$\mathcal{E}_\infty$ ring spectra built from small complexes

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Iterated mapping cones using elements of Hopf invariant 1

**Conventions and assumptions:** Everything is 2-local. Homology and cohomology will usually be taken with $\mathbb{F}_2$ coefficients, so $H_*(-) = H_*(-; \mathbb{F}_2)$ and $H^*(-) = H^*(-; \mathbb{F}_2)$.

Recall the elements of Hopf invariant 1:

- $2 \in \pi_0(S^0) \cong \mathbb{Z}$,
- $\eta \in \pi_1(S^0) \cong \mathbb{Z}/2$,
- $\nu \in \pi_3(S^0) \cong \mathbb{Z}/8$,
- $\sigma \in \pi_7(S^0) \cong \mathbb{Z}/16$.

In the cohomology of the mapping cones,

- $H^*(C_2) = \mathbb{F}_2\{t^0, t^1\}$, $Sq^1 t^0 = t^1$,
- $H^*(C_\eta) = \mathbb{F}_2\{u^0, u^2\}$, $Sq^2 u^0 = u^2$,
- $H^*(C_\nu) = \mathbb{F}_2\{v^0, v^4\}$, $Sq^4 v^0 = v_4$,
- $H^*(C_\sigma) = \mathbb{F}_2\{w^0, w^8\}$, $Sq^8 w^0 = w^8$. 
These elements satisfy algebraic relations such as $2\eta = 0 = \eta \nu$, allowing the following iterated mapping cones to be constructed:

$$S^0 \cup \eta e^2 \cup 2 e^3, \quad S^0 \cup \nu e^4 \cup \eta e^6 \cup 2 e^7, \quad S^0 \cup \sigma e^8 \cup \nu e^{12} \cup \eta e^{14} \cup 2 e^{15}.$$  

The cohomology of these as modules over the Steenrod algebra $\mathcal{A}^*$ is simple to describe. For example:

$$H^*(S^0 \cup \sigma e^8 \cup \nu e^{12} \cup \eta e^{14} \cup 2 e^{15})$$

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Lemma

\[ S^0 \cup \eta \ e^2 \cup_2 e^3 \sim H\mathbb{Z}^{[3]}, \quad S^0 \cup \nu \ e^4 \cup_2 e^6 \cup_2 e^7 \sim kO^{[7]}, \]

\[ S^0 \cup \sigma \ e^8 \cup_2 e^{12} \cup_2 e^{14} \cup_2 e^{15} \sim \text{tmf}^{[15]}. \]

Here \( X^{[n]} \) denotes the \( n \)-skeleton of a CW spectrum \( X \).

Remark: Each of these spectra is the first of a series of ‘generalised integral Brown-Gitler spectra’ associated with \( H\mathbb{Z}, kO \) and \( \text{tmf} \) (the existence of the last series is still only conjectural).
Such iterated mapping cones exist in nature unstably. The infinite loop spaces $BSO = BO\langle 2 \rangle$, $BSpin = BO\langle 4 \rangle$ and $BString = BO\langle 8 \rangle$ have the following skeleta:

$$BSO^{[3]} = S^2 \cup_2 e^3, \quad BSpin^{[7]} = S^4 \cup_\eta e^6 \cup_2 e^7,$$

$$BString^{[15]} = S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15},$$

and the associated Thom spectra over these are

$$M_{SO}^{[3]} = S^0 \cup_\eta e^2 \cup_2 e^3, \quad M_{Spin}^{[7]} = S^0 \cup_\nu S^4 \cup_\eta e^6 \cup_2 e^7,$$

$$M_{String}^{[15]} = S^0 \cup_\sigma S^8 \cup_\nu e^{12} \cup_\eta e^{14} \cup_2 e^{15}.$$  

