$\mathcal{E}_\infty$ ring spectra and Hopf invariant one elements

Andrew Baker, University of Glasgow

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Hopf invariant one elements

Conventions: Everything will be 2-local. Homology and cohomology will usually be taken with $\mathbb{F}_2$ coefficients, so $H_*(-) = H_*(-;\mathbb{F}_2)$ and $H^*(-) = H^*(-;\mathbb{F}_2)$.

Recall the elements of Hopf invariant 1:

$$2 \in \pi_0(S^0) \cong \mathbb{Z}, \qquad \eta \in \pi_1(S^0) \cong \mathbb{Z}/2,$$
$$\nu \in \pi_3(S^0) \cong \mathbb{Z}/8, \qquad \sigma \in \pi_7(S^0) \cong \mathbb{Z}/16.$$

In the cohomology of the mapping cones,

$$H^*(C_2) = \mathbb{F}_2\{t^0, t^2\}, \quad \mathrm{Sq}^1\, t^0 = t^1,$$
$$H^*(C_\eta) = \mathbb{F}_2\{u^0, u^2\}, \quad \mathrm{Sq}^2\, u^0 = u^2,$$
$$H^*(C_\nu) = \mathbb{F}_2\{v^0, v^4\}, \quad \mathrm{Sq}^4\, v^0 = v_4,$$
$$H^*(C_\sigma) = \mathbb{F}_2\{w^0, w^8\}, \quad \mathrm{Sq}^8\, w^0 = w^8.$$

These elements satisfy algebraic relations such as $2\eta = 0 = \eta\nu$, allowing the following mapping cones to be constructed:

$$S^0 \cup_\eta e^2 \cup_2 e^3, \quad S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7, \quad S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$

The cohomology of these as modules over the Steenrod algebra $\mathcal{A}^*$ are simple to describe.

$$H^*(S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15})$$

$$\begin{tikzcd} \overset{0}{\bullet} \arrow[r, bend right, "\mathrm{Sq}^8"'] & \overset{8}{\bullet} \arrow[r, bend right, "\mathrm{Sq}^4"'] & \overset{12}{\bullet} \arrow[r, bend right, "\mathrm{Sq}^2"'] & \overset{14}{\bullet} \arrow[r, bend right, "\mathrm{Sq}^1"'] & \overset{15}{\bullet} \end{tikzcd}$$

Lemma

$$S^0 \cup_\eta e^2 \cup_2 e^3 \sim H\mathbb{Z}^{[3]}, \quad S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7 \sim k\mathrm{O}^{[7]},$$
$$S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15} \sim \mathrm{tmf}^{[15]}.$$

Here $X^{[n]}$ denotes the n-skeleton of a CW spectrum X.
$H\mathbb{Z}$ is the integral *Eilenberg-Mac Lane spectrum*, $k\mathrm{O}$ is the *real connective K-theory* spectrum and tmf is the connective spectrum of *topological modular forms*.

These iterated mapping cones exist in nature. The classifying spaces $B\mathrm{SO} = B\mathrm{O}\langle 2\rangle$, $B\mathrm{Spin} = B\mathrm{O}\langle 4\rangle$ and $B\mathrm{String} = B\mathrm{O}\langle 8\rangle$ are infinite loop spaces with the following low-dimensional skeleta:

$$B\mathrm{SO}^{[3]} = S^2 \cup_2 e^3, \quad B\mathrm{Spin}^{[7]} = S^4 \cup_\eta e^6 \cup_2 e^7,$$
$$B\mathrm{String}^{[15]} = S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$

and the associated Thom spectra have skeleta

$$M\mathrm{SO}^{[3]} = S^0 \cup_\eta e^2 \cup_2 e^3, \quad M\mathrm{Spin}^{[7]} = S^0 \cup_\nu S^4 \cup_\eta e^6 \cup_2 e^7,$$
$$M\mathrm{String}^{[15]} = S^0 \cup_\sigma S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$

Each skeletal inclusion map factors through an infinite loop map,

$$\begin{tikzcd}
B\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \arrow[rr] \arrow[dr] & & B\mathrm{O}\langle 2^r\rangle \\
& \mathrm{Q}B\mathrm{O}\langle 2^r\rangle^{[2^{r+1}-1]} \arrow[ur, "j_r"] &
\end{tikzcd}$$

where $\mathrm{Q} = \Omega^\infty\Sigma^\infty$ is the free infinite loop space functor. The associated Thom spectra Mj_1, Mj_2 and Mj_3 are $\mathcal{E}_\infty$ ring spectra.

