NOTES ON HOPF ALGEBRAS

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Introduction

These notes are intended to cover some basic ideas and results on Hopf algebras, especially finite dimensional ones.

Throughout, $\Bbbk$ will be a field, although much of the theory works over a general commutative ring provided due care is taken over flatness and finiteness conditions.

We will make use of basic category theory terminology some of which will be explained as it arises. The books by Mac Lane and Riehl [ML98, Rie16] are good sources for this.

The References contain several books and expository articles that cover aspects of the theory that will be covered in these notes. Radford's book [Rad12] is probably the most complete source for the general theory of Hopf algebras, while Montgomery [Mon93] is more terse but extremely useful. The recent book by Cartier & Patras [CP21] covers examples from areas such as combinatorics and is a good introduction to the 'classical' theory. The lecture notes by Brown & Goodearl [BG02] are wide ranging although their main focus is quantum groups. Lorentz [Lor18] is an amazing book which contains a lot on Hopf algebras. Waterhouse [Wat79] is a very accessible introduction to group schemes and the functorial viewpoint in Algebraic Geometry.

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Background material on vector spaces over a field as a monoidal category

The abelian category of (left) $\Bbbk$-vector spaces $\mathbf{Vect}_{\Bbbk}$ is very simple in terms of its additive structure. For example, every short exact sequence splits

$$0 \to U \to V \to W \to 0$$

so that $V \cong U \oplus W$. As a result there is no homological algebra like there is for modules over a general ring. Of course this is a consequence of the existence of bases, which also implies that every vector space is a direct sum of 1-dimensional ones.

However, additional structure is available: $\mathbf{Vect}_{\Bbbk}$ is also a *closed symmetric monoidal category* under tensor product $\otimes = \otimes_{\Bbbk}$ and with the internal function object given by

$$\hom(-,-) = \mathrm{Hom}_{\Bbbk}(-,-) = \mathbf{Vect}_{\Bbbk}(-,-);$$

the vector space $\Bbbk$ is a unit object for these since there are functorial isomorphisms

$$\Bbbk \otimes V \cong V \cong V \otimes \Bbbk, \qquad \hom(\Bbbk, V) \cong V.$$

It is symmetric because of the functorial switch isomorphism

$$\mathrm{T}\colon U \otimes V \xrightarrow{\cong} V \otimes U; \quad \mathrm{T}(x \otimes y) = y \otimes x.$$

The tensor product is functorial in the two variables: given linear mappings $f\colon U \to U'$ and $g\colon V \to V'$ there is a linear mapping $f \otimes g\colon U \otimes V \to U' \otimes V'$ fitting into a commutative diagram of linear mappings.

$$\begin{tikzcd}
U \otimes V \arrow[r, "f\otimes \mathrm{Id}_V"] \arrow[d, "\mathrm{Id}_{U'} \otimes g"'] \arrow[dr, "f\otimes g"] & U' \otimes V \arrow[d, "\mathrm{Id}_U \otimes g"] \\
U \otimes V' \arrow[r, "f\otimes \mathrm{Id}_{V'}"'] & U' \otimes V'
\end{tikzcd}$$

The tensor product is associative in the sense that for three vector spaces U, V, W, there is an isomorphism

$$(U \otimes V) \otimes W \xrightarrow{\cong} U \otimes (V \otimes W)$$

and then for linear mappings $f\colon U \to U'$, $g\colon V \to V'$ and $h\colon W \to W'$ there is a commutative diagram.

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(f\otimes g)\otimes h"'] & U \otimes (V \otimes W) \arrow[d, "f\otimes(g\otimes h)"] \\
(U' \otimes V') \otimes W' \arrow[r, leftrightarrow, "\cong"] & U' \otimes (V' \otimes W')
\end{tikzcd}$$

Because of this we usually just write $U \otimes V \otimes W$ for $(U \otimes V) \otimes W$ and $f \otimes g \otimes h\colon U \otimes V \otimes W \to U' \otimes V' \otimes W'$ for $(f \otimes g) \otimes h\colon (U \otimes V) \otimes W \to U' \otimes V' \otimes W'$, and identify it with $f \otimes (g \otimes h)\colon U \otimes (V \otimes W) \to U' \otimes (V' \otimes W')$ using the isomorphisms and diagram above.

For any three vector spaces U, V, W there is an adjunction isomorphism

$$\mathbf{Vect}_{\Bbbk}(U \otimes V, W) \xrightarrow{\cong} \mathbf{Vect}_{\Bbbk}(U, \hom(V, W))$$

which is functorial in the variables. This means that for linear mappings $f\colon U \to U'$, $g\colon V \to V'$ and $h\colon W \to W'$ there are commutative diagrams involving these isomorphisms.

$$\begin{tikzcd}
\mathbf{Vect}_{\Bbbk}(U' \otimes V, W) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(f\otimes \mathrm{Id})^*"'] & \mathbf{Vect}_{\Bbbk}(U', \hom(V,W)) \arrow[d, "f^*"] \\
\mathbf{Vect}_{\Bbbk}(U \otimes V, W) \arrow[r, leftrightarrow, "\cong"] & \mathbf{Vect}_{\Bbbk}(U, \hom(V,W))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Vect}_{\Bbbk}(U \otimes V', W) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "(\mathrm{Id}\otimes g)^*"'] & \mathbf{Vect}_{\Bbbk}(U, \hom(V',W)) \arrow[d, "(g^*)_*"] \\
\mathbf{Vect}_{\Bbbk}(U \otimes V, W) \arrow[r, leftrightarrow, "\cong"] & \mathbf{Vect}_{\Bbbk}(U, \hom(V,W))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{Vect}_{\Bbbk}(U \otimes V, W) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "h_*"'] & \mathbf{Vect}_{\Bbbk}(U, \hom(V,W)) \arrow[d, "(h_*)_*"] \\
\mathbf{Vect}_{\Bbbk}(U \otimes V, W') \arrow[r, leftrightarrow, "\cong"] & \mathbf{Vect}_{\Bbbk}(U, \hom(V,W'))
\end{tikzcd}$$

The *dual* (*space*) of V is $V^* = \hom(V, \Bbbk)$; if V is finite dimensional, a choice of basis leads to an isomorphism $V \xrightarrow{\cong} V^*$ dependent on the basis used.

