Approaching $BP$ as a commutative $S$-algebra

Andrew Baker

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What is $BP$?

For each prime $p$, there is a $p$-local spectrum $BP$ whose cohomology as an $A^*$-module is the quotient

$$H^*(BP; \mathbb{F}_p) = A^*/(\beta)$$

(where $\beta = Sq^1$ when $p = 2$), or equivalently, when $p$ is odd

$$H_*(BP; \mathbb{F}_p) = \mathbb{F}_p[\zeta_r : r \geq 1] \subset \mathbb{F}_p[\zeta_r : r \geq 1] \otimes \Lambda(\tau_s : s \geq 0) = A_*,$$

and when $p = 2$

$$H_*(BP; \mathbb{F}_2) = \mathbb{F}_2[\zeta_r^2 : r \geq 1] \subset \mathbb{F}_p[\zeta_r : r \geq 1] = A_*.$$  

These spectra are important since Milnor showed that as an $A^*$-module, $H^*(MU; \mathbb{F}_p)$ is a coproduct of suspensions of $A^*/(\beta)$, so then $MU(p)$ is a wedge of suspensions of $BP$ provided such a spectrum exists.
Brown and Peterson constructed $BP$ by ad hoc methods, so Milnor’s result showed that there was a topological splitting of $MU_{(p)}$.

In fact there is a canonical construction due to Quillen, who showed how to define an idempotent map of commutative ring spectra $\varepsilon: MU_{(p)} \to MU_{(p)}$ which splits off $BP$ as a retract of $MU_{(p)}$. There are resulting maps of ring spectra $BP \to MU_{(p)} \to BP$ whose composition is the identity. This construction depends on the algebraic universality of $MU_\ast$ for formal group laws and the idempotent corresponds to a functorial $p$-typification operation.
Since $MU$ is an $E_\infty$ ring spectrum, or equivalently a commutative $S$-algebra, it is natural to ask whether $BP$ also has such structure. A stronger form of this question asks whether the natural maps

$$BP \to MU(p) \to BP$$

are morphisms of commutative $S$-algebras, or of $H_\infty$ ring spectra. McClure and AB both worked unsuccessfully on resolving on this in the early 1980s. Recently it has been shown by Johnson & Noel that the map $MU(p) \to BP$ is not $H_\infty$ for small primes $p$. Hu, Kriz & May showed that for all primes $BP \to MU(p)$ is not $H_\infty$. Kriz gave a sketch of a proof that $BP$ is $E_\infty$ based on TAQ, but that is widely believed to be incorrect. Other work by Basterra & Mandell, and Richter have shown that $BP$ supports some partial approximations to $E_\infty$ structures.

The difficulties stem from the fact that $BP$ has no known ‘geometric’ description, and the failure of $E_\infty$ obstruction theory methods.
Around 1980, Priddy gave a cellular construction of $BP$. Ideas in this were later resurrected by Hu, Kriz & May, then AJB & JPM et al, so that $BP$ is minimal atomic and any map $BP ightarrow MU_{(p)}$ which induces an isomorphism on $\pi_0(-)$ gives a monomorphism on $\pi_*(-)$, i.e., this map is a core for $MU_{(p)}$.

Priddy constructs a CW $p$-local spectrum $X$ so that the skeleta satisfy $X[0] = S_{(p)}$, $X[2n] = X[2n+1]$ and $X[2m+2]$ is obtained from $X[2m]$ by attaching $(2m + 2)$-cells to kill a minimal generating set of $\pi_{2m+1}X[2m]$.

