Frobenius algebras

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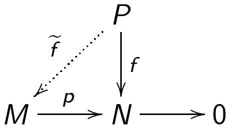
Frobenius rings

Let R be a ring and write ${}_R\mathscr{M}$, $\mathscr{M}_R$ for the categories of left and right R-modules. More generally, for a second ring S, write ${}_R\mathscr{M}_S$ for the category of $R-S$-bimodules.

A left R-module P is *projective* if every diagram of solid arrows in ${}_R\mathscr{M}$

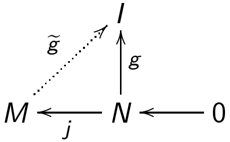


with exact row extends to a diagram of the form shown.

It is standard that every free module $F \cong \bigoplus_i R$ is projective. In fact, P is projective if and only if P is a retract of a free module, *i.e.*, for some Q, $P \oplus Q$ is free.

The category ${}_R\mathscr{M}$ *has enough projectives*, *i.e.*, for any left R-module M there is an epimorphism $P \twoheadrightarrow M$ with P projective.

There are similar notions for right R-modules and $R-S$-bimodules.
Dually, A left R-module I is *injective* if every diagram of solid arrows in ${}_R\mathscr{M}$



with exact row extends to a diagram of the form shown.
It is not immediately obvious what injectives there are. For $R = \mathbb{Z}$ (or more generally any pid) a module is injective if and only if it is divisible. Thus $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules. In fact ${}_R\mathscr{M}$ has enough injectives, *i.e.*,for any left R-module M there is a monomorphism $M \rightarrowtail I$ with I injective.
Given a ring homomorphism $S \to R$, R becomes a left S-module, and if J is an injective left S-module then $\mathrm{Hom}_S(R, J)$ is an injective left R-module. A ring R is left/right *self-injective* if it is an injective left/right R-module.
Example: For any $n \geqslant 2$, $\mathbb{Z}/n$ is self-injective.

A ring R is *quasi-Frobenius* (QF) if it is left Noetherian and right self-injective, or equivalently right Noetherian and left self-injective.

Theorem

R is QF if and only if for each R-module M,

$$M \text{ projective} \iff M \text{ injective.}$$

Frobenius algebras

Let $\Bbbk$ be a field and let A be a finite dimensional $\Bbbk$-algebra. The $\Bbbk$-linear dual of A, $A^* = \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$, is a left A-module with scalar multiplication $\cdot$ given by

$$a \cdot f(x) = f(xa) \quad (a, x \in A,\ f \in A^*).$$

Equally it is a right A-module with scalar multiplication

$$(f \cdot a)(x) = f(ax) \quad (a, x \in A,\ f \in A^*).$$

It is immediate that

- A is Artinian and Noetherian;
- A^* is an injective left/right A-module.

The pair (A, Φ) is a *Frobenius algebra* over $\Bbbk$ if

$$A \xrightarrow[\cong]{\Phi} A^*.$$

is an isomorphism of left A-modules. When Φ is understood, we sometimes say that A is a Frobenius algebra, although this can be ambiguous.

Since $A \cong A^*$ where the latter is injective, A is self-injective, giving

Theorem

Every Frobenius algebra is a QF ring.

The Frobenius isomorphism Φ is not unique, but two such isomorphisms differ by a unit of A. Associated to Φ there is a distinguished linear map, the *counit* $\varepsilon\colon A \to \Bbbk$, namely $\varepsilon = \Phi(1)$. Suppose that $z \in A$ and $\varepsilon(z) = 0$; then

$$\begin{aligned} \varepsilon(az) = 0 \text{ for all } a \in A \quad &\iff \quad z \cdot \varepsilon(a) = 0 \text{ for all } a \in A \\ &\iff \quad \Phi(z) = z \cdot \varepsilon = 0. \end{aligned}$$

Thus $\ker \varepsilon$ contains no left ideals.
There is a $\Bbbk$-bilinear form $A \otimes_{\Bbbk} A \to \Bbbk$ given by

$$\langle x|y \rangle = \varepsilon(xy).$$

This is non-degenerate in the sense that the $\Bbbk$-linear maps $A \to A^*$ with $a \mapsto \langle a|-\rangle$ and $a \mapsto \langle -|a \rangle$ are isomorphisms. It also satisfies the *Frobenius associativity relation*:

$$\langle xy|z \rangle = \langle x|yz \rangle \quad (x, y, z \in A).$$

Given such a pairing $\langle - | - \rangle$, there are corresponding ε and Φ are given by $\varepsilon(x) = \langle 1 | x \rangle = \langle x | 1 \rangle$ and $\Phi(x) = \varepsilon(x)$. This leads to three equivalent ways to define a Frobenius algebra namely as a $\mathbb{k}$-algebra together with a piece of extra structure.

