Characteristics for E_∞ ring spectra

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Characteristics of rings and algebras

For any (not necessarily commutative) ring with unity $1 \neq 0$ there is a *unit* or *characteristic* ring homomorphism $\eta\colon \mathbb{Z} \to R$. Then $\ker\eta \lhd \mathbb{Z}$ is a proper ideal and there is a monomorphism $\overline{\eta}\colon \mathbb{Z}/\ker\eta \to R$ identifying $\mathbb{Z}/\ker\eta$ with the subring $\eta\mathbb{Z} \subset R$.

The *characteristic* of R, $\operatorname{char} R$, is the unique non-negative integer such that $\ker\eta = (\operatorname{char} R) \lhd \mathbb{Z}$.

This generalises to unital $\Bbbk$-algebras over a commutative ring $\Bbbk$. For the unit $\eta\colon \Bbbk \to A$ of such an algebra, there is an ideal $\ker\eta \lhd \Bbbk$ and quotient homomorphism $\Bbbk/\ker\eta \to A$ whose image is the *characteristic subalgebra* of A.

Problem: How can we generalise this to *derived rings*?

Topological examples:
E_∞ *ring spectra* $=$ *commutative* S*-algebras*.

N.B. In this setting ideals and quotients objects may not exist.

Some desirable properties of characteristics

- A characteristic of a commutative S-algebra R will involve a factorisation of its unit $S \to R_0 \xrightarrow{\chi_R} R$.
- Homotopically well defined.
- Functorial (at least homotopically).

We may assume that R_0 is connective even if R isn't, so can replace R by its connective cover.
Our definition is for p-local commutative S-algebras, but there should be a resulting global notion in terms of local data.

Free commutative S-algebras

If X is an S-module then the free commutative S-algebra on X is

$$\mathbb{P}X = \mathbb{P}_S X = \bigvee_{r \geqslant 0} X^{(r)}/\Sigma_r$$

$\mathbb{P}\colon \mathscr{M}_S \to \mathscr{C}_S$ is left adjoint to the forgetful functor, so it preserves pushouts, and creates the model structure on $\mathscr{C}_S$. Commutative S-algebras can be characterised as the algebras over the monad $\mathbb{P}(-)$ on $\mathscr{M}_S$.
When X is cofibrant, there is a weak equivalence

$$\bigvee_{r \geqslant 0} D_r X \xrightarrow{\sim} \mathbb{P}X.$$

Basic observation: For a $\mathbb{P}X$-module E, $\mathrm{TAQ}_*(\mathbb{P}X, S; E) \cong E_*(X)$. This leads to a cellular interpretation of $\mathrm{TAQ}_*(A, S; HM)$ for a CW commutative S-algebra A where M is a $\pi_0(A)$-module.

Reduced free commutative S-algebras

Let $S^0 \to S$ be the functorial cofibrant replacement in $\mathscr{M}_S$.
If X is an S-module under S^0, $S^0 \to X$, then $\widetilde{\mathbb{P}}X$ is defined to be the pushout in

$$\begin{tikzcd} \mathbb{P}S^0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}X \arrow[d] \\ S \arrow[r] & \widetilde{\mathbb{P}}X \end{tikzcd}$$

where we use $S^0 \to S$ to define the left hand map.
Here $\widetilde{\mathbb{P}}\colon S^0/\mathscr{M}_S \to \mathscr{C}_S$ is left adjoint to the functor which sends an S-algebra A to $S^0 \to S \to A$, so it preserves pushouts and creates the model structure on the category of commutative S-algebras $\mathscr{C}_S$.

Observation: For a $\widetilde{\mathbb{P}}X$-module E, $\mathrm{TAQ}_*(\widetilde{\mathbb{P}}X, S; E) \cong E_*(X/S^0)$.