The (*strongly*) *dualisable* objects V are characterised by the coindition that for all W,

$$\hom(V,W) \cong W \otimes V^*,$$

and

$$V^{**} = \hom(V^*, \Bbbk) \cong V,$$

where the latter isomorphism can be chosen to be independent of choice of basis and functorial in V; these turn out to be preceisely the finite dimensional vector spaces. Notice also that $\Bbbk^* \cong \Bbbk$ and $\mathrm{End}_{\Bbbk}(V) \cong V \otimes V^*$. There are also functorial isomorphisms

$$\mathbf{Vect}_{\Bbbk}(U \otimes V, W) \cong \mathbf{Vect}_{\Bbbk}(U, W \otimes V^*).$$

When U and V are finite dimensional, we will make the canonical identification

$$(U \otimes V)^* \cong V^* \otimes U^* \tag{0.1}$$

not with $U^* \otimes V^*$, although these are isomorphic via the switch isomorphism; the literature has varying conventions on this and some minor differences occur as a result. Warning: when U or V is infinite dimensional there is a canonical injective linear mapping

$$V^* \otimes U^* \to (U \otimes V)^* \tag{0.2}$$

which is *not* an isomorphism.

For later use we mention an important construction. There is a *forgetful functor* $\mathbf{Vect}_{\Bbbk} \to \mathbf{Set}$ which 'forgets' the algebraic structure and just remembers the underlying set; it sends each $\Bbbk$-linear mapping to itself just viewed as a function.

Proposition 0.1. *There is a functor $\mathbb{F}\colon \mathbf{Set} \to \mathbf{Vect}_{\Bbbk}$ which is left adjoint to the forgetful functor, i.e., for every set X and vector space V there is a bijection*

$$\mathbf{Vect}_{\Bbbk}(\mathbb{F}(X), V) \cong \mathbf{Set}(X, V),$$

and this gives a natural isomorphism of bifunctors

$$\mathbf{Vect}_{\Bbbk}(\mathbb{F}(-),-) \cong \mathbf{Set}(-,-).$$

Furthermore, there is a natural isomorphism $\mathbb{F}\circ(-\times-) \cong \mathbb{F}(-)\otimes\mathbb{F}(-)$ *between the bifunctors*

$$\mathbb{F}\circ(-\times-)\colon \mathbf{Set}\times\mathbf{Set}\to\mathbf{Vect}_{\Bbbk};\quad (X,Y)\mapsto\mathbb{F}(X\times Y),$$
$$\mathbb{F}(-)\otimes\mathbb{F}(-)\colon \mathbf{Set}\times\mathbf{Set}\to\mathbf{Vect}_{\Bbbk};\quad (X,Y)\mapsto\mathbb{F}(X)\otimes\mathbb{F}(Y).$$

We usually think of the vector space $\mathbb{F}X = \mathbb{F}(X)$ as having X as a basis and it is called the *free vector space on* X. One construction is

$$\mathbb{F}(X) = \{(\alpha\colon X\to\Bbbk) : \alpha \text{ is finitely supported}\},$$

where a function is *finitely supported* if it is zero except on finitely many elements. Of course if we take $X = \{1,2,\ldots,n\}$, $\mathbb{F}X \cong \Bbbk^n$; in particular, if $n = 1$, $\mathbb{F}X \cong \Bbbk$.

Notation for adjoint functors: When discussing a pair of adjoint functors $L\colon \mathbf{C}\to\mathbf{D}$ and $R\colon \mathbf{D}\to\mathbf{C}$ for which

$$\mathbf{D}(L(-),-) \cong \mathbf{C}(-,R(-))$$

it is standard to use the *Kan turnstyle* $\dashv$ or $\perp$, where $L\dashv R$ indicates that L is the left adjoint of R, or equivalently that R is the right adjoint of L.

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "L"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, bend left, "R"] \end{tikzcd}$$

Notice that for any $X\in\mathbf{C}$, under the isomorphism $\mathbf{D}(L(X),L(X)) \cong \mathbf{C}(X,R(L(X)))$, the identity morphism $I_{L(X)}$ corresponds to a morphism $i_X\colon X\to R(L(X))$ usually called a *universal morphism*.

1. Algebras and coalgebras

Algebras. A $\mathbb{k}$-*algebra* (A, φ, η) is a *monoid* in the monoidal category $(\mathbf{Vect}_{\mathbb{k}}, \otimes)$, i.e., a $\mathbb{k}$-vector space A equipped with a $\mathbb{k}$-linear *product* $\varphi\colon A \otimes A \to A$ and *unit* $\eta\colon \mathbb{k} \longrightarrow A$, which make the following diagrams in $\mathbf{Vect}_{\mathbb{k}}$ commute.

$$
\begin{tikzcd}
(A\otimes A)\otimes A \arrow[rr, leftrightarrow, "\cong"] \arrow[d, "\varphi\otimes\mathrm{Id}"'] & & A\otimes(A\otimes A) \arrow[d, "\mathrm{Id}\otimes\varphi"] \\
A\otimes A \arrow[dr, "\varphi"'] & & A\otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{k}\otimes A \arrow[d, "\eta\otimes\mathrm{Id}"'] & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, "\mathrm{Id}"'] & A\otimes\mathbb{k} \arrow[d, "\mathrm{Id}\otimes\eta"] \\
A\otimes A \arrow[dr, "\varphi"'] & & A\otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\tag{1.1}
$$

If in addition the following diagram commutes then A is *commutative*.

$$
\begin{tikzcd}
A\otimes A \arrow[rr, leftrightarrow, "\mathrm{T}", "\cong"'] \arrow[dr, "\varphi"'] & & A\otimes A \arrow[dl, "\varphi"] \\
& A &
\end{tikzcd}
\tag{1.2}
$$

We usually set $xy = \varphi(x \otimes y)$ and $1 = \eta(1)$ when this will not lead to confusion. Of course commutativity means that for all $x, y \in A$, $xy = yx$.

Unpacking the definition we find that an algebra is a ring with the additional structure of a specified ring homomorphism $\eta\colon \mathbb{k} \to A$ whose image lies in the centre of A and makes A a $\mathbb{k}$-vector space.

A *homomorphism* $\theta\colon (A, \varphi, \eta) \to (A', \varphi', \eta')$ between two $\mathbb{k}$-algebras is a $\mathbb{k}$-linear mapping $\theta\colon A \to A'$ making the following diagrams commute.

$$
\begin{tikzcd}
A\otimes A \arrow[r, "\theta\otimes\theta"] \arrow[d, "\varphi"'] & A'\otimes A' \arrow[d, "\varphi'"] \\
A \arrow[r, "\theta"] & A'
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{k} \arrow[dl, "\eta"'] \arrow[dr, "\eta'"] & \\
A \arrow[rr, "\theta"] & & A'
\end{tikzcd}
$$

So a homomorphism is a ring homomorphism which is also a $\mathbb{k}$-linear mapping. Of course the kernel of a homomorphism θ is an ideal, $\ker\theta \lhd A$, and its image is a subalgebra. The conditions for a subspace $I \subseteq A$ to be a two-sided ideal amount to saying that there is commutative diagram of the following form.

$$
\begin{tikzcd}
A\otimes I \arrow[r, "\mathrm{id}\otimes\mathrm{inc}"] \arrow[d, dotted] & A\otimes A \arrow[d, "\varphi"] & I\otimes A \arrow[l, "\mathrm{inc}\otimes\mathrm{id}"'] \arrow[d, dotted] \\
I \arrow[r, "\mathrm{inc}"] & A & I \arrow[l, "\mathrm{inc}"']
\end{tikzcd}
$$

The image of θ is isomorphic to the quotient algebra A/I.

