Approaching BP as a commutative S-algebra

Andrew Baker

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What is BP?

For each prime p, there is a p-local spectrum BP whose cohomology as an $\mathcal{A}^*$-module is the quotient

$$H^*(BP; \mathbb{F}_p) = \mathcal{A}^*/(\beta)$$

(where $\beta = Sq^1$ when $p = 2$), or equivalently, when p is odd

$$H_*(BP; \mathbb{F}_p) = \mathbb{F}_p[\zeta_r : r \geqslant 1] \subset \mathbb{F}_p[\zeta_r : r \geqslant 1] \otimes \Lambda(\tau_s : s \geqslant 0) = \mathcal{A}_*,$$

and when $p = 2$

$$H_*(BP; \mathbb{F}_2) = \mathbb{F}_2[\zeta_r^2 : r \geqslant 1] \subset \mathbb{F}_p[\zeta_r : r \geqslant 1] = \mathcal{A}_*.$$

These spectra are important since Milnor showed that as an $\mathcal{A}^*$-module, $H^*(MU; \mathbb{F}_p)$ is a coproduct of suspensions of $\mathcal{A}^*/(\beta)$, so then $MU_{(p)}$ is a wedge of suspensions of BP provided such a spectrum exists.

Brown and Peterson constructed BP by ad hoc methods, so Milnor's result showed that there was a topological splitting of $MU_{(p)}$.

In fact there is a canonical construction due to Quillen, who showed how to define an idempotent map of commutative ring spectra $\varepsilon\colon MU_{(p)} \longrightarrow MU_{(p)}$ which splits off BP as a retract of $MU_{(p)}$. There are resulting maps of ring spectra $BP \longrightarrow MU_{(p)} \longrightarrow BP$ whose composition is the identity. This construction depends on the algebraic universality of MU_* for formal group laws and the idempotent corresponds to a functorial p-typification operation.

Further structure

Since MU is an E_∞ ring spectrum, or equivalently a commutative S-algebra, it is natural to ask whether BP also has such structure. A stronger form of this question asks whether the natural maps

$$BP \longrightarrow MU_{(p)} \longrightarrow BP$$

are morphisms of commutative S-algebras, or of H_∞ ring spectra. McClure and AB both worked unsuccessfully on resolving on this in the early 1980s. Recently it has been shown by Johnson & Noel that the map $MU_{(p)} \longrightarrow BP$ is not H_∞ for small primes p.
Hu, Kriz & May showed that for all primes $BP \longrightarrow MU_{(p)}$ is not H_∞. Kriz gave a sketch of a proof that BP is E_∞ based on TAQ, but that is widely believed to be incorrect. Other work by Basterra & Mandell, and Richter have shown that BP supports some partial approximations to E_∞ structures.
The difficulties stem from the fact that BP has no known 'geometric' description, and the failure of E_∞ obstruction theory methods.

Priddy's construction

Around 1980, Priddy gave a cellular construction of BP. Ideas in this were later resurrected by Hu, Kriz & May, then AJB & JPM et al, so that BP is minimal atomic and any map $BP \longrightarrow MU_{(p)}$ which induces an isomorphism on $\pi_0(-)$ gives a monomorphism on $\pi_*(-)$, *i.e.*, this map is a core for $MU_{(p)}$.

Priddy constructs a CW p-local spectrum X so that the skeleta satisfy $X^{[0]} = S_{(p)}$, $X^{[2n]} = X^{[2n+1]}$ and $X^{[2m+2]}$ is obtained from $X^{[2m]}$ by attaching $(2m+2)$-cells to kill a minimal generating set of $\pi_{2m+1}X^{[2m]}$.

Obstruction theory arguments imply there are maps

$$X \longrightarrow MU_{(p)} \longrightarrow X$$

extending the identity on the 0-cell. By Milnor's calculations, X has the correct cohomology as an $\mathcal{A}^*$-module.

