R) \otimes S^*(X; R) \rightarrow \text{Hom}_\mathbb{Z}(S_*(X), R); \quad \alpha \otimes \beta \mapsto \alpha \cup \beta. \]
Again, this is a cochain map.

**Theorem 2.2.** If $R$ is a commutative ring then the cup product turns $H^*(\quad; R)$ into a graded commutative ring valued contravariant functor on the homotopy category of spaces, i.e., if $u \in H^p(X; R)$ and $v \in H^q(X; R)$ then
\[ v \cup u = (-1)^{pq} u \cup v. \]
Furthermore, if $\varphi: R \rightarrow R'$ is a ring homomorphism then there is a natural transformation of graded ring valued functors extending $\varphi$ which agrees with $\varphi$ when evaluated on a point $\ast$.

We usually just write $uv$ for $u \cup v$.

**Remark 2.3.** In fact, $H^*(\quad; R)$ is really an $R$-algebra valued functor, where for each space $X$, the projection to a point $X \rightarrow \ast$ induces the unit
\[ R \cong H^*(\ast; R) \rightarrow H^*(X; R). \]
Notice that if $R$ has characteristic 2 (i.e., $2 = 0$ in $R$) then $H^*(X; R)$ is commutative in the usual sense. In general, if $p, q$ are odd and $u \in H^p(X; R)$, $v \in H^q(X; R)$, then
\[ uv + vu = 0, \quad 2a^2 = 0. \]
In particular, if $H^p(X; R)$ has no 2-torsion,
\[ u^2 = 0. \]

**Example 2.4.** Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ where $n \geq 1$. Then
\[ H^*(S^n; R) = \Lambda_R(s_n), \]
the exterior algebra over $R$ on a generator $s_n \in H^n(S^n; R)$.

**Proof.** Recall that
\[ H^k(S^n; R) \cong \begin{cases} R & \text{if } k = 0, n, \\ 0 & \text{otherwise.} \end{cases} \]

**Example 2.5.** Suppose that $X$ and $Y$ are two spaces and that $\text{pr}_X: X \times Y \rightarrow X$ and $\text{pr}_Y: X \times Y \rightarrow Y$ are the product maps. Then for $u \in H^p(X; R)$ and $v \in H^q(Y; R)$ we have $\text{pr}_X^* u \in H^p(X \times Y; R)$ and $\text{pr}_Y^* v \in H^q(X \times Y; R)$, so can form their cup product $\text{pr}_X^* u \cup \text{pr}_Y^* v \in H^{p+q}(X \times Y; R)$. If $H^*(X; R)$ and $H^*(Y; R)$ are free $R$-modules with bases $\{u_r\}$ and $\{v_s\}$ say, then $H^*(X \times Y; R)$ is also free with basis $\{\text{pr}_X^* u_r \cup \text{pr}_Y^* v_s\}$. 
Example 2.6 (Complex and quaternionic projective spaces). For \(1 \leq n \leq \infty\),

\[
H^k(\mathbb{C}P^n) = H^k(\mathbb{C}P^n; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k \leq n \text{ is even}, \\
0 & \text{otherwise}.
\end{cases}
\]

Then for any commutative ring \(R\),

\[
H^*(\mathbb{C}P^n; R) = R[x]/(x^{n+1}),
\]

where \(x \in H^2(\mathbb{C}P^n; R)\) is a generator.

Similarly,

\[
H^k(\mathbb{H}P^n) = H^k(\mathbb{H}P^n; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k \leq n \text{ is divisible by } 4, \\
0 & \text{otherwise}.
\end{cases}
\]

Then for any commutative ring \(R\),

\[
H^*(\mathbb{H}P^n; R) = R[y]/(y^{n+1}),
\]

where \(y \in H^4(\mathbb{H}P^n; R)\) is a generator.

