BP: Close encounters of the E_∞ kind

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Motivation

The notion of an E_∞ *ring spectrum* arose over thirty years ago, and was studied in depth by many people including Peter May *et al*; later it was reinterpreted in the framework of EKMM as equivalent to that of a *commutative S-algebra*. A great deal of work on the existence of E_∞ structures using various obstruction theories has led to a considerable increase in the number of known examples. Despite this, there are some gaps in our knowledge. The question that is a major motivation for this talk is

- *Does the Brown-Peterson spectrum BP for a prime p admit an E_∞ ring structure?*

This has been recognized as an important outstanding problem for almost four decades, surviving various attempts to answer it.

What is BP?

For each prime p, there is a p-local spectrum BP whose cohomology as an $\mathcal{A}^* = \mathcal{A}(p)^*$-module is the cyclic quotient

$$H^*(BP; \mathbb{F}_p) = \mathcal{A}^*/(\beta)$$

(where $\beta = Sq^1$ when $p = 2$), or equivalently, when p is odd

$$H_*(BP; \mathbb{F}_p) = \mathbb{F}_p[\zeta_r : r \geqslant 1] \subset \mathbb{F}_p[\zeta_r : r \geqslant 1] \otimes \Lambda(\overline{\tau}_s : s \geqslant 0) = \mathcal{A}_*,$$

and when $p = 2$

$$H_*(BP; \mathbb{F}_2) = \mathbb{F}_2[\zeta_r^2 : r \geqslant 1] \subset \mathbb{F}_p[\zeta_r : r \geqslant 1] = \mathcal{A}_*.$$

These spectra have been important since Milnor showed that as an $\mathcal{A}^*$-module, $H^*(MU; \mathbb{F}_p)$ is a coproduct of suspensions of $\mathcal{A}^*/(\beta)$, so then $MU_{(p)}$ is a wedge of suspensions of BP provided such a spectrum exists. Later Brown and Peterson constructed BP by ad hoc methods, showing that such a topological splitting of $MU_{(p)}$ does exist.

There is a more concrete construction due to Quillen, who showed how to define an idempotent map of commutative ring spectra $\varepsilon\colon MU_{(p)} \longrightarrow MU_{(p)}$ which splits off BP as a retract of $MU_{(p)}$. There are associated maps of ring spectra

$$BP \longrightarrow MU_{(p)} \longrightarrow BP$$

whose composition is the identity. This construction depends on the algebraic universality of MU_* for formal group laws and the idempotent corresponds to a functorial p-typification operation.

Further structure

Since MU is an E_∞ ring spectrum, or equivalently a commutative S-algebra, it is natural to ask whether BP also has such structure. A stronger form of this question asks whether the natural maps

$$BP \longrightarrow MU_{(p)} \longrightarrow BP$$

are morphisms of commutative S-algebras (or more weakly, H_∞) ring spectra.

McClure and AB both worked unsuccessfully on resolving on this in the early 1980s. Recently it has been shown by Johnson & Noel that the map $MU_{(p)} \longrightarrow BP$ is not H_∞ for small primes p.

Hu, Kriz & May showed that for each prime p, $BP \longrightarrow MU_{(p)}$ cannot be H_∞. Kriz sketched a proof that BP is E_∞ based on TAQ, but this is widely believed to be incorrect. Further work by Basterra & Mandell, and Richter showed that BP supports some partial approximations to E_∞ structures.

The difficulties stem from the fact that BP has no known geometric description, and the failure of E_∞ obstruction theory.

Priddy's construction

Around 1980, Priddy gave a cellular construction of BP. Ideas in this were later reworked by Hu, Kriz & May, then AJB & JPM et al: BP is *minimal atomic* and any map $BP \longrightarrow MU_{(p)}$ which induces an isomorphism on $\pi_0(-)$ gives a monomorphism on $\pi_*(-)$, *i.e.*, this map is a *core* for $MU_{(p)}$.