The *trivial* $\mathbb{k}$-*algebra* is $\mathbb{k}$ with the product given by the canonical isomorphism

$$\mathbb{k} \otimes \mathbb{k} \xrightarrow{\cong} \mathbb{k}$$

which on basic tensors is given by

$$r \otimes t \mapsto rs.$$

For any algebra the unit $\eta\colon \mathbb{k} \to A$ is an injective homomorphism of $\mathbb{k}$-algebras and it is usual to identify its image with $\mathbb{k}$, thus making $\mathbb{k}$ a subring of A. The exact sequence of vector spaces

$$0 \to \mathbb{k} \to A \to \operatorname{coker}\eta \to 0$$

splits, giving a linear isomorphism $A \cong \Bbbk \oplus \operatorname{coker} \eta$. However this isomorphism depends on choosing a basis of A which extends a basis of $\Bbbk$. If additional structure is present then it can sometimes be made canonical.

A $\Bbbk$-algebra A is *augmented* if there is a given homomorphism of $\Bbbk$-algebras $\varepsilon\colon A \to \Bbbk$, so $\ker \varepsilon \lhd A$. Notice that the commutative diagram of vector spaces and linear mappings

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \Bbbk \arrow[d, "\eta"'] \arrow[dr, equal] & & \\
0 \arrow[r] & \ker \varepsilon \arrow[r] \arrow[dr, dashed, "\cong"'] & A \arrow[r, "\varepsilon"'] \arrow[d] & \Bbbk \arrow[r] & 0 \\
 & & \operatorname{coker} \eta \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

has exact row and column, and the composition

$$\begin{tikzcd}
\ker \varepsilon \arrow[r] \arrow[rr, bend left] & A \arrow[r] & \operatorname{coker} \eta
\end{tikzcd}$$

is an isomorphism. Therefore there is a canonical decomposition of vector spaces

$$A \cong \Bbbk \oplus \ker \varepsilon \cong \Bbbk \oplus \operatorname{coker} \eta.$$

Given an algebra A, its *opposite algebra* A^{op} has the same underlying vector space but product

$$\varphi^{\mathrm{op}} = \varphi \circ \mathrm{T}.$$

So if we denote $a \in A$ viewed as an element of A^{op} by a^{op},

$$a^{\mathrm{op}} b^{\mathrm{op}} = \varphi^{\mathrm{op}}(a^{\mathrm{op}} \otimes b^{\mathrm{op}}) = (ba)^{\mathrm{op}}.$$

The identity function $A \to A^{\mathrm{op}}$ is an algebra homomorphism if and only if A is commutative. It is not always possible to find an isomorphism $A \to A^{\mathrm{op}}$ but it does sometimes occur.

Given two algebras (A_1, φ_1, η_1) and (A_2, φ_2, η_2), their tensor product $A_1 \otimes A_2$ becomes an algebra with product and unit given by the compositions

$$(A_1 \otimes A_2) \otimes (A_1 \otimes A_2) \xrightarrow[\cong]{\mathrm{Id}_{A_1} \otimes \mathrm{T} \otimes \mathrm{Id}_{A_2}} (A_1 \otimes A_1) \otimes (A_2 \otimes A_2) \xrightarrow{\varphi_1 \otimes \varphi_2} A_1 \otimes A_2$$

and

$$\Bbbk \xrightarrow[\cong]{} \Bbbk \otimes \Bbbk \xrightarrow{\eta_1 \otimes \eta_2} A_1 \otimes A_2.$$

So given basic tensors $a_1 \otimes a_2, b_1 \otimes b_2 \in A_1 \otimes A_2$, their product is

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2.$$

Example 1.1. Given an algebra A and its opposite algebra A^{op}, the algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ is called the *enveloping algebra of A*. It is encountered when studying bimodules over A and Hochschild (co)homology.

We can summarises the main properties of $\Bbbk$-algebras and their homomorphisms in the following.

Theorem 1.2. *Algebras and commutative algebras form symmetric monoidal categories* $\mathbf{Alg}_{\Bbbk}$ *and* ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$ *under* $\otimes$. *In* ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$, $\otimes$ *is the categorical coproduct and* $\Bbbk$ *is an initial object. In* $\mathbf{Alg}_{\Bbbk}$ *and* ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$ *the Cartesian product* $\times = \oplus$ *is the categorical product.*

Coalgebras. The dual notion to an algebra is that of a $\Bbbk$-*coalgebra*, which is a triple (C, ψ, ε), with C a $\Bbbk$-vector space, a *coproduct* $\psi\colon C \to C \otimes C$, and a *counit* $\varepsilon\colon C \to \Bbbk$ fitting into the commutative diagrams shown.

(1.3)
$$\begin{tikzcd}
(C\otimes C)\otimes C \arrow[rr, leftrightarrow, "\cong"] & & C\otimes(C\otimes C) \\
C\otimes C \arrow[u, "\psi\otimes\mathrm{Id}"] & & C\otimes C \arrow[u, "\mathrm{Id}\otimes\psi"'] \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk\otimes C \arrow[r, "\cong"] & C & C\otimes\Bbbk \arrow[l, "\cong"'] \\
C\otimes C \arrow[u, "\varepsilon\otimes\mathrm{Id}"] & & C\otimes C \arrow[u, "\mathrm{Id}\otimes\varepsilon"'] \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] \arrow[uu, "\mathrm{Id}"] &
\end{tikzcd}$$

This says that (C, ψ, ε) is a *comonoid* in $\mathbf{Vect}_{\Bbbk}$.

If the following diagram commutes then C is *cocommutative*.

