Derived equivalent K3 surfaces via conjugation

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Introduction

In his seminal paper [2] Mukai proved that the Poincare bundle on $A \times \widehat{A}$, where A is an abelian variety and $\widehat{A}$ its dual abelian variety, defines an equivalence between the derived categories of coherent sheaves on A and $\widehat{A}$. Since the two varieties are in general not isomorphic this so called *Fourier–Mukai transform* provides a new geometrical invariant. Generalizing this special case one calls two varieties X and Y *Fourier–Mukai partners* (FMP) or *derived equivalent* if there is an exact $\mathbb{C}$-linear equivalence between the derived categories of coherent sheaves on X resp. Y, denoted by $\mathrm{D^b}(X)$ resp. $\mathrm{D^b}(Y)$. Usually only smooth projective varieties are considered since in this case the derived category is reasonably big and by a theorem of Orlov every equivalence is induced by an object on the product. So from now on every variety will be assumed to be smooth and projective.
Similarly to the example above one expects a FMP of a variety X to be somehow geometrically related to X or, more precisely, be a moduli space of certain objects on X. Here are some cases where this relation has been completely understood.

- Two curves are derived equivalent if and only if they are isomorphic.
- If the canonical bundle of X is ample or anti-ample then any FMP of X is isomorphic to X.
- Two abelian varieties A and B are derived equivalent if and only if there exists an isomorphism between $A \times \widehat{A}$ and $B \times \widehat{B}$ (respecting an additional quadratic form).

Another important and much investigated case is that of K3 surfaces and this is the topic of my current work. My goal is to understand whether a certain geometric construction provides a FMP of a given K3 surface.

By definition, a *K3 surface* is a connected compact complex surface X such that $\omega_X \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. A concrete example, the so called Fermat quartic, is given by the set of points in $\mathbb{P}^3$ satisfying the equation $x_0^4 + \ldots + x_3^4 = 0$. Another important class of K3 surfaces are *Kummer surfaces* which are constructed by blowing up a two-dimensional complex torus in the 16 2-torsion points and then dividing out the induced involution.
K3 surfaces are usually studied by their periods, i.e. the second cohomology group with integral coefficients $H^2(X, \mathbb{Z})$ with its Hodge structure, and we have the Torelli theorem which says that two K3 surfaces are isomorphic if and only if their second cohomology groups are Hodge-isometric.

Derived equivalences of K3 surfaces

We have the following theorem due to Orlov, see [4].

Theorem 1 *The following conditions are equivalent (the trivial case $X \cong Y$ is omitted):*

- *Two K3 surfaces X and Y are derived equivalent.*
- *There exists a Hodge isometry between $T(X)$ and $T(Y)$.*
- *There exists a Hodge isometry between $H^*(X, \mathbb{Z})$ and $H^*(Y, \mathbb{Z})$.*
- *Y is a fine moduli space of stable vector bundles on X with respect to a certain polarization.*

Here $T(X)$ and $H^*(X, \mathbb{Z})$ denote the transcendental lattice and the full cohomology group of X (with a certain Hodge-structure) respectively and the third condition can be reformulated by saying that the periods of X and Y lie in the same orbit of a certain group $O(\tilde{\Gamma})$.
Furthermore in some cases one can compute the number of non-isomorphic FM-partners of a given K3 surface X, see [1].

