Mathematics 2Q – Solutions for Chapter 2

2.1. (a) We have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 1 & 2 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}$$

(b) There are two disjoint cycles

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 1, \quad 4 \longrightarrow 5 \longrightarrow 6 \longrightarrow 4,$$

so

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} = (1\ 2\ 3)(4\ 5\ 6) = (4\ 5\ 6)(1\ 2\ 3).$$

(c) From (b) we have

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} = (1\ 2\ 3)(4\ 5\ 6) = (1\ 3)(1\ 2)(4\ 6)(4\ 5).$$

Using the multiplicativity of sgn and its value on a transposition, we have

$$\operatorname{sgn}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} = \operatorname{sgn}(1\ 3)\operatorname{sgn}(1\ 2)\operatorname{sgn}(4\ 6)\operatorname{sgn}(4\ 5) = (-1)^4 = 1.$$

2.2. We have $(2\ 3\ 5\ 6)(1\ 6\ 2\ 3) = (1\ 2\ 5\ 6\ 3)$, $(2\ 3)(1\ 6\ 2)(5\ 6\ 2\ 4) = (1\ 6)(2\ 4\ 5\ 3)$ and $(5\ 6\ 2\ 4)^{-1} = (2\ 6\ 5\ 4)$.

2.3. (a) There are five anti-clockwise rotations:

$$\begin{aligned}
\text{rotation through } 0 \quad &= \iota, \\
\text{rotation through } 2\pi/5 &= (A\ B\ C\ D\ E), \\
\text{rotation through } 4\pi/5 &= (A\ C\ E\ B\ D), \\
\text{rotation through } 6\pi/5 &= (A\ D\ B\ E\ C), \\
\text{rotation through } 8\pi/5 &= (A\ E\ D\ C\ B).
\end{aligned}$$

There are five reflections in lines through O and a vertex:

$$\begin{aligned}
\text{reflection in } OA &= (B\ E)(C\ D), \\
\text{reflection in } OB &= (A\ C)(B\ D), \\
\text{reflection in } OC &= (A\ E)(B\ D), \\
\text{reflection in } OD &= (A\ B)(C\ E), \\
\text{reflection in } OE &= (A\ D)(B\ C).
\end{aligned}$$

(b) From (a), these are represented by $(B\ E)(C\ D)$ and $(A\ E)(B\ D)$. The compositions are

$$\begin{aligned}(B\ E)(C\ D)(A\ E)(B\ D) &= (A\ B\ C\ D\ E),\\ (A\ E)(B\ D)(B\ E)(C\ D) &= (A\ E\ D\ C\ B),\end{aligned}$$

which correspond to rotations through 1/5 of a turn in the anti-clockwise and clockwise directions.
(c) From (a), these are represented by $(B\ E)(C\ D)$ and $(A\ D\ B\ E\ C)$. The compositions are

$$\begin{aligned}(B\ E)(C\ D)(A\ D\ B\ E\ C) &= (A\ C)(B)(D\ E) = (A\ C)(D\ E),\\ (A\ D\ B\ E\ C)(B\ E)(C\ D) &= (A\ D)(B\ C)(E) = (A\ D)(B\ C),\end{aligned}$$

which correspond to reflections in the lines OB and OE.

2.4. On labelling the vertices A–E in each figure, (i) has essentially the same symmetry group as the regular pentagon of the previous question. On the other hand, symmetries of (ii) must preserve the direction of the arrows and so only rotational symmetries occur, giving 5 in all, including the identity.

2.5. (a) If $\alpha, \beta \in \Gamma_{S,P}$ then

$$\begin{aligned}\alpha\beta(P) &= \alpha(\beta(P)) = \alpha(P) = P,\\ \mathrm{Id}(P) &= P,\\ \alpha^{-1}(P) &= \alpha^{-1}(\alpha(P)) = \alpha^{-1}\alpha(P) = \mathrm{Id}(P) = P.\end{aligned}$$

This shows that $\alpha\beta, \mathrm{Id}, \alpha^{-1} \in \Gamma_{S,P}$, so $\Gamma_{S,P} \leqslant \mathrm{Euc}(2)_S$.
(b) From Example 2.24 in the Notes,

$$\Gamma_{S,D} = \left\{ \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix}, \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}. \right\} = \left\{ \iota, \begin{pmatrix} A & B & C & D \\ C & B & A & D \end{pmatrix}. \right\}$$

(c) From Example 2.25 in the Notes,

$$\Gamma_{S,D} = \left\{ \iota = \begin{pmatrix} A & B & C & D \\ A & B & C & D \end{pmatrix} \right\} = \{\iota\}.$$

2.6. The direct symmetries of $\mathbb{T}$ are all possible rotations about the origin, $\mathrm{Rot}_{O,\theta}$ with $\theta \in [0, 2\pi)$, while the indirect symmetries are reflections in lines of the form

$$x \sin\varphi - y\cos\varphi = 0$$

for $\varphi \in [0, \pi)$. The rotations form the subgroup $\mathrm{SO}(2) \leqslant \mathrm{Euc}(2)$, while the reflections form the subset

$$\left\{ \begin{bmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{bmatrix} : \varphi \in [0, \pi) \right\} \subseteq \mathrm{Euc}(2).$$

It is easily checked that each dihedral group D_{2n} is a finite subgroup $\mathrm{D}_{2n} \leqslant \mathrm{Euc}(2)$.