(1.4)
$$\begin{tikzcd}
C\otimes C \arrow[rr, leftrightarrow, "\mathrm{T}", "\cong"'] & & C\otimes C \\
& C \arrow[ul, "\psi"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

A *homomorphism* $\theta\colon (C, \psi, \varepsilon) \to (C', \psi', \varepsilon')$ between two $\Bbbk$-coalgebras is a $\Bbbk$-linear mapping $\theta\colon C \to C'$ making the following diagrams commute.

$$\begin{tikzcd}
C\otimes C \arrow[r, "\theta\otimes\theta"] & C'\otimes C' \\
C \arrow[u, "\psi"] \arrow[r, "\theta"] & C' \arrow[u, "\psi'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
& \Bbbk & \\
C \arrow[ur, "\varepsilon"] \arrow[rr, "\theta"] & & C' \arrow[ul, "\varepsilon'"']
\end{tikzcd}$$

The kernel of θ is a coideal, where a subspace $J \subseteq C$ is a *coideal* if the coproduct ψ restricts to give a map $J \to C \otimes J + J \otimes C \subseteq C \otimes C$. Then there is a commutative diagram

$$\begin{tikzcd}
J \arrow[r] \arrow[d, hook] & C\otimes J + J\otimes C \arrow[d, hook] \\
C \arrow[r, "\psi"] \arrow[d] & C\otimes C \arrow[d] \\
C/J \arrow[r, dotted, "\overline{\psi}"] & C/J\otimes C/J
\end{tikzcd}$$

The image of θ is a *subcoalgebra* isomorphic to the quotient C/J equipped with the induced coproduct $\overline{\psi}\colon C/J \to C/J \otimes C/J$.

The *trivial* $\Bbbk$-*coalgebra* is $\Bbbk$ with the coproduct given by the isomorphism

$$\Bbbk \xrightarrow{\cong} \Bbbk \otimes_{\Bbbk} \Bbbk = \Bbbk \otimes \Bbbk$$

which on basic tensors is just

$$t \mapsto t \otimes 1 = 1 \otimes t.$$

For any coalgebra the counit $\varepsilon\colon C \to \Bbbk$ is a homomorphism of $\Bbbk$-coalgebras.

A $\Bbbk$-coalgebra C is *(co)augmented* if there is a homomorphism of $\Bbbk$-coalgebras $\eta\colon \Bbbk \to C$. The commutative diagram of vector spaces and linear mappings has exact row and column

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ker\varepsilon \arrow[d] \arrow[dr, dashed, "\cong"] & & \\
0 \arrow[r] & \Bbbk \arrow[r, "\eta"] \arrow[dr, equal] & C \arrow[r] \arrow[d, "\varepsilon"] & \operatorname{coker}\eta \arrow[r] & 0 \\
 & & \Bbbk \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

and there is a canonical decomposition of vector spaces

$$C \cong \Bbbk \oplus \ker\varepsilon \cong \Bbbk \oplus \operatorname{coker}\eta.$$

A coalgebra C has an *opposite coalgebra* C^{op} with coproduct

$$\psi^{\mathrm{op}} = \mathrm{T} \circ \psi.$$

So if $\psi(c) = \sum_i c'_i \otimes c''_i$,

$$\psi^{\mathrm{op}}(c^{\mathrm{op}}) = \sum_i (c''_i)^{\mathrm{op}} \otimes (c'_i)^{\mathrm{op}}.$$

The identity function $C \to C^{\mathrm{op}}$ is a coalgebra homomorphism if and only if C is cocommutative.

Remark 1.3 (Sweedler notation)**.** Coalgebraists often use the notations

$$\psi(c) = \sum c_{(1)} \otimes c_{(2)} = \sum c_1 \otimes c_2$$

and even drop the summation sign (this is like the *Einstein summation convention* used with tensors in Applied Mathematics). This notation is quite convenient in calculations especially as an alternative to working with huge commutative diagrams. For example, the coassociativity condition is equivalent to the calculation

$$(\psi\otimes\mathrm{Id})\circ\psi(c) = \sum (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)} = (\mathrm{Id}\otimes\psi)\circ\psi(c), \tag{1.5}$$

while the counit conditions become

$$\sum \varepsilon(c_{(1)})c_{(2)} = c = \sum \varepsilon(c_{(2)})c_{(1)}. \tag{1.6}$$

For coalgebras C_1, C_2, their tensor product $C_1 \otimes C_2$ becomes a coalgebra whose coproduct and counit are the compositions

$$C_1 \otimes C_2 \xrightarrow{\psi_1\otimes\psi_2} (C_1 \otimes C_1) \otimes (C_2 \otimes C_2) \xrightarrow[\cong]{\mathrm{Id}_{C_1}\otimes\mathrm{T}\otimes\mathrm{Id}_{C_2}} (C_1 \otimes C_2) \otimes (C_1 \otimes C_2),$$

$$C_1 \otimes C_2 \xrightarrow{\varepsilon_1\otimes\varepsilon_2} \Bbbk \otimes \Bbbk \xrightarrow[\cong]{} \Bbbk.$$

So if $c' \in C_1$ and $c'' \in C_2$ with coproducts

$$\psi_{C_1}(c') = \sum c'_{(1)} \otimes c'_{(2)}, \quad \psi_{C_2}(c'') = \sum c''_{(1)} \otimes c''_{(2)},$$

the coproduct on $c' \otimes c'' \in C_1 \otimes C_2$ is

$$\psi_{C_1\otimes C_2}(c' \otimes c'') = \sum (c'_{(1)} \otimes c''_{(1)}) \otimes (c'_{(2)} \otimes c''_{(2)}) \in (C_1 \otimes C_2) \otimes (C_1 \otimes C_2).$$

Dually to Theorem 1.2 we have

Theorem 1.4. *Coalgebras and cocommutative coalgebras form symmetric monoidal categories* $\mathbf{Coalg}_{\Bbbk}$ *and* ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$ *under* $\otimes$. *In* ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$, $\otimes$ *is the categorical product and* $\Bbbk$ *is a terminal object. In* $\mathbf{Coalg}_{\Bbbk}$ *and* ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$ *the Cartesian product* $\times = \oplus$ *is the categorical coproduct.*

Dualising between algebras and coalgebras. The diagrams satisfied by the structure morphisms of algebras and coalgebras are dual in the sense that they are related by 'reversing all the arrows'. We can exploit this categorical symmetry to dualise algebras to coalgebras and sometimes coalgebras to algebras.

