Morava K-theory of BG: the good, the bad and the MacKey

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15th May 2012

Recollections on Galois extensions of commutative rings

Let R, S be commutative rings with a ring monomorphism $R \longrightarrow S$ making S an R-algebra, which we indicate by writing S/R. Suppose also that a finite group G acts faithfully on S by R-algebra automorphisms. Then S/R is a G-Galois extension if

- $S^G = R$;
- the unramified condition holds: the adjoint of the multiplication map induces a ring isomorphism

$$S \otimes_R S \xrightarrow{\cong} \mathrm{Map}(G, S) \cong \prod_G S.$$

The unramified condition implies that S is

- a finitely generated projective R-module,
- a separable (in fact étale) R-algebra,
- a faithfully flat R-module, *i.e.*,

$$S \otimes_R M = 0 \quad \iff \quad M = 0.$$

Some history

This version of Galois theory was developed by Auslander, Chase, Goldman, Harrison and Rosenburg in the 1960s. Much of the theory for fields has analogues, although there are some interesting differences.

For example, there is a trivial G-Galois extension $\prod_G R$ which is almost never a field. For G abelian the isomorphism classes of G-Galois extensions for an abelian group, and even when R is a field, the product of field extensions need not be a field.

The topological version of Galois theory is mainly due to John Rognes and we will discuss this next.

Moving to topology: good categories of spectra

- We must work in a good category of spectra with strictly associative and unital smash product *before passage to its derived category*. The category of S-modules $\mathscr{M}_S$ has this property. Both $\mathscr{M}_S$ and the derived category $\mathscr{D}_S$ are symmetric monoidal under $\wedge = \wedge_S$ with S as unit.
- Since S is not cofibrant in $\mathscr{M}_S$, to define cellular objects we use use free objects $\mathbb{F}S^n$ as cofibrant spheres. We write X for $\mathbb{F}X$, and if Y is a space, write $\mathbb{F}Y$ for $\mathbb{F}\Sigma^\infty Y$.
- A monoid R in $\mathscr{M}_S$ is an *S-algebra*, and a commutative monoid is a *commutative S-algebra*. An S-algebra gives a monoid in $\mathscr{D}_S$, traditionally called a *ring spectrum*.
- For a commutative S-algebra R we can also define R-modules and a symmetric monoidal category $\mathscr{M}_R$ with a smash product $\wedge_R$. A monoid A in $\mathscr{M}_R$ is an *R-algebra*, and when commutative it is a *commutative R-algebra*. The category of commutative R-algebras $\mathscr{C}_R$ is also a model category.

Galois extensions for commutative S-algebras

An extension of commutative S-algebras $A \longrightarrow B$ with a finite group G of automorphisms is a *Galois extension* if it satisfies the conditions

- the natural map $A \longrightarrow B^{hG}$ is a weak equivalence, where the target is the homotopy fixed point spectrum of the G-action, $B^{hG} = F(EG_+, B)^G$;
- the adjoint of the action map on the right hand factor induces a weak equivalence

$$B \wedge_A B \longrightarrow F(G_+, B) \cong \prod_G B.$$

The extension is *faithful* if B is a faithful A-module, *i.e.*, for any A-module M,

$$B \wedge_A M \sim * \quad \Longrightarrow \quad M \sim *.$$

Some examples of Galois extensions

- Let T/R be a G-Galois extension of commutative rings. Then there is a G-Galois extension HT/HR (Eilenberg-Mac Lane embedding).
- The morphism of commutative S-algebras $KO \longrightarrow KU$ induced from complexification of bundles makes KU/KO a faithful C_2-Galois extension. This example shows that for a topological G-Galois extension B/A, it may not be true that $\pi_* B/\pi_* A$ need not be G-Galois. For example, $\pi_* KU$ is not a projective $\pi_* KO$-module.
- Let p be a prime. John Rognes proved that for any finite group G which acts nilpotently on $\mathbb{F}_p[G]$,

$$F(BG_+, H\mathbb{F}_p) \longrightarrow F(EG_+, H\mathbb{F}_p) \sim H\mathbb{F}_p$$

  is a G-Galois extension; in particular this is true for nilpotent groups which include p-groups.

