Mathematics 2Q – Solutions for Chapter 1

1.1. (a) The point $(-1,-1)$ is on this line; since $(2,-3)\cdot(3,2)=0$, the vector $(3,2)$ is parallel to it. So a parametric equation is $\mathbf{x}=(3t-1,2t-1)$.
(b) Writing $(x,y)=(t-1,3t+1)$ we have

$$3x-y=(3t-3)-(3t+1)=-4,$$

so implicit equations are $3x-y=-4$ or $(3,-1)\cdot\mathbf{x}=-4$.
(c) A parametric equation is

$$\mathbf{x}=t(1,1)+(1,-1)=(t+1,t-1),$$

while an implicit equation is $x-y=2$.
(d) If $(s+1,s-1)$ is the point of intersection of $\mathcal{L}_1$ and $\mathcal{L}_3$, then

$$2(s+1)-3(s-1)=1,$$

so $-s+5=1$, *i.e.*, $s=4$. So the point $P(5,3)$ is the point of intersection. To find the angle θ, notice that $(3,2)$ is parallel to $\mathcal{L}_1$ and $(1,1)$ is parallel to $\mathcal{L}_3$, so

$$\theta=\cos^{-1}\frac{(3,2)\cdot(1,1)}{|(3,2)|\,|(1,1)|}=\cos^{-1}\frac{3+2}{\sqrt{13}\,\sqrt{2}}=\cos^{-1}\frac{5}{\sqrt{26}}=\cos^{-1}\frac{5\sqrt{26}}{26}.$$

N.B. $\pi-\theta$ is also an acceptable answer.

1.2. Let $\mathbf{u}=(5,0)$ and $\mathbf{v}=(2,-1)$.
(a) The angle is

$$\cos^{-1}\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}=\cos^{-1}\hat{\mathbf{u}}\cdot\hat{\mathbf{v}}=\cos^{-1}\frac{2}{\sqrt{5}}=\cos^{-1}\frac{2\sqrt{5}}{5}\in[0,\pi/2].$$

(b) First find the unit vector

$$\hat{\mathbf{v}}=\frac{1}{|\mathbf{v}|}\mathbf{v}=\frac{1}{\sqrt{5}}(2,-1),$$

then the projection is

$$(\mathbf{u}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}=\frac{1}{\sqrt{5}}(10+0)\frac{1}{\sqrt{5}}(2,-1)=\frac{1}{5}(20,-10)=(4,-2).$$

(c) First find the unit vector

$$\hat{\mathbf{u}}=\frac{1}{|\mathbf{u}|}\mathbf{u}=\frac{1}{5}(5,0)=(1,0),$$

then the projection is

$$(\mathbf{v}\cdot\hat{\mathbf{v}})\hat{\mathbf{u}}=(2+0)(1,0)=(2,0).$$

1.3. The vector $(1,1)$ is perpendicular to $\mathcal{L}_1$ and the line $\mathbf{x}=t(1,1)+(1,0)=(t+1,t)$ meets $\mathcal{L}_1$ when $(t+1)+t=2$, so $t=1/2$ and the intersection is at $M_1(3/2,1/2)$. Then

$$\mathrm{Refl}_{\mathcal{L}_1}(1,0)=(1,0)+2\times(1/2)(1,1)=(2,1).$$

The vector $(1,-1)$ is perpendicular to $\mathcal{L}_2$ and the line $\mathbf{x}=t(1,-1)+(1,0)=(t+1,-t)$ meets $\mathcal{L}_2$ when $(t+1)+t=2$, so $t=1/2$ and the intersection is at $M_2(3/2,-1/2)$. Then

$$\mathrm{Refl}_{\mathcal{L}_2}(1,0)=(1,0)+2\times(1/2)(1,-1)=(2,-1).$$

1.4. Following the proof of Proposition 1.6, we choose O as a point on $\mathcal{L}_1$ and then take the point of form $t(2,1)+(0,0)=(2t,t)$ which lies on $\mathcal{L}_2$. For this point, $2(2t)+t=2$ which gives $5t=2$, hence $t=2/5$, so the point on $\mathcal{L}_2$ is $\mathbf{v}=(4/5,2/5)$. Then