Given a coalgebra (C, ψ, ε) the dual space $C^* = \hom(C, \Bbbk)$ becomes an algebra by defining the product $C^* \otimes C^* \to C^*$ to be the following composition.

$$\begin{tikzcd}
C^* \otimes C^* \arrow[r, hook] \arrow[rr, bend left] & (C \otimes C)^* \arrow[r, "\psi^*"'] \arrow[d, equal] & C^* \arrow[d, equal] \\
& \hom(C \otimes C, \Bbbk) & \hom(C, \Bbbk)
\end{tikzcd}$$

On elements, for $\alpha, \beta \in C^*$ and $c \in C$,

$$(\alpha\beta)(c) = \sum \alpha(c_{(2)}) \otimes \beta(c_{(1)})$$

where we use Sweedler notation for the coproduct on c; notice the switch in order of the indices which is a consequence of our definition of the dual of a tensor product (0.2). The unit is the dual of the counit ε^*,

$$\varepsilon^* \colon \Bbbk \to C^*; \quad \varepsilon^*(t) = t\varepsilon.$$

To see why this product on C^* is associative, for $\alpha, \beta, \gamma \in C^*$ and $c \in C$, using (1.5) we find

$$\begin{aligned}
\big((\alpha\beta)\gamma\big)(c) &= \sum (\alpha\beta)(c_{(2)})\gamma(c_{(1)}) \\
&= \sum \alpha((c_{(2)})_{(2)})\beta((c_{(2)})_{(1)})\gamma(c_{(1)}) \\
&= \sum \alpha(c_{(2)})\beta((c_{(1)})_{(2)})\gamma((c_{(1)})_{(1)}) \\
&= \sum \alpha(c_{(2)})(\beta\gamma)(c_{(1)}) \\
&= \big(\alpha(\beta\gamma)\big)(c),
\end{aligned}$$

showing that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$; alternatively we could do this with a humongous commutative diagram. A similar calculation using (1.6) shows that ε^* is a unit. Also, if C is cocommutative then C^* is commutative. An important fact is that the algebra C^* acts on C to make it a left C^*-module, as we will see later.

We summarise this discussion in a result.

Proposition 1.5. *Given a coalgebra* $(C, \psi_C, \varepsilon_C)$, *there is a dual algebra* $(C^*, \psi_C^*, \varepsilon_C^*)$; C^* *is commutative if and only if* C *is cocommutative.*

A homomorphism of coalgebras $\theta \colon (C, \psi_C, \varepsilon_C) \to (C', \psi_{C'}, \varepsilon_{C'})$ *induces a homomorphism of algebras* $\theta^* \colon ((C')^*, \psi_{C'}^*, \varepsilon_{C'}^*) \to (C^*, \psi_C^*, \varepsilon_C^*)$.

Proof. The last part is left as an exercise. □

If (A, φ, η) is an algebra which is finite dimensional then we can similarly dualise to get a coalgebra (A^*, φ^*, η^*). However, if A is infinite dimensional we need to modify the notion of dual appropriately to make this work. Actually there are two ways to do this: the more drastic one involves introducing linearly topologised vector spaces and a notion of completeness, the other leads to a more 'algebraic' outcome by suitably restricting the elements in the dual space. We will take the latter approach.

Suppose that V is a vector space. Then a subspace $U \subseteq V$ is *cofinite* or has *finite codimension* if $\dim_{\Bbbk} V/U < \infty$.

For an algebra A, we define its *finite* or *restricted dual* by

$$A^\circ = \{\alpha \in A^* : \text{there is a cofinite } I \lhd A \text{ such that } I \subseteq \ker\alpha\} \subseteq A^*.$$

So $\alpha \in A^*$ is in A° if it factors through a finite dimensional quotient algebra A/I. Of course when A is finite dimensional, $A^\circ = A^*$, but otherwise $A^\circ \subsetneqq A^*$. As before we can 'dualise' the algebra structure on A to obtain a coproduct $\varphi^*\colon A^* \to (A\otimes A)^*$ but in order to land in a tensor product we need to restrict it to A° using the fact that there is a commutative diagram

$$\begin{tikzcd}
A^\circ \arrow[r, dotted, "\varphi^\circ"] \arrow[d, hook] & A^\circ \otimes A^\circ \arrow[r, leftrightarrow, "\cong"] \arrow[d, hook] & (A\otimes A)^\circ \\
A^* \arrow[r, "\varphi^*"] & (A\otimes A)^* &
\end{tikzcd}$$

and the dotted arrow is defined to be the coproduct $\varphi^\circ\colon A^\circ \to A^\circ \otimes A^\circ$. There is a counit $\eta^\circ\colon A^\circ \to \Bbbk$ obtained by precomposing with η. Then $(A^\circ, \varphi^\circ, \eta^\circ)$ is a coalgebra.

Proposition 1.6. *Given an algebra (A, φ_A, η_A), there is a dual coalgebra $(A^\circ, \varphi_A^\circ, \eta^\circ)$; A° is cocommutative if and only if A is commutative.*

A homomorphism of algebras $\theta\colon (A, \varphi_A, \eta_A) \to (A', \varphi_{A'}, \eta_{A'})$ induces a homomorphism of coalgebras $\theta^\circ\colon ((A')^\circ, \varphi_{A'}^\circ, \eta_{A'}^\circ) \to (A^\circ, \varphi_A^\circ, \eta_A^*)$.

Convolution monoids. In order to define Hopf algebras we will require a construction that can be made using an algebra and a coalgebra as ingredients.

Let (A, φ, η) be a $\Bbbk$-algebra and (C, ψ, ε) a $\Bbbk$-coalgebra. The vector space $\hom(C, A)$ can be given a product $*$ called the *convolution product*: for $f, g \in \hom(C, A)$,

$$f * g = \varphi \circ (f \otimes g) \circ \psi$$

and since this function $C \to A$ is a composition of linear mappings it is an element of $\hom(C, A)$.

Proposition 1.7. *With the multiplication $*$, $\hom(C, A)$ becomes a monoid with identity element*

$$\mathbf{1} = \varepsilon \circ \eta \in \hom(C, A).$$

When A is commutative and C is cocommutative, this monoid is also commutative.

Proof. The main thing is to verify that $*$ is associative and that $\mathbf{1}$ acts as the unity, thus showing $\hom(C, A)$ a monoid.

Associativity and coassociativity of A and C imply the commutativity of the following diagram which gives $(f*g)*h = f*(g*h)$.

$$\begin{tikzcd}
 & & C \arrow[lld, bend right, "\psi"'] \arrow[rrd, bend left, "\psi"] \arrow[dddd, dotted] & & \\
C\otimes C \arrow[r, "\psi\otimes\mathrm{Id}"] \arrow[d, "(f*g)\otimes h"'] & C\otimes C\otimes C \arrow[rr, leftrightarrow, "\cong"] \arrow[d, "(f\otimes g)\otimes h"'] & & C\otimes C\otimes C \arrow[d, "f\otimes(g\otimes h)"] & C\otimes C \arrow[l, "\mathrm{Id}\otimes\psi"'] \arrow[d, "f\otimes(g*h)"] \\
A\otimes A \arrow[rrdd, bend right, "\varphi"'] & A\otimes A\otimes A \arrow[l, "\varphi\otimes\mathrm{Id}"] \arrow[rr, leftrightarrow, "\cong"'] \arrow[u, phantom, ""] & (f*g)*h=f*(g*h) & A\otimes A\otimes A \arrow[r, "\mathrm{Id}\otimes\varphi"'] & A\otimes A \arrow[lldd, bend left, "\varphi"] \\
 & & & & \\
 & & A & &
\end{tikzcd}$$

