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Real multiplication on K3 surfaces

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Hodge structures of K3 type with real multiplication

a) Hodge structures of K3 type. The complex singular cohomology of smooth, projective varieties admits the Hodge decomposition. Motivated by this, one defines a *rational Hodge structure of weight* k to be a finite-dimensional rational vector space V together with a decomposition

$$V \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. A *polarization* of a Hodge structure is a bilinear form on V with certain compatibility properties with the Hodge structure. This notion just mimics the cup-product of two primitive cohomology classes with an appropriate power of the class of an ample line bundle.

Definition. A *Hodge strucure of K3 type* consists of an irreducible, polarized rational Hodge structure (T, q) of weight 2 such that $\dim_{\mathbf{C}} T^{2,0} = 1$.

Example. Consider a complex, projective K3 surface S. Denote by $\mathrm{NS}(S)_{\mathbf{Q}}$ the $\mathbf{Q}$-linear span of the image of the first Chern class $c_1 : \mathrm{Pic}(S) \to H^2(S, \mathbf{Q})$. Then we get an orthogonal decomposition

$$H^2(S, \mathbf{Q}) = \mathrm{NS}(S)_{\mathbf{Q}} \oplus^{\perp} T(S).$$

The space $T(S)$ is called the rational transcendental lattice of S, it is a sub-Hodge structure because $\mathrm{NS}(S)_{\mathbf{Q}}$ is purely of type $(1,1)$. The rational transcendental lattice of a K3 surface together with the quadratic form induced by the cup-product on S is a Hodge structure of K3 type.

b) Endomorphisms of K3 type Hodge structures. Since Hodge structures of K3 type are irreducible, their endomorphisms form a division algebra. Zarhin (see [Z]) used Albert's classification to obtain the following

Theorem (Zarhin). *Let (T, q) be a polarized Hodge structure of K3 type and denote by $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ its algebra of endomorphisms of Hodge structures. Then either E is a totally real number field (we say (T, q) has* real multiplication*) or E is a purely imaginary quadratic extension of a totally real number field (we say (T, q) has* complex multiplication (CM)*).*

Somewhat surprisingly, K3 surfaces with complex multiplication (i.e. whose transcendental lattices have complex multiplication) are better understood than those with real multiplication which a priori have less endomorphisms of Hodge structures. The reason for this is that in case S has CM, $E = \mathrm{End}_{\mathrm{Hdg}}(T(S))$ can be shown to be spanned by isometries of $T(S)$. Mukai's theory of moduli spaces of sheaves on K3 surfaces predicts that self-isometries should correspond to (possibly twisted) Fourier–Mukai partners of S.

In the case of real multiplication very little is known. It is easy to see that there is a countable number of positive-dimensional subvarieties of the moduli space of polarized K3 surfaces which parametrize surfaces with real multiplication by a field different from $\mathbf{Q}$. The purpose of my work is to search for a better understanding of K3 surfaces with real multiplication.

Kuga–Satake varieties of K3 type Hodge structures

a) Kuga–Satake varieties. Let A be an Abelian surface and $S = K(A)$ the minimal resolution of its Kummer surface which is a K3 surface. Then there is a natural isomorphism $T(S) \simeq T(A)$.
Kuga and Satake found a way to generalize this. They associate to any K3 type Hodge structure (T, q) an isogeny class of Abelian varieties, in other words a polarizable rational Hodge structure V of weight one, such that there exists an inclusion of Hodge structures

$$T \hookrightarrow \bigwedge^2 V.$$

Any Abelian variety in the isogeny class of V is called a *Kuga–Satake variety* for (T, q). Unfortunately, only for very few examples of K3 surfaces, the Kuga–Satake variety has been shown to be geometrically related to the surface (see below). In general, the Kuga–Satake construction is purely Hogde-theoretical.

We want to prove that for K3 type Hodge structures with real multiplication there is a natural decomposition of the Kuga–Satake variety which enables us to calculate the endomorphism algebra of V in that case. To formulate the results we need some more preparations.

b) Corestriction of algebras. Let E be a number field, denote by $\tilde{E}$ its normal closure and by $G := \mathrm{Gal}(\tilde{E}/\mathbf{Q})$ the Galois group of $\tilde{E}$ over $\mathbf{Q}$. Let H be the subgroup of G fixing E, such that the set of cosets G/H parametrizes the different embeddings of E in $\tilde{E}$. The corestriction is a construction which associates with an E-algebra in a natural way a $\mathbf{Q}$-algebra.
Let A be a finite-dimensional E-algebra. For any coset σH denote by $A^{\sigma H}$ the twisted E-algebra

$$A^{\sigma H} = A \otimes_E \tilde{E}$$

where $e \in E$ acts by multiplication with $\sigma(e)$ on the second tensor factor. There is a natural G-action by $\mathbf{Q}$-algebra homomorphisms on $\bigotimes_{\sigma H \in G/H} A^{\sigma H}$. Then one defines the corestriction of the E-algebra A to $\mathbf{Q}$ to be the $\mathbf{Q}$-algebra

$$\mathrm{Cores}_{E/\mathbf{Q}}(A) := \left(\bigotimes_{\sigma H \in G/H} A^{\sigma H} \right)^G .$$

c) The quadratic form Q on a K3 type Hodge structure with real multiplication. Let (T, q) be a K3 type Hodge structure, assume that $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ is a totally real number field. Then T is in a natural way an E-vector space. There exists an E-bilinear form