Priddy constructs a CW p-local spectrum X so that the skeleta satisfy $X^{[0]} = S_{(p)}$, $X^{[2n]} = X^{[2n+1]}$ and $X^{[2m+2]}$ is obtained from $X^{[2m]}$ by attaching $(2m+2)$-cells to kill a minimal generating set of $\pi_{2m+1}X^{[2m]}$.

Obstruction theory arguments imply there are maps

$$X \longrightarrow MU_{(p)} \longrightarrow X$$

extending the identity on the 0-cell. By Milnor's calculations, X has the correct cohomology as an $\mathcal{A}^*$-module.

For the prime $p = 2$, Hu, Kriz & May identified a core of $MU_{(2)}$ in the homotopy category of 2-local commutative S-algebras, namely $(MSp/U)_{(2)} \longrightarrow MU_{(2)}$, where MSp/U is the Thom spectrum over the fibre in the fibration sequence of infinite loop spaces

$$Sp/U \longrightarrow BU \longrightarrow BSp.$$

For an odd prime p, the analogous core seems not to be known!

Commutative S-algebras and E_∞ ring spectra

Commutative S-algebras are essentially the same thing as E_∞ *ring spectra* which can be defined using extended power functors. To make sense of this we need to work in a context such as the model category of S-modules of EKMM. This has a symmetric monoidal structure with smash product $\wedge = \wedge_S$. In this setting, commutative S-algebras are the commutative monoids. Untangling the underlying structure of the smash product leads to the connection with the other notion of E_∞ ring spectra.
For a spectrum X,

$$D_n X = E\Sigma_n \ltimes_{\Sigma_n} X^{(n)}.$$

When $X = \Sigma^\infty Z_+$,

$$D_n \Sigma^\infty Z_+ = \Sigma^\infty (E\Sigma_n \times_{\Sigma_n} Z^n)_+$$

Then E is an E_∞ *ring spectrum* if there are suitably compatible maps $\mu_n \colon D_n E \longrightarrow E$ extending a product map

$$\mu \colon E^{(2)} \longrightarrow D_2 E \xrightarrow{\mu_2} E.$$

Killing homotopy the E_∞ way

Now we recall the idea of attaching E_∞ cells to a commutative S-algebra, and use various obstructions involving free commutative S-algebras.

If X is an S-module then the free commutative S-algebra on X is

$$\mathbb{P}X = \mathbb{P}_S X = \bigvee_{r \geqslant 0} X^{(r)}/\Sigma_r$$

When X is cofibrant the natural map is a weak equivalence

$$D_r X = E\Sigma_r \ltimes_{\Sigma_r} X^{(r)} \xrightarrow{\sim} X^{(r)}/\Sigma_r.$$

Let E be a commutative S-algebra, and let $f\colon \bigvee_i S^n \longrightarrow E$ be a map from a finite wedge of n-spheres. Then there is a unique extension of f to a morphism of commutative S-algebras $\widetilde{f}\colon \mathbb{P}(\bigvee_i S^n) \longrightarrow E$.

The pushout diagram of commutative S-algebras

$$\begin{tikzcd}
\mathbb{P}(\bigvee_i S^n) \arrow[r, "\widetilde{f}"] \arrow[d, "\mathbb{P}(\mathrm{inc})"'] \arrow[dr, phantom, "\ulcorner"] & E \arrow[d] \\
\mathbb{P}(\bigvee_i D^{n+1}) \arrow[r] & E/\!/f
\end{tikzcd}$$

defines $E/\!/f$ which we can regard as obtained from E by attaching E_∞ cells. In fact, we can take

$$E/\!/f = \mathbb{P}(\bigvee_i D^{n+1}) \wedge_{\mathbb{P}(\bigvee_i S^n)} E$$

where $\mathbb{P}(\bigvee_i D^{n+1})$ and E are $\mathbb{P}(\bigvee_i S^n)$-algebras in the evident way.
Note: $E/\!/f$ is weakly equivalent to $E/\!/g$ if f is homotopic to g.
So we sometimes write $E/\!/\alpha$ where α is the homotopy class of f.