Each skeletal inclusion map factors through an infinite loop map,

$$BO\langle 2^r \rangle^{[2^{r+1}-1]} \xrightarrow{j_r} BO\langle 2^r \rangle$$

where $Q = \Omega^\infty \Sigma^\infty$ is the free infinite loop space functor. The associated Thom spectra $Mj_1$, $Mj_2$ and $Mj_3$ are $\mathcal{E}_\infty$ ring spectra.
Theorem
The homology rings of the Thom spectra \( M_{j_r} \) are given by

\[
H_*(M_{j_1}) = \mathbb{F}_2[Q^I x_2, Q^J x_3 : I, J \text{ admissible, } \text{exc}(I) > 2, \text{exc}(J) > 3],
\]
\[
H_*(M_{j_2}) = \mathbb{F}_2[Q^I x_4, Q^J x_6, Q^K x_7 : I, J, K \text{ admissible, } \text{exc}(I) > 4, \text{exc}(J) > 6, \text{exc}(K) > 7],
\]
\[
H_*(M_{j_3}) = \mathbb{F}_2[Q^I x_8, Q^J x_{12}, Q^K x_{14}, Q^L x_{15} : I, J, K, L \text{ admiss.}, \text{exc}(I) > 8, \text{exc}(J) > 12, \text{exc}(K) > 14, \text{exc}(L) > 15].
\]

There are \( \mathcal{E}_\infty \) morphisms \( M_{j_r} \rightarrow H\mathbb{F}_2 \) inducing algebra homomorphisms \( H_*(M_{j_r}) \rightarrow A_* \) with images

\[
\mathbb{F}_2[\zeta_1^2, \zeta_2, \zeta_3, \ldots] \cong H_*(H\mathbb{Z}), \quad \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \ldots] \cong H_*(kO), \\
\mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \zeta_5, \ldots] \cong H_*(tmf).
\]
In this result,

\[ x_2 \mapsto \zeta_1^2, \quad x_3 \mapsto \zeta_2, \]
\[ x_4 \mapsto \zeta_1^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3, \]
\[ x_8 \mapsto \zeta_1^8, \quad x_{12} \mapsto \zeta_2^4, \quad x_{14} \mapsto \zeta_3^2, \quad x_{15} \mapsto \zeta_4. \]

**Known result:** Mark Steinberger showed that the spectrum \( Mj_1 \) is a wedge of suspensions of \( H\mathbb{Z} \) and \( H\mathbb{Z}/2^k \) for various \( k \geq 1 \). It seems plausible that analogous splittings should exist for the others.

**Conjectures:**

- \( Mj_2 \) is a wedge of \( kO \) module spectra.
- \( Mj_3 \) is a wedge of \( \text{tmf} \) module spectra.

We will describe some algebraic analogues for such conjectural splittings.
Let $\mathbb{k}$ be a field of characteristic 2. We work with (graded) $\mathbb{k}$-vector spaces and set $\otimes = \otimes_{\mathbb{k}}$.

Let $A$ be a commutative Hopf algebra over $\mathbb{k}$, and let $B$ be a quotient Hopf algebra of $A$. We denote the (left) coaction on a comodule $M$ by $\psi_M$. We can induce a left $B$-comodule structure by composing.

$$\begin{array}{ccc}
M & \xrightarrow{\psi_M} & A \otimes M \\
& & \downarrow \\
& & B \otimes M
\end{array}$$
The cotensor product $L \square_B M$ of a right and a left $B$-comodule is the equaliser of the following diagram.

\[
\begin{array}{c}
L \otimes M \ar[r]^{	ext{Id} \otimes \psi'_M} & L \otimes B \otimes M \\
& \psi_L \otimes \text{Id} \ar@{_{(}->}[u] & \end{array}
\]

The cotensor product $A \square_B k \subseteq A \otimes k$ can be identified with a subalgebra of $A$ using the canonical isomorphism $A \otimes k \xrightarrow{\cong} A$. If $L$ or $M$ is extended (or cofree) we have

\[
(U \otimes B) \square_B M \cong U \otimes M, \quad L \square_B (B \otimes V) \cong L \otimes V.
\]
**Lemma**

Suppose that $C$ is a commutative $B$-comodule algebra and $D$ is a commutative $A$-comodule algebra. There is an isomorphism of $A$-comodule algebras

$$(A \Box_B C) \otimes D \cong A \Box_B (C \otimes D),$$

where the domain has the diagonal left $A$-coaction and $C \otimes D$ has the diagonal left $B$-coaction.

