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**Conventions:** Everything will be 2-local. Homology and cohomology will usually be taken with \(\mathbb{F}_2\) coefficients, so \(H_*(-) = H_*(-; \mathbb{F}_2)\) and \(H^*(-) = H^*(-; \mathbb{F}_2)\).

Recall the elements of Hopf invariant 1:

\[
2 \in \pi_0(S^0) \cong \mathbb{Z}, \quad \eta \in \pi_1(S^0) \cong \mathbb{Z}/2, \\
\nu \in \pi_3(S^0) \cong \mathbb{Z}/8, \quad \sigma \in \pi_7(S^0) \cong \mathbb{Z}/16.
\]

In the cohomology of the mapping cones,

\[
H^*(C_2) = \mathbb{F}_2\{t^0, t^2\}, \quad \text{Sq}^1 t^0 = t^1, \\
H^*(C_\eta) = \mathbb{F}_2\{u^0, u^2\}, \quad \text{Sq}^2 u^0 = u^2, \\
H^*(C_\nu) = \mathbb{F}_2\{v^0, v^4\}, \quad \text{Sq}^4 v^0 = v_4, \\
H^*(C_\sigma) = \mathbb{F}_2\{w^0, w^8\}, \quad \text{Sq}^8 w^0 = w^8.
\]
These elements satisfy algebraic relations such as \(2\eta = 0 = \eta \nu\), allowing the following mapping cones to be constructed:

\[
S^0 \cup_\eta e^2 \cup_2 e^3, \quad S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7, \quad S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.
\]

The cohomology of these as modules over the Steenrod algebra \(\mathcal{A}^*\) are simple to describe.

\[
H^*(S^0 \cup_\sigma e^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15})
\]
Lemma

\[ S^0 \cup \eta e^2 \cup \nu e^3 \sim H\mathbb{Z}[3], \quad S^0 \cup \nu e^4 \cup \eta e^6 \cup \nu e^7 \sim k\mathbb{O}[7], \]
\[ S^0 \cup \sigma e^8 \cup \nu e^{12} \cup \eta e^{14} \cup \nu e^{15} \sim \text{tmf}[15]. \]

Here \( X^{[n]} \) denotes the \( n \)-skeleton of a CW spectrum \( X \). \( H\mathbb{Z} \) is the integral \textit{Eilenberg-Mac Lane spectrum}, \( k\mathbb{O} \) is the \textit{real connective K-theory} spectrum and \( \text{tmf} \) is the connective spectrum of \textit{topological modular forms}. 
These iterated mapping cones exist in nature. The classifying spaces $B \text{SO} = BO \langle 2 \rangle$, $B \text{Spin} = BO \langle 4 \rangle$ and $B \text{String} = BO \langle 8 \rangle$ are infinite loop spaces with the following low-dimensional skeleta:

$$B \text{SO}^{[3]} = S^2 \cup_2 e^3, \quad B \text{Spin}^{[7]} = S^4 \cup_\eta e^6 \cup_2 e^7,$$

$$B \text{String}^{[15]} = S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$ 

and the associated Thom spectra have skeleta

$$M \text{SO}^{[3]} = S^0 \cup_\eta e^2 \cup_2 e^3, \quad M \text{Spin}^{[7]} = S^0 \cup_\nu S^4 \cup_\eta e^6 \cup_2 e^7,$$

$$M \text{String}^{[15]} = S^0 \cup_\sigma S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$ 

Each skeletal inclusion map factors through an infinite loop map,

$$BO \langle 2^r \rangle^{[2^{r+1}-1]} \xrightarrow{j_r} B \text{SO} \langle 2^r \rangle \xrightarrow{Q} B \text{SO} \langle 2^r \rangle^{[2^{r+1}-1]}$$

where $Q = \Omega^\infty \Sigma^\infty$ is the free infinite loop space functor. The associated Thom spectra $Mj_1$, $Mj_2$ and $Mj_3$ are $\mathcal{E}_\infty$ ring spectra.
Theorem
The homology rings of the Thom spectra $M^r_j$ are given by

