NOTES ON BASIC HOMOLOGICAL ALGEBRA

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1. Chain complexes and their homology

Let R be a ring and $\mathbf{Mod}_R$ the category of *right* R-modules; a very similar discussion can be had for the category of *left* R-modules ${}_R\mathbf{Mod}$ also makes sense and is left to the reader. Then a sequence of R-module homomorphisms

$$L \xrightarrow{f} M \xrightarrow{g} N$$

is said to be *exact* if $\operatorname{Ker} g = \operatorname{Im} f$. Of course this implies that $gf = 0$. More generally, a sequence of homomorphisms

$$\cdots \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \cdots$$

is *exact* if for each n, the sequence

$$M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1}$$

is exact. An exact sequence of the form

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called *short exact*. Such a sequence is *split exact* if there is a homomorphism $r\colon M \longrightarrow L$ (or equivalently $j\colon N \longrightarrow M$) so that $rf = \mathrm{id}_L$ (respectively $gj = \mathrm{id}_N$).

$$\begin{tikzcd}
& & L & & \\
0 \arrow[r] & L \arrow[ur, "\mathrm{id}"] \arrow[r, "f"'] & M \arrow[u, dotted, "r"'] \arrow[r, "g"] & N \arrow[r] & 0 \\
& & N \arrow[u, dotted, "j"] \arrow[ur, "\mathrm{id}"'] & &
\end{tikzcd}$$

These equivalent conditions imply that $M \cong L \oplus N$. Such homomorphisms r and g are said to be *retractions*, while L and N are said to be *retracts* of M.

A sequence of homomorphisms

$$\cdots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

is called a *chain complex* if for each n,

$$d_n d_{n+1} = 0, \tag{1.1}$$

or equivalently,

$$\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n.$$

An *exact* or *acyclic* chain complex is one which each segment

$$C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1}$$

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is exact. We write (C_*, d) for such a chain complex and refer to the d_n and *boundary homomorphisms*. We symbolically write $d^2 = 0$ to indicate that (1.1) holds for all n. For clarity we sometimes (C_*, d^C) to indicate which boundary is being used.

If a chain complex is of finite length finite we often pad it out to a doubly infinite complex by adding in trivial modules and homomorphisms. In particular, if M is a R-module we can view it as the chain complex with $M_0 = M$ and $M_n = 0$ whenever $n \neq 0$. It is often useful to consider the *null complex* $\mathbf{0} = (\{0\}, 0)$.

(C_*, d) is called *bounded below* if there is an n_1 such that $C_n = 0$ whenever $n < n_1$. Similarly, (C_*, d) is called bounded above if there is an n_2 such that $C_n = 0$ whenever $n > n_2$. (C_*, d) is called *bounded* if it is bounded both below and above.

Given a complex (C_*, d), its *homology* is defined to be the complex $(H_*(C_*, d), 0)$ where

$$H_n(C_*, d) = \operatorname{Ker} d_n / \operatorname{Im} d_{n+1}.$$

The homology of a complex measures its deviation from exactness; in particular, (C_*, d) is exact if and only if $H_*(C_*, d) = 0$. Notice that there are exact sequences

$$\begin{aligned} 0 \to \operatorname{Im} d_{n+1} &\longrightarrow \operatorname{Ker} d_n \longrightarrow H_n(C_*, d) \to 0, \\ 0 \to \operatorname{Ker} d_n &\longrightarrow C_n \longrightarrow \operatorname{Im} d_n \to 0. \end{aligned} \tag{1.2}$$

Example 1.1. *Consider the complex of $\mathbb{Z}$-modules where*

$$C_n = \mathbb{Z}/4, \quad d\colon \mathbb{Z}/4 \longrightarrow \mathbb{Z}/4;\ d(\bar{t}) = \overline{2t}.$$

Then $\operatorname{Ker} d_n = 2\,\mathbb{Z}/4 = \operatorname{Im} d_n$ *and $H_n(C_*, d) = 0$, hence (C_*, d) is acyclic.*

A *homomorphism of chain complexes* or *chain homomorphism* $h\colon (C_*, d^C) \longrightarrow (D_*, d^D)$ is a sequence of homomorphisms $h_n\colon C_n \longrightarrow D_n$ for which the following diagram commutes.

$$\begin{array}{ccc} C_n & \xrightarrow{d_n^C} & C_{n-1} \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_{n-1}} \\ D_n & \xrightarrow{d_n^D} & D_{n-1} \end{array}$$

We often write $h\colon C_* \longrightarrow D_*$ when the boundary homomorphisms are clear from the context.

A chain homomorphism for which each $h_n\colon C_n \longrightarrow D_n$ is an isomorphism is called a *chain isomorphism* and admits an inverse chain homomorphism $D_* \longrightarrow C_*$ consisting of the inverse homomorphisms $h_n^{-1}\colon D_* \longrightarrow C_*$.

The *category of chain complexes in* $\mathbf{Mod}_R$, $\mathbf{Ch}_{\mathbb{Z}}(\mathbf{Mod}_R)$, has chain complexes as its objects and chain homomorphisms as its morphisms. Like $\mathbf{Mod}_R$, it is an *abelian category*.

Let $h\colon (C_*, d^C) \longrightarrow (D_*, d^D)$ be a chain homomorphism. If $u \in \operatorname{Ker} d_n^C$ we have

$$d_n^D(h_n(u)) = h_n(d_n^C(u)) = 0,$$

and if $v \in C_{n+1}$,

$$h_n(d_{n+1}^C(v)) = d_{n+1}^D(h_{n+1}(v)).$$

Together these allow us to define for each n a well-defined homomorphism

$$h_* = H_n(h)\colon H_n(C_*, d^C) \longrightarrow H_n(D_*, d^D); \quad h_*(u + \operatorname{Im} d_{n+1}^C) = h_n(u) + \operatorname{Im} d_{n+1}^D.$$

If $g\colon (B_*, d^B) \longrightarrow (C_*, d^C)$ is another morphism of chain complexes, it is easy to check that

$$(hg)_* = h_* g_*,$$

while for the identity morphism $\mathrm{id}\colon (C_*, d^C) \longrightarrow (C_*, d^C)$ we have

$$\mathrm{id}_* = \mathrm{id}\,.$$

Thus each H_n is a *covariant functor* from chain complexes to R-modules.