Explicitly, this isomorphism has the following effect on

$$\sum_r u_r \otimes v_r \otimes x \in (A \Box_B C) \otimes D \subseteq A \otimes C \otimes D,$$

$$\sum_r u_r \otimes v_r \otimes x \mapsto \sum_r \sum_i u_r a_i \otimes v_r \otimes x_i,$$

where $\psi_D x = \sum_i a_i \otimes x_i$. 
Lemma
Suppose that $M$ is a left $A$-comodule and $N$ is a left $B$-comodule. Then there is a natural isomorphism

$$\text{Comod}_B(M, N) \xrightarrow{\cong} \text{Comod}_A(M, A \Box BN); \quad f \mapsto \tilde{f},$$

where $\tilde{f}$ is the unique factorisation of $(\text{Id} \otimes f)\psi_M: M \to A \otimes N$ through $A \Box BN$. When $M$ is an $A$-comodule algebra and $N$ is a $B$-comodule algebra, if $f$ is an algebra homomorphism, so is $\tilde{f}$. 

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Some quotient Hopf algebras of the dual Steenrod algebra

We write $H = H\mathbb{F}_2$ and note that $H_*(H) = A_*$ and $H^*(H) = A^*$. Recall that

$$A_* = \mathbb{F}_2[\xi_r : r \geq 1] = \mathbb{F}_2[\zeta_r : r \geq 1],$$

where $\xi_r, \zeta_r \in A_{2r-1}$ and the coproduct and antipode satisfy

$$\psi(\xi_r) = \sum_{0 \leq i \leq r} \xi_{r-i}^2 \otimes \xi_i, \quad \psi(\zeta_r) = \sum_{0 \leq i \leq r} \zeta_i \otimes \zeta_{r-i}^2, \quad \zeta_r = \chi(\xi_r).$$

For $n \geq 0$, the ideal

$$\mathcal{I}(n) = (\zeta_1^{2n+1}, \zeta_2^n, \ldots, \zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3}, \ldots) \triangleleft A_* \quad (n \geq 0)$$

is a Hopf ideal and the quotient algebra $A(n)_* = A_*/\mathcal{I}(n)$ is a finite dimensional commutative Hopf algebra. The dual of $A(n)_*$ is the subalgebra

$$A(n)^* = \mathbb{F}_2\langle Sq^1, Sq^2, Sq^4, \ldots, Sq^{2n+1} \rangle \subseteq A^*$$

generated by the listed Steenrod operations.
We have

$$\mathcal{A}_* \boxtimes A(n)_* \mathbb{F}_2 = \mathbb{F}_2[\zeta_1^{2n+1}, \zeta_2^{2n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \zeta_{n+3}, \ldots] \subseteq \mathcal{A}_*,$$

where \( A(n)_* = \mathcal{A}_*/(\mathcal{A}_* \boxtimes A(n)_* \mathbb{F}_2). \)

**Theorem**

The natural morphisms of \( E_\infty \) ring spectra \( H\mathbb{Z} \to H, \) \( kO \to H \) and \( \text{tmf} \to H \) induce monomorphisms in \( H_*(-) \) whose images are \( \mathcal{A}_* \boxtimes A(0)_* \mathbb{F}_2, \mathcal{A}_* \boxtimes A(1)_* \mathbb{F}_2 \) and \( \mathcal{A}_* \boxtimes A(2)_* \mathbb{F}_2. \)
Theorem

For each $n \geq 0$, $\mathcal{A}_*$ is an extended right $\mathcal{A}(n)_*$-comodule:

$$\mathcal{A}_* \cong (\mathcal{A}_* \Box \mathcal{A}(n)_* \mathbb{F}_2) \otimes \mathcal{A}(n)_*.$$

This implies that for any left $\mathcal{A}(n)_*$-comodule $M$,

$$\mathcal{A}_* \Box \mathcal{A}(n)_* M \cong (\mathcal{A}_* \Box \mathcal{A}(n)_* \mathbb{F}_2) \otimes M.$$

This can be viewed as an isomorphism of left $\mathcal{A}_*$-comodules using a suitable comodule structure on the right hand side.
Coactions and Dyer-Lashof operations

For an $\mathcal{E}_\infty$ ring spectrum $X$, the intertwining of right action of the Steenrod algebra and the left action of the Dyer-Lashof operations on $H_*(X)$ is described using the *Nishida relations*. However, the interaction of the coaction $\psi : H_*(X) \rightarrow A_* \otimes H_*(X)$ can also be described. It is better to twist this into a right coaction $\tilde{\psi} : H_*(X) \rightarrow H_*(X) \otimes A_*$: for $x \in H_m(X)$ and $r \geq m$,

$$\tilde{\psi} Q^s(x) = \sum_{k=m}^s Q^k(\tilde{\psi}(x)) \left[ \left( \frac{\zeta(t)}{t} \right)^k \right]_{t^{s-k}}.$$