Attaching cells the E_∞ way

Let E be a commutative S-algebra, and let $f\colon \bigvee_i S^n \to E$ be a map from a finite wedge of n-spheres. There is a unique extension of f to a morphism $\widetilde{f}\colon \mathbb{P}(\bigvee_i S^n) \to E$ in $\mathscr{C}_S$. The pushout diagram in $\mathscr{C}_S$

$$\begin{tikzcd} \mathbb{P}(\bigvee_i S^n) \arrow[r, "\mathbb{P}(\mathrm{inc})"] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "\ulcorner"] & \mathbb{P}(\bigvee_i D^{n+1}) \arrow[d] \\ E \arrow[r] & E/\!/f \end{tikzcd}$$

defines $E/\!/f$, and we say it is obtained from E by attaching E_∞ $(n+1)$-cells. In fact,

$$E/\!/f \cong \mathbb{P}(\bigvee_i D^{n+1}) \wedge_{\mathbb{P}(\bigvee_i S^n)} E,$$

where $\mathbb{P}(\bigvee_i D^{n+1})$ and E are viewed as $\mathbb{P}(\bigvee_i S^n)$-algebras.
If f is homotopic to g, then $E/\!/f$ is weakly equivalent to $E/\!/g$, so we write $E/\!/\alpha$ where α is the homotopy class of f.

Throughout we work with connective finite type p-local commutative S-algebras.

Definition

A *characteristic* for R is a nuclear CW commutative S-algebra $j^0 \colon S \to T$ for which there is a *characteristic morphism* $j \colon T \to R$, where the E_∞ skeleta $T^{\langle n \rangle}$ ($n \geqslant 1$) are defined inductively using E_∞ attaching maps induced from maps

$$f^n \colon \bigvee_i S^n \to S \to T^{\langle n \rangle},$$

which factor through the unit of $T^{\langle n \rangle}$ and satisfy

$$\begin{aligned} \operatorname{im} f^n_* = \operatorname{im}[j^0_* \colon \pi_n(S) \to \pi_n(T^{\langle n \rangle})] \\ \cap \ker[j^{\langle n \rangle}_* \colon \pi_n(T^{\langle n \rangle}) \to \pi_n(R)]. \end{aligned}$$

This definition begs the question of whether characteristics are in any sense unique. Notice also that the attaching maps of the E_∞ cells originate as maps into the sphere spectrum S.

Theorem

Characteristics exist, and enjoy the following properties.

- *Suppose that T and T' are characteristics for R and R' respectively. If there is a morphism $R \to R'$, then there is a morphism $T \to T'$.*
- *Suppose that T_1 and T_2 are two characteristics for R. Then there is a homotopy equivalence of commutative S-algebras $T_1 \xrightarrow{\simeq} T_2$. Therefore characteristics are unique up to homotopy equivalence.*
- *Let $h\colon R \to R'$ be a weak equivalence and let $k\colon T \to R'$ be a characteristic morphism. Then there is a morphism $j\colon T \to R$ such that $g \circ j \simeq k$ and j is a characteristic morphism for R.*

Nuclear CW algberas

Suppose that $p > 0$ is a prime. We work with connective p-local commutative S-modules and S-algebras, where S is the p-local sphere spectrum.

A connective CW commutative S-algebra A is *nuclear* if it is defined by requiring that its $(n+1)$-cells are attached so that for each $n \geqslant 0$,

$$\ker[\pi_n(\bigvee_i S^n) \to \pi_n(A^{\langle n \rangle})] \subseteq p\,\pi_n(\bigvee_i S^n).$$

Nuclear algebras are *minimal atomic*, and in turn these are characterised by the property that their positive degree homotopy is not detected by the Hurewicz homomorphism in $\mathrm{TAQ}_*(-, S; \mathbb{F}_p)$.

There is a similar notion of *nuclear CW S-module under S^0*.

Let R be a connective p-local commutative S-algebra.

Theorem

Form a nuclear complex X by inductively attaching cells to S^0 so as to kill

$$\mathrm{im}[\pi_n(S) \to \pi_n(X^{[n]})] \cap \ker[\pi_n(X^{[n]}) \to \pi_n R].$$

Then there is a an morphism of commutative S-algebras $\widetilde{\mathbb{P}}X \to R$; furthermore, if $\widetilde{\mathbb{P}}X$ is minimal atomic, this is a homotopy equivalence.

Remark: In general for a CW complex $S^0 \to Y$, $\widetilde{\mathbb{P}}Y$ minimal atomic implies Y minimal atomic, but the converse need not be true. For $p = 2$, an example is provided by $Y = \Sigma^{-2}\Sigma^\infty\mathbb{C}\mathrm{P}^\infty$.

The homology of extended powers has been studied extensively. In particular, life is simple rationally.