A *cochain complex* is a collection of R-modules C^n together with *coboundary homomorphisms* $d^n\colon C^n \longrightarrow C^{n+1}$ for which $d^{n+1}d^n = 0$; the *cohomology* of this complex is $(H^*(C^*, d), 0)$ where

$$H^n(C^*, d) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1}.$$

Given a chain complex (C_n, d) we can re-index so that $C^n = C_{-n}$ and form the cochain complex (C^n, d); similarly each cochain complex gives rise to a chain complex. We then have

$$H^n(C^*, d) = H_{-n}(C_*, d).$$

We mainly focus on chain complexes, but everything can be reworked for cochain complexes using this correspondence.

There are some useful constructions on complexes. Given (C_*, d), for any integer t we can form $(C[t]_*, d[t])$ with

$$C[t]_n = C_{n+t}, \quad d[t] = (-1)^t d.$$

Then

$$H_n(C[t]_*, d[t]) = H_{n+t}(C_*, d).$$

Viewing an R-module M as a complex concentrated in degree 0, we have a complex $M[t]$ which is a copy of M in degree t.

2. Short exact sequences of chain complexes

Given a morphism of chain complexes $h\colon (C_*, d) \longrightarrow (D_*, d)$ we may define two new chain complexes

$$\operatorname{Ker} h = ((\operatorname{Ker} h)_*, d), \quad \operatorname{Im} h = ((\operatorname{Im} h)_*, d),$$

where

$$(\operatorname{Ker} h)_n = \operatorname{Ker} h\colon C_n \longrightarrow D_n, \quad (\operatorname{Im} h)_n = \operatorname{Im} h\colon C_n \longrightarrow D_n.$$

The boundary homomorphisms are the restrictions of d to each of these.

Let $h\colon (L_*, d) \longrightarrow (M_*, d)$ and $k\colon (M_*, d) \longrightarrow (N_*, d)$ be chain homomorphisms and suppose that

$$0 \to (L_*, d) \xrightarrow{h} (M_*, d) \xrightarrow{k} (N_*, d) \to 0$$

is *short exact*, *i.e.*,

$$\operatorname{Ker} h = 0, \quad \operatorname{Im} k \cong (N_*, d), \quad \operatorname{Ker} k \cong \operatorname{Im} h.$$

It is natural to ask about the relationship between the three homology functors $H_*(L_*, d)$, $H_*(M_*, d)$ and $H_*(N_*, d)$.

Example 2.1. *Consider the following exact sequence of chain complexes of $\mathbb{Z}$-modules (written vertically).*

$$\begin{array}{cccccccc}
 & & L_* & & M_* & & N_* & \\
*=2 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
*=1 & 0 \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 0} & \\
*=0 & 0 \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
*=-1 & & 0 & & 0 & & 0 &
\end{array}$$

On taking homology we obtain

$$
\begin{array}{ccccccc}
 & H_*(L_*) & & H_*(M_*) & & H_*(N_*) \\
*=1 & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 \\
*=0 & \mathbb{Z}/2 & \xrightarrow{0} & \mathbb{Z}/2 & \xrightarrow{=} & \mathbb{Z}/2
\end{array}
$$

Neither of the rows with $*=0,1$ are exact, however there is an isomorphism

$$H_1(N_*)=\mathbb{Z}/2 \longrightarrow H_0(L_*)=\mathbb{Z}/2.$$

Theorem 2.2. *There is a long exact sequence of the form*

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(N_*,d) \arrow[dll, "\partial_{n+1}"'] \\
H_n(L_*,d) \arrow[r, "h_*"] & H_n(M_*,d) \arrow[r, "k_*"] & H_n(N_*,d) \arrow[dll, "\partial_n"'] \\
H_{n-1}(L_*,d) \arrow[r] & \cdots &
\end{tikzcd}
$$

and this is natural in the following sense: given a commutative diagram of short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & (L_*,d) \arrow[r, "h"] \arrow[d, "p"] & (M_*,d) \arrow[r, "k"] \arrow[d, "q"] & (N_*,d) \arrow[r] \arrow[d, "r"] & 0 \\
0 \arrow[r] & (L'_*,d) \arrow[r, "h'"] & (M'_*,d) \arrow[r, "k'"] & (N'_*,d) \arrow[r] & 0
\end{tikzcd}
$$

there is a commutative diagram

$$
\begin{tikzcd}
\cdots H_{n+1}(N_*,d) \arrow[dr, "\partial_{n+1}"] \arrow[dd, "r_*"] & & & & \\
 & H_n(L_*,d) \arrow[r, "h_*"] \arrow[dd, "p_*"] & H_n(M_*,d) \arrow[r, "k_*"] \arrow[dd, "q_*"] & H_n(N_*,d) \arrow[dr, "\partial_n"] \arrow[dd, "r_*"] & \\
\cdots H_{n+1}(N'_*,d) \arrow[dr, "\partial_{n+1}"] & & & & H_{n-1}(L_*,d)\cdots \arrow[dd, "p_*"] \\
 & H_n(L'_*,d) \arrow[r, "h'_*"] & H_n(M_*,d) \arrow[r, "k'_*"] & H_n(N'_*,d) \arrow[dr, "\partial_n"] & \\
 & & & & H_{n-1}(L'_*,d)\cdots
\end{tikzcd}
$$

In Example 2.1, the boundary map ∂_1 induces the isomorphism $H_1(N_*)\cong H_0(L_*)$ and there is a long exact sequence

$$0\to H_1(N_*) \xrightarrow[\cong]{\partial_1} H_0(L_*) \xrightarrow{0} H_0(M_*) \xrightarrow[\cong]{} H_0(N_*)\to 0.$$

For cochain complexes we get a similar result for a short exact sequence of cochain complexes

$$0\to (L^*,d)\xrightarrow{h}(M^*,d)\xrightarrow{k}(N^*,d)\to 0,$$

but the ∂_n are replaced by homomorphisms

$$\delta^n \colon H^n(N^*, d) \longrightarrow H^{n+1}(L^*, d).$$

3. Chain homotopies and quasi-isomorphisms

Given two chain complexes (C_*, d), (D_*, d) and chain homomorphisms $f, g \colon C_* \longrightarrow D_*$, a chain homotopy between f and g is a sequence of R-module homomorphisms $s_n \colon C_n \longrightarrow D_{n+1}$ for which the following functional equation holds for all n:

$$ds_n + s_{n-1}d = f - g.$$

We say that f and g are *chain homotopic* or just *homotopic* if there is such a chain homotopy. In particular, f is *null homotopic* if $f \simeq 0$.