\[ H_*(M^r_1) = F_2[Q^I x_2, Q^J x_3 : I, J \text{ admissible, } \text{exc}(I) > 2, \text{exc}(J) > 3], \]
\[ H_*(M^r_2) = F_2[Q^I x_4, Q^J x_6, Q^K x_7 : I, J, K \text{ admissible, } \text{exc}(I) > 4, \text{exc}(J) > 6, \text{exc}(K) > 7], \]
\[ H_*(M^r_3) = F_2[Q^I x_8, Q^J x_{12}, Q^K x_{14}, Q^L x_{15} : I, J, K, L \text{ admissible, } \text{exc}(I) > 8, \text{exc}(J) > 12, \text{exc}(K) > 14, \text{exc}(L) > 15]. \]

There are $E_\infty$ morphisms $M^r_j \to HF_2$ inducing homomorphisms $H_*(M^r_j) \to A_*$ with images

\[ F_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \cong H_*(H\mathbb{Z}), \quad F_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \cong H_*(kO), \]
\[ F_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \cong H_*(\text{tmf}). \]
In this result,

\[ x_2 \mapsto \zeta_1^2, \quad x_3 \mapsto \zeta_2, \]
\[ x_4 \mapsto \zeta_3^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3, \]
\[ x_8 \mapsto \zeta_1^8, \quad x_{12} \mapsto \zeta_2^4, \quad x_{14} \mapsto \zeta_3^2, \quad x_{15} \mapsto \zeta_4. \]

**Known result:** Mark Steinberger showed that the spectrum \( M_{j_1} \) is a wedge of suspensions of \( H\mathbb{Z} \) and \( H\mathbb{Z}/2^k \) for various \( k \geq 1 \). It seems plausible that analogous splittings should exist for the others.

**Conjecture:** \( M_{j_2} \) is a wedge of \( kO \) module spectra, and \( M_{j_3} \) is a wedge of \( \text{tmf} \) module spectra.

We will describe some algebraic analogues of these statements.
Some coalgebra

We work over a field $\mathbb{k}$ of characteristic 2 with $\mathbb{k}$-vector spaces (possibly graded). We will set $\otimes = \otimes_\mathbb{k}$.

Let $A$ be a commutative Hopf algebra over $\mathbb{k}$, and let $B$ be a quotient Hopf algebra of $A$. We denote the product on $A$ by $\varphi_A$ and the (left) coaction on a comodule $M$ by $\psi_M$. We can induce a left $B$-comodule structure by composing.

\[
\begin{array}{c}
M \xrightarrow{\psi_M} A \otimes M \xrightarrow{\psi_M} B \otimes M
\end{array}
\]
The cotensor product $L \Box_B M$ of a right and a left $B$-comodule is the equaliser of the following diagram.

\[
\begin{array}{ccc}
L \otimes M & \xrightarrow{\text{Id} \otimes \psi'_M} & L \otimes B \otimes M \\
\psi_L \otimes \text{Id} & \downarrow & \\
& & L \otimes B \otimes M
\end{array}
\]

The cotensor product $A \Box_B k \subseteq A \otimes k$ can be identified with a subalgebra of $A$ using the canonical isomorphism $A \otimes k \xrightarrow{\cong} A$. If $L$ or $M$ is extended (or cofree) we have

\[(U \otimes B) \Box_B M \cong U \otimes M, \quad L \Box_B (B \otimes V) \cong L \otimes V.\]
Lemma

Suppose that $C$ is a commutative $B$-comodule algebra and $D$ is a commutative $A$-comodule algebra. There is an isomorphism of $A$-comodule algebras

$$(A □_B C) \otimes D \xrightarrow{\cong} A □_B (C \otimes D),$$

where the domain has the diagonal left $A$-coaction and $C \otimes D$ has the diagonal left $B$-coaction.

Explicitly, this isomorphism has the following effect on

$$\sum_r u_r \otimes v_r \otimes x \in (A □_B C) \otimes D \subseteq A \otimes C \otimes D,$$

$$\sum_r u_r \otimes v_r \otimes x \mapsto \sum_r \sum_i u_r a_i \otimes v_r \otimes x_i,$$

where $\psi_D x = \sum_i a_i \otimes x_i$. 
We also need a way of generated comodule homomorphisms.