Here is one way to produce topological Galois extensions. Start with a commutative S-algebra A. Then $A_* = \pi_* A$ is a commutative (graded) ring. Now let B_*/A_* be an algebraic G-Galois extension.

Theorem

There is an essentially unique realization of this as $\pi_* B/A_*$, where $A \longrightarrow B$ is a faithful G-Galois extension of commutative S-algebras.

Some non-faithful Galois extensions

Ben Wieland observed that the C_2-Galois extension

$$F(BC_{2+}, H\mathbb{F}_2) \longrightarrow F(EC_{2+}, H\mathbb{F}_2) \sim H\mathbb{F}_2$$

which is not faithful. This depends on the algebraic fact that

$$\pi_*(F(BC_{2+}, H\mathbb{F}_2)) = H^{-*}(BC_2; \mathbb{F}_2)$$

is a polynomial algebra and so it is a regular ring. For the Tate spectrum $t_{C_2}H\mathbb{F}_2 \nsim *$ we have

$$H\mathbb{F}_2 \wedge_{F(BC_{2+}, H\mathbb{F}_2)} t_{C_2}H\mathbb{F}_2 \sim *.$$

This can be shown using the Künneth spectral sequence which is trivial because

$$\pi_* F(BC_{2+}, H\mathbb{F}_2) = \mathbb{F}_2[z], \quad \pi_* t_{C_2}H\mathbb{F}_2 = \mathbb{F}_2[z, z^{-1}],$$
$$\mathrm{Tor}_{*,*}^{\mathbb{F}_2[z]}(\mathbb{F}_2, \mathbb{F}_2[z, z^{-1}]) = 0.$$

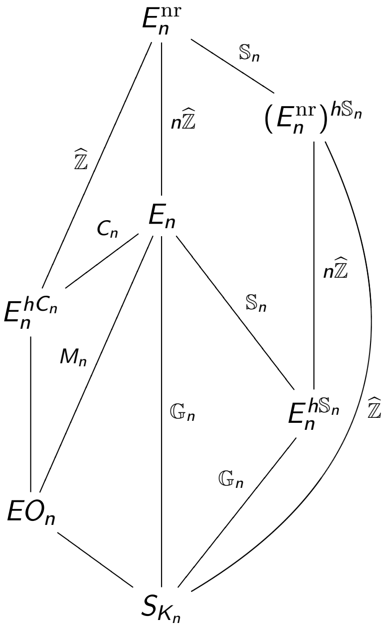
Analogous arguments work for C_p for any odd prime p.

Lubin-Tate and Morava K-theory spectra

If p is a prime and $1 \leqslant n < \infty$, there is a Lubin-Tate spectrum E_n which is a commutative S-algebra in an essentially unique way.
The associated Morava K-theory spectrum K_n is an E_n-algebra.
These spectra are 2-periodic and have homotopy rings

$$\begin{aligned} \pi_* E_n &= W\mathbb{F}_{p^n}[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}], \\ \pi_* K_n &= \mathbb{F}_{p^n}[u, u^{-1}]. \end{aligned}$$

It is useful to think of K_n as a residue skew-field of E_n (it is not even homotopy commutative when $p = 2$).
$n = 1$: $E_1 = KU_p$, p-adic complex K-theory, $K_1 = KU/p$, mod p complex K-theory.
$n = 2$: E_2 is related to *elliptic cohomology* and *topological modular forms*.
These theories are central in modern day stable homotopy and capture the *chromatic periodicity*. Working with K_n-local spectra, there is a Galois theory diagram capturing information in the n-th monochromatic layer of p-local-homotopy theory of S.