2.7. Since every similarity transformation is obtained by composing an isometry with a dilation it is sufficient to verify this when H is a dilation. Expressing H as a Seitz symbol,

$$H = (\delta I \mid (1-\delta)\mathbf{c}).$$

Then if P, Q, R are three distinct points we have

$$H(\mathbf{p}) = \delta\mathbf{p} + (1-\delta)\mathbf{c},\ H(\mathbf{q}) = \delta\mathbf{q} + (1-\delta)\mathbf{c},\ H(\mathbf{r}) = \delta\mathbf{r} + (1-\delta)\mathbf{c}.$$

Then

$$H(\mathbf{q}) - H(\mathbf{p}) = \delta(\mathbf{q} - \mathbf{p}), \quad H(\mathbf{r}) - H(\mathbf{p}) = \delta(\mathbf{r} - \mathbf{p}),$$

so the cosine of the angle between $H(P)H(Q)$ and $H(P)H(R)$ is

$$\begin{aligned}\cos\angle H(Q)H(P)H(R) &= \frac{(H(\mathbf{q})-H(\mathbf{p}))\cdot(H(\mathbf{r})-H(\mathbf{p}))}{|H(\mathbf{q})-H(\mathbf{p})|\,|H(\mathbf{r})-H(\mathbf{p})|}\\ &= \frac{\delta(\mathbf{q}-\mathbf{p})\cdot\delta(\mathbf{r}-\mathbf{p})}{|\delta(\mathbf{q}-\mathbf{p})|\,|\delta(\mathbf{r}-\mathbf{p})|}\\ &= \frac{\mathbf{q}-\mathbf{p}\cdot\mathbf{r}-\mathbf{p}}{|\mathbf{q}-\mathbf{p}|\,|\mathbf{r}-\mathbf{p}|} = \cos\angle QPR.\end{aligned}$$

2.8. The Seitz symbol of a similarity transformation has the form $(\delta A \mid \mathbf{s})$, where A is orthogonal, $\delta > 0$ and $\mathbf{s} \in \mathbb{R}^2$. Given two such transformations $(\delta_1 A_1 \mid \mathbf{s}_1)$ and $(\delta_2 A_2 \mid \mathbf{s}_2)$, we have

$$\begin{aligned}(\delta_1 A_1 \mid \mathbf{s}_1)(\delta_2 A_2 \mid \mathbf{s}_2) &= ((\delta_1\delta_2)(A_1A_2) \mid \mathbf{s}_1 + \delta_1 A_1 \mathbf{s}_2),\\ \mathrm{Id} &= (1\cdot I_2 \mid \mathbf{0}),\\ (\delta A \mid \mathbf{s})^{-1} &= ((1/\delta)A^{-1} \mid -(1/\delta)A^{-1}\mathbf{s}) = ((1/\delta)A^T \mid -(1/\delta)A^T\mathbf{s}).\end{aligned}$$

Each of these is a similarity transformation, so $(\Sigma(2), \circ)$ is a group. Since $\mathrm{Euc}(2) \subseteq \Sigma(2)$ it is clearly a subgroup. Another subgroup consists of all the scalings centred at the origin,

$$\{(\delta I_2 \mid \mathbf{0}) : \delta > 0\} \leqslant \Sigma(2).$$

2.9. If the lines are parallel, then following the ideas of Example 2.29, let $\mathbf{p}_1, \mathbf{p}_2$ be the position vectors of points P_1, P_2 on these lines. Setting

$$\mathbf{t} = \mathbf{p}_2 - \mathbf{p}_1 = \overrightarrow{P_1P_2},$$

we have

$$\mathrm{Refl}_{\mathcal{L}_2} = \mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Trans}_{-\mathbf{t}},$$

so $\mathrm{Refl}_{\mathcal{L}_2}$ is similar to $\mathrm{Refl}_{\mathcal{L}_1}$.

If the lines are not parallel then they meet at a point M say. There is then a rotation about M, $\mathrm{Rot}_{M,\theta}$ say, which maps $\mathcal{L}_1$ into $\mathcal{L}_2$. Then

$$\mathrm{Refl}_{\mathcal{L}_2} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ \mathrm{Rot}_{M,-\theta} = \mathrm{Rot}_{M,\theta} \circ \mathrm{Refl}_{\mathcal{L}_1} \circ {\mathrm{Rot}_{M,\theta}}^{-1}.$$

2.10. From the Notes, the Seitz symbol of $\mathrm{Rot}_{C,\theta}$ is $(R \mid (I-R)\mathbf{c})$ where $R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$. Then the Seitz symbol of $\mathrm{Trans}_{\mathbf{t}} \circ \mathrm{Rot}_{C,\theta}$ is

$$(I_2 \mid \mathbf{t})(R \mid (I-R)\mathbf{c}) = (R \mid \mathbf{t} + (I-R)\mathbf{c}).$$