$$\mathrm{Refl}_{\mathcal{L}_2}\circ\mathrm{Refl}_{\mathcal{L}_1}=\mathrm{Trans}_{2\mathbf{v}}=\mathrm{Trans}_{(8/5,4/5)},\quad \mathrm{Refl}_{\mathcal{L}_1}\circ\mathrm{Refl}_{\mathcal{L}_2}=\mathrm{Trans}_{-2\mathbf{v}}=\mathrm{Trans}_{(-8/5,-4/5)}\,.$$

1.5. (a) The relevant formula is

$$(x_1,y_1)\cdot(x_2,y_2)=x_1x_2+y_1y_2=\begin{bmatrix}x_1 & y_1\end{bmatrix}\begin{bmatrix}x_2\\ y_2\end{bmatrix}=\begin{bmatrix}x_1\\ y_1\end{bmatrix}^T\begin{bmatrix}x_2\\ y_2\end{bmatrix}.$$

(b) We have

$$U\mathbf{x}=\mathbf{u}(\mathbf{u}^T\mathbf{x})=\mathbf{u}(\mathbf{u}\cdot\mathbf{x})=(\mathbf{u}\cdot\mathbf{x})\mathbf{u}.$$

As $\mathbf{u}$ is a unit vector, $\mathbf{u}\cdot(t\mathbf{u})=t\mathbf{u}\cdot\mathbf{u}=t$. The result follows.
(c) Reflection in $\mathcal{L}$ has the effect

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{x})=\begin{cases}\mathbf{x} & \text{if } \mathbf{u}\cdot\mathbf{x}=0,\\ -\mathbf{x} & \text{if } \mathbf{x}=t\mathbf{u} \text{ for some } t\in\mathbb{R}.\end{cases}$$

By the result of (b),

$$\begin{aligned}U'\mathbf{x}=\mathbf{x}-2U\mathbf{x}&=\begin{cases}\mathbf{x} & \text{if } \mathbf{u}\cdot\mathbf{x}=0,\\ \mathbf{x}-2\mathbf{x} & \text{if } \mathbf{x}=t\mathbf{u} \text{ for some } t\in\mathbb{R},\end{cases}\\ &=\begin{cases}\mathbf{x} & \text{if } \mathbf{u}\cdot\mathbf{x}=0,\\ -\mathbf{x} & \text{if } \mathbf{x}=t\mathbf{u} \text{ for some } t\in\mathbb{R}.\end{cases}\end{aligned}$$

Since every vector can be expressed as a sum $\mathbf{x}=\mathbf{x}'+\mathbf{x}''$ with $\mathbf{u}\cdot\mathbf{x}'=0$ and $\mathbf{x}''=t\mathbf{u}$, the result follows using linearity,

$$\mathrm{Refl}_{\mathcal{L}}(\mathbf{x}'+\mathbf{x}'')=\mathrm{Refl}_{\mathcal{L}}(\mathbf{x}')+\mathrm{Refl}_{\mathcal{L}}(\mathbf{x}''),\quad U'(\mathbf{x}'+\mathbf{x}'')=U'(\mathbf{x}')+U'(\mathbf{x}'').$$

1.6. (a) This is rotation through $-\pi/4$ anticlockwise (or equivalently through $\pi/4$ clockwise) about the point with position vector

$$\begin{aligned}\begin{bmatrix}1-1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1-1/\sqrt{2}\end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}&=\frac{1}{(1-1/\sqrt{2})^2+1/2}\begin{bmatrix}1-1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\\ &=\frac{1}{2-\sqrt{2}}\begin{bmatrix}1-1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\\ &=\frac{1}{2(\sqrt{2}-1)}\begin{bmatrix}\sqrt{2}-1 & 1\\ -1 & \sqrt{2}-1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\\ &=\frac{\sqrt{2}+1}{2}\begin{bmatrix}\sqrt{2}\\ \sqrt{2}-2\end{bmatrix}=\begin{bmatrix}1+1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}\end{aligned}$$