Theorem

The homology rings of the Thom spectra Mj_r are given by

$$\begin{aligned}
&H_*(Mj_1) = \\
&\quad \mathbb{F}_2[\mathrm{Q}^I x_2, \mathrm{Q}^J x_3 : I, J \text{ admissible, } \mathrm{exc}(I) > 2,\ \mathrm{exc}(J) > 3], \\
&H_*(Mj_2) = \mathbb{F}_2[\mathrm{Q}^I x_4, \mathrm{Q}^J x_6, \mathrm{Q}^K x_7 : \\
&\quad I, J, K \text{ admissible, } \mathrm{exc}(I) > 4,\ \mathrm{exc}(J) > 6,\ \mathrm{exc}(K) > 7], \\
&H_*(Mj_3) = \mathbb{F}_2[\mathrm{Q}^I x_8, \mathrm{Q}^J x_{12}, \mathrm{Q}^K x_{14}, \mathrm{Q}^L x_{15} : I, J, K, L \text{ admissible,} \\
&\qquad \mathrm{exc}(I) > 8,\ \mathrm{exc}(J) > 12,\ \mathrm{exc}(K) > 14,\ \mathrm{exc}(L) > 15].
\end{aligned}$$

There are $\mathcal{E}_\infty$ morphisms $Mj_r \to H\mathbb{F}_2$ inducing homomorphisms $H_*(Mj_r) \to \mathcal{A}_*$ with images

$$\mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \cong H_*(H\mathbb{Z}), \quad \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \cong H_*(k\mathrm{O}),$$
$$\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \cong H_*(\mathrm{tmf}).$$

In this result,

$$x_2 \mapsto \zeta_1^2, \quad x_3 \mapsto \zeta_2,$$

$$x_4 \mapsto \zeta_1^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3,$$

$$x_8 \mapsto \zeta_1^8, \quad x_{12} \mapsto \zeta_2^4, \quad x_{14} \mapsto \zeta_3^2, \quad x_{15} \mapsto \zeta_4.$$

Known result: Mark Steinberger showed that the spectrum Mj_1 is a wedge of suspensions of $H\mathbb{Z}$ and $H\mathbb{Z}/2^k$ for various $k \geqslant 1$. It seems plausible that analogous splittings should exist for the others.

Conjecture: Mj_2 is a wedge of $k\mathrm{O}$ module spectra, and Mj_3 is a wedge of tmf module spectra.

We will describe some algebraic analogues of these statements.

Some coalgebra

We work over a field $\Bbbk$ of characteristic 2 with $\Bbbk$-vector spaces (possibly graded). We will set $\otimes = \otimes_{\Bbbk}$.

Let A be a commutative Hopf algebra over $\Bbbk$, and let B be a quotient Hopf algebra of A. We denote the product on A by φ_A and the (left) coaction on a comodule M by ψ_M. We can induce a left B-comodule structure by composing.

$$\begin{tikzcd} M \arrow[r, "\psi_M"'] \arrow[rr, bend left, "\psi'_M"] & A \otimes M \arrow[r] & B \otimes M \end{tikzcd}$$

The *cotensor product* $L\Box_B M$ of a right and a left B-comodule is the equaliser of the following diagram.

$$L \otimes M \underset{\psi_L \otimes Id}{\overset{\mathrm{Id} \otimes \psi'_M}{\rightrightarrows}} L \otimes B \otimes M$$

The cotensor product $A\Box_B \Bbbk \subseteq A \otimes \Bbbk$ can be identified with a subalgebra of A using the canonical isomorphism $A \otimes \Bbbk \xrightarrow{\cong} A$. If L or M is *extended* (or *cofree*) we have

$$(U \otimes B)\Box_B M \cong U \otimes M, \quad L\Box_B(B \otimes V) \cong L \otimes V.$$

Lemma

Suppose that C is a commutative B-comodule algebra and D is a commutative A-comodule algebra. There is an isomorphism of A-comodule algebras

$$(A\Box_B C)\otimes D \xrightarrow{\cong} A\Box_B(C\otimes D),$$

where the domain has the diagonal left A-coaction and $C\otimes D$ has the diagonal left B-coaction.

Explicitly, this isomorphism has the following effect on

$$\sum_r u_r\otimes v_r\otimes x \in (A\Box_B C)\otimes D \subseteq A\otimes C\otimes D,$$

$$\sum_r u_r\otimes v_r\otimes x \longmapsto \sum_r\sum_i u_r a_i\otimes v_r\otimes x_i,$$

where $\psi_D x = \sum_i a_i\otimes x_i$.