**Lemma**

*Suppose that $M$ is a left $A$-comodule and $N$ is a left $B$-comodule. Then there is a natural isomorphism*

\[
\text{Comod}_B(M, N) \xrightarrow{\sim} \text{Comod}_A(M, A \Box_B N); \quad f \mapsto \tilde{f},
\]

*where $\tilde{f}$ is the unique factorisation of $(\text{Id} \otimes f)\psi_M : M \to A \otimes N$ through $A \Box_B N$.

*If $M$ is an $A$-comodule algebra and $N$ is a $B$-comodule algebra, then if $f$ is an algebra homomorphism, so is $\tilde{f}$.**
Some quotient Hopf algebras of the dual Steenrod algebra

Recall that

\[ A_* = \mathbb{F}_2[\xi_r : r \geq 1] = \mathbb{F}_2[\zeta_r : r \geq 1], \]

where \( \xi_r, \zeta_r \in A_{2^r - 1} \). The coproduct and antipode are given by

\[ \psi(\xi_r) = \sum_{0 \leq i \leq r} \xi_{r-i}^2 \otimes \xi_i, \quad \psi(\zeta_r) = \sum_{0 \leq i \leq r} \zeta_i \otimes \zeta_{r-i}^2, \quad \zeta_r = \chi(\xi_r). \]

For \( n \geq 0 \), the ideal

\[ \mathcal{I}(n) = (\zeta_1^{2n+1}, \zeta_2^n, \ldots, \zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}, \ldots) \triangleleft A_* \quad (n \geq 0) \]

is a Hopf ideal and the quotient algebra \( A(n)_* = A_* / \mathcal{I}(n) \) is a finite dimensional commutative Hopf algebra. The dual of \( A(n)_* \) is the subalgebra

\[ A(n)^* = \mathbb{F}_2 \langle \text{Sq}^1, \text{Sq}^2, \text{Sq}^4, \ldots, \text{Sq}^{2n+1} \rangle \subseteq A^* \]

generated by the listed Steenrod operations.
We also have the cotensor product $\mathcal{A}_* \boxtimes \mathcal{A}(n)_* \mathbb{F}_2 \subseteq \mathcal{A}_*$ and the polynomial sub-Hopf algebra

$$
\mathcal{A}_* \boxtimes \mathcal{A}(n)_* \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{2n+1}, \zeta_2, \ldots, \zeta_n, \zeta_{n+2}, \zeta_{n+3}, \ldots],
$$

where $\mathcal{A}(n)_* = \mathcal{A}_*/(\mathcal{A}_* \boxtimes \mathcal{A}(n)_* \mathbb{F}_2)$.

Write $H = H\mathbb{F}_2$ and note that $H_*(H) = \mathcal{A}_*$.

**Theorem**

The natural morphisms of $\mathcal{E}_\infty$ ring spectra $H\mathbb{Z} \to H$, $kO \to H$ and $tmf \to H$ induce monomorphisms in $H_*(-)$ whose images are $\mathcal{A}_* \boxtimes \mathcal{A}(0)_* \mathbb{F}_2$, $\mathcal{A}_* \boxtimes \mathcal{A}(1)_* \mathbb{F}_2$ and $\mathcal{A}_* \boxtimes \mathcal{A}(2)_* \mathbb{F}_2$.

**Note:** the Hopf invariant 1 Theorem implies that these are the only $\mathcal{A}(n)_*$ that can be realised in this way.
Here is an important fact about these quotient Hopf algebras. A right $A$-comodule $L$ over a Hopf algebra $A$ is extended (or cofree) if

$$L \cong W \otimes A$$

where the coaction is $\text{Id}_W \otimes \psi_A$.