By contemplating the following commutative diagram we see that $\mathbf{1} * f = f = f * \mathbf{1}$.

$$
\begin{tikzcd}
 & C \arrow[dl, "\psi"'] \arrow[dr, "\psi"] \arrow[dd, "\mathrm{Id}"] \arrow[ddddd, bend right=90, "\mathbf{1}*f"'] \arrow[ddddd, bend left=90, "f*\mathbf{1}"] & \\
C \otimes C \arrow[d, "\varepsilon\otimes\mathrm{Id}"'] & & C \otimes C \arrow[d, "\mathrm{Id}\otimes\varepsilon"] \\
\Bbbk \otimes C \arrow[d, "\mathrm{Id}\otimes f"'] & C \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "f"] & C \otimes \Bbbk \arrow[d, "f\otimes\mathrm{Id}"] \\
\Bbbk \otimes A \arrow[d, "\eta\otimes\mathrm{Id}"'] & A \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, "\mathrm{Id}"] & A \otimes \Bbbk \arrow[d, "\mathrm{Id}\otimes\eta"] \\
A \otimes A \arrow[dr, "\varphi"'] & & A \otimes A \arrow[dl, "\varphi"] \\
 & A &
\end{tikzcd}
$$

The commutativity result is left as an exercise. □

When we combine $*$ with the vector space structure on $\hom(C, A)$ we get an algebra.

Corollary 1.8. *The vector space* $\hom(C, A)$ *becomes a* $\Bbbk$*-algebra with product* $*$ *and unity* $\mathbf{1}$*.*

Given an algebra homomorphism $\alpha\colon A \to A'$ and a coalgebra homomorphism $\gamma\colon C' \to C$, there are $\Bbbk$-linear mappings

$$
\begin{aligned}
\alpha_*\colon \hom(C, A) &\to \hom(C, A'); \quad \alpha_*(f) = \alpha \circ f, \\
\gamma^*\colon \hom(C, A) &\to \hom(C', A); \quad \gamma^*(f) = f \circ \gamma.
\end{aligned}
$$

It is easy to verify that these are monoid homomorphisms and so algebra homomorphisms, i.e., for $f, g \in \hom(C, A)$,

$$
\begin{aligned}
\alpha_*(f * g) &= \alpha_*(f) * \alpha_*(g), & \gamma^*(f * g) &= \gamma^*(f) * \gamma^*(g), \\
\alpha_*(\mathbf{1}) &= \mathbf{1}, & \gamma^*(\mathbf{1}) &= \mathbf{1}.
\end{aligned}
$$

In a monoid elements need not have inverses, but sometimes they do. If $f \in \hom(C, A)$ then $\overline{f} \in \hom(C, A)$ is an inverse for f if

$$\overline{f} * f = \mathbf{1} = f * \overline{f},$$

or more explicitly if for every $c \in C$, using Sweedler notation in A we have

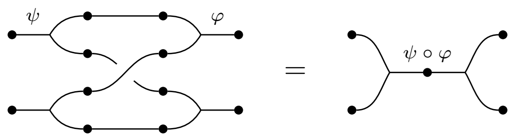
$$\sum \overline{f}(c_{(1)}) f(c_{(2)}) = \varepsilon(c) = \sum f(c_{(1)}) \overline{f}(c_{(2)}).$$

Of course such a two-sided inverse for f is unique.

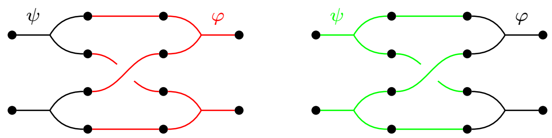
Notice that when $A = \Bbbk$, $C^* = \hom(C, \Bbbk)$ and $\mathbf{1} = \varepsilon^*$, and the algebra $(C^*, *, \mathbf{1})$ agrees with $(C^*, \psi^*, \varepsilon^*)$ discussed earlier.

In order to define a *bialgebra* (also known as a *bigèbre* in French) we need a vector space equipped with both an algebra and a coalgebra structure, (B, φ, η) and (B, ψ, ε), which interact appropriately.

Recall that we can give $B \otimes B$ an algebra stucture and a coalgebra structure, so it makes sense to ask if $\varphi\colon B \otimes B \to B$ and $\eta\colon \Bbbk \to B$ are coalgebra homomorphisms or if $\psi\colon B \to B \otimes B$ and $\varepsilon\colon B \to \Bbbk$ are algebra homomorphisms. Either of these amounts to requiring that chasing around the following diagrams read from left to right gives the same output.



Here the product for $B \otimes B$ is shown in red, the coproduct in green.



We also have two commutative diagrams for the unit and counit.

$$\begin{tikzcd}
\Bbbk \arrow[r, "\cong"] \arrow[d, "\eta"'] & \Bbbk \otimes \Bbbk \arrow[d, "\eta\otimes\eta"] \\
B \arrow[r, "\psi"] & B \otimes B
\end{tikzcd}
\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\varphi"] \arrow[d, "\varepsilon\otimes\varepsilon"'] & B \arrow[d, "\varepsilon"] \\
\Bbbk \otimes \Bbbk \arrow[r, "\cong"] & \Bbbk
\end{tikzcd}$$

If necessary, we denote the structure maps in a bialgebra by writing $(B, \varphi, \eta, \psi, \varepsilon)$.

Of course a homomorphism of bialgebras should be simultaneously an algebra and a coalgebra homomorphism, and there is a category of bialgebras $\mathbf{Bialg}_{\Bbbk}$ with full subcategories of commutative and cocommutative bialgebras. If a bialgebra is both commutative and cocommutative then it is called *bicommutative*.

Example 2.1 (The Quantum Plane)**.** Let $1 \neq q \in \Bbbk$. Then the *Quantum Plane* is the non-commutative bialgebra

$$\mathcal{O}_q(\Bbbk^2) = \Bbbk\langle X, Y\rangle/(YX - qXY).$$

We will denote the residue classes of X and Y by x and y, so these satisfy $yx = qxy$; notice that the monomials $x^i y^j$ form a basis of $\mathcal{O}_q(\Bbbk^2)$. There is a coproduct ψ and counit ε given by

$$\psi(x) = x \otimes x, \quad \psi(y) = y \otimes 1 + x \otimes y, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.$$

This bialgebra is neither commutative nor cocommutative so it is a *quantum monoid*.