For example, if $\psi(x) = \sum_i a_i \otimes x_i$ then $\tilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i)$ and

$$\tilde{\psi} Q^{m+1}(x) = \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i Q^{m+1}(x_i \otimes \chi(a_i))$$

$$= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \sum_j Q^{m+1-j} x_i \otimes Q^j \chi(a_i)).$$
The Dyer-Lashof action on $\mathcal{A}_*$ was determined by Kochman (implicitly) and Steinberger. For example,
\[
Q^{2s} \zeta_s = \zeta_{s+1}.
\]
Another useful formula is
\[
Q^{2s} \xi_s = \xi_{s+1} + \xi_1 \xi_s^2.
\]
Here is a formula for the homology action of the Milnor primitive $q^s \in \mathcal{A}_{2^{s+1} - 1}$. If $x \in H_n(X)$, $s \geq 0$ and $r > n$, then
\[
q^s \ast Q^r x = (r + 1)Q^{r-2^{s+1}+1} x + \sum_{0 \leq k \leq s-1} Q^{r-2^{s+1}+2k+1} (q^k x).
\]
Theorem

For $r = 1, 2, 3$ there is a regular sequence $X_{r,s} \in H_*(Mjr)$ ($s \geq 1$) so that the ideal $I_r = (X_{r,s} : s \geq 1) \triangleleft H_*(Mjr)$ is $A(r-1)_*$-invariant. Furthermore, the top composition is an isomorphism in the commutative diagram of commutative $A_*$-comodule algebras.

$$
\begin{array}{ccc}
H_*(Mjr) & \longrightarrow & A_* \boxtimes A(1-1)_* \\
& \searrow & \downarrow \\
& \psi & \\
A_* \otimes H_*(Mjr) & \longrightarrow & A_* \otimes H_*(Mjr)/I_r
\end{array}
$$
Explicit formulae

\[ X_{1,s} = \begin{cases} 
  x_2 & \text{if } s = 1, \\
  x_3 & \text{if } s = 2, \\
  Q^{(2^{s-1}, \ldots, 2^4, 2^3, 2^2)}x_3 & \text{if } s \geq 3. 
\end{cases} \]

\[ X_{2,s} = \begin{cases} 
  x_4 & \text{if } s = 1, \\
  x_6 & \text{if } s = 2, \\
  x_7 & \text{if } s = 3, \\
  Q^8x_7 + Q^9x_6 & \text{if } s = 4, \\
  Q^{(2^{s-1}, \ldots, 2^5, 2^4)}(Q^8x_7 + Q^9x_6) & \text{if } s \geq 5. 
\end{cases} \]
\[ x_{3,s} = \begin{cases} 
  x_8 & \text{if } s = 1, \\
  x_{12} & \text{if } s = 2, \\
  x_{14} & \text{if } s = 3, \\
  x_{15} & \text{if } s = 4, \\
  Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12} & \text{if } s = 5, \\
  Q^{(2^{s-1}, \ldots, 2^6, 2^5)}(Q^{16}x_{15} + Q^{17}x_{14} + Q^{19}x_{12}) & \text{if } s \geq 6. 
\end{cases} \]
For $r = 1, 2, 3$, the trivial algebra homomorphism $\mathbb{F}_2 \to H_*(Mj_r)/I_r$ is an $\mathcal{A}(r - 1)_*$-comodule algebra homomorphism which induces an $\mathcal{A}_*$-comodule algebra homomorphism

$$\mathcal{A}_* \boxtimes A(r-1)_* \mathbb{F}_2 \to \mathcal{A}_* \boxtimes A(r-1)_* H_*(Mj_r)/I_r \cong H_*(Mj_r).$$

There are morphisms of $\mathcal{E}_\infty$ ring spectra $Mj_1 \to H\mathbb{Z}$, $Mj_2 \to kO$ and $Mj_3 \to \text{tmf}$ which are surjective on $H_*(-)$. Using the last result we see that there are splittings of $\mathcal{A}_*$-algebras of the form

$$\xymatrix{ \mathcal{A}_* \boxtimes A(r-1)_* \mathbb{F}_2 \ar[rr]^R \ar[dr] \ar[dd] & & \mathcal{A}_* \boxtimes A(r-1)_* \mathbb{F}_2 \ar[dl] \ar[dd] \\ & H_*(Mj_r) & }$$
Some applications