Example 2.7 (Real projective spaces). Recall that for \(1 \leq n \leq \infty\),

\[
H^k(\mathbb{R}P^n) = H^k(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{if } k = 0, \\
\mathbb{Z}/2 & \text{if } n \text{ and } k \text{ are even and } k \leq n, \\
\mathbb{Z}/2 & \text{if } n \text{ is odd or } \infty, \text{ } k \text{ is even and } k < n, \\
\mathbb{Z} & \text{if } n \text{ is odd and } k = n, \\
0 & \text{otherwise}.
\end{cases}
\]

(a) If \(R = \mathbb{F}_2\) or any \(\mathbb{F}_2\)-algebra, then for \(n \leq \infty\),

\[
H^*(\mathbb{R}P^n; R) = R[z]/(z^{n+1}),
\]

where \(z \in H^1(\mathbb{R}P^n; R) \cong R\) is a generator.

(b) If \(n \leq \infty\) then

\[
H^*(\mathbb{R}P^n; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}[y]/(2y, y^{n+1}) & \text{if } n = \infty \text{ or } n < \infty \text{ is even}, \\
\mathbb{Z}[y]/(2y, y^{n+1}) \otimes \Lambda_n(w_n) & \text{if } n < \infty \text{ is odd},
\end{cases}
\]

where \(y \in H^2(\mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}/2\) (and \(w_n \in H^n(\mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}\) if \(n\) is odd) are generators.

Application 2.8. Let \(f: \mathbb{R}P^m \to \mathbb{R}P^n\) be a map such that \(f_*: \pi_1(\mathbb{R}P^m) \to \pi_1(\mathbb{R}P^n)\) is non-trivial. Then \(m \leq n\).

Proof. The Hurewicz Theorem implies that \(f_*: H_1(\mathbb{R}P^m; \mathbb{Z}/2) \to H_1(\mathbb{R}P^n; \mathbb{Z}/2)\) is also non-zero and therefore so is \(f^*: H^1(\mathbb{R}P^n; \mathbb{F}_2) \to H^1(\mathbb{R}P^m; \mathbb{F}_2)\). Writing \(z_m \in H^1(\mathbb{R}P^m; \mathbb{F}_2)\) for the generator, we have \(f^* z_m = z_m\) and so

\[
f^* z_m^m = z_m^m \neq 0.
\]

If \(m > n\) then \(z_m^n = 0\), giving a contradiction, so we must have \(m \leq n\). 

A map \(f: S^m \to S^n\) is called antipodal if for all \(x \in S^m\),

\[
f(-x) = -f(x).
\]

Such a map induces a map \(\overline{f}: \mathbb{R}P^m \to \mathbb{R}P^n\) fitting into a commutative diagram

\[
\begin{array}{ccc}
S^m & \xrightarrow{f} & S^n \\
\downarrow & & \downarrow \\
\mathbb{R}P^m & \xrightarrow{\overline{f}} & \mathbb{R}P^n
\end{array}
\]
and it can be shown that $\tilde{F} : \pi_1(\mathbb{RP}^m) \to \pi_1(\mathbb{RP}^n)$ is non-trivial. Hence we have

**Application 2.9.** Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be an antipodal map. Then $m \leq n$.

**Application 2.10 (Borsuk-Ulam).** If $n \geq 1$, for every map $g : \mathbb{S}^n \to \mathbb{R}^n$, there is an $x \in \mathbb{S}^n$ for which $g(-x) = g(x)$.

**Proof.** If not then $g(-x) \neq g(x)$ for every $x$. The map $f : \mathbb{S}^n \to \mathbb{S}^n$; $f(x) = \frac{g(-x) - g(x)}{|g(-x) - g(x)|}$, is an antipodal map, contradicting 2.9. □

Calculating cup products is not very straightforward in general and tends to require geometric input rather than just homotopy theoretic input. In the next section we discuss some important examples of the latter type.