$$Q : T \times T \to E$$

having the property that for $e \in E$ and $t, t' \in T$

$$q(et, t') = \mathrm{tr}_{E/\mathbf{Q}}(eQ(t, t')).$$

Kuga–Satake varieties and real multiplication

a) Decomposition of the Kuga–Satake variety and real multiplication. Again, let (T, q) be a K3 type Hodge structure with $E = \mathrm{End}_{\mathrm{Hdg}}(T)$ a totally real number field of degree d. Denote by V the associated Kuga–Satake Hodge structure.
Denote by $C^0(Q)$ the even Clifford algebra of Q. Van Geemen showed in [vG] that there is a natural inclusion of Hodge structures $\mathrm{Cores}_{E/\mathbf{Q}}(C^0(Q)) \hookrightarrow V$. This result can be improved to the following

Theorem (US). *Under the above assumptions we get the following:*

(i) The special Mumford–Tate group of V is $\mathrm{Res}_{E/\mathbf{Q}}(Spin(Q))$.

(ii) There is a natural decomposition into sub-Hodge structures

$$V = \mathrm{Cores}_{E/\mathbf{Q}}(C^0(Q))^{\oplus 2^{d-1}}.$$

(iii) The endomorphism algebra of a Kuga–Satake variety of T is

$$\mathrm{End}_{\mathrm{Hdg}}(V) = \mathrm{Mat}_{2^{d-1}}(\mathrm{Cores}_{E/\mathbf{Q}}(C^0(Q))).$$

b) An application. As mentioned above, in general the Kuga–Satake construction is not geometrically understood. However, there is a four-dimensional family of projective K3 surfaces for which there exists a very nice geometric explanation. This is due to work of Paranjape (see [P]).
Let S be a K3 surface with a morphism $\pi : S \to \mathbf{P}^2$ such that the branch locus of π in $\mathbf{P}^2$ is the union of six lines in general position. Then Paranjape shows that there exists a triple

$$(C, E, f : C \to E)$$

where C is a genus five curve, E an elliptic curve and f a $(4:1)$ map such that $\mathrm{Prym}(f)^4$ is a Kuga–Satake variety for S. Further, S can be obtained as the resolution of a certain quotient of $C \times C$. This establishes that the algebraicity of the Kuga–Satake inclusion

$$T(S) \hookrightarrow H^2((\mathrm{Prym}(f))^4, \mathbf{Q}). \qquad (1)$$

As mentioned, the family of K3 surfaces which are double covers of $\mathbf{P}^2$ ramified along six lines is four-dimensional. This is, because the parameter space of six plane lines in general position is four-dimensional (any four lines in general position can be transformed in given four lines by a linear transformation, so the space is a subspace of $(\mathbf{P}^2)^* \times (\mathbf{P}^2)^*$).
Paranjape's proof somehow goes the other way round, he constructs out of a triple (C, E, f) a K3 surface which is a double cover of $\mathbf{P}^2$ ramified along six lines, then he shows that varying the triples, with this construction he gets precisely the four-dimensional family of K3 surfaces of this type.

Double covers of $\mathbf{P}^2$ ramified along six lines

In the four-dimensional moduli space of these K3 surfaces we find the following endomorphism types:

- Real multiplication by $\mathbf{Q}$. This is the generic case.
- Real multiplication by a quadratic extension $\mathbf{Q}(\sqrt{d})$ for some $d > 0$. These surfaces appear in one-dimensional families, there is a countable number of such curves in the moduli space.
- Complex multiplication by a quadratic extension $\mathbf{Q}(\sqrt{-d})$ for some $d > 0$. These surfaces can appear either as isolated points or in one-dimensional families. Again, there are countably many of such subvarieties of the moduli space.
- Complex multiplication by a CM-field of degree four over $\mathbf{Q}$. These surfaces are isolated points in the moduli space.

With this analysis we are now able to show

Corollary (US). *Let S be a K3 surface which is a double cover of $\mathbf{P}^2$ ramified along six lines. Then the Hodge Conjecture holds for $S \times S$.*

Sketch of proof: One only has to check the algebraicity of the classes in $\mathrm{End}_{\mathrm{Hdg}}(T(S))$. Using Mukai's results on moduli spaces of sheaves on K3s, in the above list only the case of real multiplication by a quadratic extension of $\mathbf{Q}$ has to be studied.
So assume now that $T(S)$ has real multiplication by $E = \mathbf{Q}(\sqrt{d})$ for some $d > 0$. The above theorem allows us to calculate the endomorphism algebra of a Kuga–Satake variety A of $T(S)$ as $\mathrm{End}_{\mathbf{Q}}(A) = \mathrm{Mat}_4(D)$ where D is a definite quaternion algebra over $\mathbf{Q}$, so $A \sim B^4$ for some Abelian fourfold B with $\mathrm{End}_{\mathbf{Q}}(B) = D$. This B can be shown to be of Weil type for the field $\mathbf{Q}(i)$, its discriminant is 1. Then by a theorem of van Geemen the Weil cycles of such Abelian fourfolds are algebraic. Using this, a theorem of Abdulali (see [A]) shows that any self-product of B satisfies the Hodge Conjecture. This combined with the algebraicity of (1) implies our result.

References

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