We also need a way of generated comodule homomorphisms.

Lemma

Suppose that M is a left A-comodule and N is a left B-comodule. Then there is a natural isomorphism

$$\mathrm{Comod}_B(M, N) \xrightarrow{\cong} \mathrm{Comod}_A(M, A\Box_B N); \quad f \mapsto \widetilde{f},$$

where $\widetilde{f}$ is the unique factorisation of $(\mathrm{Id}\otimes f)\psi_M \colon M \to A \otimes N$ through $A\Box_B N$.
If M is an A-comodule algebra and N is a B-comodule algebra, then if f is an algebra homomorphism, so is $\widetilde{f}$.

Some quotient Hopf algebras of the dual Steenrod algebra

Recall that

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r : r \geqslant 1] = \mathbb{F}_2[\zeta_r : r \geqslant 1],$$

where $\xi_r, \zeta_r \in \mathcal{A}_{2^r-1}$. The coproduct and antipode are given by

$$\psi(\xi_r) = \sum_{0 \leqslant i \leqslant r} \xi_{r-i}^{2^i} \otimes \xi_i, \quad \psi(\zeta_r) = \sum_{0 \leqslant i \leqslant r} \zeta_i \otimes \zeta_{r-i}^{2^i}, \quad \zeta_r = \chi(\xi_r).$$

For $n \geqslant 0$, the ideal

$$\mathcal{I}(n) = (\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots) \lhd \mathcal{A}_* \quad (n \geqslant 0)$$

is a Hopf ideal and the quotient algebra $\mathcal{A}(n)_* = \mathcal{A}_*/\mathcal{I}(n)$ is a finite dimensional commutative Hopf algebra. The dual of $\mathcal{A}(n)_*$ is the subalgebra

$$\mathcal{A}(n)^* = \mathbb{F}_2\langle \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^{n+1}} \rangle \subseteq \mathcal{A}^*$$

generated by the listed Steenrod operations.

We also have the cotensor product $\mathcal{A}_*\square_{\mathcal{A}(n)_*}\mathbb{F}_2 \subseteq \mathcal{A}_*$ and the polynomial sub-Hopf algebra

$$\mathcal{A}_*\square_{\mathcal{A}(n)_*}\mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots],$$

where $\mathcal{A}(n)_* = \mathcal{A}_*//(\mathcal{A}_*\square_{\mathcal{A}(n)_*}\mathbb{F}_2)$.
Write $H = H\mathbb{F}_2$ and note that $H_*(H) = \mathcal{A}_*$.

Theorem

The natural morphisms of $\mathcal{E}_\infty$ ring spectra $H\mathbb{Z} \to H$, $k\mathrm{O} \to H$ and $\mathrm{tmf} \to H$ induce monomorphisms in $H_*(-)$ whose images are $\mathcal{A}_*\square_{\mathcal{A}(0)_*}\mathbb{F}_2$, $\mathcal{A}_*\square_{\mathcal{A}(1)_*}\mathbb{F}_2$ and $\mathcal{A}_*\square_{\mathcal{A}(2)_*}\mathbb{F}_2$.

Note: the Hopf invariant 1 Theorem implies that these are the only $\mathcal{A}(n)_*$ that can be realised in this way.

Here is an important fact about these quotient Hopf algebras. A right A-comodule L over a Hopf algebra A is *extended* (or *cofree*) if

$$L \cong W \otimes A$$

where the coaction is $\mathrm{Id}_W \otimes \psi_A$.

Theorem

For each $n \geqslant 0$, $\mathcal{A}_*$ is an extended right $\mathcal{A}(n)_*$-comodule:

$$\mathcal{A}_* \cong (\mathcal{A}_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2) \otimes \mathcal{A}(n)_*.$$

This implies that for any left $\mathcal{A}(n)_*$-comodule M,

$$\mathcal{A}_* \square_{\mathcal{A}(n)_*} M \cong (\mathcal{A}_* \square_{\mathcal{A}(n)_*} \mathbb{F}_2) \otimes M.$$

This can be viewed as an isomorphism of left $\mathcal{A}_*$-comodules using a suitable comodule structure on the right hand side.

$\mathcal{E}_\infty$ ring spectra and power operations

To understand these we must work in a context such as the model category $\mathscr{M}_S$ of S-modules of EKMM, which has a symmetric monoidal structure with smash product $\wedge = \wedge_S$.