Proposition 3.1. *$\simeq$ is an equivalence relation on the set of chain homomorphisms $C_* \longrightarrow D_*$.*

The equivalences classes of $\simeq$ are called *chain homotopy classes.*

Proposition 3.2. *Suppose that two chain homomorphisms $f, g \colon C_* \longrightarrow D_*$ are chain homotopic. Then the induced homomorphisms $f_*, g_* \colon H_*(C_*, d) \longrightarrow H_*(D_*, d)$ are equal.*

Proof. Let $n \in \mathbb{Z}$. Then for $[z] \in H_n(C_*, d)$ we have $dz = 0$, hence

$$f(z) - g(z) = (ds_n + s_{n-1}d)(z) = ds_n(z) + s_{n-1}(0) = d(s_n(z)).$$

Therefore

$$g_*[z] = [g(z)] = [f(z) - d(s_n(z))] = [f(z)] = f_*[z]. \qquad \square$$

The idea behind this definition comes from the geometric *mapping cone* of a continuous map $f \colon X \longrightarrow Y$. Here is the algebraic version of this construction. Given a chain homomorphism $f \colon (C_*, d) \longrightarrow (D_*, d)$, its (*mapping*) *cone* is the complex $\operatorname{Cone} f$ with

$$(\operatorname{Cone} f)_n = C_{n-1} \oplus D_n, \quad d(u, v) = (-du, dv - fu).$$

There is an exact sequence

$$0 \to D_* \xrightarrow{j} (\operatorname{Cone} f)_* \xrightarrow{q} C[-1]_* \to 0$$

in which

$$j(v) = (0, v), \quad q(u, v) = -u.$$

A particular case of this is the *cone on* (C_*, d), $\operatorname{Cone} C_* = \operatorname{Cone}(C, d)_* = \operatorname{Cone} \mathrm{id}_{C_*}$, for which

$$(\operatorname{Cone} C)_n = C_{n-1} \oplus C_n.$$

Proposition 3.3. *A chain homomorphism $f \colon C_* \longrightarrow D_*$ is null homotopic if and only if there is a factorisation*

$$\begin{tikzcd}
C_* \arrow[rr, "f"] \arrow[dr, "j"'] & & D_* \\
& (\operatorname{Cone} C)_* \arrow[ur] &
\end{tikzcd}$$

There is also a closely related notion of (*mapping*) *cylinder* $\operatorname{Cyl} C_*$ based on the topological cylinder construction $X \times I$ for a space X, where $I = [0, 1]$ is the unit interval.

A chain homomorphism $f \colon C_* \longrightarrow D_*$ is a *chain homotopy equivalence* if there is a chain homomorphism $h \colon D_* \longrightarrow C_*$ for which $hf \simeq \mathrm{id}_{C_*}$ and $fh \simeq \mathrm{id}_{D_*}$. Notice that these conditions imply the equations

$$h_*f_* = \mathrm{id} \colon H_*(C_*, d) \longrightarrow H_*(D_*, d), \quad f_*h_* = \mathrm{id} \colon H_*(D_*, d) \longrightarrow H_*(C_*, d),$$

hence f_* and h_* are mutually inverse isomorphisms. Note that chain homotopy equivalences can identify very different complexes.

Example 3.4. *Consider the complex* (C_*, d) *of* $\mathbb{Z}$*-modules in* Example 1.1. *Then the chain homomorphism* $q\colon (C_*, d) \longrightarrow \mathbf{0}$ *is a chain homotopy equivalence which is not an chain isomorphism.*

More generally, a chain homomorphism $f\colon (C_*, d) \longrightarrow (D_*, d)$ which induces an isomorphism $f_*\colon H_*(C_*, d) \longrightarrow H_*(D_*, d)$ is called a *quasi-isomorphism* and does not have to be a chain homotopy equivalence.

4. Split complexes

A complex (C_*, d) is called *split* if for each $n \in \mathbb{Z}$ there is a $s_n\colon C_n \longrightarrow C_{n+1}$ such that

$$d = ds_{n-1}d\colon C_n \longrightarrow C_{n-1}.$$

(C_*, d) is called *split exact* if it is split and exact.

Example 4.1. *Suppose that* F *is a field and* (V_*, d) *is a complex of* F*-vector spaces. Then* (V_*, d) *is split.*

Proof. Consider the exact sequences of (1.2),

$$\begin{aligned} 0 \to \operatorname{Im} d_{n+1} &\longrightarrow \operatorname{Ker} d_n \longrightarrow H_n(V_*, d) \to 0,\\ 0 \to \operatorname{Ker} d_n &\longrightarrow V_n \longrightarrow \operatorname{Im} d_n \to 0. \end{aligned}$$

For each n, choose vector subspaces $B_n \subseteq V_n$, $H_n \subseteq \operatorname{Ker} d_n$ for which $V_n = \operatorname{Ker} d_n \oplus B_n$, $d_n\colon B_n \xrightarrow{\cong} \operatorname{Im} d_n$ and $\operatorname{Ker} d_n = \operatorname{Im} d_{n+1} \oplus H_n$. Then there is an evident isomorphism $V_n \cong B_{n+1} \oplus H_n \oplus B_n$ under which d corresponds to the linear mapping

$$d'\colon (u, v, w) \longmapsto (w, 0, 0).$$

Now define

$$s'_n\colon B_{n+1} \oplus H_n \oplus B_n \longrightarrow B_{n+2} \oplus H_{n+1} \oplus B_{n+1}; \quad s'_n(u, v, w) = (0, 0, u).$$

It is easy to see that $d's'_{n-1}d' = d'$ which implies that the corresponding mapping $s_n\colon V_n \longrightarrow V_{n+1}$ satisfies $ds_{n-1}d = d$. □

The arguments used in Example 4.1 apply to any complex of R-modules (C_*, d).