For the prime $p = 2$, Hu, Kriz & May identified a core of $MU_{(2)}$ in the homotopy category of 2-local commutative S-algebras, namely $(MSp/U)_{(2)} \longrightarrow MU_{(2)}$, where MSp/U is the Thom spectrum over the fibre in the fibration sequence of infinite loop spaces

$$Sp/U \longrightarrow BU \longrightarrow BSp.$$

Commutative S-algebras and E_∞ ring spectra

Commutative S-algebras are essentially the same thing as E_∞ ring spectra, and to describe these we need to use the extended power functors. For a spectrum X,

$$D_n X = E\Sigma_n \ltimes_{\Sigma_n} X^{(n)}.$$

When $X = \Sigma^\infty Z_+$,

$$D_n \Sigma^\infty Z_+ = \Sigma^\infty (E\Sigma_n \times_{\Sigma_n} Z^n)_+$$

Then E is an E_∞ *ring spectrum* if there are suitably compatible maps $\mu_n \colon D_n E \longrightarrow E$ extending a product map

$$\mu \colon E^{(2)} \longrightarrow D_2 E \xrightarrow{\mu_2} E.$$

It turns out that such an E_∞ ring structure is equivalent to the product μ making E into a commutative S-algebra.

Power operations and the Adams spectral sequence

We will work 2-locally from now on. However, most of what we discuss has analogues for other primes.

Given an E_∞ ring spectrum E, there are various types of power operations that can be defined. We will use operations based on D_2E, but relations between these depend on the D_nE for $n > 2$.

Given $\alpha \in \pi_k D_2 S^n$ (so we can realise α as a map $S^k \longrightarrow D_2 S^n$) there is an operation $\alpha^* \colon \pi_n E \longrightarrow \pi_k E$ which for $x \colon S^n \longrightarrow E$ is given by

$$\alpha^* x \colon S^k \xrightarrow{\alpha} D_2 S^n \xrightarrow{D_2 x} D_2 E \xrightarrow{\mu_2} E.$$

To understand elements of $\pi_* D_2 S^n$ it helps to notice that

$$D_2 S^n \sim \Sigma^n \mathbb{R}\mathrm{P}_n^\infty \sim \Sigma^n (\text{Thom spectrum of } n\rho_1 \downarrow \mathbb{R}\mathrm{P}^\infty).$$

The cell structure of this is simple, with one cell in each degree from $2n$ up. The Steenrod module structure for $H^* D_2 S^n$ can be found using the Wu fomulae. Although $n \in \mathbb{Z}$ makes sense in this context, we will assume that E is connective.

Theorem

Suppose that E is a connective commutative S-algebra for which $0 = \eta 1 \in \pi_1 E$. Then for $k \geqslant 1$, the operation $\mathcal{P}^{2^{k+1}-1}$ is defined on $\pi_{2^{k+1}-2} E$, giving a map

$$\mathcal{P}^{2^{k+1}-1} \colon \pi_{2^{k+1}-2} E \longrightarrow \pi_{2^{k+2}-3} E.$$

Moreover, the indeterminacy is trivial and the operation $2\mathcal{P}^{2^{k+1}-1}$ is trivial.

The next result shows how this works in the mod 2 Adams spectral sequence converging to $\pi_* E$ in good situations.

Lemma

With same assumptions, if $w \in \pi_{2^{k+1}-2} E$ is detected in the 1-line of the ASS by $W \in \mathrm{Ext}^{1,2^{k+1}-1}_{\mathcal{A}(2)_*}(\mathbb{F}_2, H_* E)$, then $\mathcal{P}^{2^{k+1}-1} w$ is detected in the 1-line by

$$\mathcal{P}^{2^{k+1}-1} W \in \mathrm{Ext}^{1,2^{k+2}-2}_{\mathcal{A}_*}(\mathbb{F}_2, H_* E),$$

where $\mathcal{P}^{2^{k+1}-1}$ is the algebraic Steenrod operation of May et al.