- (A, Φ), where $\Phi \colon A \xrightarrow{\cong} A^*$ is an isomorphism of left/right A-modules.
- (A, ε), where $\varepsilon \colon A \to \mathbb{k}$ is $\mathbb{k}$-linear and $\ker \varepsilon$ contains no non-zero left, or equivalently right, ideals.
- $(A, \langle - | - \rangle)$, where $\langle - | - \rangle : A \otimes_{\mathbb{k}} A \to \mathbb{k}$ is a non-degenerate $\mathbb{k}$-linear pairing which satisfies the Frobenius associativity condition.

A Frobenius algebra is *symmetric* if the pairing $\langle - | - \rangle$ is symmetric, *i.e.*, $\langle x | y \rangle = \langle y | x \rangle$ for all $x, y \in A$. This applies when A is commutative but also in other cases such as $n \times n$ matrices over a field with the usual trace map.

Some examples of Frobenius algebras

Let $\Bbbk$ be a field and G a finite group. Then the group algebra $\Bbbk[G]$ and the dual group algebra

$$\Bbbk[G]^* = \mathrm{Hom}_{\Bbbk}(\Bbbk[G], \Bbbk) = \mathrm{Map}(G, \Bbbk)$$

are both Frobenius algebras. Note that $\Bbbk[G]$ is commutative if and only if G is, while $\Bbbk[G]^*$ is always commutative. As choices of ε we have

$$\varepsilon\colon \Bbbk[G] \to \Bbbk; \quad \varepsilon(\sum_{g\in G} t_g g) = t_1,$$
$$\varepsilon\colon \Bbbk[G]^* \to \Bbbk; \quad \varepsilon(f) = \sum_{g\in G} f(g).$$

More generally, if H is any finite dimensional Hopf algebra over $\Bbbk$, then H is a Frobenius algebra by the Larson-Sweedler theorem, as is its dual H^*.

For a field $\mathbb{k}$, the $n \times n$ matrices form a $\mathbb{k}$-algebra $M_n(\mathbb{k})$ with trace map $\mathrm{tr}\colon M_n(\mathbb{k}) \longrightarrow \mathbb{k}$ giving a Frobenius form. This also works with $M_n(D)$ for a division algebra D which is finite dimensional over its centre $\mathbb{k}$, using the composition ($\mathrm{redtr}\colon D \to \mathbb{k}$ is the reduced trace)

$$M_n(D) \xrightarrow{\mathrm{tr}} D \xrightarrow{\mathrm{redtr}} \mathbb{k}.$$

Any finite dimensional separable $\mathbb{k}$-algebra has a trace map with which it becomes a Frobenius algebra. In particular, any finite dimensional field extension separable field extension has such a trace map. In fact for any finite dimensional field extension L of $\mathbb{k}$, any $\mathbb{k}$-linear map $L \to \mathbb{k}$ maps L into a Frobenius algebra since L has no non-trivial proper ideals.

For an n-dimensional oriented compact connected manifold M and for any field coefficients $\mathbb{k}$, the cohomology $H^*(M;\mathbb{k})$ is a (graded) commutative Frobenius algebra with bilinear pairing given by cup product and evaluation on the fundamental class, $\langle x|y\rangle = xy[M]$. When $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$, can be taken to be de Rham cohomology, and this formula can be interpreted in terms of integrals of forms.

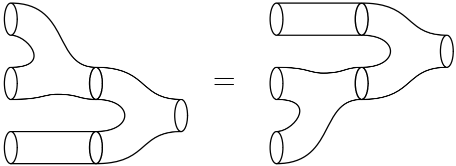
Cobordism diagrams

Let A be a Frobenius algebra over $\Bbbk$ with structure maps Φ,... as above. It is useful to represent the various structure maps and their relations with *cobordism diagrams*. Here a circle always represents a copy of A, two circles represents $A \otimes_{\Bbbk} A$, and so on. An empty circle represents $A^0 = \Bbbk$. Here are the interpretations of the basic diagrams.

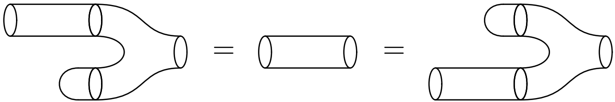
	identity function	$\mathrm{id}\colon A \longrightarrow A$
	multiplication/product	$\varphi\colon A \otimes A \to A$
	unit	$\eta\colon \Bbbk \to A$
	comultiplication/coproduct	$\psi\colon A \to A \otimes A$
	counit	$\varepsilon\colon A \to \Bbbk$
	bilinear pairing	$\langle - \vert - \rangle : A \otimes_{\Bbbk} A \to \Bbbk$

Here are some examples of diagrammatic representations of algebraic identities.