In $\mathscr{M}_S$, the *commutative S-algebras* are the commutative monoids.

Untangling the underlying structure of the smash product leads to the connection with the older notion of *$\mathcal{E}_\infty$ ring spectra*.

For a spectrum X,

$$D_n X = E\Sigma_n \ltimes_{\Sigma_n} X^{(n)}.$$

When X is cofibrant the natural map is a weak equivalence

$$D_r X = E\Sigma_r \ltimes_{\Sigma_r} X^{(r)} \xrightarrow{\sim} X^{(r)}/\Sigma_r.$$

Then E is an *$\mathcal{E}_\infty$ ring spectrum* if there are suitably compatible maps $\mu_n \colon D_n E \longrightarrow E$ which extend a product map

$$\mu \colon E^{(2)} \longrightarrow D_2 E \xrightarrow{\mu_2} E.$$

The homology of D_2E can be described in terms of group homology,

$$H_*(D_2E) \cong \mathrm{H}^{\mathrm{alg}}_*(\Sigma_2; H_*(E^{\otimes 2})).$$

For $x \in H_m(E)$, there are elements $e_i \otimes x \otimes x \in H_{i+2m}(D_2E)$ and the *r-th Dyer-Lashof operation* Q^r is defined by the rule

$$\mathrm{Q}^r x = (\mu_2)_*(e_{r-m} \otimes x \otimes x) \in H_{m+r}(E).$$

These are natural for morphisms of $\mathcal{E}_\infty$ ring spectra and satisfy properties similar to the Steenrod operations. There are Adem relations of the form

$$\mathrm{Q}^r\mathrm{Q}^s = \sum_i \binom{i-s-1}{2i-r}\mathrm{Q}^{r+s-i}\mathrm{Q}^i$$

whenever $r > 2s$; these lead to notions of *admissible sequences* and *excess conditions*.

Coactions and Dyer-Lashof operations

Usually the right action of the Steenrod algebra is intertwined with the Dyer-Lashof operations using the *Nishida relations*. However, for an $\mathcal{E}_\infty$ ring spectrum X, the interaction of the coaction $\psi\colon H_*(X) \to \mathcal{A}_* \otimes H_*(X)$ can also be described.

It is better to twist this into a *right* coaction $\widetilde{\psi}\colon H_*(X) \to H_*(X) \otimes \mathcal{A}_*$. Then if $x \in H_m(X)$ and $r \geqslant m$,

$$\widetilde{\psi}\mathrm{Q}^s(x) = \sum_{k=m}^{s} \mathrm{Q}^k(\widetilde{\psi}(x)) \left[\left(\frac{\zeta(t)}{t} \right)^k \right]_{t^{s-k}}.$$

For example, if $\psi(x) = \sum_i a_i \otimes x_i$ then $\widetilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i)$ and

$$\begin{aligned}\widetilde{\psi}\mathrm{Q}^{m+1}(x) &= \sum_i x_i^2 \otimes \chi(a_i)^2\zeta_1 + \sum_i \mathrm{Q}^{m+1}(x_i \otimes \chi(a_i)) \\ &= \sum_i x_i^2 \otimes \chi(a_i)^2\zeta_1 + \sum_i \sum_j \mathrm{Q}^{m+1-j}x_i \otimes \mathrm{Q}^j\chi(a_i)).\end{aligned}$$

The Dyer-Lashof on $\mathcal{A}_*$ was determined by Kochman and Steinberger. For example,

$$\mathrm{Q}^{2^s}\zeta_s = \zeta_{s+1}.$$

Also, I showed recently that

$$\mathrm{Q}^{2^s}\xi_s = \xi_{s+1} + \xi_1\xi_s^2.$$

One way to pin down the action is by using symmetric function formula.

Some cotensor product decompositions

We would like to describe $H_*(Mj_r)$ in terms of $\mathcal{A}(r-1)_*$.