Proposition

For $n \in \mathbb{N}$, let $x_m \in H_m(\mathbb{P}S^m; \mathbb{Q})$ be the image of the homology generator of $H_m(S^m; \mathbb{Q})$. Then

$$H_*(\mathbb{P}S^{2n-1}; \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x_{2n-1}), \quad H_*(\mathbb{P}S^{2n}; \mathbb{Q}) = \mathbb{Q}[x_{2n}].$$

In positive characteristic, the next result is fundamental.

Theorem

If X is connective then for a prime p, $H_*(\mathbb{P}X; \mathbb{F}_p)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by elements $Q^I x_j$, where x_j for $j \in J$ gives a basis for $H_*(X; \mathbb{F}_p)$, and $I = (\varepsilon_1, i_1, \varepsilon_2, \ldots, \varepsilon_\ell, i_\ell)$ is admissible with $\mathrm{excess}(I) + \varepsilon_1 > |x_j|$.

So for $p = 2$, $I = (i_1, i_2, \ldots, i_\ell)$ and we have

$$H_*(\mathbb{P}X; \mathbb{F}_2) = \mathbb{F}_2[Q^I x_j : j \in J, \text{excess}(I) > |x_j|\,].$$

Theorem

Assume that X is p-local, connective and that $\pi_0(X)$ is a cyclic $\mathbb{Z}_{(p)}$-module, and let $S^0 \to X$ give a generator of $\pi_0(X)$. Then $H_*(\widetilde{\mathbb{P}}X; \mathbb{F}_p)$ is the free commutative graded $\mathbb{F}_p$-algebra generated by elements $Q^I x_j$, where x_j for $j \in J$ gives a basis for $H_*^+(X; \mathbb{F}_p)$, and $I = (\varepsilon_1, i_1, \varepsilon_2, \ldots, \varepsilon_\ell, i_\ell)$ is admissible with $\text{excess}(I) + \varepsilon_1 > |x_j|$.

The idea here is that the generator $x_0 \in H_*(\mathbb{P}X; \mathbb{F}_p)$ coming from the bottom cell has been identified with 1.

Examples

Markus Syzmik: Let p be a prime. Then for any p-local commutative S-algebra R for which $\pi_0(R)$ is an $\mathbb{F}_p$-algebra, there is a morphism $S//p \to R$ which is in fact a characteristic. The proof depends on a result of Mark Steinberger.

Theorem

If A is a commutative S-algebra for which $\pi_0(A)$ is an $\mathbb{F}_p$-algebra, then A is a wedge of suspensions of $H\mathbb{F}_p$.

The theorem applies to $S//p$ itself, so it is a wedge of suspensions of $H\mathbb{F}_p$. Hence $\pi_*(S) \to \pi_*(S//p)$ is trivial in positive degrees as well as p itself.
A morphism $S//p \to R$ so that $S \to S//p \to R$ kills all positive degree homotopy.

Generalisation: Replace p by p^r for $r > 1$. Steinberger showed that if $\operatorname{char}\pi_0(A) = p^r$ and $\beta\mathcal{P}^1$ (Sq^3 if $p = 2$) acts non-trivially on $H^0(A)$ then A is a wedge of suspensions of $H\mathbb{Z}/p^s$ for $1 \leqslant s \leqslant r$.
To ensure this condition holds we need to form $S//p^r, \alpha_1, v_1$ ($S//2^r, \eta, v_1$ if $p = 2$).
Conjecture: For p an odd prime, $S//p^r, \alpha_1$ is a characteristic for $H\mathbb{Z}/p^r$.
For $p = 2$, $S//2^r, \eta, \sigma$ is a characteristic for $H\mathbb{Z}/2^r$.

More speculation

Let p be an odd prime. What should $S//\alpha_1$ be a characteristic of? There is a morphism $S//\alpha_1 \to \ell$ (the Adams summand of $ku_{(p)}$). This induces an epimorphism $\pi_*(S//\alpha_1) \to \pi_*(\ell)$ whose kernel is torsion; it does not induce an epimorphism in $H_*(-)$. If $\pi_n(S) \to \pi_n(S//\alpha_1)$ is trivial for all $n > 0$ then $S//\alpha_1 \to \ell$ is a characteristic morphism. Could replace ℓ by $MU_{(p)}$ or $H\mathbb{Z}_{(p)}$.