There are morphisms of $E_\infty$ ring spectra $Mj_1 \rightarrow MSO$, $Mj_2 \rightarrow MSpin$ and $Mj_3 \rightarrow MString$ so there are splittings of $A_*$-algebras of the form

$$A_* \square A(r-1)_* \mathbb{F}_2 \xrightarrow{\sim} A_* \square A(r-1)_* \mathbb{F}_2 \xrightarrow{\sim} H_*(M?)$$

where $M?$ is one of the Thom spectra $Mj_r$, $MSO$, $MSpin$ or $MString$.

Such algebraic results have a long history. The case of $MString$ was proved by Bahri & Mahowald using the $E_2$-space $\Omega^2 \Sigma^2 BString^{[15]}$. 
An example related to connective $K$-theory

There is a map $S^2 \vee B\text{Spin}^{[7]} \rightarrow (B\text{Spin}^c)^{[7]}$ which factors through an infinite loop map

$$
\begin{array}{ccc}
S^2 \vee B\text{Spin}^{[7]} & \longrightarrow & B\text{Spin}^c \\
\downarrow & & \downarrow \ j^c \\
Q(S^2 \vee B\text{Spin}^{[7]}) & \longrightarrow & B\text{Spin}^c
\end{array}
$$

where $j^c$ is a 7-equivalence. The Thom spectrum $Mj^c$ is $E_\infty$ and $H_*(Mj^c)$ is polynomial on suitable elements $Q^I x_2, Q^J x_4, Q^K x_6, Q^L x_7$. There is an $E_\infty$ morphism $Mj^c \rightarrow kU$ inducing an epimorphism $H_*(Mj^c) \rightarrow H_*(kU)$ under which

$$
x_2 \mapsto \zeta_1^2, \quad x_4 \mapsto \zeta_1^4, \quad x_6 \mapsto \zeta_2^2, \quad x_7 \mapsto \zeta_3.
$$
The element $x_4 + x_2^2$ is spherical an attaching an $\mathcal{E}_\infty$ cell to kill the homotopy element $w$ detected by it gives $\mathcal{E}_\infty$ morphisms

$$Mj^c \rightarrow Mj^c \sslash w \rightarrow kU$$

where $H_*(Mj^c \sslash w)$ is a regular quotient of $H_*(Mj^c)$ generated by suitable elements $Q^I x_2$, $Q^J x_6$, $Q^K x_7$.

**Theorem**

*There is an isomorphism of $A_\ast$-comodule algebras*

$$H_*(Mj^c \sslash w) \cong A_\ast \square \mathcal{E}(1,2)_\ast H_*(Mj^c \sslash w)/I^c,$$

*where $\mathcal{E}(1,2)_\ast = A_\ast/\langle \zeta_1^2, \zeta_2^2, \zeta_3, \ldots \rangle$ and $I^c \triangleleft H_*(Mj^c \sslash w)$ is a regular $\mathcal{E}(1,2)_\ast$-comodule ideal with generators $X_2, X_6, X_7, X_{15}, \ldots$*
The 7-skeleta of $Mj^c$ and $Mj^c//w$
Observations and questions

- It is known that $\pi_*(Mj_1) \to \pi_*(HZ)$, $\pi_*(Mj_2) \to \pi_*(kO)$ and $\pi_*(Mj^c//w) \to \pi_*(kU)$ are surjective. Furthermore, $\pi_k(Mj_3) \to \pi_k(tmf)$ is an isomorphism for $k \leq 16$. Is $\pi_*(Mj_3) \to \pi_*(tmf)$ surjective?

- Is $Mj_2$ a wedge of $kO$ module spectra? Is $Mj_3$ a wedge of $tmf$ module spectra? Is $Mj^c//w$ a wedge of $kU$ module spectra?

- The Thom spectra $Mj_r$ are equivalent to reduced free algebras, i.e.,

$$Mj_1 \sim \widetilde{P}HZ^{[3]}, \quad Mj_2 \sim \widetilde{P}kO^{[7]}, \quad Mj_3 \sim \widetilde{P}tmf^{[15]}.$$  

This useful for giving $\mathcal{E}_\infty$ maps to $HZ$, $kO$ and $tmf$ which avoids using orientations for $MSO$, $MSpin$ and $MString$. 