Proposition 2.2. *Suppose that $(B, \varphi, \eta, \psi, \varepsilon)$ is a bialgebra.*
(a) *If (B, ψ, ε) is a cocommutative coalgebra, then (B, φ, η) is a monoid in ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$. In particular, φ and η are coalgebra homomorphisms.*
(b) *If (B, φ, η) is a commutative algebra then (B, ψ, ε) is a comonoid in ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$. In particular, ψ and ε are algebra homomorphisms.*

Proof. (a) In the category ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$, $\otimes$ is the categorical product and $\Bbbk$ is a terminal object. Now expand the diagrams of (1.1) for (B,φ,η) with $A=B$ and interpret them as being in ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$.
(b) In the category ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$, $\otimes$ is the categorical coproduct and $\Bbbk$ is an initial object. Now expand the diagrams of (1.3) for (B,ψ,ε) with $C=B$ and interpret them as being in ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$. $\square$

Before introducing Hopf algebras, we note a result on inverses in convolution monoids for bialgebras.

Lemma 2.3. *Suppose that B is a bialgebra.*
(a) *If A is an algebra and $f\colon B\to A$ is an algebra homomorphism which has a convolution inverse $\overline{f}$ in* $\hom(B,A)$*, then $\overline{f}$ is an algebra homomorphism $B\to A^{\mathrm{op}}$.*
(b) *If C is a coalgebra and $g\colon C\to B$ is a coalgebra homomorphism which has a convolution inverse $\overline{g}$ in* $\hom(C,B)$*, then $\overline{g}$ is a coalgebra homomorphism $C^{\mathrm{op}}\to B$.*

Proof. (a) Let $B\otimes B$ with its product $\varphi_{B\otimes B}$ which is also a coalgebra homomorphism with respect to its coproduct $\psi_{B\otimes B}$. This means that $\varphi^*_{B\otimes B}\colon \hom(B,A)\to\hom(B\otimes B,A)$ is a monoid homomorphism and in particular $\varphi^*_{B\otimes B}(f)\in\hom(B\otimes B,A)$ has inverse $\varphi^*_{B\otimes B}(\overline{f})$.

Now define $\ell=\varphi_A\circ(\overline{f}\otimes\overline{f})\circ\mathrm{T}\colon B\otimes B\to A$, given on elments by

$$\ell(x\otimes y)=\overline{f}(y)\overline{f}(x).$$

We will show that ℓ is also a left inverse for $\varphi^*_{B\otimes B}(f)$ and therefore it agrees with $\varphi^*_{B\otimes B}(\overline{f})$. To verify this we calculate: for $x,y\in B$,

$$\begin{aligned}
(\ell*\varphi^*_{B\otimes B}(f))(x\otimes y) &= \sum\sum \ell(x_{(1)}\otimes y_{(1)})\varphi^*_{B\otimes B}(f)(x_{(2)}\otimes y_{(2)})\\
&= \sum\sum \overline{f}(y_{(1)})\overline{f}(x_{(1)})f(x_{(2)}y_{(2)})\\
&= \sum\sum \overline{f}(y_{(1)})\overline{f}(x_{(1)})f(x_{(2)})f(y_{(2)})\\
&= \sum \overline{f}(y_{(1)})(\overline{f}*f)(x)f(y_{(2)})\\
&= \sum \overline{f}(y_{(1)})\varepsilon(x)f(y_{(2)})\\
&= \varepsilon(x)\sum \overline{f}(y_{(1)})f(y_{(2)})\\
&= \varepsilon(x)(\overline{f}*f)(y)\\
&= \varepsilon(x)\varepsilon(y)=\varepsilon(xy).
\end{aligned}$$

So $(\ell*\varphi^*_{B\otimes B}(f))=1$ and ℓ is the inverse of $\varphi^*_{B\otimes B}(f)$.

The proof of (b) is similar. $\square$

In particular, when B is a bialgebra, the identity function $\mathrm{Id}_B\colon B\to B$ is both an algebra homomorphism and a coalgebra homomorphism; so if it has a convolution inverse $\overline{\mathrm{Id}_B}\in\hom(B,B)$, this is both an algebra isomorphism $B\to B^{\mathrm{op}}$ and a coalgebra isomorphism $B^{\mathrm{op}}\to B$.

Lemma 2.4. *Suppose that $(B,\varphi,\eta,\psi,\varepsilon)$ is a bialgebra which is either commutative or cocommutative and that $\overline{\mathrm{Id}_B}$ exists. Then $\overline{\mathrm{Id}_B}\colon B\to B$ is self-inverse, i.e.,*

$$\overline{\mathrm{Id}_B}\circ\overline{\mathrm{Id}_B}=\mathrm{Id}_B\,.$$

Proof. We will give the proof when B is commutative, the other case is similar. So Id_B is an isomorphism $B\cong B^{\mathrm{op}}$ and by Lemma 2.3(a), $\overline{\mathrm{Id}_B}\colon B\to B$ is an algebra homomorphism, hence $\varphi\circ(\overline{\mathrm{Id}_B}\otimes\overline{\mathrm{Id}_B})=\overline{\mathrm{Id}_B}\circ\varphi$. To identify $\overline{\mathrm{Id}_B}\circ\overline{\mathrm{Id}_B}$ it is sufficient to show that

$$(\overline{\mathrm{Id}_B}\circ\overline{\mathrm{Id}_B})*\overline{\mathrm{Id}_B}=\mathbf{1}.$$

We have

$$\begin{aligned}(\overline{\mathrm{Id}_B} \circ \overline{\mathrm{Id}_B}) * \overline{\mathrm{Id}_B} &= \varphi \circ ((\overline{\mathrm{Id}_B} \circ \overline{\mathrm{Id}_B}) \otimes \overline{\mathrm{Id}_B}) \circ \psi \\ &= \varphi \circ (\overline{\mathrm{Id}_B} \otimes \overline{\mathrm{Id}_B}) \circ (\mathrm{Id}_B \otimes \overline{\mathrm{Id}_B}) \circ \psi \\ &= \overline{\mathrm{Id}_B} \circ \varphi \circ (\mathrm{Id}_B \otimes \overline{\mathrm{Id}_B}) \circ \psi \\ &= \overline{\mathrm{Id}_B} \circ (\mathrm{Id}_B * \overline{\mathrm{Id}_B}) \\ &= \overline{\mathrm{Id}_B} \circ \mathbf{1} \\ &= \overline{\mathrm{Id}_B} \circ \eta \circ \varepsilon = \mathbf{1},\end{aligned}$$

and so $\overline{\mathrm{Id}_B} \circ \overline{\mathrm{Id}_B} = \mathrm{Id}_B$ as required. □

Definition 2.5. If $(H, \varphi, \eta, \psi, \varepsilon)$ is a bialgebra for which $\chi = \overline{\mathrm{Id}_H}$ exists then it is called the *antipode* of H and $(H, \varphi, \eta, \psi, \varepsilon, \chi)$ is called a *Hopf algebra.* In many sources χ is denoted by S.