### 3. The Hopf Invariant

Let $f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}$ be a map (here $n \geq 1$). The mapping cone $\text{Cone}(f)$ is a 2-cell complex, 

$\text{Cone}(f) = \mathbb{S}^{2n} \cup_f e^{4n}$,

whose cohomology has the form

$$H^k(\text{Cone}(f)) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 2n, 4n, \\ 0 & \text{otherwise.} \end{cases}$$

Choose generators $u_{2n} \in H^{2n}(\text{Cone}(f))$ and $u_{2n} \in H^{4n}(\text{Cone}(f))$. We can ‘normalise’ these by requiring that in the long exact sequence associated with the cofibration sequence $\mathbb{S}^{2n} \to \text{Cone}(f) \to \mathbb{S}^{2n}$, we have

$$H^{2n}(\text{Cone}(f)) \cong H^{2n}(\mathbb{S}^{2n}); \quad u_{2n} \mapsto s_{2n}$$

$$H^{4n}(\mathbb{S}^{2n}) \cong H^{4n}(\text{Cone}(f)); \quad s_{4n} \mapsto u_{4n}.$$ 

Then we must have

$$u_{2n}^2 = h(f)u_{4n},$$

where $h(f) \in \mathbb{Z}$ depends on the homotopy class of $f$, i.e., is well defined on $\pi_{4n-1}\mathbb{S}^{2n}$. $h(f)$ is the Hopf invariant of $f$.

Now recall that for a map $\ell : \mathbb{S}^m \to \mathbb{S}^m$, the degree of $\ell$ is defined to be the integer $\deg \ell$ for which

$$\ell^* : H^m(\mathbb{S}^m) \to H_m(\mathbb{S}^m); \quad \ell^*(s_m) = (\deg \ell)s_m.$$ 

We could equally well have used homology here.

**Proposition 3.1.** For $n \geq 1$, the Hopf invariant gives a function $h : \pi_{4n-1}\mathbb{S}^{2n} \to \mathbb{Z}$ with the following properties.

(a) $h : \pi_{4n-1}\mathbb{S}^{2n} \to \mathbb{Z}$ is a group homomorphism.

(b) If $g : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ and $k : \mathbb{S}^{4n-1} \to \mathbb{S}^{4n-1}$ are maps, then

$$h(g \circ f \circ k) = (\deg g)^2(\deg k)h(f).$$

(c) There is an element $w_{2n} \in \pi_{4n-1}\mathbb{S}^{2n}$ for which $h(w_{2n}) = 2$. 

Properties (a) and (b) follow from the definition. For (c) here is an explicit construction of such a map $w_{2n}$. Recall that $S^m \times S^m$ has a cell structure of the following form:

$$S^m \times S^m = (S^m \lor S^m) \cup_{\tilde{w}_m} S^{2m}.$$  

The fold map $\nabla: S^m \lor S^m \rightarrow S^m$ can be composed with the attaching map $\tilde{w}_m$ to give a map $w_m = \nabla \circ \tilde{w}_m: S^{2m-1} \rightarrow S^m$ and there is a map $\tilde{\nabla}: S^m \times S^m \rightarrow \text{Cone}(w_m)$ extending $\nabla$. It's easy to see that

$$H^2m(S^m \times S^m) = H^2m(S^m) \oplus H^2m(S^m)$$

with generators obtained by pulling back the generator $s_m$ under the two projections to $S^m$. Then we have

$$\tilde{\nabla}^*(s_m) = (s_m, s_m).$$

Also,

$$\tilde{\nabla}^*: H^2m(S^m \times S^m) \cong H^2m(\text{Cone}(w_m)).$$

Now take $m = 2n$. Then an easy calculation in $H^{4n}(S^{2n} \times S^{2n}) \cong \mathbb{Z}$ gives

$$\tilde{\nabla}^*(s_{2n}^2) = \tilde{\nabla}^*(s_{2n})^2$$

$$= (s_{2n}, s_{2n})^2$$

$$= [(s_{2n}, 0) + (0, s_{2n})]^2$$

$$= 2(s_{2n}, 0) \cup (0, s_{2n})$$

$$= 2 \times \text{(generator)},$$

since $(s_m, 0) \cup (0, s_m) = \text{(generator)}$. Using this it follows that $h(w_{2m}) = 2$.

**Question 3.2.** For which $n$ is there a map $f: S^{4n-1} \rightarrow S^{2n}$ with Hopf invariant 1, $h(f) = 1$?

Here are some examples.