Obstruction theory arguments imply there are maps

$$X \rightarrow MU_{(p)} \rightarrow X$$

extending the identity on the 0-cell. By Milnor’s calculations, $X$ has the correct cohomology as an $A^*$-module.
For the prime $p = 2$, Hu, Kriz & May identified a core of $MU(2)$ in the homotopy category of 2-local commutative $S$-algebras, namely $(MSp/U)(2) \rightarrow MU(2)$, where $MSp/U$ is the Thom spectrum over the fibre in the fibration sequence of infinite loop spaces

$$Sp/U \rightarrow BU \rightarrow BSp.$$
Commutative $S$-algebras and $E_\infty$ ring spectra

Commutative $S$-algebras are essentially the same thing as $E_\infty$ ring spectra, and to describe these we need to use the extended power functors. For a spectrum $X$,

$$D_nX = E\Sigma_n \ltimes \Sigma_n X^{(n)}.$$  

When $X = \Sigma^\infty \mathbb{Z}_+$,

$$D_n\Sigma^\infty \mathbb{Z}_+ = \Sigma^\infty (E\Sigma_n \times \Sigma_n \mathbb{Z}^n)_+.$$  

Then $E$ is an $E_\infty$ ring spectrum if there are suitably compatible maps $\mu_n : D_nE \to E$ extending a product map

$$\mu : E^{(2)} \to D_2E \xrightarrow{\mu_2} E.$$  

It turns out that such an $E_\infty$ ring structure is equivalent to the product $\mu$ making $E$ into a commutative $S$-algebra.
We will work 2-locally from now on. However, most of what we discuss has analogues for other primes.

Given an $E_\infty$ ring spectrum $E$, there are various types of power operations that can be defined. We will use operations based on $D_2E$, but relations between these depend on the $D_nE$ for $n > 2$. Given $\alpha \in \pi_k D_2 S^n$ (so we can realise $\alpha$ as a map $S^k \rightarrow D_2 S^n$) there is an operation $\alpha^* : \pi_n E \rightarrow \pi_k E$ which for $x : S^n \rightarrow E$ is given by

$$\alpha^* x : S^k \xrightarrow{\alpha} D_2 S^n \xrightarrow{D_2 x} D_2 E \xrightarrow{\mu_2} E.$$  

To understand elements of $\pi_* D_2 S^n$ it helps to notice that

$$D_2 S^n \sim \Sigma^n \mathbb{R}P^n \sim \Sigma^n (\text{Thom spectrum of } n \rho_1 \downarrow \mathbb{R}P^n).$$

The cell structure of this is simple, with one cell in each degree from $2n$ up. The Steenrod module structure for $H^* D_2 S^n$ can be found using the Wu formulae. Although $n \in \mathbb{Z}$ makes sense in this context, we will assume that $E$ is connective.
Theorem

Suppose that $E$ is a connective commutative $S$-algebra for which $0 = \eta 1 \in \pi_1 E$. Then for $k \geq 1$, the operation $P^{2^{k+1}-1}$ is defined on $\pi_{2k+1-2}E$, giving a map

$$P^{2^{k+1}-1} : \pi_{2k+1-2}E \to \pi_{2k+2-3}E.$$  

Moreover, the indeterminacy is trivial and the operation $2P^{2^{k+1}-1}$ is trivial.
The next result shows how this works in the mod 2 Adams spectral sequence converging to $\pi_* E$ in good situations.

**Lemma**

With same assumptions, if $w \in \pi_{2^{k+1}-2} E$ is detected in the 1-line of the ASS by $W \in \text{Ext}^{1, 2^{k+1}-1}_{A(2)_*}(\mathbb{F}_2, H_* E)$, then $\mathcal{P}^{2^{k+1}-1} w$ is detected in the 1-line by

$$\mathcal{P}^{2^{k+1}-1} W \in \text{Ext}^{1, 2^{k+2}-2}_{A_*}(\mathbb{F}_2, H_* E),$$

where $\mathcal{P}^{2^{k+1}-1}$ is the algebraic Steenrod operation of May et al.
Suppose that $R$ is a commutative $S$-algebra and that $f : Z \to R$ is a map. There is a unique extension to a map of commutative $S$-algebras $\tilde{f} : \mathbb{P} Z \to R$, where $\mathbb{P}(\cdot)$ is the free commutative $S$-algebra functor. We can form a pushout diagram of commutative $S$-algebras