(b) This represents reflection in the line $\mathcal{L}\colon\ \sin(\pi/8)x-\cos(\pi/8)y=0$ followed with translation by a vector. It is therefore a glide reflection obtained by reflecting in a line parallel to $\mathcal{L}$ followed with translation by a vector $\mathbf{u}$ also parallel to $\mathcal{L}$. To determine $\mathbf{u}$ and the reflection line, use the unit vector $(\sin(\pi/8),-\cos(\pi/8))$ perpendicular to $\mathcal{L}$ and find the component of $\mathbf{t}$ parallel to it,

$$\begin{aligned}&(\,(\sin(\pi/8),-\cos(\pi/8))\cdot(1,-1)\,)\,(\sin(\pi/8),-\cos(\pi/8))\\ &\qquad=(\sin(\pi/8)+\cos(\pi/8))(\sin(\pi/8),-\cos(\pi/8)).\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{v} &= \frac{(\sin(\pi/8)+\cos(\pi/8))}{2}(\sin(\pi/8), -\cos(\pi/8)) \\
&= \frac{1}{2}(\sin^2(\pi/8)+\cos(\pi/8)\sin(\pi/8), -\cos(\pi/8)\sin(\pi/8)-\cos^2(\pi/8)) \\
&= \frac{1}{4}(2\sin^2(\pi/8)+2\cos(\pi/8)\sin(\pi/8), -2\cos(\pi/8)\sin(\pi/8)-2\cos^2(\pi/8)) \\
&= \frac{1}{4}(1-\cos(\pi/4)+\sin(\pi/4), -\sin(\pi/4)-1-\cos(\pi/4)) \\
&= \frac{1}{4}(1-1/\sqrt{2}+1/\sqrt{2}, -2/\sqrt{2}-1) = \frac{1}{4}(1, -\sqrt{2}-1), \\
\mathbf{u} &= \mathbf{t}-2\mathbf{v} = (1,-1) - \frac{1}{2}(1,-\sqrt{2}-1) = (1/2, (\sqrt{2}-1)/2).
\end{aligned}$$

So this is a reflection in the line

$$\sin(\pi/8)x - \cos(\pi/8)y = \frac{(\sin(\pi/8)+\cos(\pi/8))}{2},$$

followed by translation by $(1/2, (\sqrt{2}-1)/2)$.
(c) This is another glide reflection obtained by reflection in the x-axis (*i.e.*, $y = 0$) followed by translation by $(0, 1)$ which is perpendicular to this line. So taking $\mathbf{v} = (0, 1/2)$ and $\mathbf{u} = (0, 0)$, we see that it is a reflection in the line $y = 1/2$.

For $(A \mid \mathbf{t})^2 = (A^2 \mid A\mathbf{t}+\mathbf{t})$, the three cases give

(a) $$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad A\mathbf{t}+\mathbf{t} = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix};$$

(b) $$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \qquad A\mathbf{t}+\mathbf{t} = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix};$$

(c) $$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \qquad A\mathbf{t}+\mathbf{t} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So (b) and (c) give translations, while (a) gives a clockwise rotation through angle $\pi/2$ about the point with position vector

$$(I_2 - A^2)^{-1}\begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}.$$

1.7. (a) By the product formula for determinants, valid for any pair of $n \times n$ matrices A, B, $\det(AB) = \det A \det B$. In particular, if A is orthogonal we have $A^T A = I_n$ and so

$$(\det A)^2 = \det A \det A = \det A^T \det A = \det(A^T A) = \det I_n = 1,$$

which implies that $\det A = \pm 1$.
(b) Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. By the orthogonality condition $A^T A = I_2$,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+c^2 & ab+cd \\ ba+dc & b^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and by (a) we also have $\det A = ad - bc = 1$. This gives four distinct equations

$$\begin{cases} a^2+c^2 = 1, \\ b^2+d^2 = 1, \\ ab+cd = 0 \\ ad-bc = 1. \end{cases}$$