Proposition 4.2. *Let* (C_*, d) *be a chain complex.* (C_*, d) *is split if and only if it has the form*

$$C_n = B_{n+1} \oplus H_n \oplus B_n$$

with differential

$$d\colon C_n \longrightarrow C_{n+1}; \quad d(u, v, w) = (w, 0, 0).$$

Furthermore, (C_*, d) *is split exact if and only if* $H_n = 0$.

Example 4.3. *For any complex* (C_*, d)*, the cone* $\operatorname{Cone}(C_*, d) = \operatorname{Cone} C_*$ *is split exact, hence* $H_*(C_*, d) = 0$.

5. Projective and injective resolutions

Let R be a ring. A right R-module F is called *free* if there is a set of elements $\{b_\lambda : \lambda \in \Lambda\} \subseteq F$ such that every element $x \in F$ can be uniquely expressed as

$$x = \sum_{\lambda\in\Lambda} b_\lambda t_\lambda$$

for elements $t_\lambda \in R$. We say that the b_λ form a *basis for F over R*. We can make a similar definition for left modules.

For example, if $n \geqslant 1$,

$$R^n = \{(t_1, \dots, t_n) : t_1, \dots, t_n \in R\}$$

is free on the basis consisting of the standard elements

$$e_1 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1).$$

This works whether we view R^n as a left or right module. More generally, if we have a set A, then there is a free module

$$\bigoplus_{\alpha\in A} R = \{s \colon A \longrightarrow R : s \text{ finitely supported}\},$$

where 's finitely supported' means that $s(\alpha) \neq 0$ for only finitely many $\alpha \in A$.

Notice that for any module M, there is always a surjection $F \longrightarrow M$ from some free module. For example, if we take a set of generators m_α ($\alpha \in A$) for M, then there is a surjection

$$\bigoplus_{\alpha\in A} R \longrightarrow M; \quad (x_\alpha) \mapsto \sum_{\alpha\in A} x_\alpha m_\alpha.$$

An R-module P is called *projective* if given an exact sequence $M \xrightarrow{f} N \to 0$ and a homomorphism $p \colon P \longrightarrow N$, there is a (not usually unique) homomorphism $\widetilde{p} \colon P \longrightarrow M$ for which $f\widetilde{p} = p$.

$$\begin{tikzcd} & M \arrow[d, "f"] \\ P \arrow[ur, dotted, "\exists\widetilde{p}"] \arrow[r, "p"] & N \arrow[d] \\ & 0 \end{tikzcd}$$

In particular, every free R-module is projective. More generally, we have the following characterisation.

Proposition 5.1. *An R-module P is projective if and only if it is a retract of a free module.*

Proof. Suppose that P is projective. Then there is a surjective homomorphism $g \colon F \longrightarrow P$ from a free module F. Consider the diagram of solid arrows

$$\begin{tikzcd} & F \arrow[d, "g"] \\ P \arrow[ur, dotted] \arrow[r, "\mathrm{id}"'] & P \arrow[d] \\ & 0 \end{tikzcd}$$

in which the vertical sequence is exact. By projectivity there is a homomorphism $k \colon P \longrightarrow F$ corresponding to the dotted arrow and making the whole diagram commute. Then g is a retraction and $F \cong P \oplus \operatorname{Ker} g$.

Now suppose that for some module Q, $P \oplus Q$ is a free module with basis e_α ($\alpha \in A$) say. Suppose there is a diagram

$$\begin{tikzcd} & M \arrow[d, "f"] \\ P \arrow[r, "p"] & N \arrow[d] \\ & 0 \end{tikzcd}$$

in which the vertical sequence is exact. Then we can extend p to the homomorphism $p' \colon P \oplus Q \longrightarrow N$ given by $p'(x, y) = p(x)$. Then the elements $p'(e_\alpha) \in N$ can be lifted to elements $u_\alpha \in M$ for which $f(u_\alpha) = p'(e_\alpha)$, and there is a unique homomorphism $\widetilde{p}' \colon P \oplus Q \longrightarrow M$ for which $\widetilde{p}'(e_\alpha) = u_\alpha$. Now restricting this to $P \subseteq P \oplus Q$ gives the required $\widetilde{p}$ satisfying $f\widetilde{p} = p$. $\square$

An exact complex

$$\cdots \longrightarrow F_k \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \to 0 \tag{5.1}$$

is called a *resolution* of M, and denoted $F_* \longrightarrow M \to 0$. Here we view M as the (-1)-term and 0 as the (-2)-term. If each F_k is free (resp. projective) over R then it is called a *free* (resp. *projective*) resolution of M.

Proposition 5.2. *Let M be an R-module.*
(a) *There is a free (hence projective) resolution $F_* \longrightarrow M \to 0$.*
(b) *If $f \colon M \longrightarrow M'$ is a homomorphism and $P_* \longrightarrow M \to 0$, $P'_* \longrightarrow M' \to 0$ are projective resolutions, then there is a chain homomorphism between these which is f in degree -1. Furthermore, any two such chain homomorphisms are homotopic.*
(c) *If $P_* \longrightarrow M \to 0$ and $Q_* \longrightarrow M \to 0$ are two projective resolutions, then they are chain homotopy equivalent by a chain map which is the identity in degree -1.*

An R-module J is called *injective* if given an exact sequence $0 \to K \xrightarrow{g} L$ and a homomorphism $q \colon K \longrightarrow J$, there is a (not usually unique) homomorphism $\widetilde{q} \colon L \longrightarrow J$ for which $\widetilde{q}g = q$.

$$\begin{tikzcd} 0 \arrow[d] & \\ K \arrow[r, "q"] \arrow[d, "g"] & J \\ L \arrow[ur, dotted, "\exists\,\widetilde{q}"'] & \end{tikzcd}$$

It is easy to see that if J is injective, then $\mathrm{Hom}_R(\ , J)$ is right exact.

Injective modules are less easy to characterise than projective modules.