E_n^{nr}
$\mathbb{S}_n$
$(E_n^{\mathrm{nr}})^{h\mathbb{S}_n}$
$n\widehat{\mathbb{Z}}$
$\widehat{\mathbb{Z}}$
E_n
C_n
$E_n^{hC_n}$
$n\widehat{\mathbb{Z}}$
$\mathbb{S}_n$
M_n
$\mathbb{G}_n$
$E_n^{h\mathbb{S}_n}$
$\widehat{\mathbb{Z}}$
$\mathbb{G}_n$
EO_n
S_{K_n}

Warning: In this diagram most of the groups are not finite but profinite. Also, $C_n = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) = \widehat{\mathbb{Z}}$, and $M_n \leqslant \mathbb{G}_n$ is a maximal finite subgroup.
The Galois extension E_n^{nr}/E_n is obtained by adjoining roots of unity of order prime to p. Then $\pi_0(E_n^{\mathrm{nr}})/\pi_0(E_n)$ is a maximal unramified extension with Galois group $n\widehat{\mathbb{Z}} = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^n}) \leqslant \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$.

Theorem

There are no non-trivial connected Galois extensions of E_n^{nr}. Hence E_n^{nr} is the separable closure of S_{K_n}.

Lubin-Tate and Morava theory for classifying spaces

Let G be a finite group. It is known that $K_n^*(BG_+)$ is a finite dimensional K_n^*-vector space, and also a local ring, thus it is an Artinian local K_n^*-algebra. It follows that $E_n^*(BG_+)$ is a finitely generated E_n^*-module. If $K_n^{\mathrm{odd}}(BG_+) = 0$, then $E_n^{\mathrm{odd}}(BG_+) = 0$ and $E_n^*(BG_+)$ is a finitely generated free E_n^*-module.
The spectra $E_n^{BG} = F(BG_+, E_n)$ and $K_n^{BG} = F(BG_+, K_n)$ are K_n-local E_n-algebras, with $F(BG_+, E_n)$ commutative.
The extension

$$E_n^{BG} = F(BG_+, E_n) \longrightarrow F(EG_+, E_n) \sim E_n$$

is a candidate for being a G-Galois extension. We will consider this in the K_n-local setting. First we will consider faithfulness.

Finite groups and faithfulness

Let G be a finite group with a p-Sylow subgroup $G' \leqslant G$. The transfer map associated with $BG' \longrightarrow BG$ induces retractions

$$\begin{tikzcd} E^{BG'} \arrow[r, bend left, "\mathrm{Tr}^*"] & E^{BG} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} K^{BG'} \arrow[r, bend left, "\mathrm{Tr}^*"] & K^{BG} \arrow[l] \end{tikzcd}$$

therefore we can often focus on p-groups.

From now on we set $E = E_n$ and $K = K_n$. As E-algebras,

$$K^{BG} \cong K \wedge_E E^{BG}.$$

Lemma

For any E^{BG}-module M, there is isomorphism of K-modules

$$K \wedge_{E^{BG}} M \cong (K \wedge_E E) \wedge_{K \wedge_E E^{BG}} (K \wedge_E M).$$

In particular,

$$K \wedge_{E^{BG}} E \cong K \wedge_{K^{BG}} K.$$

Theorem

Let G be a finite group. Then E and K are faithful K-local E^{BG}-modules.

The Lemma allows reduction to the case of K^{BG}-modules, so it suffices to show that K is a faithful K^{BG}-module. For these we use some algebraic theory for modules over Artinian algebras, and a topological version of the *socle series* of a finitely generated module.

Algebraic recollections

If R is an Artinian local ring then its Jacobson radical $\operatorname{rad} R$ is maximal and nilpotent, say $(\operatorname{rad} R)^{e-1} \neq 0 = (\operatorname{rad} R)^e$, and $R/\operatorname{rad} R$ is a division ring.

For a non-trivial left R-module M, the socle series

$$0 \subsetneq \operatorname{soc}^1 M \subsetneq \operatorname{soc}^2 M \subsetneq \cdots \subsetneq \operatorname{soc}^e M = M.$$

is defined recursively by

$$\operatorname{soc}^1 M = \operatorname{soc} M = \{x \in M : (\operatorname{rad} R)x = 0\},$$

and the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{soc}^{k-1} M \arrow[r] \arrow[d] & \operatorname{soc}^k M \arrow[r] \arrow[d] & \operatorname{soc}(M/\operatorname{soc}^{k-1} M) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{soc}^{k-1} M \arrow[r] & M \arrow[r] & M/\operatorname{soc}^{k-1} M \arrow[r] & 0
\end{tikzcd}$$

Here each quotient $\operatorname{soc}^k M/\operatorname{soc}^{k-1} M$ is a vector space over the division ring $R/\operatorname{rad} R$.