Now set $a = \cos\theta$, $c = \sin\theta$, $b = \sin\varphi$ and $d = \cos\varphi$ for real numbers θ and φ. Then the third equation gives

$$\cos\theta\sin\varphi + \sin\theta\cos\varphi = 0$$

and using the double angle formula we obtain

$$\sin(\theta+\varphi) = 0.$$

The fourth equation gives

$$\cos\theta\cos\varphi - \sin\theta\sin\varphi = 1$$

which yields the equation

$$\cos(\theta+\varphi) = 1$$

which has solution $\varphi = -\theta + 2k\pi$ for $k \in \mathbb{Z}$. Thus we obtain

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin(-\theta) \\ \sin\theta & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(c) By the product formula for determinants we have $\det C = 1$, so by (b) C has the form

$$C = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

and so

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = C\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

(d) Suppose that $P^TP = I_n = Q^TQ$. Then

$$(PQ)^T(PQ) = (Q^TP^T)(PQ) = Q^T(P^TP)Q = Q^TI_nQ = Q^TQ = I_n.$$

1.8. Composing these two we have

$$(A \mid \mathbf{t})(A^T \mid -A^T\mathbf{t}) = (AA^T \mid \mathbf{t} - AA^T\mathbf{t}) = (I_2 \mid \mathbf{t} - \mathbf{t}) = (I_2 \mid \mathbf{0})$$

since $A^TA = I_2 = AA^T$, and

$$(A^T \mid -A^T\mathbf{t})(A \mid \mathbf{t}) = (A^TA \mid A^T\mathbf{t} - A^T\mathbf{t}) = (I_2 \mid \mathbf{0}).$$

Thus we have

$$(A \mid \mathbf{t})(A^T \mid -A^T\mathbf{t}) = \mathrm{Id}_{\mathbb{R}^2} = (A^T \mid -A^T\mathbf{t})(A \mid \mathbf{t}).$$

1.9. (a) Direct computation shows that

$$G(\mathbf{0}) = \mathrm{Trans}_{-\mathbf{p}} \circ F(\mathbf{0}+\mathbf{p}) = \mathrm{Trans}_{-\mathbf{p}}(F(\mathbf{p})) = \mathrm{Trans}_{-\mathbf{p}}(\mathbf{p}) = \mathbf{0}.$$

If F is a rotation about P then G is a rotation about O through the same angle; if F is reflection in a line through P then G is reflection in the line through O parallel to it.
(b) This is similar to (a), with Q replacing O in the description of G.

1.10. (a) We have

$$\begin{aligned}(A \mid \mathbf{t})(s\mathbf{x} + (1-s)\mathbf{y}) &= A(s\mathbf{x} + (1-s)\mathbf{y}) + \mathbf{t} \\ &= A(s\mathbf{x}) + A((1-s)\mathbf{y}) + \mathbf{t} \\ &= sA\mathbf{x} + A(1-s)\mathbf{y} + s\mathbf{t} + (1-s)\mathbf{t} \\ &= s(A \mid \mathbf{t})\mathbf{x} + (1-s)(A \mid \mathbf{t})\mathbf{y}.\end{aligned}$$

(b) [*N.B. This result is used in Chapter 2 when discussing finite subgroups of* Euc(2).]
We have

$$\begin{aligned}(A \mid \mathbf{t})(s_1\mathbf{x}_1 + \cdots + s_n\mathbf{x}_n) &= A(s_1\mathbf{x}_1 + \cdots + s_n\mathbf{x}_n) + \mathbf{t}\\ &= s_1A\mathbf{x}_1 + s_2A\mathbf{x}_2 + \cdots + s_nA\mathbf{x}_n + (s_1 + \cdots + s_n)\mathbf{t}\\ &= s_1(A\mathbf{x}_1 + \mathbf{t}) + s_2(A\mathbf{x}_2 + \mathbf{t}) + \cdots + s_n(A\mathbf{x}_n + \mathbf{t})\\ &= s_1(A \mid \mathbf{t})\mathbf{x}_1 + s_2(A \mid \mathbf{t})\mathbf{x}_2 + \cdots + s_n(A \mid \mathbf{t})\mathbf{x}_n.\end{aligned}$$