Theorem 2 *Denote by $NS(X) = \mathrm{Pic}(X)$ the Picard group and by $\rho(X)$ the Picard number of X, i.e. the rank of $\mathrm{Pic}(X)$.*

- *If $\rho(X) \geq 12$ then X does not have any non-trivial FM-partners.*
- *If $\rho(X) \geq 3$ and the determinant of $\mathrm{Pic}(X)$ is square-free then X does not have any non-trivial FM-partners.*
- *If $\rho(X) = 1$, so $NS(X) = \mathbb{Z}H$ with $H^2 = 2n > 0$, then the number of non-isomorphic FM-partners of X is equal to $2^{\tau(n)-1}$, where $\tau(n)$ is the number of distinct primes dividing n.*

However explicit constructions of derived equivalent K3 surfaces are known essentially only in two cases:

- In [5] all possible FM-partners of a K3 surface of Picard rank 1 were computed by using the fourth condition of the above theorem and describing the moduli spaces arising in this case. Lattice theory gives that the degree of the polarization is the same for all of these.
- In [3] and later in [5] it was proved that for a given natural number N there are at least N non-isomorphic K3 surfaces which are FM-partners. This result is proved in the second paper by describing N non-isometric rank 2 lattices L_i, which are the Picard groups of the wanted surfaces X_i, having Hodge-isometric transcendental lattices, i.e. $T(X_i)$. In contrast to the Picard rank 1 case these K3 surfaces have different polarizations.

Conjugate K3 surfaces

Let X be a K3 surface and consider σ, an arbitrary automorphism of $\mathbb{C}$. Then we have the *conjugate* K3 surface X^σ given by the fibre square

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

As abstract schemes X and X^σ are isomorphic (in particular their Picard groups are isomorphic), so their categories of coherent sheaves are equivalent and therefore there is an exact equivalence between $\mathrm{D^b}(X)$ and $\mathrm{D^b}(X^\sigma)$ but this equivalence is a priori only $\mathbb{Q}$-linear. Since the above process changes the complex structure (consider e.g. the conjugation and observe that it just changes the complex structure on X by a sign) the question whether there exists a $\mathbb{C}$-linear equivalence is interesting and this is exactly what I am thinking about:

Does there exist an example of a K3 surface X and σ, an automorphism of $\mathbb{C}$, such that X and X^σ are derived equivalent but not isomorphic as complex varieties?

Thus we want to understand the connection between the actions of the groups $\mathrm{Aut}(\mathbb{C})$ and $O(\tilde{\Gamma})$ on the moduli space of projective K3 surfaces.

Theorem 2 above provides some restrictions on the K3 surfaces which can be considered. Moreover, in this special case of conjugation we can not take into considerations K3 surfaces X having an uni-modular Picard group since in this case $H^2(X, \mathbb{Z}) \cong \mathrm{Pic}(X) \oplus T(X)$ and any derived equivalent X^σ is isomorphic to X by the Torelli theorem.

In the following I will sketch two possibilities to approach the above question:

- Consider a K3 surface over a number field $\mathbb{K}$. By using two different embeddings α_1, α_2 of $\mathbb{K}$ into $\overline{\mathbb{Q}}$ we get two K3 surfaces over $\overline{\mathbb{Q}}$ and then over $\mathbb{C}$. Using this we can search for a K3 surface over $\mathbb{K}$ such that there exists a conjugate non-isomorphic derived equivalent K3 surface X^φ over $\mathbb{K}$ and try to lift this situation to $\mathbb{C}$ (in the above notation $\alpha_2 = \alpha_1 \circ \varphi$).
- Consider the moduli space of all K3 surfaces with a fixed polarization and try to find a curve C in this moduli space such that there exists an element g in the above mentioned group $O(\tilde{\Gamma})$ which leaves C invariant but acts non-trivially on it. Geometrically this means that we have two families of K3 surfaces which generically have derived equivalent fibers. Of course, one wants non-isomorphic FM-partners so C must fulfill another condition which can also be stated in terms of the action of a certain group. In fact, we want this curve to be defined over $\overline{\mathbb{Q}}$ and the idea is that over the generic point of C we would then get two K3 surfaces which are conjugate, non-isomorphic and derived equivalent over $\overline{\mathbb{Q}}(t)$ (or some function field) and through different embeddings $\overline{\mathbb{Q}}(t) \hookrightarrow \mathbb{C}$ we would then get the desired conjugate K3 surfaces over $\mathbb{C}$.

References

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