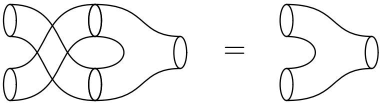
Associativity $\qquad \varphi(\varphi \otimes \mathrm{id}) = \varphi(\mathrm{id} \otimes \varphi)$



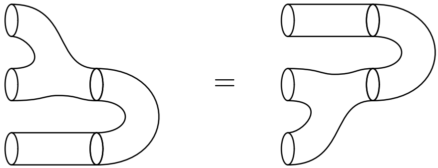
Unit $\varphi(\mathrm{id} \otimes \eta) = \mathrm{id} = \varphi(\eta \otimes \mathrm{id})$



Commutativity $\varphi(\mathrm{switch}) = \varphi$

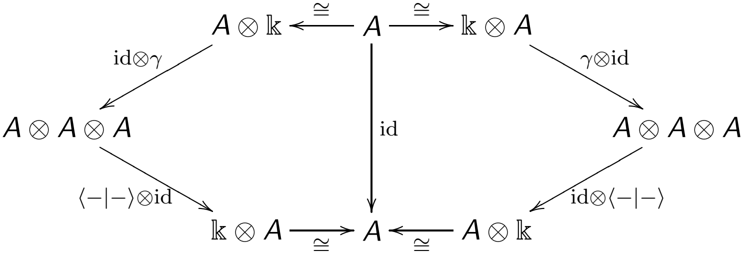


Frobenius associativity $\langle \varphi | \mathrm{id} \rangle = \langle \mathrm{id} | \varphi \rangle$

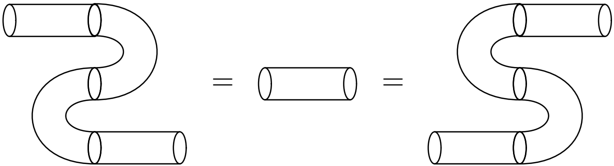


Consequences of duality

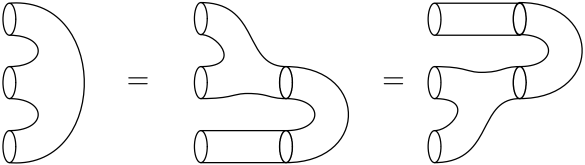
From now on set $\otimes = \otimes_{\Bbbk}$ and $\mathrm{Hom} = \mathrm{Hom}_{\Bbbk}$.
By standard linear algebra, the non-degenerate bilinear form $\langle - | - \rangle$ implies the existence of a unique *coform* $\gamma\colon \Bbbk \to A \otimes_{\Bbbk} A$ 
making the following diagram commute.



These relations are captured in following cobordism diagrams and are sometimes known as the *Snake relations*.

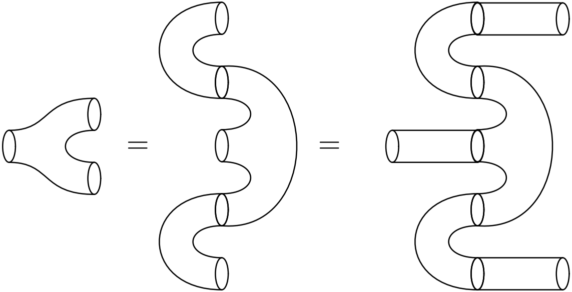


We will define a *coproduct* $\psi\colon A \to A \otimes A$  with ε as its counit. It is useful to introduce the *three point function* $\theta\colon \varepsilon(\varphi \otimes \mathrm{id}) = \varepsilon(\mathrm{id} \otimes \varphi)\colon A \otimes A \otimes A \to \Bbbk$ suggested by the Frobenius associativity relation.



The coproduct

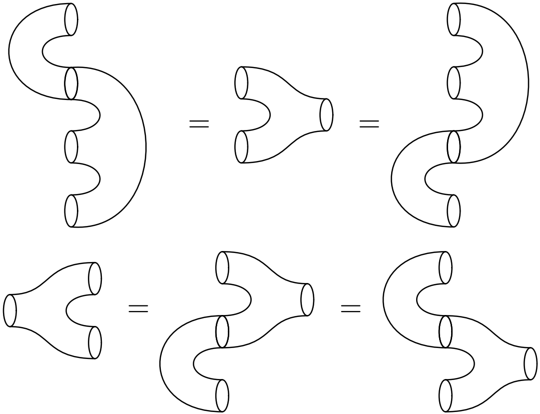
We define ψ as a function suggested by the diagram



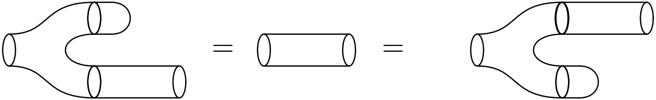
which represents the following composition.