Killing homotopy the E_∞ way

Suppose that R is a commutative S-algebra and that $f\colon Z \longrightarrow R$ is a map. There is a unique extension to a map of commutative S-algebras $\widetilde{f}\colon \mathbb{P}Z \longrightarrow R$, where $\mathbb{P}(-)$ is the free commutative S-algebra functor. We can form a pushout diagram of commutative S-algebras

$$\begin{tikzcd}
\mathbb{P}Z \arrow[r, "\widetilde{f}"] \arrow[d] & R \arrow[d] \\
\mathbb{P}CZ \arrow[r] & R//f
\end{tikzcd}$$

where the left hand arrow is induced by the inclusion of Z into the cone CZ. In fact,

$$R//f = R \wedge_{\mathbb{P}Z} \mathbb{P}CZ.$$

When Z is an m-sphere or wedge of m-spheres, $R//f$ is said to be obtained from R by attaching E_∞ $(m+1)$-cells to kill the homotopy class of f.

If R is connective then we can build a CW commutative S-algebra R' and a weak equivalence $R' \longrightarrow R$ by inductively attaching E_∞ cells starting with the unit map $S \longrightarrow R$.

Now we proceed to inductively construct a sequence of 2-local connective commutative S-algebras

$$S = R(0) \longrightarrow R(1) \longrightarrow \cdots \longrightarrow R(n-1) \longrightarrow R(n) \longrightarrow \cdots$$

where $R(n)$ is obtained from by $R(n-1)$ by attaching a single E_∞ $(2^{n+1}-2)$-cell.
The first step involves killing the generator $\eta \in \pi_1 S$, and taking $R(1) = S//\eta$. Then in the ASS with standard cobar complex notation, η is represented by

$$[\zeta_1^2 \otimes 1] \in \mathrm{Ext}^{1,2}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2).$$

Homological calculations

We will write $H_*(-)$ for mod 2 ordinary homology.
Suppose that R is a 2-local connective commutative S-algebra and that $\alpha \in \pi_{2n-1}R$ is non-trivial and has Adams filtration 1. Let its representative in the ASS be $[w] \in \mathrm{Ext}^{1,2}_{\mathcal{A}_*}(\mathbb{F}_2, H_*R)$.

Theorem

The homology of $R//\alpha$ has the form

$$H_*(R//\alpha) = H_*R[Q^I s : I \text{ is admissible with excess } e(I) > 2n],$$

for a generator $s \in H_{2n}(R//\alpha)$ with coaction $\psi s = 1 \otimes s + w$. The rational homology is

$$H_*(R//\alpha; \mathbb{Q}) = H_*(R; \mathbb{Q})[S],$$

where S is the image of a lift of s to integral homology $H_*(R//\alpha; \mathbb{Z}_{(2)})$.

The inductive construction

Now we proceed to build the $R(n)$ starting with $R(0) = S$ and η to give $R(1) = R//\eta$. By the last Theorem, the homology of $R(1)$. Since η has order 2, there is a commutative diagram of S-modules

$$\begin{tikzcd}
 & & S^2 \arrow[d, dotted] \arrow[dr, "2"] \arrow[dd, dashed, bend right] & \\
S^1 \arrow[r, "\eta"] & R(0) \arrow[r] & C_\eta \arrow[r] \arrow[d] & S^2 \\
 & & R(1) &
\end{tikzcd}$$

in which the dashed arrow provides a homotopy class $u_1 \in \pi_2 R(1)$ of infinite order. The representative of this element is

$$[\zeta_1 \otimes s + \zeta_2 \otimes 1] \in \mathrm{Ext}^{1,3}_{\mathcal{A}_*}(\mathbb{F}_2, H_* R(1)).$$

We can use the power operation $\mathcal{P}^3$ to obtain a homotopy element $\mathcal{P}^3 u_1 \in \pi_5 R(1)$ of order 2 and represented in the ASS by

$$[\zeta_1^2 \otimes s^2 + \zeta_2^2 \otimes 1] \in \mathrm{Ext}^{1,6}(\mathbb{F}_2, H_* R(1)).$$