The homology of extended powers has been well studied. In particular, life is simple rationally.

Proposition

For $n \in \mathbb{N}$, we have

$$H_*(\mathbb{P}S^{2n-1};\mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_{2n-1}), \quad H_*(\mathbb{P}S^{2n};\mathbb{Q}) = \mathbb{Q}[x_{2n}],$$

where $x_m \in H_m(\mathbb{P}S^m;\mathbb{Q})$ is the image of the homology generator of $H_m(S^m;\mathbb{Q})$.

In positive characteristic, the next result is fundamental.

Theorem

If X is connective then for a prime p, $H_*(\mathbb{P}X;\mathbb{F}_p)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by elements $Q^I x_j$, where x_j for $j \in J$ gives a basis for $H_*(X;\mathbb{F}_p)$, and $I = (\varepsilon_1, i_1, \varepsilon_2, \ldots, \varepsilon_\ell, i_\ell)$ is admissible with $\mathrm{excess}(I) + \varepsilon_1 > |x_j|$.

So for $p = 2$, $I = (i_1, i_2, \ldots, i_\ell)$ and we have

$$H_*(\mathbb{P}X;\mathbb{F}_2) = \mathbb{F}_2[Q^I x_j : j \in J,\ \mathrm{excess}(I) + i_1 > |x_j|\ ].$$

For ease of explanation we'll focus on the case $p = 2$, but the case of odd primes is similar, although the Künneth spectral sequence has a non-trivial differential d^{p-1}.
Suppose that R is a connective 2-local commutative S-algebra and that $\alpha \in \pi_{2n-1} R$ has Adams filtration 1 with representative $[w] \in \mathrm{Ext}^{1,2n}_{\mathcal{A}_*}(\mathbb{F}_2, H_*(R; \mathbb{F}_2))$ in the ASS.

Theorem

The homology of $R//\alpha$ has the form

$$H_*(R//\alpha; \mathbb{F}_2) = H_*(R; \mathbb{F}_2)[Q^I s : I \textit{ admissible, } \mathrm{excess}(I) > 2n],$$

for a generator $s \in H_{2n}(R//\alpha; \mathbb{F}_2)$ with coaction $\psi s = 1 \otimes s + w$. The rational homology is

$$H_*(R//\alpha; \mathbb{Q}) = H_*(R; \mathbb{Q})[S],$$

where S is the image of a lift of s to $H_*(R//\alpha; \mathbb{Z}_{(2)})$.

Power operations and the Adams spectral sequence

We will describe the 2-local case in detail. However, most of what we discuss has analogues for other primes. We set $H_*(-) = H_*(-;\mathbb{F}_2)$ and $H^*(-) = H^*(-;\mathbb{F}_2)$.

Given an E_∞ ring spectrum E, there are various types of power operations that can be defined. We will use operations based on D_2E, but relations between these depend on the D_nE for $n > 2$.

Given $\alpha \in \pi_k D_2 S^n$ (so we can realise α as a map $S^k \longrightarrow D_2 S^n$) there is an operation $\alpha^* \colon \pi_n E \longrightarrow \pi_k E$ given on $x \colon S^n \longrightarrow E$ by

$$\alpha^* x \colon S^k \xrightarrow{\alpha} D_2 S^n \xrightarrow{D_2 x} D_2 E \xrightarrow{\mu_2} E.$$

To understand elements of $\pi_* D_2 S^n$ it helps to notice that

$$D_2 S^n \sim \Sigma^n \mathbb{R}\mathrm{P}^\infty_n \sim \Sigma^n(\text{Thom spectrum of } n\rho_1 \downarrow \mathbb{R}\mathrm{P}^\infty).$$

The cell structure of this is simple, with one cell in each degree from $2n$ up. The $\mathcal{A}^*$-module structure of $H^* D_2 S^n$ can be found using the Wu formulae. Although $n \in \mathbb{Z}$ makes sense here, since we will assume that E is connective we will only need $n \geqslant 0$.

