# Mathematics 2Q – Solutions for Chapter 3

3.1. The vectors

$$\mathbf{u} = (0, 2, 1) - (1, 1, -2) = (-1, 1, 3), \quad \mathbf{v} = (1, -1, 2) - (1, 1, -2) = (0, -2, 4),$$

are parallel to $\mathcal{P}$ and linearly independent. Their vector product is

$$\mathbf{w} = \mathbf{u} \times \mathbf{v} = (10, 4, 2)$$

and this vector is normal to $\mathcal{P}$. Then an implicit equation for $\mathcal{P}$ is

$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w} \cdot (1, 1, -2) = 10 + 4 - 4 = 10$$

since $A(1, 1, -2)$ is in $\mathcal{P}$. This gives

$$10x + 4y + 2z = 10.$$

For a parametric form we have

$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} + (1, 1, -2) \quad (s, t \in \mathbb{R})$$

which is equivalent to

$$(x, y, z) = (-s + t + 1, s - 2t + 1, 3s + 4t - 2) \quad (s, t \in \mathbb{R}).$$

Of course, the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ could all be replaced by suitable unit vectors.

3.2. (a) Let the matrix be $R$. It is easily checked that $R$ is orthogonal, *i.e.*, $R^T R = I_3 = RR^T$. Then expanding along the third row we have

$$\begin{aligned}
\det R &= \frac{1}{\sqrt{2}} \begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} \end{vmatrix} - \frac{(-1)}{\sqrt{2}} \begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} \end{vmatrix} + 0 \\
&= \frac{2}{\sqrt{2}} \begin{vmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} \end{vmatrix} \\
&= \sqrt{2} \left( \frac{-2}{\sqrt{3}\sqrt{6}} - \frac{1}{\sqrt{3}\sqrt{6}} \right) = -\frac{3\sqrt{2}}{3\sqrt{2}} = -1.
\end{aligned}$$

So there must be a plane of reflection. Consider the equation $A\mathbf{u} = -\mathbf{u}$ or the equivalent equation $(I_3 + A)\mathbf{u} = \mathbf{0}$. Writing $\mathbf{u} = (u_1, u_2, u_3)$, we have

$$\begin{bmatrix} 1 + 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1 + 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and solving by row reduction the general solution turns out to be

$$\mathbf{u} = t(\sqrt{6} + 3 - 2\sqrt{2} - 2\sqrt{3}, 1, -\sqrt{3} - \sqrt{2} + 2 + \sqrt{6}) \quad (t \in \mathbb{R}).$$

Thus the reflecting plane is $\mathbf{u} \cdot \mathbf{x} = 0$.
(b) Let $S$ be the matrix. This time we find that $S$ is orthogonal and $\det S = 1$ so there must be a line of rotation. To find this solve the equation $S\mathbf{x} = \mathbf{x}$, or equivalently $(S - I_3)\mathbf{x} = \mathbf{0}$. The general solution turns out to be

$$\mathbf{x} = t(-2\sqrt{2} + 2\sqrt{3} + 3 - \sqrt{6}, 1, -2 + \sqrt{6} + \sqrt{2} - \sqrt{3}) \quad (t \in \mathbb{R}),$$

which is the line of rotation in parametric form.

3.3. First we need to check that

$$\det S = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{vmatrix} = -1,$$

so $(S \mid \mathbf{0})$ does correspond to a reflection in a plane through the origin. Now look for vectors satisfying $S\mathbf{x} = -\mathbf{x}$ or equivalently $(I + S)\mathbf{x} = \mathbf{0}$. Setting $\mathbf{x} = (x, y, z)$ we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

whose solution is $x = -t$, $y = t$, $z = t$ $(t \in \mathbb{R})$. Taking $t = -1/\sqrt{3}$ we obtain the unit vector

$$\mathbf{w} = (1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$$

normal to the reflecting plane which has implicit equation $x - y - z = 0$.

For the translation vector we have $(1, 0, 1) \cdot \mathbf{w} = 0$, hence $\mathbf{t}$ is parallel to the reflecting plane of $(S \mid \mathbf{0})$. Therefore,

$$(S \mid \mathbf{t}) = (I_3 \mid (1, 0, 1))(S \mid \mathbf{0}),$$

the composition of reflection in $x - y - z = 0$ with translation by $(1, 0, 1)$ parallel to this plane.

3.4. We have

$$\det R = \begin{vmatrix} -1/2 & 0 & -\sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{vmatrix} = 1,$$

so $(R \mid \mathbf{0})$ represents a rotation. Notice that

$$R \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

hence $(R \mid \mathbf{0})$ corresponds to a rotation about the $y$-axis. If we take the vectors

$$\mathbf{v}_1 = (0, 1, 0), \quad \mathbf{v}_2 = (0, 0, 1), \quad \mathbf{v}_3 = (1, 0, 0),$$

these form a right handed orthonormal system and we find that

$$R\mathbf{v}_2 = -\frac{1}{2}\mathbf{v}_2 - \frac{\sqrt{3}}{2}\mathbf{v}_3, \quad R\mathbf{v}_3 = \frac{\sqrt{3}}{2}\mathbf{v}_2 - \frac{1}{2}\mathbf{v}_3,$$

so we have

$$R(x_1'\mathbf{v}_1 + x_2'\mathbf{v}_2 + x_3'\mathbf{v}_3) = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} \begin{bmatrix}1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2\end{bmatrix} \begin{bmatrix}x_1' \\ x_2' \\ x_3'\end{bmatrix}$$

From Equation 3.3, the angle of rotation is $\theta = \cos^{-1}(-1/2)$ since $\sin\theta = \sqrt{3}/2 > 0$, so $\theta = 2\pi/3$.

We also have

$$\mathbf{t} = (1,1,0) = \mathbf{v}_1 + \mathbf{v}_3 = \mathbf{v}_1 + 0\mathbf{v}_2 + \mathbf{v}_3.$$

We need to find the vector $\mathbf{c} = c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for which $(I_3 - R)\mathbf{c} = \mathbf{v}_3$. From the Notes, this is given by

$$\begin{aligned}\begin{bmatrix}c_2 \\ c_3\end{bmatrix} &= \begin{bmatrix}3/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2\end{bmatrix}^{-1} \begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}\sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2\end{bmatrix}^{-1} \begin{bmatrix}0 \\ 1\end{bmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix}\sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2\end{bmatrix} \begin{bmatrix}0 \\ 1\end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix}-1/2 \\ \sqrt{3}/2\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/6 \\ 1/2\end{bmatrix}.\end{aligned}$$

So the axis of rotation is the line parallel to the $y$-axis which contain the point

$$\mathbf{c} = (0,0,-\sqrt{3}/6) + (1/2,0,0) = (1/2,0,-\sqrt{3}/6),$$

*i.e.*, the line with parametric equation

$$\mathbf{x} = t(0,1,0) + (1/2,0,-\sqrt{3}/6) = (1/2,t,-\sqrt{3}/6) \quad (t \in \mathbb{R}).$$

The translation vector parallel to this axis of rotation is $\mathbf{v}_1 = (0,1,0)$.