**Theorem**

For each $n \geq 0$, $\mathcal{A}_*$ is an extended right $\mathcal{A}(n)_*$-comodule:

$$\mathcal{A}_* \cong (\mathcal{A}_* \Box \mathcal{A}(n)_* \mathbb{F}_2) \otimes \mathcal{A}(n)_*.$$

This implies that for any left $\mathcal{A}(n)_*$-comodule $M$,

$$\mathcal{A}_* \Box \mathcal{A}(n)_* M \cong (\mathcal{A}_* \Box \mathcal{A}(n)_* \mathbb{F}_2) \otimes M.$$

This can be viewed as an isomorphism of left $\mathcal{A}_*$-comodules using a suitable comodule structure on the right hand side.
To understand these we must work in a context such as the model category $\mathcal{M}_S$ of $S$-modules of EKMM, which has a symmetric monoidal structure with smash product $\wedge = \wedge_S$.

In $\mathcal{M}_S$, the *commutative S-algebras* are the commutative monoids. Untangling the underlying structure of the smash product leads to the connection with the older notion of $\mathcal{E}_\infty$ *ring spectra*. For a spectrum $X$,

$$D_nX = E\Sigma_n \ltimes \Sigma_n X^{(n)}.$$

When $X$ is cofibrant the natural map is a weak equivalence

$$D_rX = E\Sigma_r \ltimes \Sigma_r X^{(r)} \xrightarrow{\sim} X^{(r)}/\Sigma_r.$$

Then $E$ is an $\mathcal{E}_\infty$ *ring spectrum* if there are suitably compatible maps $\mu_n : D_nE \to E$ which extend a product map

$$\mu : E^{(2)} \to D_2E \xrightarrow{\mu_2} E.$$
The homology of $D_2E$ can be described in terms of group homology,

$$H_*(D_2E) \cong H^\text{alg}_*(\Sigma_2; H_*(E \otimes^2)).$$

For $x \in H_m(E)$, there are elements $e_i \otimes x \otimes x \in H_{i+2m}(D_2E)$ and the $r$-th Dyer-Lashof operation $Q^r$ is defined by the rule

$$Q^r x = (\mu_2)_*(e_{r-m} \otimes x \otimes x) \in H_{m+r}(E).$$

These are natural for morphisms of $\mathcal{E}_\infty$ ring spectra and satisfy properties similar to the Steenrod operations. There are Adem relations of the form

$$Q^r Q^s = \sum_i \binom{i-s-1}{2i-r}Q^{r+s-i}Q^i$$

whenever $r > 2s$; these lead to notions of admissible sequences and excess conditions.
Coactions and Dyer-Lashof operations

Usually the right action of the Steenrod algebra is intertwined with the Dyer-Lashof operations using the Nishida relations. However, for an $E_\infty$ ring spectrum $X$, the interaction of the coaction $\psi: H_*(X) \to A_* \otimes H_*(X)$ can also be described.

It is better to twist this into a right coaction $\tilde{\psi}: H_*(X) \to H_*(X) \otimes A_*$. Then if $x \in H_m(X)$ and $r \geq m$,

$$\tilde{\psi} Q^s(x) = \sum_{k=m}^s Q^k(\tilde{\psi}(x)) \left[ \left( \frac{\zeta(t)}{t} \right)^k \right]_{t^{s-k}}.$$ 

For example, if $\psi(x) = \sum_i a_i \otimes x_i$ then $\tilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i)$ and

$$\tilde{\psi} Q^{m+1}(x) = \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i Q^{m+1}(x_i \otimes \chi(a_i))$$

$$= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \sum_j Q^{m+1-j} x_i \otimes Q^j \chi(a_i)).$$
The Dyer-Lashof on $A_*$ was determined by Kochman and Steinberger. For example,

$$Q^{2s} \zeta_s = \zeta_{s+1}.$$  

Also, I showed recently that

$$Q^{2s} \xi_s = \xi_{s+1} + \xi_1 \xi_s^2.$$  

One way to pin down the action is by using symmetric function formula.
Some cotensor product decompositions

We would like to describe $H_*(M_{jr})$ in terms of $A(r - 1)_*$. 