$$
\begin{array}{ccc}
\mathbb{P} Z & \xrightarrow{\tilde{f}} & R \\
\downarrow & & \downarrow \\
\mathbb{P} C Z & \xrightarrow{} & R//f
\end{array}
$$

where the left hand arrow is induced by the inclusion of $Z$ into the cone $CZ$. In fact,

$$R//f = R \wedge_{\mathbb{P} Z} \mathbb{P} C Z.$$ 

When $Z$ is an $m$-sphere or wedge of $m$-spheres, $R//f$ is said to be obtained from $R$ by attaching $E_\infty (m+1)$-cells to kill the homotopy class of $f$. 
If $R$ is connective then we can build a CW commutative $S$-algebra $R'$ and a weak equivalence $R' \longrightarrow R$ by inductively attaching $E_\infty$ cells starting with the unit map $S \longrightarrow R$.

Now we proceed to inductively construct a sequence of 2-local connective commutative $S$-algebras

$$S = R(0) \longrightarrow R(1) \longrightarrow \cdots \longrightarrow R(n - 1) \longrightarrow R(n) \longrightarrow \cdots$$

where $R(n)$ is obtained from $R(n - 1)$ by attaching a single $E_\infty (2^{n+1} - 2)$-cell.

The first step involves killing the generator $\eta \in \pi_1 S$, and taking $R(1) = S//\eta$. Then in the ASS with standard cobar complex notation, $\eta$ is represented by

$$[\zeta^2_1 \otimes 1] \in \text{Ext}^{1,2}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2).$$
Homological calculations

We will write $H_*(-)$ for mod 2 ordinary homology. Suppose that $R$ is a 2-local connective commutative $S$-algebra and that $\alpha \in \pi_{2n-1}R$ is non-trivial and has Adams filtration 1. Let its representative in the ASS be $[w] \in \text{Ext}_{A_*}^{1,2}(F_2, H_*R)$.

**Theorem**

The homology of $R//\alpha$ has the form

$$H_*(R//\alpha) = H_*R[Q^I s : I is admissible with excess e(I) > 2n],$$

for a generator $s \in H_{2n}(R//\alpha)$ with coaction $\psi s = 1 \otimes s + w$. The rational homology is

$$H_*(R//\alpha; \mathbb{Q}) = H_*(R; \mathbb{Q})[S],$$

where $S$ is the image of a lift of $s$ to integral homology $H_*(R//\alpha; \mathbb{Z}(2)).
The inductive construction

Now we proceed to build the $R(n)$ starting with $R(0) = S$ and $\eta$ to give $R(1) = R//\eta$. By the last Theorem, the homology of $R(1)$. Since $\eta$ has order 2, there is a commutative diagram of $S$-modules

\[
\begin{array}{ccc}
S^2 & \overset{2}{\longrightarrow} & S^2 \\
\downarrow & & \downarrow \\
S^1 & \overset{\eta}{\longrightarrow} & R(0) \\
\downarrow & & \downarrow \\
C_\eta & \longrightarrow & S^2 \\
\downarrow & & \downarrow \\
R(1) & \longrightarrow & 
\end{array}
\]

in which the dashed arrow provides a homotopy class $u_1 \in \pi_2 R(1)$ of infinite order. The representative of this element is

\[ [\zeta_1 \otimes s + \zeta_2 \otimes 1] \in \text{Ext}_{\mathbb{A}_*}^{1,3}(\mathbb{F}_2, H_* R(1)) \].
We can use the power operation $P^3$ to obtain a homotopy element $P^3 u_1 \in \pi_5 R(1)$ of order 2 and represented in the ASS by

$$[\zeta_1^2 \otimes s^2 + \zeta_2^2 \otimes 1] \in \text{Ext}^{1,6}(\mathbb{F}_2, H_* R(1)) .$$