A topological analogue

Suppose that A is an S-algebra so that $A_* = \pi_*(A)$ is a local Artinian graded ring and M is an finitely generated A-module. If we can realize the quotient $A_*/\operatorname{rad} A_*$ as $\pi_*(D)$ for some A-module D, then we can define a topological socle series

$$\operatorname{soc}^1 M \longrightarrow \operatorname{soc}^2 M \longrightarrow \cdots \longrightarrow \operatorname{soc}^e M = M$$

so that

$$\pi_*(\operatorname{soc}^k M) = \operatorname{soc}^k \pi_*(M).$$

Here $\operatorname{soc}^k M$ is well-defined up to isomorphism in the homotopy category $\mathscr{D}_A$.

For the case of $A = K^{BG}$ the residue ring is realized by K, so any K^{BG}-module M has a socle series.

There is a commutative diagram of S-algebras

$$\begin{tikzcd} K \arrow[rr, "="] \arrow[dr] & & K \\ & K^{BG} \arrow[ur] & \end{tikzcd}$$

and K is a left and right module over each of these algebras, so on smashing two copies over this diagram we obtain another

$$\begin{tikzcd} K \wedge_K K \arrow[rr, "="] \arrow[dr] & & K \wedge_K K \\ & K \wedge_{K^{BG}} K \arrow[ur] & \end{tikzcd}$$

from which it follows that $K \wedge_{K^{BG}} K \not\sim *$.

Now for a arbitrary K^{BG}-module with $\pi_* M \neq 0$, the socle series can be used to show that $K \wedge_{K^{BG}} M \not\sim *$, hence K is a faithful K^{BG}-module.

Theorem

For any finite group G,

$$F(BG_+, E) \longrightarrow F(EG_+, E) \sim E$$

is a faithful extension of K-local commutative E-algebras.

Now we consider the unramified condition which says that there is a weak equivalence

$$\Theta\colon F(BG_+, E) \wedge_{E^{BG}} F(BG_+, E) \longrightarrow F(G_+, E)$$

and therefore a weak equivalence

$$E \wedge_{E^{BG}} E \xrightarrow{\sim} \prod_G E.$$

In particular, $\pi_*(E \wedge_{E^{BG}} E)$ is concentrated in even degrees. We will call such an extension *ramified* if it is not unramified.

Theorem

For each $r \geqslant 1$, the extension

$$E^{BC_{p^r}} = F(BC_{p^r}{}_+, E) \longrightarrow F(EC_{p^r}{}_+, E)$$

is ramified and hence it is not C_{p^r}-Galois.

Sketch of proof: Recall that

$$(E^{BC_{p^r}})_* = E^*[[y]]/([p^r]y),$$

where $y \in (E^{BC_{p^r}})_0 = E^0(BC_{p^r}{}_+)$. The p-series $[p]y$ satisfies

$$[p]y \equiv y^{p^n} \mod \mathfrak{m},$$

and for each $r \geqslant 1$ the p^r-series is inductively given by

$$[p^r]y = [p]([p^{r-1}]y) = p^r y + \cdots + y^{p^{rn}} + \cdots \equiv y^{p^{rn}} \mod \mathfrak{m}.$$