Let R be a commutative integral domain. Then an R-module M is said to be *divisible* if for every non-zero $r \in R$, the multiplication by r homomorphism $r \cdot \colon M \longrightarrow M$ is surjective. For example, if the fraction field of R is F, then for any vector space over F is an R-module and is divisible. Another example is the quotient R-module F/R.

Proposition 5.3. *Let R be a pid. Then an R-module is injective if and only if it is divisible. Furthermore, for every R-module M, there is an injective homomorphism $j \colon M \longrightarrow J$ with J injective.*

Thus the injective $\mathbb{Z}$-modules (= abelian groups) are the divisible groups. Any torsion-free abelian group can be viewed as a subgroup of a $\mathbb{Q}$-vector space. On the other hand the torsion group $\mathbb{Z}/n$ is isomorphic to the subgroup of $\mathbb{Q}/\mathbb{Z}$ consisting of all elements of order dividing n.

Now let R be any ring and let M be a right R-module. We may view M as a $\mathbb{Z}$-module and choose and injection of $\mathbb{Z}$-modules $i\colon M \longrightarrow I$. Now consider $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ which is right R-module with the action of $r \in R$ on $f \in \mathrm{Hom}_{\mathbb{Z}}(R, I)$ given by

$$(f \cdot r)(x) = f(rx).$$

Then there is a homomorphism of R-modules

$$\theta\colon M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I); \qquad \theta(m)(x) = i(mx) \quad (x \in R).$$

Since $i(m) = 0$ implies $m = 0$, this is an injection. In fact, $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is an injective R-module since for any right R-module N, there is a natural isomorphism

$$\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(R, I)) \cong \mathrm{Hom}_{\mathbb{Z}}(N, I).$$

An exact cochain complex $0 \to M \longrightarrow J^*$ in which each J^n is injective is called an *injective resolution* of M.

Proposition 5.4. *Let M be an R-module.*
(a) *There is an injective resolution $0 \to M \longrightarrow J^*$.*
(b) *If $f\colon M \longrightarrow M'$ is a homomorphism and $0 \to M \longrightarrow J^*$, $0 \to M' \longrightarrow K^*$ are injective resolutions, then there is a chain homomorphism between these which is f in degree -1. Furthermore, any two such chain homomorphisms are homotopic.*
(c) *If $0 \to M \longrightarrow J^*$ and $0 \to M \longrightarrow K^*$ are two injective resolutions, then they are chain homotopy equivalent by a chain map which is the identity in degree -1.*

6. Hom and Ext

Let M, N be two right R-modules. Then we define

$$\mathrm{Hom}_R(M, N) = \{h : M \longrightarrow N : h \text{ is a homomorphism of } R\text{-modules}\}.$$

If R is commutative then $\mathrm{Hom}_R(M, N)$ is also an R-module, otherwise it may only be an abelian group.

If $f\colon M \longrightarrow M'$ and $g\colon N \longrightarrow N'$ are homomorphisms of R-modules, then there are functions

$$f^*\colon\ \mathrm{Hom}_R(M', N) \longrightarrow \mathrm{Hom}_R(M, N); \quad f^*h = h \circ f,$$
$$g_*\colon\ \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, N'); \quad g_*h = g \circ h.$$

These are group homomorphisms and homomorphisms of R-modules if R is commutative.

Proposition 6.1. *Let M, N be right R-modules.*

(a) *Given a short exact sequence of R-modules*

$$0 \to M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3 \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M_3, N) \xrightarrow{f_2^*} \mathrm{Hom}_R(M_2, N) \xrightarrow{f_1^*} \mathrm{Hom}_R(M_1, N)$$

is exact.

(b) *Given a short exact sequence of R-modules*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M, N_1) \xrightarrow{g_{1*}} \mathrm{Hom}_R(M, N_2) \xrightarrow{g_{2*}} \mathrm{Hom}_R(M, N_3)$$

is exact.

These results show that $\mathrm{Hom}_R(\ ,N)$ and $\mathrm{Hom}_R(M,\ )$ are *left exact*.

If P is projective, then $\mathrm{Hom}_R(P,\ )$ is right exact, since for given a short exact sequence

$$0 \to L \longrightarrow M \longrightarrow N \to 0,$$

every homomorphism $P \longrightarrow N$ lifts to a homomorphism $P \longrightarrow M$.

Now suppose that $P_* \longrightarrow M \to 0$ is a *resolution* of M by projective modules (for example, each P_n could be free). This means that the complex $P_* \longrightarrow M \to 0$ is exact. We can forget the M and just look at $P_* \to 0$, whose homology is just M in degree 0. Then for any N we can form the cochain complex $\mathrm{Hom}_R(P_*, N)$ whose n-th term is $\mathrm{Hom}_R(P_n, N)$. The n-th cohomology group of this is

$$\mathrm{Ext}^n_R(M,N) = H^n(\mathrm{Hom}_R(P_*, N)).$$

It turns out that this is independent of the choice of projective resolution of M. Notice also that

$$\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N).$$

Now let $0 \to M \longrightarrow J^*$ be an injective resolution of M. Then we may form the cochain complex $\mathrm{Hom}_R(M, J^*)$ whose n-term is $\mathrm{Hom}_R(M, J^n)$. The n-th cohomology group of this complex is

$$\mathrm{rExt}^n_R(M,N) = H^n(\mathrm{Hom}_R(M, J^*)).$$

It turns out that this is independent of the choice of injective resolution of N. Notice also that

$$\mathrm{rExt}^0_R(M,N) = \mathrm{Hom}_R(M,N).$$

Proposition 6.2. *For R-modules M, N there is a natural isomorphism*

$$\mathrm{rExt}^n_R(M,N) \cong \mathrm{Ext}^n_R(M,N).$$

Thus $\mathrm{Ext}^n_R(\ ,\ )$ is a *balanced* functor of its two variables.