This represents rotation through angle θ about the point with position vector $\mathbf{c}'$ where

$$(R \mid \mathbf{t} + (I-R)\mathbf{c})\mathbf{c}' = \mathbf{c}',$$

which gives

$$R\mathbf{c}' + \mathbf{t} + (I-R)\mathbf{c} = \mathbf{c}',$$

and solving this we obtain

$$\mathbf{c}' = (I-R)^{-1}\mathbf{t} + \mathbf{c}.$$

Expanding out we find

$$\begin{aligned}
(I-R)^{-1} &= \begin{bmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 2\sin^2(\theta/2) & 2\sin(\theta/2)\cos(\theta/2) \\ -2\sin(\theta/2)\cos(\theta/2) & 2\sin^2(\theta/2) \end{bmatrix}^{-1} \\
&= \frac{1}{2\sin(\theta/2)} \begin{bmatrix} \sin(\theta/2) & \cos(\theta/2) \\ -\cos(\theta/2) & \sin(\theta/2) \end{bmatrix}^{-1} \\
&= \frac{1}{2\sin(\theta/2)} \begin{bmatrix} \sin(\theta/2) & -\cos(\theta/2) \\ \cos(\theta/2) & \sin(\theta/2) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & -\cot(\theta/2) \\ \cot(\theta/2) & 1 \end{bmatrix}.
\end{aligned}$$

2.11. (a) Notice that

$$C = \begin{bmatrix} \cos(4\pi/3) & -\sin(4\pi/3) \\ \sin(4\pi/3) & \cos(4\pi/3) \end{bmatrix},$$

from which it easily follows that $(C^2 + C + I) = O$. Then Γ has the three distinct elements

$$\begin{aligned}
\gamma &= (C \mid \mathbf{w}), \\
\gamma^2 &= (C \mid \mathbf{w})^2 = (C^2 \mid (C+I)\mathbf{w}), \\
\gamma^3 &= (C^3 \mid (C^2 + C + I)\mathbf{w}) = (I \mid \mathbf{0}).
\end{aligned}$$

Here

$$C^2 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}, \quad (C+I)\mathbf{w} = \begin{bmatrix} \sqrt{3}/2 \\ 3/2 \end{bmatrix}$$

(b) Following the proof of Theorem 2.32 with $\mathbf{p} = \mathbf{0}$ we obtain the fixed point

$$\mathbf{p}_0 = \frac{1}{3}\left(\mathbf{w} + (C+I)\mathbf{w} + \mathbf{0}\right) = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This is the only fixed point for Γ since γ, γ^2 are non-trivial rotations which fix only their common centre $\mathbf{p}_0$.

(c) Take $\psi = \mathrm{Trans}_{-\mathbf{p}_0}$.

2.12. Consider the finite subgroup $\Gamma \leqslant \mathrm{Euc}(2)$ of order 8 generated by the isometries $\alpha = (A \mid \mathbf{u})$ and $\beta = (B \mid \mathbf{v})$ where

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(a) Writing $\alpha^r = (A_r \mid \mathbf{u}_r)$, we have

$$\begin{array}{llll}
A_1 = A, & \mathbf{u}_1 = \mathbf{u}, & A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, & \mathbf{u}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \\
A_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \mathbf{u}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, & A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{u}_4 = \mathbf{0}.
\end{array}$$

Similarly, writing $\alpha^r\beta = (B_r \mid \mathbf{v}_r)$,

$$\begin{array}{llll}
B_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{v}_1 = \mathbf{0}, & B_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, & \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
B_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, & \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, & B_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.
\end{array}$$

(b) Following the proof of Theorem 2.32 with $\mathbf{p} = \mathbf{0}$ we obtain the fixed point

$$\mathbf{p}_0 = \frac{1}{8}\left(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 + \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\right) = \frac{1}{8}\begin{bmatrix}0\\8\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}.$$

This is the only fixed point of Γ since α, α^3 are distinct rotations about their common centre.
(c) Take $\varphi = \mathrm{Trans}_{-\mathbf{p}_0}$.

2.13. (a) The composition $F \circ G$ is a non-trivial translation which has no fixed points. Hence Γ has no common fixed points and so cannot be finite.
(b) The only fixed point of the rotation G is $\mathbf{p}$, but F does not fix $\mathbf{p}$. Hence F and G have no common fixed points, so nor does Γ which therefore cannot be finite.

2.14. If $\mathbf{p}$ is fixed by F and G, it is also fixed by their inverses since

$$F^{-1}(\mathbf{p}) = F^{-1}(F(\mathbf{p})) = \mathbf{p}, \quad G^{-1}(\mathbf{p}) = G^{-1}(G(\mathbf{p})) = \mathbf{p}.$$

Also, any power of F or G fixes $\mathbf{p}$. Hence since any element of Γ is a product of powers of F and G it fixes $\mathbf{p}$.

2.15. A=6; B=2; C=4; D=6; E=3; F=7.

2.16. (a) pmm; (b) cmm; (c) pm (this is the diagram in the Notes but rotated through $\pi/2$); (d) p2g.