By the Weierstrass preparation theorem, there is a polynomial

$$\langle p^r \rangle\, y = p^r + \cdots + y^{p^{rn}-1} \equiv y^{p^{rn}-1} \mod \mathfrak{m}$$

for which $[p^r]y = y\,\langle p^r \rangle\, y(1 + yf_r(y))$ for some $f_r(y) \in E^*[[y]]$. Then

$$(E^{BC_{p^r}})_* = E^*[[y]]/(y\,\langle p^r \rangle\, y).$$

The $(E^{BC_{p^r}})_*$-module E_* has the periodic minimal free resolution

$$0 \leftarrow E_* \longleftarrow (E^{BC_{p^r}})_* \xleftarrow{y} (E^{BC_{p^r}})_* \xleftarrow{\langle p^r \rangle y} (E^{BC_{p^r}})_* \xleftarrow{y} \cdots$$

so $\mathrm{Tor}_{*,*}^{(E^{BC_{p^r}})_*}(E_*, E_*)$ is the homology of the complex

$$\begin{aligned} 0 \leftarrow E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* &\xleftarrow{1\otimes y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \\ &\xleftarrow{1\otimes\langle p^r \rangle y} E_* \otimes_{(E^{BC_{p^r}})_*} (E^{BC_{p^r}})_* \xleftarrow{1\otimes y} \cdots \end{aligned}$$

which is equivalent to

$$0 \leftarrow E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \xleftarrow{0} E_* \xleftarrow{p^r} E_* \longleftarrow \cdots.$$

Since E_* is torsion-free, for $s \geqslant 0$ we get

$$\mathrm{Tor}_{s,*}^{(E^{BC_{p^r}})_*}(E_*, E_*) = \begin{cases} E_* & \text{if } s = 0, \\ E_*/p^r E_* & \text{if } s \text{ is odd}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus in the Künneth spectral sequence

$$\mathrm{E}^2_{s,t} = \mathrm{Tor}_{s,t}^{(E^{BC_{p^r}})_*}(E_*, E_*) \Longrightarrow \pi_{s+t}(E \wedge_{E^{BC_{p^r}}} E)$$

there can be no non-trivial differentials since for degree reasons the only possibilities involve E_*-module homomorphisms of the form

$$d^{2k-1} \colon \mathrm{E}^2_{2k-1,t} = E_t/p^r E_t \longrightarrow \mathrm{E}^2_{0,t+2k-2} = E_{t+2k-2},$$

with torsion-free target. This shows that the odd degree terms in $\pi_*(E \wedge_{E^{BC_{p^r}}} E)$ are non-zero, contradicting the unramified condition for a Galois extension.

We can extend this Theorem to arbitrary p-groups.

Theorem

Let G be a non-trivial p-group. Then the extension

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

is ramified, hence it is not G-Galois.

Sketch of proof: Choose a non-trivial epimorphism $G \longrightarrow C_p$. For some $k \geqslant 1$ there is a factorization

$$\begin{tikzcd} C_{p^k} \arrow[r, tail] \arrow[rr, bend left, two heads] & G \arrow[r, two heads] & C_p \end{tikzcd}$$

inducing morphisms between the associated Künneth spectral sequences

$$\mathrm{E}^r_{**}(C_p) \longrightarrow \mathrm{E}^r_{**}(G) \longrightarrow \mathrm{E}^r_{**}(C_{p^k}).$$

As in the proof of the Theorem above, the two outer spectral sequences have trivial differentials. We will analyze the composite morphism $\mathrm{E}^2_{**}(C_p) \longrightarrow \mathrm{E}^2_{**}(C_{p^k})$. On choosing generators appropriately, the canonical epimorphism $C_{p^k} \longrightarrow C_p$ induces the E_*-algebra monomorphism

$$(E^{BC_p})_* = E_*[[y]]/([p]y) \longrightarrow (E^{BC_{p^k}})_* = E_*[[y]]/([p^k]y);$$
$$y \mapsto [p^{k-1}]y,$$

so the induced map between the two resolutions of the form seen earlier is

$$\begin{tikzcd}
(E^{BC_p})_* \arrow[d, "\rho_0"] & (E^{BC_p})_* \arrow[l, "y"'] \arrow[d, "\rho_1"] & (E^{BC_p})_* \arrow[l, "\langle p\rangle y"'] \arrow[d, "\rho_2"] & \cdots \arrow[l, "y"'] \\
(E^{BC_{p^k}})_* & (E^{BC_{p^k}})_* \arrow[l, "y"'] & (E^{BC_{p^k}})_* \arrow[l, "\langle p^k\rangle y"'] & \cdots \arrow[l, "y"']
\end{tikzcd}$$

where the vertical maps are given by

$$\rho_{2s}\colon g(y) \mapsto g([p^{k-1}]y), \quad \rho_{2s-1}\colon h(y) \mapsto h([p^{k-1}]y)\left\langle p^{k-1}\right\rangle y.$$