Theorem

For $r = 1, 2, 3$ there is a regular sequence $X_{r,s} \in H_*(Mj_r)$ $(s \geqslant 1)$ so that the ideal $I_r = (X_{r,s} : s \geqslant 1) \lhd H_*(Mj_r)$ is $\mathcal{A}(r-1)_*$-invariant. Furthermore, the top composition is an isomorphism in the commutative diagram of commutative $\mathcal{A}_*$-comodule algebras.

$$\begin{tikzcd}
H_*(Mj_r) \arrow[r] \arrow[rr, bend left, "\cong"] \arrow[dr, bend right, "\psi"'] & \mathcal{A}_*\square_{\mathcal{A}(r-1)_*} H_*(Mj_r) \arrow[r] \arrow[d, hook] & \mathcal{A}_*\square_{\mathcal{A}(r-1)_*} H_*(Mj_r)/I_r \arrow[d, hook] \\
& \mathcal{A}_* \otimes H_*(Mj_r) \arrow[r] & \mathcal{A}_* \otimes H_*(Mj_r)/I_r
\end{tikzcd}$$

Explicit formulae

$$X_{1,s} = \begin{cases} x_2 & \text{if } s = 1, \\ x_3 & \text{if } s = 2, \\ \mathrm{Q}^4 x_3 & \text{if } s = 3, \\ \mathrm{Q}^{(2^{s-1},\ldots,2^4,2^3,2^2)} x_3 & \text{if } s \geqslant 4. \end{cases}$$

$$X_{2,s} = \begin{cases} x_4 & \text{if } s = 1, \\ x_6 & \text{if } s = 2, \\ x_7 & \text{if } s = 3, \\ \mathrm{Q}^8 x_7 + \mathrm{Q}^9 x_6 & \text{if } s = 4, \\ \mathrm{Q}^{(2^{s-1},\ldots,2^5,2^4)}(\mathrm{Q}^8 x_7 + \mathrm{Q}^9 x_6) & \text{if } s \geqslant 5. \end{cases}$$

$$X_{3,s} = \begin{cases} x_8 & \text{if } s = 1, \\ x_{12} & \text{if } s = 2, \\ x_{14} & \text{if } s = 3, \\ x_{15} & \text{if } s = 4, \\ \mathrm{Q}^{16}x_{15} + \mathrm{Q}^{17}x_{14} + \mathrm{Q}^{19}x_{12} & \text{if } s = 5, \\ \mathrm{Q}^{(2^{s-1},\ldots,2^6,2^5)}(\mathrm{Q}^{16}x_{15} + \mathrm{Q}^{17}x_{14} + \mathrm{Q}^{19}x_{12}) & \text{if } s \geqslant 6. \end{cases}$$

For each $r = 1, 2, 3$, the trivial algebra homomorphism

$$\mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \to H_*(Mj_r)/I_r$$

is actually an $\mathcal{A}(r-1)_*$-comodule algebra homomorphism. Therefore it induces an $\mathcal{A}_*$-comodule algebra homomorphism

$$\mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \to \mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}H_*(Mj_r)/I_r \cong H_*(Mj_r).$$

There are morphisms of $\mathcal{E}_\infty$ ring spectra $Mj_1 \to H\mathbb{Z}$, $Mj_2 \to k\mathrm{O}$ and $Mj_3 \to \mathrm{tmf}$ which are surjective on $H_*(-)$. Using the last result we see that there are splittings of $\mathcal{A}_*$-algebras of the form

$$\begin{tikzcd}
\mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \\
& H_*(Mj_r)) \arrow[ur] &
\end{tikzcd}$$

Some applications

There are morphisms of $\mathcal{E}_\infty$ ring spectra $Mj_1 \to M\mathrm{SO}$, $Mj_2 \to M\mathrm{Spin}$ and $Mj_3 \to M\mathrm{String}$ so we obtain splittings of $\mathcal{A}_*$-algebras of the form

$$\begin{tikzcd} \mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \arrow[rr, "\cong"] \arrow[dr] & & \mathcal{A}_*\Box_{\mathcal{A}(r-1)_*}\mathbb{F}_2 \\ & H_*(M?)) \arrow[ur] & \end{tikzcd}$$

where $M?$ is one of the Thom spectra Mj_r or one of $M\mathrm{SO}$, $M\mathrm{Spin}$, $M\mathrm{String}$.
Such results have along history. The case of $M\mathrm{String}$ was proved by Bahri & Mahowald using the $\mathcal{E}_2$-space $\Omega^2\Sigma^2 B\mathrm{String}$[15].

Observations and questions

- It is known that $\pi_*(Mj_1) \to \pi_*(H\mathbb{Z})$ and $\pi_*(Mj_2) \to \pi_*(k\mathrm{O})$ are surjective, and $\pi_k(Mj_3) \to \pi_k(\mathrm{tmf})$ is an isomorphism for $k \leqslant 16$. Is $\pi_*(Mj_3) \to \pi_*(\mathrm{tmf})$ surjective?
- Is Mj_2 a wedge of $k\mathrm{O}$ module spectra? Is Mj_3 a wedge of tmf module spectra?