**Theorem**

For $r = 1, 2, 3$ there is a regular sequence $X_{r,s} \in H_*(M_{jr})$ ($s \geq 1$) so that the ideal $I_r = (X_{r,s} : s \geq 1) \triangleleft H_*(M_{jr})$ is $A(r - 1)_*$-invariant. Furthermore, the top composition is an isomorphism in the commutative diagram of commutative $A_*$-comodule algebras.

\[
\begin{array}{ccc}
H_*(M_{jr}) & \xrightarrow{\cong} & A_* \square A(r-1)_* H_*(M_{jr}) \\
& \downarrow \psi & \downarrow \\
& A_* \otimes H_*(M_{jr}) & \xrightarrow{=} A_* \otimes H_*(M_{jr})/I_r \\
& \downarrow & \downarrow \\
& A_* \square A(r-1)_* H_*(M_{jr})/I_r & \xrightarrow{\cong} A_* \square A(r-1)_* H_*(M_{jr})/I_r
\end{array}
\]
Explicit formulae

\[ X_{1,s} = \begin{cases} 
  x_2 & \text{if } s = 1, \\
  x_3 & \text{if } s = 2, \\
  Q^4 x_3 & \text{if } s = 3, \\
  Q^{(2^{s-1}, \ldots, 2^4, 2^3, 2^2)} x_3 & \text{if } s \geq 4. 
\end{cases} \]

\[ X_{2,s} = \begin{cases} 
  x_4 & \text{if } s = 1, \\
  x_6 & \text{if } s = 2, \\
  x_7 & \text{if } s = 3, \\
  Q^8 x_7 + Q^9 x_6 & \text{if } s = 4, \\
  Q^{(2^{s-1}, \ldots, 2^5, 2^4)}(Q^8 x_7 + Q^9 x_6) & \text{if } s \geq 5. 
\end{cases} \]
\[ X_{3,s} = \begin{cases} 
  x_8 & \text{if } s = 1, \\
  x_{12} & \text{if } s = 2, \\
  x_{14} & \text{if } s = 3, \\
  x_{15} & \text{if } s = 4, \\
  Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12} & \text{if } s = 5, \\
  Q^{(2s-1, 2^6, 2^5)}(Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12}) & \text{if } s \geq 6.
\]
For each $r = 1, 2, 3$, the trivial algebra homomorphism

$$A_* \square A_{(r-1)_*} \mathbb{F}_2 \to H_*(Mj_r)/I_r$$

is actually an $A(r-1)_*$-comodule algebra homomorphism. Therefore it induces an $A_*$-comodule algebra homomorphism

$$A_* \square A_{(r-1)_*} \mathbb{F}_2 \to A_* \square A_{(r-1)_*} H_*(Mj_r)/I_r \cong H_*(Mj_r).$$

There are morphisms of $E_\infty$ ring spectra $Mj_1 \to H\mathbb{Z}$, $Mj_2 \to kO$ and $Mj_3 \to tmf$ which are surjective on $H_*(-)$. Using the last result we see that there are splittings of $A_*$-algebras of the form

$$A_* \square A_{(r-1)_*} \mathbb{F}_2 \quad \text{and} \quad H_*(Mj_r))$$
Some applications

There are morphisms of $\mathcal{E}_\infty$ ring spectra $Mj_1 \to M_{SO}$, $Mj_2 \to M_{Spin}$ and $Mj_3 \to M_{String}$ so we obtain splittings of $A_*$-algebras of the form

$$A_* \square A(r-1)_* \mathbb{F}_2 \xrightarrow{\mathcal{R}} A_* \square A(r-1)_* \mathbb{F}_2$$

$$\xrightarrow{} H_*(M?)$$

where $M?$ is one of the Thom spectra $Mj_r$ or one of $M_{SO}$, $M_{Spin}$, $M_{String}$.

Such results have a long history. The case of $M_{String}$ was proved by Bahri & Mahowald using the $\mathcal{E}_2$-space $\Omega^2 \Sigma^2 B_{String}^{[15]}$. 

Observations and questions

- It is known that $\pi_*(Mj_1) \to \pi_*(H\mathbb{Z})$ and $\pi_*(Mj_2) \to \pi_*(kO)$ are surjective, and $\pi_k(Mj_3) \to \pi_k(tmf)$ is an isomorphism for $k \leq 16$. Is $\pi_*(Mj_3) \to \pi_*(tmf)$ surjective?
- Is $Mj_2$ a wedge of $kO$ module spectra? Is $Mj_3$ a wedge of $tmf$ module spectra?