Applying $E_* \otimes_{(E^{BC_{p^r}})_*} (-)$ with $r = 1, k$, we obtain a map of chain complexes

$$\begin{tikzcd}
0 & E_* \arrow[l] \arrow[d, "\rho'_0", "="'] & E_* \arrow[l, "0"'] \arrow[d, "\rho'_1 = p^{k-1}\cdot"] & E_* \arrow[l, "p"'] \arrow[d, "\rho'_2", "="'] & \cdots \arrow[l, "0"'] \\
0 & E_* \arrow[l] & E_* \arrow[l, "0"'] & E_* \arrow[l, "p^k"'] & \cdots \arrow[l, "0"']
\end{tikzcd}$$

where

$$\rho'_{2s} = \mathrm{id}, \quad \rho'_{2s-1} = p^{k-1} \cdot \ .$$

Applying this to the odd degree terms found in the proof of the previous Theorem we see that the induced map

$$E_*/pE_* \xrightarrow{p^{k-1}\cdot} E_*/p^k E_*$$

is always a monomorphism, so the first of the induced morphisms

$$\mathrm{E}^2_{**}(C_p) \longrightarrow \mathrm{E}^r_{**}(G) \longrightarrow \mathrm{E}^r_{**}(C_{p^k})$$

is a monomorphism. There can be no higher differentials killing elements in its image because they map to non-trivial elements of $\mathrm{E}^2_{**}(C_{p^k})$ which survive the right hand spectral sequence.

This shows that $\mathrm{E}^{\infty}_{**}(G)$ contains elements of odd degree, and as in the cyclic group case this is incompatible with the unramified condition.

We can extend this result to the class of p-nilpotent groups. A finite group G is p-nilpotent if each p-Sylow subgroup $P \leqslant G$ has a normal p-complement, *i.e.*, there is a normal subgroup $N \triangleleft G$ with $p \nmid |N|$ and $G = PN = P \ltimes N$.

Corollary

If G is a p-nilpotent group for which p divides $|G|$, then the extension

$$F(BG_+, E) \longrightarrow F(EG_+, E)$$

is ramified and so is not G-Galois.

Sketch of proof: By a result of John Tate, G being p-nilpotent is equivalent to the restriction homomorphism giving an isomorphism

$$H^*(BG; \mathbb{F}_p) \xrightarrow{\cong} H^*(BP; \mathbb{F}_p),$$

and it is sufficient that this holds in degree 1. Comparison of the Serre spectral sequences for $K^*(BG_+)$ and $K^*(BP_+)$ shows that

$$K^*(BG_+) \xrightarrow{\cong} K^*(BP_+).$$

It now follows that

$$E^*(BG_+) \xrightarrow{\cong} E^*(BP_+).$$

and the result can be deduced from our second Theorem.

Implications for Eilenberg-Moore spectral sequences

Our results show that the E-theory Eilenberg-Moore spectral sequence with E^2-term

$$ {}^{\text{L-T}}\mathrm{E}^2_{s,t} = \mathrm{Tor}^{(E^{BC_{p^r}})_*}(E_*, E_*) $$

does not converge to its expected target $\pi_*(\prod_{C_{p^r}} E)$. This is also true for any non-trivial p-group with $K^*(BG_+)$ concentrated in even degrees. Tilman Bauer has shown that the analogue based on Morava K-theory does converge correctly, at least when $G = C_p$.