The prime $p = 2$.
The Hopf invariant elements η, ν, σ are analogous to α_1. The unit $S \to S//\eta$ induces $\pi_3(S) \to \pi_3(S//\eta)$ which kills ν because of the exact sequence

$$\pi_3(S^1) \xrightarrow{\eta} \pi_3(S^0) \to \pi_3(C_\eta) \to \pi_2(S^1) \xrightarrow[\cong]{\eta} \pi_2(S^0)$$

together with $Sq^4_*(x_2^2) = 1$ in $H_*(S//\eta; \mathbb{F}_2)$. This means that there is a morphism $S//\nu \to S//\eta$.
John Rognes: $\pi_7(S) \to \pi_7(S//\eta)$ does not kill σ. So it makes sense to form $S//\eta, \sigma$.
Conjecture: For $n > 0$, $\pi_n(S) \to \pi_n(S//\eta, \sigma)$ is trivial, hence $S//\eta, \sigma$ is a characteristic for $ku_{(2)}$, $H\mathbb{Z}_{(2)}$, etc.

If we consider $S//\nu$, then σ does not die in $\pi_7(S//\nu)$. This is harder to see since although in $H_*(S//\nu;\mathbb{F}_2)$ we have $Sq^8_*(x_4^2) = 1$, there is an exact sequence

$$0 \to \pi_7(S) \to \pi_7(C_\nu) \to \pi_7(S^4) \xrightarrow{\nu} \pi_7(S^1) \to 0$$

and $\pi_7(C_\nu) \cong \mathbb{Z}/8\,\sigma \oplus \mathbb{Z}/4\,\widetilde{2\nu}$.
Peter Eccles: The cell of the homology class x_4^2 is attached to C_ν by $\sigma + 2\nu$, hence $\sigma \neq 0$ in $\pi_7(S//\nu)$.
Consequence: There is a morphism $S//\nu, \sigma \to MSp_{(2)}$ which is an 8-equivalence.
Can calculate

$$\mathrm{TAQ}_*(S//\nu, \sigma, S; \mathbb{F}_2) = H_*(S^4 \vee S^8; \mathbb{F}_2),$$
$$\mathrm{TAQ}_*(MSp, S; \mathbb{F}_2) = H_*(\Sigma^4 ko; \mathbb{F}_2) = \Sigma^4\mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots],$$

and the above morphism induces

$$x_4 \mapsto \Sigma^4 1, \quad x_8 \mapsto \Sigma^4\zeta_1^4.$$

N.B. $MSp_{(2)}$ is minimal atomic so it is equivalent to a nuclear CW S-algebra.

What about $S//\nu,\sigma \to ko_{(2)}$? A Toda bracket argument shows that this induces an epimorphism on $\pi_*(-)$.

Conjecture: In positive degrees,

$$\begin{aligned}\ker[\pi_*(S) \to \pi_*(S//\nu,\sigma)] &= \ker[\pi_*(S) \to \pi_*(kO)] \\ &= \ker[\pi_*(S) \to \pi_*(MSp)].\end{aligned}$$

So the image of $\pi_*(S) \to \pi_*(S//\nu,\sigma)$ maps isomorphically to the image of $\pi_*(S) \to \pi_*(kO)$, i.e., the image of the μ-family. This would show that $S//\nu,\sigma$ is a characteristic for kO and MSp.
Stan Kochman proved the second equality.

There is a morphism $S//\sigma \to tmf$.
Conjecture:

$$\ker[\pi_*(S) \to \pi_*(S//\sigma)] = \ker[\pi_*(S) \to \pi_*(tmf)]$$

therefore $S//\sigma \to tmf$ is a characteristic of tmf.

More on these examples

- The natural maps $S//\eta,\sigma \to kU$, $S//\nu,\sigma \to kO$, and $S//\sigma \to tmf$ cannot split as maps of spectra since they do not induce epimorphisms in $H_*(-)$.
- The first non-trivial homotopy in the fibre of $S//\eta \to kU$ is a $\mathbb{Z}/4$ in degree 5 detected by $\mathrm{Q}^3 x_2$.
- Working $K(1)$-locally, $S//\eta$ is equivalent to a wedge of copies of kU .
- Does $S//\sigma \to tmf$ induce an epimorphism on $\pi_*(-)$?