$$A \xrightarrow{\cong} \Bbbk \otimes A \otimes \Bbbk \xrightarrow{\gamma \otimes \mathrm{id} \otimes \gamma} A \otimes A \otimes A \otimes A \otimes A \xrightarrow{\mathrm{id} \otimes \theta \otimes \mathrm{id}} A \otimes \Bbbk \otimes A \xrightarrow{\cong} A \otimes A$$

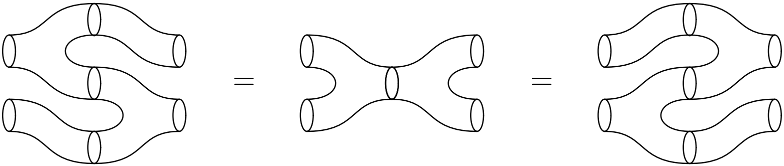
It is useful to note the following consequences of the Snake relations.



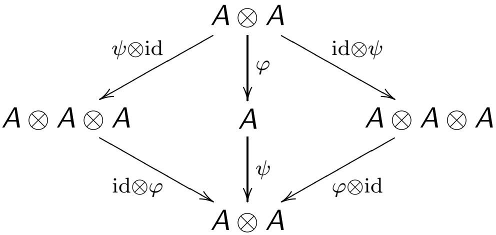
The counit condition for ε now follows from



Coassociativity can also be proved using diagrams.
There is also the important *Frobenius relation/condition*



which says that the following diagram commutes.



Some examples

Take $A = \Bbbk[G]$, the group algebra of a finite group. The bilinear pairing on $g, h \in G$ is given by

$$\langle g|h\rangle = \begin{cases} 1 & \text{if } h = g^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

The isomorphism $\Phi\colon \Bbbk[G] \to \Bbbk[G]^*$ is given by

$$\Phi(g) = \langle -|g\rangle$$

which is a left $\Bbbk[G]$-module homomorphism since

$$\Phi(hg) = \langle -|hg\rangle = \langle (-)h|g\rangle \,.$$

The Frobenius coproduct is given by: for $g \in G$,

$$\psi(g) = \sum_{h\in G} h \otimes h^{-1}g.$$

For the Frobenius relation, notice that for $g, h \in G$,

$$\begin{aligned}(\mathrm{id} \otimes \varphi)(\psi(g) \otimes h) &= \sum_{k \in G} k^{-1} \otimes kgh \\ &= \psi(gh),\end{aligned}$$

while

$$\begin{aligned}(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes \psi)(g \otimes h) &= \sum_{\ell \in G} g\ell \otimes \ell^{-1} h \\ &= \sum_{\ell \in G} g\ell \otimes \ell^{-1} g^{-1} gh \\ &= \sum_{k \in G} k \otimes k^{-1} gh \\ &= \psi(gh).\end{aligned}$$

The Nakayama automorphism

Let $(A, \langle - | - \rangle)$ be a Frobenius algebra over $\Bbbk$, where $\langle - | - \rangle$ is the non-degenerate associative bilinear form. Note that there are two $\Bbbk$-linear isomorphisms $A \xrightarrow{\cong} A^*$, namely

$$a \mapsto \langle a | - \rangle , \quad a \mapsto \langle - | a \rangle .$$

So for $a \in A$, the function $\langle a | - \rangle : A \longrightarrow \Bbbk$ can also be represented as $\langle - | a' \rangle : A \longrightarrow \Bbbk$ for some $a' \in A$. This defines a $\Bbbk$-linear automorphism $\sigma \colon A \xrightarrow{\cong} A$ with $\sigma(a) = a'$ and so $\langle a | - \rangle = \langle - | \sigma(a) \rangle$. Viewed as a linear operator on A, right multiplication by $\sigma(a)$ is the *adjoint* of a with respect to $\langle - | - \rangle$. Then for $x \in A$,

$$\langle ax | 1 \rangle = \langle a | x \rangle = \langle x | \sigma(a) \rangle = \langle x\sigma(a) | 1 \rangle .$$

We also find that for $b \in A$,

$$\langle abx | 1 \rangle = \langle x\sigma(ab) | 1 \rangle = \langle x\sigma(a)\sigma(b) | 1 \rangle ,$$

giving $\sigma(ab) = \sigma(a)\sigma(b)$. So σ is a $\Bbbk$-algebra automorphism, called the *Nakayama automorphism* of A.

The choice of $\langle - | - \rangle$ affects the definition of σ, however

Theorem

The Nakayama automorphism is well-defined up to inner automorphisms. In particular, if A is commutative then the Nakayama automorphism is independent of the choice of bilinear form.

What about symmetric Frobenius algebras, where $\langle x|y \rangle = \langle y|x \rangle$?

Theorem

A Frobenius algebra is symmetric if and only if its Nakayama automorphism for any choice of bilinear form is an inner automorphism.