The main properties of Ext^*_R are summarised in the following result. Let R_0 be the centre of R, *i.e.*, the subring

$$R_0 = \{r \in R : rs = sr \text{ for all } s \in R\}.$$

Theorem 6.3. *Let M and N be right R-modules.*
(a) *For each $n \geqslant 0$, $\mathrm{Ext}^n_R(M,\ )$ is a covariant functor $\mathbf{Mod}_R \longrightarrow \mathbf{Mod}_{R_0}$.*
(b) *For each $n \geqslant 0$, $\mathrm{Ext}^n_R(\ ,N)$ is a contravariant functor $\mathbf{Mod}_R \longrightarrow \mathbf{Mod}_{R_0}$.*
(c) *For short exact sequences*

$$0 \to N_1 \longrightarrow N_2 \longrightarrow N_3 \to 0, \quad 0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \to 0,$$

there are natural long exact sequences

$$0 \to \mathrm{Ext}^0_R(M,N_1) \longrightarrow \mathrm{Ext}^0_R(M,N_2) \longrightarrow \mathrm{Ext}^0_R(M,N_3) \longrightarrow \cdots$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}^{n-1}_R(M,N_1) \arrow[dll] \\
\mathrm{Ext}^n_R(M,N_1) \arrow[r] & \mathrm{Ext}^n_R(M,N_2) \arrow[r] & \mathrm{Ext}^n_R(M,N_3) \arrow[dll] \\
\mathrm{Ext}^{n+1}_R(M,N_1) \arrow[r] & \cdots &
\end{tikzcd}$$

and

$$0 \to \mathrm{Ext}^0_R(M_3,N) \longrightarrow \mathrm{Ext}^0_R(M_2,N) \longrightarrow \mathrm{Ext}^0_R(M_1,N) \longrightarrow \cdots$$

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Ext}^{n-1}_R(M_1,N) \arrow[dll] \\
\mathrm{Ext}^n_R(M_3,N) \arrow[r] & \mathrm{Ext}^n_R(M_2,N) \arrow[r] & \mathrm{Ext}^n_R(M_1,N) \arrow[dll] \\
\mathrm{Ext}^{n+1}_R(M_3,N) \arrow[r] & \cdots &
\end{tikzcd}$$

7. Tensor products and Tor

Let M be any right R-module and N be *left* R-module. Then we can form the *tensor product* $M \otimes_R N$ which is an abelian group. If R is commutative, $M \otimes_R N$ is also an R-module. The definition involves forming the free $\mathbb{Z}$-module $F(M,N)$ with basis consisting of all the pairs (m,n) where $m \in M$ and $n \in N$. Then

$$M \otimes_R N = F(M,N)/S(M,N),$$

where $S(M,N) \leqslant F(M,N)$ is the subgroup generated by all the elements of form

$$(m_1+m_2,n)-(m_1,n)-(m_2,n), \quad (m,n_1+n_2)-(m,n_1)-(m,n_2), \quad (mr,n)-(m,rn),$$

where $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $r \in R$. We usually denote the coset of (m,n) by $m \otimes n$; such elements generate the group $M \otimes_R N$.

The tensor product $M \otimes_R N$ has an important *universal property* which characterizes it up to isomorphism. Write $q\colon M \times N \longrightarrow M \otimes_R N$ for the quotient function. Let $f\colon M \times N \longrightarrow V$ be a function into an abelian group V which satisfies

$$\begin{aligned}
f(m_1+m_2,n) &= f(m_1,n)+f(m_2,n),\\
f(m,n_1+n_2) &= f(m,n_1)+f(m,n_2),\\
f(mr,n) &= f(m,rn)
\end{aligned}$$

for all $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$, $r \in R$, there is a unique homomorphism $\widetilde{f}\colon M \otimes_R N \longrightarrow V$ for which $f = \widetilde{f} \circ q$.

$$\begin{tikzcd}
M \times N \arrow[r, "q"] \arrow[d, "f"'] & M \otimes_R N \arrow[dl, dotted, "\exists!\, \widetilde{f}"] \\
V &
\end{tikzcd}$$

Proposition 7.1. *If $f\colon M_1 \longrightarrow M_2$ and $g\colon N_1 \longrightarrow N_2$ are homomorphisms of R-modules, there is a group homomorphism*

$$f \otimes g\colon M_1 \otimes_R N_1 \longrightarrow M_2 \otimes_R N_2$$

for which

$$f \otimes g(m \otimes n) = f(m) \otimes g(n).$$

Proposition 7.2. *Given a short exact sequence of left R-modules*

$$0 \to N_1 \xrightarrow{g_1} N_2 \xrightarrow{g_2} N_3 \to 0,$$

there is an exact sequence

$$M \otimes_R N_1 \xrightarrow{1 \otimes g_1} M \otimes_R N_2 \xrightarrow{1 \otimes g_2} M \otimes_R N_3 \to 0.$$

Because of this property, we say that $M \otimes_R (\ )$ is *right exact*. Obviously it would be helpful to understand $\operatorname{Ker} 1 \otimes g_1$ which measures the deviation from *left exactness* of $M \otimes_R (\ )$.

Every M admits a projective resolution $P_* \longrightarrow M \to 0$, and given a left R-module N, we can form a new complex $P_* \otimes_R N \to 0$, where the boundary maps are obtained by tensoring those of P_* with the identity on N an taking $P_0 \otimes_R N \longrightarrow 0$ rather than using the original map $P_0 \longrightarrow M$. Here $P_n \otimes_R N$ is in degree n. We define

$$\mathrm{Tor}_n^R(M, N) = H_n(P_* \otimes_R N).$$

It is easy to see that

$$\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N.$$

Of course we could also form a projective resolution of N, tensor it with M and then take homology.

Theorem 7.3. Tor_*^R *has the following properties.*

i) $\mathrm{Tor}_*^R(M, N)$ *can be computed by using projective resolutions of either variable and the answers agree up to isomorphism.*

ii) *Given R-module homomorphisms $f\colon M_1 \longrightarrow M_2$ and $g\colon N_1 \longrightarrow N_2$ there are homomorphisms*

$$(f \otimes g)_* = f_* \otimes g_* \colon\ \mathrm{Tor}_n^R(M_1, N_1) \longrightarrow \mathrm{Tor}_n^R(M_2, N_2)$$

generalizing $f_* \otimes g_* \colon M_1 \otimes_R N_1 \longrightarrow M_2 \otimes_R N_2$.