Theorem

Suppose that E is a connective commutative S-algebra for which $0 = \eta 1 \in \pi_1 E$. Then for $k \geqslant 1$, an operation $\mathcal{P}^{2^{k+1}-1}$ is defined on $\pi_{2^{k+1}-2}E$, giving a map

$$\mathcal{P}^{2^{k+1}-1} \colon \pi_{2^{k+1}-2}E \longrightarrow \pi_{2^{k+2}-3}E.$$

Moreover, the indeterminacy is trivial and the operation $2\mathcal{P}^{2^{k+1}-1}$ is trivial.

The next result shows how this works in the mod 2 Adams spectral sequence converging to $\pi_* E$ in good situations.

Lemma

With same assumptions, if $w \in \pi_{2^{k+1}-2} E$ is detected in the 1-line of the ASS by $W \in \mathrm{Ext}^{1,2^{k+1}-1}_{\mathcal{A}_*}(\mathbb{F}_2, H_* E)$, then $\mathcal{P}^{2^{k+1}-1} w$ is detected in the 1-line by

$$\mathcal{P}^{2^{k+1}-1} W \in \mathrm{Ext}^{1,2^{k+2}-2}_{\mathcal{A}_*}(\mathbb{F}_2, H_* E),$$

where $\mathcal{P}^{2^{k+1}-1}$ is the algebraic Steenrod operation of May et al.

We can calculate $\mathcal{P}^{2^{k+1}-1}$ on $\mathrm{Ext}^{1,2^{k+1}-1}_{\mathcal{A}_*}(\mathbb{F}_2, H_* E)$ by applying $Sq^1_* Q^{2^{k+1}-1} = Q^{2^{k+1}-1}$ to the cobar representatives to obtain explicit formulae.

The inductive construction

We will construct a sequence of connective 2-local commutative S-algebras

$$S = R(0) \longrightarrow R(1) \longrightarrow \cdots \longrightarrow R(n-1) \longrightarrow R(n) \longrightarrow \cdots$$

where $R(n)$ is obtained from by $R(n-1)$ by attaching a single E_∞ $(2^{n+1}-2)$-cell.
We start with $R(0) = S$ and $w_0 = \eta$ to give $R(1) = R//w_0$. In the ASS with standard cobar complex notation, η is represented by

$$[\zeta_1^2 \otimes 1] \in \mathrm{Ext}^{1,2}_{\mathcal{A}_*}(\mathbb{F}_2, \mathbb{F}_2).$$

As η has order 2, there is a commutative diagram of S-modules

$$\begin{tikzcd}
& & S^2 \arrow[d, dotted] \arrow[dr, "2"] \arrow[dd, dashed, bend right=60] & \\
S^1 \arrow[r, "\eta"] & R(0) \arrow[r] & C_\eta \arrow[r] \arrow[d] & S^2 \\
& & R(1) &
\end{tikzcd}$$

where the dashed arrow provides a homotopy class $u_1 \in \pi_2 R(1)$ of infinite order. The representative of this element is

$$[\zeta_1 \otimes s + \zeta_2 \otimes 1] \in \mathrm{Ext}^{1,3}_{\mathcal{A}_*}(\mathbb{F}_2, H_* R(1)).$$

We can use the power operation $\mathcal{P}^3$ to obtain a homotopy element $\mathcal{P}^3 u_1 \in \pi_5 R(1)$ of order 2 and represented in the ASS by

$$[\zeta_1^2 \otimes s^2 + \zeta_2^2 \otimes 1] \in \mathrm{Ext}^{1,6}_{\mathcal{A}_*}(\mathbb{F}_2, H_* R(1)).$$