For $n = 1$, $\mathbb{C}P^2 = \text{Cone}(\eta)$ where $\eta: S^3 \rightarrow S^2$ can be constructed geometrically as the complex Hopf map. Then from Example 2.6,

$$H^*(\mathbb{C}P^2) = \mathbb{Z}[u_2]/(u_2^3),$$

so we have $h(\eta) = 1$. This construction depends on existence and properties of the complex numbers.

For $n = 2$, $\mathbb{H}P^2 = \text{Cone}(\nu)$ where $\nu: S^7 \rightarrow S^4$ can be constructed geometrically as the quaternionic Hopf map. Then from 2.6,

$$H^*(\mathbb{H}P^2) = \mathbb{Z}[u_2]/(u_2^3),$$

we have $h(\eta) = 1$. This construction depends on existence and properties of the quaternions.

If $n = 4$, there is a Cayley projective plane $\mathbb{O}P^2 = \text{Cone}(\sigma)$ where $\sigma: S^{15} \rightarrow S^8$ is a certain map. Then

$$H^*(\mathbb{O}P^2) = \mathbb{Z}[u_8]/(u_8^3)$$

and we have $h(\sigma) = 1$.

This question is definitively answered in the following major result.

**Theorem 3.3** (Adams' Hopf invariant 1 theorem). If $n \neq 1, 2, 4$ there are no maps of Hopf invariant 1.

This also implies that the only spheres that are $H$-spaces (i.e., groups up to homotopy) are $S^0, S^1, S^3$ and allowing non-associativity, $S^7$. 

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**CUP PRODUCTS, COHOMOLOGY RINGS AND THOM ISOMORPHISMS**

5
4. Thom isomorphisms

Let \( \xi \downarrow B \) be vector bundle of dimension \( n \) over a connected space \( B \). Let the total space be \( E(\xi) \) and view \( B \) as a subspace by identifying it with its image under the 0-section \( B \rightarrow E(\xi) \). Let \( E_0(\xi) = E(\xi) - B \) which also gives a fibre bundle over \( B \) with fibre \( \mathbb{R}^n = \mathbb{R}^n - \{0\} \). Consider the pair \( (E(\xi), E_0(\xi)) \). If \( \xi \) admits a Riemannian structure then we can also consider the unit disc bundle \( D(\xi) \downarrow B \) and unit sphere bundle \( S(\xi) \downarrow B \) and the pair \( (D(\xi), S(\xi)) \). Then there is an homotopy equivalence of pairs \( (D(\xi), S(\xi)) \rightarrow (E(\xi), E_0(\xi)) \). The quotient space \( M\xi = D(\xi)/S(\xi) \) is the Thom space of \( \xi \).

There is a ‘diagonal map’

\[
E(\xi) \rightarrow E(\xi) \times E(\xi) \xrightarrow{\sim} B \times E(\xi)
\]

which induces a map

\[
\psi: M\xi \rightarrow (B_+) \wedge M\xi.
\]

For any commutative ring \( k \), this gives rise to a product map

\[
H^*(B; k) \otimes \tilde{H}^*(M\xi; k) \rightarrow \tilde{H}^*((B_+) \wedge M\xi; k) \xrightarrow{\psi^*} \tilde{H}^*(M\xi; k),
\]

making \( \tilde{H}^*(M\xi; k) \) into a module over \( H^*(B; k) \). We write \( x \cdot y \) for the product of \( x \in H^*(B; k) \) and \( y \in \tilde{H}^*(M\xi; k) \).

It can be seen that if \( B \) is a CW complex then so is \( M\xi \) with each \( k \)-cell in \( B \) corresponding to an \((k + n)\)-cell in \( M\xi \). In particular, the bottom cell of \( M\xi \) can be taken to be an \( n \)-sphere which can be thought of as the sphere obtained by collapsing to a point the boundary of the disc above a point of \( B \).