Now we can iterate. At each stage we have $R(n)$ with an infinite order element $u_n \in \pi_{2^{n+1}-2} R(n)$ and an element $w_n = \mathcal{P}^{2^{n+1}-1} u_n \in \pi_{2^{n+2}-3} R(n)$ of order 2. We can form $R(n+1) = R(n)//w_n$, and rationally we have

$$H_*(R(n); \mathbb{Q}) = \mathbb{Q}[S_1, \ldots, S_n]$$

where S_r is a lift of a certain homology element $s_r \in H_{2^{r+1}-2} R(n)$. Making the maps $R(n) \longrightarrow R(n+1)$ into cofibrations we can form the limit $R(\infty) = \mathrm{colim}_n R(n)$ so that

$$\pi_* R(\infty) = \operatorname*{colim}_n \pi_* R(n).$$

Explicit formulae can be found for representatives of these homotopy elements in the ASS:

$$\begin{aligned} u_n = [&\zeta_1 \otimes s_n + \zeta_2 \otimes s_{n-1}^2 + \zeta_3 \otimes s_{n-2}^{2^2} \\ &+ \cdots + \zeta_r \otimes s_{n-r+1}^{2^{r-1}} + \cdots + \zeta_{n+1} \otimes 1] \\ w_n = [&\zeta_1^2 \otimes s_{n-1}^2 + \zeta_2^4 \otimes s_{n-2}^2 + \zeta_3^3 \otimes s_{n-3}^{2^2} \\ &+ \cdots + \zeta_r^2 \otimes s_{n-r}^{2^{r-1}} + \cdots + \zeta_n^2 \otimes 1] \end{aligned}$$

The commutative S-algebra MU has torsion-free homotopy, so there is a morphism $R(\infty) \longrightarrow MU$. The natural map $MU \longrightarrow BP$ is a map of ring spectra.

Lemma

The composition $R(\infty) \longrightarrow MU \longrightarrow BP$ induces a surjection on $H_*(-)$ and is a rational equivalence.

Now we could proceed to kill the torsion in $\pi_* R(\infty)$ by attaching E_∞ cells. But in order to preserve the rational homotopy type, instead we use an idea of Tyler Lawson and attach E_∞ cones on Moore spectra $S^m \cup_{p^k} D^{m+1}$ to kill elements of order p^k. The resulting spectrum R has torsion-free homotopy and comes equipped with a map of commutative S-algebras $R(\infty) \longrightarrow R$ and a map of commutative ring spectra $BP \longrightarrow R$, both of which are rational equivalences. It is even true that $H_*(BP; \mathbb{Z}_{(2)}) \longrightarrow H_*(R; \mathbb{Z}_{(2)})$ is a split monomorphism. But it is not clear whether there is a map of ring spectra $R \longrightarrow BP$. If such a map were to exist then $R \sim BP$ so BP would have an E_∞ structure.

In the other direction the following holds.

Theorem

If BP is a commutative S-algebra then there is a weak equivalence of commutative S-algebras $R \longrightarrow BP$.

Here is another construction that gives a close approximation to BP. Start with the minimal atomic commutative S-algebra MSp/U. Then there are generators $x_{2k-1} \in \pi_{4k-2}MSp/U$ for which

$$\pi_* MSp/U = \mathbb{Z}_{(2)}[x_{2k-1} : k \geqslant 1].$$

We can inductively kill the generators x_{2k-1} for $k \neq 2^s$ to obtain a commutative S-algebra $T(\infty)$. Killing the torsion we get T with

$$\pi_* T \otimes \mathbb{Q} \cong \pi_* BP \otimes \mathbb{Q}.$$

Again it is not clear if there is a map of ring spectra $T \longrightarrow BP$, but there is a map $BP \longrightarrow T$ which is a rational equivalence.

Theorem

If BP is a commutative S-algebra then there is a weak equivalence of commutative S-algebras $T \longrightarrow BP$.