$K^*(B(-))$ as a Green functor

We will discuss the functor which assigns $K^*(BG)$ to each finite group G. Since $K^*(BG) = K^*$ if and only if $p \nmid |G|$, this is a *globally defined Green functor* for the pair $\mathcal{X}, \mathcal{Y}$, where $\mathcal{X}$ consists of all finite groups and $\mathcal{Y}$ consists of all groups with $p \nmid |G|$. This means that for every group homomorphism $f\colon G \longrightarrow H$ there is a homomorphism of K^*-algebras $f^*\colon K^*(BH) \longrightarrow K^*(BG)$, and if $\ker f \in \mathcal{Y}$ there is a $K^*(BH)$-module homomorphism $f_*\colon K^*(BG) \longrightarrow K^*(BH)$. These should satisfy the conditions of a Mackey functor, and then the module homomorphism property of f_* makes it a (graded) Green functor.

It is convenient to regarded the grading as a $\mathbb{Z}/2$-grading.

Note that if p is an odd prime then $K^*(BG)$ is a commutative graded K^*-algebra; when $p = 2$ it is quasi-commutative, so for u, v of odd degree,

$$uv - vu = Q(u)Q(v),$$

where Q is a certain operation.

Green functor properties

We'll set $\mathbf{K}(G) = K^*(BG)$, $\Bbbk = K^*$, and consider the main properties of $\mathbf{K}(-)$. Actually there are two different structures to consider, namely the contravariant functor $(\mathbf{K}(-), (-)^*)$, and the covariant $(\mathbf{K}(-), (-)_*)$ where f_* is only defined when $p \nmid |\ker f|$. We also have $\mathbf{K}(\{1\}) = \Bbbk$.

A) $(\mathbf{K}(-), (-)^*, (-)_*)$ is a Green functor with values in $\Bbbk$-algebras.

B) $\mathbf{K}(G)$ is a finite dimensional local $\Bbbk$-algebra. Hence $\mathbf{K}(G)$ is Artinian and $\dim \operatorname{soc} \mathbf{K}(G) = 1$.

C) $\mathbf{K}(G)$ is a Frobenius algebra.

D) Let $i \colon \{1\} \longrightarrow G$ be the unit for any finite group G. Then $i_* 1 \neq 0$.

E) $(\mathbf{K}(-), (-)^*)$ is a Künneth functor.

F) For a finite abelian group A, $\mathbf{K}(A)$ is a bicommutative Hopf algebra. The finite group schemes $\operatorname{Spec} \mathbf{K}(C_{p^r})$ $(r \geqslant 1)$ with homomorphisms induced by canonical inclusions and quotients induces a p-divisible group.

Implications of these properties

The above properties imply the following.

1) If $H \leqslant K$ and $p \nmid |K : H|$, then $\mathrm{inc}_*(1) \in \mathbf{K}(K)^\times$ and $\mathrm{inc}^* \colon \mathbf{K}(K) \longrightarrow \mathbf{K}(H)$ is split monic.
2) For any group, $i \colon \{1\} \longrightarrow G$ has $0 \neq i_*1 \in \operatorname{soc} \mathbf{K}(G)$, so i_*1 is a basis for $\operatorname{soc} \mathbf{K}(G)$, and a linear form $\lambda \colon \mathbf{K}(G) \longrightarrow \Bbbk$ is a Frobenius form if and only if $\lambda(i_*1) \neq 0$.
3) If $K \lhd G$ and $p \nmid |K|$, then the canonical quotient $q \colon G \longrightarrow G/K$ indices and isomorphism $q^* \colon \mathbf{K}(G/K) \longrightarrow \mathbf{K}(G)$. Hence $\mathbf{K}(G) = \Bbbk$ if and only if $p \nmid |G|$.
4) For any p-group G, every epimorphism $q \colon G \longrightarrow A$ onto an abelian group induces a monomorphism $q^* \colon \mathbf{K}(A) \longrightarrow \mathbf{K}(G)$. Every monomorphism $j \colon B \longrightarrow C$ of finite abelian groups induces an epimorphism $j^* \colon \mathbf{K}(C) \longrightarrow \mathbf{K}(B)$
5) For a finite group G, the Mackey functor obtained by restricting $\mathbf{K}(-)$ to the subgroups of G is projective with respect to the p-subgroups. This implies versions of the stable elements formula.