We say that \( \xi \) is \( k \)-orientable if there is an element \( u \in \tilde{H}^n(M\xi; k) \) which restricts to the natural choice of generator \( s_n \in \tilde{H}^n(\mathbb{S}^n; k) \cong k \). Such a \( u \) is called a \( k \)-orientation.

A vector bundle \( \xi \downarrow B \) of dimension \( n \) is said to be orientable if its structure group can be taken to be \( \text{SL}_n(\mathbb{R}) \leq \text{GL}_n(\mathbb{R}) \), i.e., if there is a principal \( \text{SL}_n(\mathbb{R}) \)-bundle \( P \downarrow B \) for which

\[
\xi \cong P \times \mathbb{R}^n.
\]

**Proposition 4.1.** Let \( \xi \downarrow B \) be a vector bundle.

(a) \( \xi \) is \( \mathbb{F}_2 \)-orientable.

(b) \( \xi \) is \( \mathbb{Z} \)-orientable if and only if it is orientable.

**Theorem 4.2** (Thom Isomorphism Theorem). If \( \xi \downarrow B \) is \( k \)-orientable then \( \tilde{H}^n(M\xi; k) \) is a free \( H^*(B; k) \)-module of rank 1, with any orientation \( u \in \tilde{H}^n(M\xi; k) \) as a generator. Thus for each orientation \( u \in \tilde{H}^n(M\xi; k) \) there is an isomorphism of \( H^*(B; k) \)-modules

\[
\Phi_u: H^*(B; k) \rightarrow \tilde{H}^n(M\xi; k); \quad x \mapsto x \cdot u.
\]

This means that given an orientation \( u \in \tilde{H}^n(M\xi; k) \), every element \( w \in \tilde{H}^{k+n}(M\xi; k) \) can be uniquely expressed in the form \( w = v \cdot u \) for some \( v \in \tilde{H}^k(M\xi; k) \).

**Example 4.3.** Recall that there is a canonical line bundle \( \lambda_n \downarrow \mathbb{R}P^n \), where

\[
(\lambda_n)|_x = \{tx : t \in \mathbb{R}\}.
\]

Then

\[
\mathbb{R}P^n \subseteq \mathbb{R}P^{n+1}; [x] \leftrightarrow [x, 0]
\]

and there is a homeomorphism

\[
E(\lambda_n) \cong \mathbb{R}P^{n+1} - \{[0, 1]\}; \quad tx \in (\lambda_n)|_x \leftrightarrow [x, t],
\]

where we always take \(|x| = 1\). Consequently, there is a homeomorphism

\[
M\lambda_n \cong \mathbb{R}P^{n+1}.
\]
Hence there is an isomorphism of $H^*(\mathbb{R}P^n; \mathbb{F}_2)$-modules

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) \xrightarrow{\delta} H^*(\mathbb{R}P^{n+1}; \mathbb{F}_2).$$

In particular, we can inductively prove Example 2.7(a),

$$H^*(\mathbb{R}P^n; \mathbb{F}_2) = \mathbb{F}_2[z_n]/(z_n+1).$$

There are various properties of Thom isomorphisms that are useful. For example, if $\xi_1 \downarrow B_1$ and $\xi_2 \downarrow B_2$ then

$$M(\xi_1 \times \xi_2) = M\xi_1 \wedge M\xi_2,$$

and we can combine orientations $\alpha_1 \in \tilde{H}^*(M\xi_1; \mathbb{F}_2)$ to obtain an orientation for $\xi_1 \times \xi_2$.

(4.1)

$$u_1 \wedge u_2 \in \tilde{H}^*(M\xi_1 \wedge M\xi_2; \mathbb{F}_2).$$

Another important invariant of a $k$-orientable bundle of dimension $n$ is the Euler class

$$e_\alpha(\xi) = s_0^\alpha u \in H^n(B; \mathbb{F}_2),$$

where $s_0: B \rightarrow D\xi \rightarrow M\xi$ is the zero-section. Then the following relation holds in $H^*(M\xi; \mathbb{F}_2)$:

$$u^2 = e_\alpha(\xi) \cdot u.$$ 

It follows that if $\xi$ has a non-vanishing section, then $e_\alpha(\xi) = 0$. Thus $e_\alpha(\xi)$ is an obstruction to $\xi$ having a non-vanishing section.