Now we can proceed inductively. At each stage we have

- a commutative S-agebra $R(n)$ with an infinite order element $u_n \in \pi_{2^{n+1}-2}R(n)$,
- an element $w_n = \mathcal{P}^{2^{n+1}-1}u_n \in \pi_{2^{n+2}-3}R(n)$ of order 2,
- homology elements $S_r \in H_{2^{r+1}-2}(R(n);\mathbb{Q})$ reducing to $s_r \in H_{2^{r+1}-2}R(n)$ so that

$$H_*(R(n);\mathbb{Q}) = \mathbb{Q}[S_1,\ldots,S_n].$$

We then form $R(n+1) = R(n)//w_n$ where $R(n) \longrightarrow R(n+1)$ is a cofibrations. So for the homotopy colimit $R(\infty) = \operatorname{colim}_n R(n)$,

$$\pi_*R(\infty) = \operatorname*{colim}_n \pi_*R(n), \quad H_*(R(\infty);\mathbb{k}) = \operatorname*{colim}_n H_*(R(n);\mathbb{k})$$

for any coefficient ring $\mathbb{k}$.

Here are some observations on these constructions.

Lemma

The element u_n is in the Toda bracket $\langle 2, w_{n-1}, 1\rangle \subseteq \pi_{2^{n+1}-2}R(n)$, and in the Adams spectral sequence it has filtration 1 with cobar representative

$$\zeta_1 \otimes s_n + \zeta_2 \otimes s_{n-1}^2 + \zeta_3 \otimes s_{n-2}^{2^2} + \cdots + \zeta_n \otimes s_1^{2^{n-1}} + \zeta_{n+1} \otimes 1,$$

where ζ_j denotes the conjugate of the Milnor generator generator $\xi_j \in \mathcal{A}_{2^j-1}$.

Here we view $R(n)$ as a left $R(n-1)$-module and treat the first two variables of $\langle -, -, -\rangle$ as associated to $R(n-1)$, while the last is associated with $R(n)$.

Explicit formulae can be found for representatives of these homotopy elements in the ASS:

$$\begin{aligned} u_n &= [\zeta_1 \otimes s_n + \zeta_2 \otimes s_{n-1}^2 + \zeta_3 \otimes s_{n-2}^{2^2} \\ &\qquad\qquad + \cdots + \zeta_r \otimes s_{n-r+1}^{2^{r-1}} + \cdots + \zeta_{n+1} \otimes 1] \\ w_n &= [\zeta_1^2 \otimes s_n^2 + \zeta_2^4 \otimes s_{n-1}^4 + \zeta_3^3 \otimes s_{n-2}^{2^3} \\ &\qquad\qquad + \cdots + \zeta_r^2 \otimes s_{n-r}^{2^{r+1}} + \cdots + \zeta_{n+1}^2 \otimes 1]. \end{aligned}$$

The latter is obtained using a Dyer-Lashof operation to evaluate an Adams representative for $\mathcal{P}^{2^{n+1}-1} u_{n-1}$.

The commutative S-algebra MU has torsion-free homotopy, so there is a morphism $R(\infty) \longrightarrow MU$. The natural map $MU \longrightarrow BP$ is a map of ring spectra and $H_*MU \longrightarrow H_*BP \longrightarrow H_*H\mathbb{F}_p = \mathcal{A}_*$ is compatible with the Dyer-Lashof operations.

Lemma

The composition $R(\infty) \longrightarrow MU \longrightarrow BP$ induces surjections on $\pi_*(-), H_*(-;\Bbbk)$, and is a rational equivalence.

Sketch proof.