iii) *For a projective right (resp. left) R-module P (resp. Q) and $n > 0$, we have*

$$\mathrm{Tor}_n^R(P, N) = 0 = \mathrm{Tor}_n^R(M, Q).$$

iv) *Associated to a short exact sequence of right R-modules*

$$0 \to M_1 \longrightarrow M_2 \longrightarrow M_3 \to 0$$

there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}_{n+1}^R(M_3, N) \arrow[dll] \\
\mathrm{Tor}_n^R(M_1, N) \arrow[r] & \mathrm{Tor}_n^R(M_2, N) \arrow[r] & \mathrm{Tor}_n^R(M_3, N) \arrow[dll] \\
\mathrm{Tor}_{n-1}^R(M_1, N) \arrow[r] & \cdots &
\end{tikzcd}$$

$$\cdots \longrightarrow M_1 \otimes_R N \longrightarrow M_2 \otimes_R N \longrightarrow M_3 \otimes_R N \to 0$$

and associated to a short exact sequence of left R-modules

$$0 \to N_1 \longrightarrow N_2 \longrightarrow N_3 \to 0$$

there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathrm{Tor}^R_{n+1}(M,N_3) \arrow[dll] \\
\mathrm{Tor}^R_n(M,N_1) \arrow[r] & \mathrm{Tor}^R_n(M,N_2) \arrow[r] & \mathrm{Tor}^R_n(M,N_3) \arrow[dll] \\
\mathrm{Tor}^R_{n-1}(M,N_1) \arrow[r] & \cdots &
\end{tikzcd}$$

$$\cdots \longrightarrow M \otimes_R N_1 \longrightarrow M \otimes_R N_2 \longrightarrow M \otimes_R N_3 \to 0$$

Corollary 7.4. *Let Q be a left R-module for which $\mathrm{Tor}^R_n(M,Q) = 0$ for all $n > 0$ and M. Then for any exact complex (C_*, d), the complex $(C_* \otimes_R Q, d \otimes 1)$ is exact, and*

$$H_n(C_* \otimes_R Q, d \otimes 1) \cong H_n(C_*, d) \otimes_R Q.$$

An R-module M for which $\mathrm{Tor}^R_n(M,N) = 0$ for all $n > 0$ and left R-module N is called *flat*. Given a module M, it is always possible to find a resolution $F_* \longrightarrow M \to 0$ for which each F_k is flat (for example a projective resolution).

Proposition 7.5. *If $F_* \longrightarrow M \to 0$ is a flat resolution, then*

$$\mathrm{Tor}^R_n(M,N) = H_n(F_* \otimes_R N, d \otimes 1).$$

8. The Künneth Theorem

Suppose that (C_*, d) is a chain complex of flat (for example projective) right R-modules. For any left R-module N we have another chain complex $(C_* \otimes_R N, d \otimes 1)$ with homology $H_*(C_* \otimes_R N, d \otimes 1)$. We would like to understand the connection between this homology and $H_*(C_*, d) \otimes_R N$.

Begin by taking a flat resolution of N, $F_* \longrightarrow N \to 0$. For each n the complex

$$C_n \otimes_R F_* \longrightarrow C_n \otimes_R N \to 0$$

is still exact since C_n is flat. The *double complex* $C_* \otimes_R F_*$ has two compatible families of boundaries, namely the 'horizontal' ones coming from the boundaries maps d tensored with the identity, $d \otimes 1$, and the 'vertical' ones coming from the identity tensored with the boundary maps δ of F_*, $1 \otimes \delta$. We can take the two types of homology in different orders to obtain

$$H^{\mathrm{v}}_*(H^{\mathrm{h}}_*(C_* \otimes_R F_*)) = H^{\mathrm{v}}_* = \mathrm{Tor}^R_*(H_*(C_*, d), N),$$
$$H^{\mathrm{h}}_*(H^{\mathrm{v}}_*(C_* \otimes_R F_*)) = H^{\mathrm{h}}_*(C_* \otimes_R N)) = H_*(C_* \otimes_R N, d \otimes 1).$$

In general, the precise relationship between these two involves a *spectral sequence*, however there are situations where the relationship is more direct.

Suppose that N has a flat resolution of the form

$$0 \to F_1 \longrightarrow F_0 \longrightarrow N \to 0.$$

This will always happen when $R = \mathbb{Z}$ or any (commutative) pid and for semi-simple rings. Then for any right R-module M and $n > 1$,

$$\mathrm{Tor}^R_n(M,N) = 0.$$

Now consider what happens when we tensor C_* with such a resolution. We obtain a short exact sequence of chain complexes

$$0 \to C_* \otimes_R F_1 \longrightarrow C_* \otimes_R F_0 \longrightarrow C_* \otimes_R N \to 0$$

and on taking homology, an associated long exact sequence as in Theorem 2.2.

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{n+1}(C_* \otimes_R N) \arrow[dll, "\partial_{n+1}"'] \\
H_n(C_* \otimes_R F_1) \arrow[r] & H_n(C_* \otimes_R F_0) \arrow[r] & H_n(C_* \otimes_R N) \arrow[dll, "\partial_n"'] \\
H_{n-1}(C_* \otimes_R F_1) \arrow[r] & \cdots &
\end{tikzcd}$$

As F_0 and F_1 are flat, in each case Corollary 7.4 gives

$$H_n(C_* \otimes_R F_i) \cong H_n(C_*) \otimes_R F_i,$$

so our long exact sequence becomes

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{n+1}(C_* \otimes_R N) \arrow[dll, "\partial_{n+1}"'] \\
H_n(C_*) \otimes_R F_1 \arrow[r] & H_n(C_*) \otimes_R F_0 \arrow[r] & H_n(C_* \otimes_R N) \arrow[dll, "\partial_n"'] \\
H_{n-1}(C_*) \otimes_R F_1 \arrow[r] & \cdots &
\end{tikzcd}$$

The segment

$$H_n(C_*) \otimes_R F_1 \longrightarrow H_n(C_*) \otimes_R F_0$$

is part of the complex used to compute $\operatorname{Tor}^R_*(H_n(C_*), N)$ and so we have the sequence

$$0 \to \operatorname{Tor}^R_1(H_n(C_*), N) \longrightarrow H_n(C_*) \otimes_R F_1 \longrightarrow H_n(C_*) \otimes_R F_0 \longrightarrow \operatorname{Tor}^R_0(H_n(C_*), N) \to 0$$

in which

$$\operatorname{Tor}^R_0(H_n(C_*), N) = H_n(C_*) \otimes_R N.$$

For $\partial_n \colon H_n(C_* \otimes_R N) \longrightarrow H_{n-1}(C_* \otimes_R F_1)$ we find that

$$\operatorname{Ker} \partial_n \cong \operatorname{Tor}^R_0(H_n(C_*), N), \quad \operatorname{Im} \partial_n \cong \operatorname{Tor}^R_1(H_{n-1}(C_*), N).$$