5. Characteristic classes defined using the Thom isomorphism

Let $\xi \downarrow B$ be a vector bundle of dimension $n$, where $B$ is connected. Then for an orientation $\alpha \in \tilde{H}^n(M\xi; \mathbb{F}_2)$ (in fact this is unique) we have a Thom isomorphism

$$\Phi_\alpha: H^*(B; \mathbb{F}_2) \rightarrow \tilde{H}^*(M\xi; \mathbb{F}_2).$$

Recall the Steenrod operations $Sq^k$ for $k \geq 0$. We may define elements

$$w_k(\xi) = \Phi_\alpha^{-1} Sq^k u \in H^k(B; \mathbb{F}_2).$$

Of course, $w_0(\xi) = 1$. Notice that for $k > n$ we have $Sq^k u = 0$ and so $w_k(\xi) = 0$. Notice also that since $Sq^n u = u^2$, $w_n(\xi) = e_\alpha(\xi)$, the Euler class. We also have from Equation (4.1) and the Cartan Formula

$$w_k(\xi_1 \times \xi_2) = \sum_{j=0}^{k} w_j(\xi_1) \times w_{k-j}(\xi_2).$$

This can be internalised to a Cartan Formula for a direct sum of bundles $\xi' \oplus \xi''$:

$$w_k(\xi' \oplus \xi'') = \sum_{j=0}^{k} w_j(\xi') w_{k-j}(\xi'').$$

Using Example 4.3 we can also show that for $\lambda_n \downarrow \mathbb{R}P^n$, $w_1(\lambda_n) = z_n$. Combining all of these we find that $w_k(\xi)$ agree with the Seifel-Whitney classes defined by other methods.
6. Manifolds and duality

Let $M$ be a compact connected manifold of dimension $m$. Then the tangent bundle $TM \downarrow M$ has dimension $m$. If $TM$ is $k$-orientable we say that $M$ is $k$-orientable. So every manifold is $\mathbb{F}_2$-orientable, while $M$ is $\mathbb{Z}$-orientable if and only if it is orientable.

If $M$ is $k$-orientable then for any choice of orientation $u \in \tilde{H}^m(TM; k)$ there is a $k$-Euler class

\[ e_u(M) = e_u(TM) \in H^m(M; k). \]

It also follows that

\[ H^m(M; k) \cong k \]

and under this isomorphism, $e_u(M) \leftrightarrow \chi_u(M) \in k$, where $\chi_u(M)$ is the associated Euler characteristic.

For example, when $M$ is orientable and $k = \mathbb{Z}$, $\chi_u(M)$ is the geometric Euler characteristic (up to sign) and this vanishes of and only if there is a non-vanishing section of $TM$.

Now take $k$ to be a field and let $M$ be $k$-orientable. Then for each $k$,

\[ H^k(M; k) = \text{Hom}_k(H_k(M; k), k), \quad H_k(M; k) \cong \text{Hom}_k(H^k(M; k), k). \]

Then if $x \in H^k(M; k)$ we can consider $x$ as defining a $k$-linear mapping

\[ H^{m-k}(M; k) \longrightarrow H^m(M; k) \cong k; \quad y \mapsto x \cup y \]

and so it corresponds to an element of $H_{m-k}(M; k)$. For each $k$, this sets up an isomorphism

\[ H^k(M; k) \cong H_{m-k}(M; k) \]

called the Poincaré duality isomorphism.

A particular case of this occurs when $m = 4n$ and $k = 2n$. Then we obtain a pairing

\[ H^{2n}(M; k) \otimes_k H^{2n}(M; k) \longrightarrow H^{4n}(M; k) \cong k; \quad x \otimes y \mapsto x \cup y. \]

This is a non-degenerate bilinear form and is an important invariant of $M$. When $k = \mathbb{Z}$, it can be used to essentially classify manifolds. For example, when $m = 4$ and $M$ is simply connected, this is the main topological invariant.