It is easy to see that the map $H_*R(1) \longrightarrow H_*BP$ sends s_1 to the standard generator t_1 in $H_*BP = \mathbb{F}_p[t_r : r \geqslant 1]$. Using Dyer-Lashof operations we also see that the remaining generators t_r are in the image. It follows that

$$H_*(R(\infty); \mathbb{Z}_{(p)}) \longrightarrow H_*(BP; \mathbb{Z}_{(p)})$$

is surjective. The argument for homotopy uses the Toda brackets to see that the Hazewinkel generators are in the image. $\square$

Now we could proceed to kill the torsion in $\pi_* R(\infty)$ by attaching E_∞ cells. But in order to preserve the rational homotopy type, we use an idea of Tyler Lawson and attach E_∞ cones on Moore spectra $S^m \cup_{p^k} D^{m+1}$ to kill elements of order p^k. The resulting spectrum R has torsion-free homotopy and comes equipped with a map of commutative S-algebras $R(\infty) \longrightarrow R$ and a map of commutative ring spectra $BP \longrightarrow R$, both of which are rational equivalences. But it is not clear whether there is a map of ring spectra $R \longrightarrow BP$ (or indeed any map which is an equivalence on the bottom cell). If such a map were to exist then $R \sim BP$ so BP would have an E_∞ structure.

In the other direction the following holds.

Theorem

If BP is a commutative S-algebra then there is a weak equivalence of commutative S-algebras $R \longrightarrow BP$.

Idea of proof.

Assume that we have a morphism of commutative S-algebras $R' \longrightarrow BP$ inducing an surjection on $\pi_*(-)$ (this is crucial to the argument taht follows) and which is a rational equivalence. Then the homotopy groups of the homotopy fibre are torsion. Now if $f\colon S^k \longrightarrow R'$ is a map of finite order, we can factor it through a Moore spectrum $\widetilde{f}\colon S^k \cup_{p^r} D^{k+1} \longrightarrow R'$ and then extend it to a map

$$R' \cup_{\widetilde{f}} C(S^k \cup_{p^r} D^{k+1}) \longrightarrow BP.$$

In fact this extends to a morphism of commutative S-algebras $R'//\widetilde{f} \longrightarrow BP$. Using this repeatedly we can obtain a morphism of commutative S-algebras $R \longrightarrow BP$ which is surjective on $\pi_*(-)$ and is a rational equivalence. $\square$

Some speculation

By construction, $R(\infty)$ is a nuclear commutative S-algebra and hence is a minimal atomic. However is not clear if R is minimal atomic. We can produce a core $R^c \longrightarrow R$, *i.e.*, a morphism of commutative S-algebras with R^c nuclear and which induces a monomorphism on $\pi_*(-)$. In particular, $\pi_*(R^c)$ is torsion-free.

Lemma

Let A be a connective p-local commutative S-algebra for which $\pi_*(A)$ is torsion-free. Then there is a morphism of commutative S-algebras $R(\infty) \longrightarrow A$. In particular, the natural morphism $R(\infty) \longrightarrow R$ admits a factorisation through any core $R^c \longrightarrow R$.

$$\begin{tikzcd} R(\infty) \arrow[r] \arrow[rr, bend left] & R^c \arrow[r] & R \end{tikzcd}$$

As R, and more generally any core R^c, have torsion-free homotopy concentrated in even degrees, standard arguments show that there are morphisms of ring spectra $BP \longrightarrow R$ and $BP \longrightarrow R^c$ associated with complex orientations with p-typical formal group laws. Our earlier arguments show that these are rational weak equivalences. Of course we have not shown that $BP \sim R$ even as (ring) spectra. One way to prove this would be to produce any map of spectra $R \longrightarrow BP$ that is an equivalence on the bottom cell, for then the composition $BP \longrightarrow R \longrightarrow BP$ would be a weak equivalence, therefore so would each of the maps $BP \longrightarrow R$ and $R \longrightarrow BP$. It is tempting to conjecture that R (or equivalently R^c) is always weak equivalent to BP but we have no hard evidence for this beyond what we have described above.