Theorem 8.1 (Künneth Theorem)**.** *Let (C_*, d) be a chain complex of flat right R-modules and N a left R-module. For each n there is an exact sequence*

$$0 \to H_n(C_*) \otimes_R N \longrightarrow H_*(C_* \otimes_R N) \longrightarrow \operatorname{Tor}^R_1(H_{n-1}(C_*), N) \to 0.$$

9. Some examples of resolutions and calculations

9.1. **Regular sequences.** Let R be a commutative ring and suppose that $u \in R$ is not a zero-divisor. Then there is an exact sequence of R-modules

$$0 \to R \xrightarrow{u} R \longrightarrow R/(u) \to 0.$$

This is a free resolution of $R/(u)$ with $F_0 = F_1 = R$ and all other terms zero. For any R-module N we have the complex

$$0 \to \operatorname{Hom}_R(P_0, N) \xrightarrow{u} \operatorname{Hom}_R(P_1, N) \to 0$$

whose cohomology is $\operatorname{Ext}^*_R(R/(u), N)$. Hence

$$\operatorname{Ext}^0_R(R/(u), N) = \operatorname{Ker}(N \xrightarrow{u} N), \quad \operatorname{Ext}^1_R(R/(u), N) = N/uN.$$

Similarly,

$$\operatorname{Tor}^R_0(R/(u), N) = R/(u) \otimes_R N = N/uN, \quad \operatorname{Tor}^R_1(R/(u), N) = \operatorname{Ker}(N \xrightarrow{u} N).$$

This can be generalised. Again let R be a commutative ring. Suppose that $u_1, \ldots, u_n \in R$ is a *regular sequence*, *i.e.*, for each $k = 1, \ldots, n$, the residue class $u_k + (u_1, \ldots, u_{k-1}) \in R/(u_1, \ldots, u_{k-1})$ is not a zero divisor. Now form the graded R-algebra

$$K(u_1, \ldots, u_n)_* = \Lambda_R(e_1, \ldots, e_n)_*,$$

where e_k has degree 1 and

$$e_k e_\ell = (-1)^{\ell k} e_\ell e_k.$$

This has a differential d which drops degree by 1 and is a *derivation*, *i.e.*, for $x \in \Lambda_R(e_1, \ldots, e_n)_r$ and $y \in \Lambda_R(e_1, \ldots, e_n)_s$ it satisfies

$$d(xy) = (dx)y + (-1)^r x dy,$$

and also

$$de_r = u_r.$$

Then there is an exact sequence of R-modules

$$\cdots \xrightarrow{d} \Lambda_R(e_1, \ldots, e_n)_s \xrightarrow{d} \Lambda_R(e_1, \ldots, e_n)_{s-1} \xrightarrow{d} \cdots$$
$$\xrightarrow{d} \Lambda_R(e_1, \ldots, e_n)_1 \xrightarrow{d} \Lambda_R(e_1, \ldots, e_n)_0 \longrightarrow R/(u_1, \ldots, u_n) \to 0.$$

This is a free resolution of $R/(u_1, \ldots, u_n)$ and $\Lambda_R(e_1, \ldots, e_n)_*$ is called the *Koszul resolution*. Then

$$\mathrm{Ext}^*_R(R/(u_1, \ldots, u_n), N) = H^*(\mathrm{Hom}_R(\Lambda_R(e_1, \ldots, e_n)_*, N))$$

and

$$\mathrm{Tor}^R_*(R/(u_1, \ldots, u_n), N) = H_*(\Lambda_R(e_1, \ldots, e_n)_* \otimes_R N).$$

For example, if $N = R/(u_1, \ldots, u_n)$, then

$$\mathrm{Tor}^R_*(R/(u_1, \ldots, u_n), R/(u_1, \ldots, u_n)) = H_*(\Lambda_R(e_1, \ldots, e_n)_* \otimes_R R/(u_1, \ldots, u_n)).$$

Here the boundary is trivial so

$$\mathrm{Tor}^R_*(R/(u_1, \ldots, u_n), R/(u_1, \ldots, u_n)) = \Lambda_{R/(u_1, \ldots, u_n)}(e_1, \ldots, e_n)_*.$$

9.2. **Exterior algebras.** Let $\Bbbk$ be a commutative ring and let $R = \Lambda_\Bbbk(\varepsilon)$, so $\varepsilon^2 = 0$. Sending ε to 0 gives a ring homomorphism $R \longrightarrow \Bbbk$ and using this $\Bbbk$ becomes an R-module. There is a free resolution of $\Bbbk$,

$$\cdots \xrightarrow{\varepsilon} R\gamma_s \xrightarrow{\varepsilon} R\gamma_{s-1} \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} R\gamma_0 \longrightarrow \Bbbk \to 0,$$

where $R\gamma_s \xrightarrow{\varepsilon} R\gamma_{s-1}$ is the obvious multiplication by ε map.

Books on homological algebra

Here are some textbooks that cover homological algebra from rather different times and perspectives. Anyone seriously learning about the subject as it now exists would probably find [3] the most sensible book to use since it covers topics such as derived categories that are absent from the others. However, it is certainly much more sophisticated than [2]. Although older, [1] is still a good source for traditional parts of the subject and contains a number of specialist topics that are not as well treated in other texts.

There are many other books that aim to cover the whole or part of the subject, and algebraic topology, algebraic geometry, and algebra books tend to cover at least the basic notions.

References

[1] H. Cartan & S. Eilenberg, Homological Algebra, Princeton University Press (1956).
[2] J. J. Rotman, An introduction to Homological Algebra, Academic Press (1979).
[3] C. A. Weibel, An Introduction to Homological Algebra, Cambridge University Press (1994).