Now we can iterate. At each stage we have $R(n)$ with an infinite order element $u_n \in \pi_{2n+1} R(n)$ and an element $w_n = P^{2n+1} u_n \in \pi_{2n+2} R(n)$ of order 2. We can form $R(n+1) = R(n) \wedge w_n$, and rationally we have

$$H_*(R(n); \mathbb{Q}) = \mathbb{Q}[S_1, \ldots, S_n]$$

where $S_r$ is a lift of a certain homology element $s_r \in H_{2r+1} R(n)$. Making the maps $R(n) \to R(n+1)$ into cofibrations we can form the limit $R(\infty) = \colim_n R(n)$ so that

$$\pi_* R(\infty) = \colim_n \pi_* R(n).$$
Explicit formulae can be found for representatives of these homotopy elements in the ASS:

\[ u_n = [\zeta_1 \otimes s_n + \zeta_2 \otimes s_{n-1}^2 + \zeta_3 \otimes s_{n-2}^2 + \cdots + \zeta_r \otimes s_{n-r+1}^{2^{r-1}} + \cdots + \zeta_{n+1} \otimes 1] \]

\[ w_n = [\zeta_1^2 \otimes s_{n-1}^2 + \zeta_2^4 \otimes s_{n-2}^2 + \zeta_3^3 \otimes s_{n-3}^2 + \cdots + \zeta_r^2 \otimes s_{n-r}^{2^{r-1}} + \cdots + \zeta_n^2 \otimes 1] \]
The commutative $S$-algebra $MU$ has torsion-free homotopy, so there is a morphism $R(\infty) \longrightarrow MU$. The natural map $MU \longrightarrow BP$ is a map of ring spectra.

**Lemma**

The composition $R(\infty) \longrightarrow MU \longrightarrow BP$ induces a surjection on $H_{\ast}(-)$ and is a rational equivalence.

Now we could proceed to kill the torsion in $\pi_{\ast}R(\infty)$ by attaching $E_\infty$ cells. But in order to preserve the rational homotopy type, instead we use an idea of Tyler Lawson and attach $E_\infty$ cones on Moore spectra $S^m \cup_{p^k} D^{m+1}$ to kill elements of order $p^k$. The resulting spectrum $R$ has torsion-free homotopy and comes equipped with a map of commutative $S$-algebras $R(\infty) \longrightarrow R$ and a map of commutative ring spectra $BP \longrightarrow R$, both of which are rational equivalences. It is even true that $H_{\ast}(BP; \mathbb{Z}(2)) \longrightarrow H_{\ast}(R; \mathbb{Z}(2))$ is a split monomorphism. But it is not clear whether there is a map of ring spectra $R \longrightarrow BP$. If such a map were to exist then $R \sim BP$ so $BP$ would have an $E_\infty$ structure.
In the other direction the following holds.

**Theorem**
*If* $BP$ *is a commutative S-algebra then there is a weak equivalence of commutative S-algebras* $R \xrightarrow{\sim} BP$.

Here is another construction that gives a close approximation to $BP$. Start with the minimal atomic commutative $S$-algebra $MSp/U$. Then there are generators $x_{2k-1} \in \pi_{4k-2} MSp/U$ for which

$$\pi_* MSp/U = \mathbb{Z}(2)[x_{2k-1} : k \geq 1].$$

We can inductively kill the generators $x_{2k-1}$ for $k \neq 2^s$ to obtain a commutative $S$-algebra $T(\infty)$. Killing the torsion we get $T$ with

$$\pi_* T \otimes \mathbb{Q} \cong \pi_* BP \otimes \mathbb{Q}.$$ 

Again it is not clear if there is a map of ring spectra $T \longrightarrow BP$, but there is a map $BP \longrightarrow T$ which is a rational equivalence.

**Theorem**
*If* $BP$ *is a commutative S-algebra then there is a weak equivalence of commutative S-algebras* $T \longrightarrow BP$. 