The antipode χ has to satisfy some conditions which we can encode in the following commutative diagram.

(2.1)
$$\begin{tikzcd}
& H \arrow[dl, "\psi"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\psi"] & \\
H \otimes H \arrow[dd, "\chi\otimes\mathrm{Id}"'] & & H \otimes H \arrow[dd, "\mathrm{Id}\otimes\chi"] \\
& \Bbbk \arrow[dd, "\eta"] & \\
H \otimes H \arrow[dr, "\varphi"'] & & H \otimes H \arrow[dl, "\varphi"] \\
& H &
\end{tikzcd}$$

On an element $h \in H$ this expands to give

(2.2)
$$\sum \chi(h_{(1)})h_2 = \varepsilon(h) = \sum h_1 \chi(h_{(2)}).$$

In general $\chi\colon H \to H$ is not a bijective function, however if it is bijective then its inverse χ^{-1} fits into the two equivalent commutative diagrams

(2.3)
$$\begin{tikzcd}
& H \arrow[dl, "\psi"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\psi"] & \\
H \otimes H \arrow[d, "\mathrm{T}"'] & & H \otimes H \arrow[d, "\mathrm{T}"] \\
H \otimes H \arrow[d, "\chi^{-1}\otimes\mathrm{Id}"'] & \Bbbk \arrow[dd, "\eta"] & H \otimes H \arrow[d, "\mathrm{Id}\otimes\chi^{-1}"] \\
H \otimes H \arrow[dr, "\varphi"'] & & H \otimes H \arrow[dl, "\varphi"] \\
& H &
\end{tikzcd}
\qquad
\begin{tikzcd}
& H \arrow[dl, "\psi"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\psi"] & \\
H \otimes H \arrow[d, "\mathrm{Id}\otimes\chi^{-1}"'] & & H \otimes H \arrow[d, "\chi^{-1}\otimes\mathrm{Id}"] \\
H \otimes H \arrow[d, "\mathrm{T}"'] & \Bbbk \arrow[dd, "\eta"] & H \otimes H \arrow[d, "\mathrm{T}"] \\
H \otimes H \arrow[dr, "\varphi"'] & & H \otimes H \arrow[dl, "\varphi"] \\
& H &
\end{tikzcd}$$

which expand to give

(2.4)
$$\sum \chi^{-1}(h_{(2)})h_{(1)} = \varepsilon(h) = \sum h_{(2)}\chi^{-1}(h_{(1)}).$$

Example 2.6 (The localised Quantum Plane)**.** We can modify the Quantum Plane of Example 2.1 to give a Hopf algebra by forcing x to have an inverse. Let

$$\mathcal{O}_q(\Bbbk^2)[x^{-1}] = \Bbbk\langle X, Y, Z\rangle/(YX - qXY, XZ - 1, ZX - 1).$$

The coproduct and counit of $\mathcal{O}_q(\Bbbk^2)$ extend to $\mathcal{O}_q(\Bbbk^2)[x^{-1}]$ so that

$$\psi(x^{-1}) = x^{-1} \otimes x^{-1}, \quad \varepsilon(x^{-1}) = 1,$$

and the antipode is given by

$$\chi(x) = x^{-1}, \quad \chi(x^{-1}) = x, \quad \chi(y) = -x^{-1}y.$$

This is a Hopf algebra which is neither commutative nor cocommutative. It has interesting finite dimensional quotient Hopf algebras when q takes special values; these are called *Taft algebras.*

Definition 2.7. A *homomorphism* of Hopf algebras $\theta\colon (H, \varphi, \eta, \psi, \varepsilon, \chi) \to (H', \varphi', \eta', \psi', \varepsilon', \chi')$ is a $\Bbbk$-linear mapping $\theta\colon H \to H'$ which is both an algebra and a coalgebra homomorphism. A homomorphism which is invertible is called an *isomorphism.*

Just as a group homomorphism maps inverses to inverses, such a homomorphism also satisfies

$$\theta \circ \chi = \chi' \circ \theta.$$

The kernel of a homomorphism of Hopf algebras θ is both an ideal and a coideal, which is also closed under the restriction of the antipode of the domain. Such an ideal in a Hopf algebra is called a *Hopf ideal.* It is easy to see if $J \lhd H$ is a Hopf ideal then there are unique algebra and coalgebra structures on H/J so that the quotient map $H \to H/J$ is a homomorphism of Hopf algebras; then H/J is called the *quotient Hopf algebra of H with respect to J.*

Proposition 2.8. *Let $\theta\colon (H, \varphi, \eta, \psi, \varepsilon, \chi) \to (H', \varphi', \eta', \psi', \varepsilon', \chi')$ be a homomorphism of Hopf algebras. Then*
(a) $\chi' \circ \theta = \theta \circ \chi$*;*
(b) $\ker\theta \lhd H$ *is a Hopf ideal and the image of θ is a subHopf algebra of H' is isomorphic to the quotient Hopf algebra $H/\ker\theta$.*

Proof. (a) The idea is to show that in the convolution monoid $\hom(H, H')$ the elements $\chi' \circ \theta$ and $\theta \circ \chi$ satisfy .

$$(\chi' \circ \theta) * \theta = \eta' \circ \varepsilon = \theta * (\chi' \circ \theta)$$

and

$$(\theta \circ \chi) * \theta = \eta' \circ \varepsilon = \theta * (\theta \circ \chi)$$

where $\chi' \circ \theta$ is the identity element. This shows that these elements are both inverses of θ and so must be equal by uniqueness of inverses. Here is a sample, the others follow by similar calculations:

$$\begin{aligned}
(\chi' \circ \theta) * \theta &= \varphi' \circ \big((\chi' \circ \theta) \otimes \theta\big) \circ \psi \\
&= \varphi' \circ (\chi' \otimes \mathrm{Id}) \circ (\theta \otimes \theta) \circ \psi \\
&= \varphi' \circ (\chi' \otimes \mathrm{Id}) \circ \psi' \circ \theta \\
&= (\chi' * \mathrm{Id}) \circ \theta \\
&= \eta' \circ \varepsilon' \circ \theta \\
&= \eta' \circ \varepsilon.
\end{aligned}$$

(b) This is a consequence of earlier results about homomorphisms of algebras and coalgebras. $\square$

Remark 2.9. Of course Hopf algebras over $\Bbbk$ and their homomorphisms define a category $\mathbf{HA}_{\Bbbk}$ which has the null (i.e., initial and terminal) object $\Bbbk$. There are three obvious full subcategories whose objects are the commutative, the cocommutative and the bicommutative Hopf algebras. In the first two, $\otimes$ is the categorical coproduct and product respectively. The category of bicommutative Hopf algebras (also known as abelian Hopf algebras) has many features possessed by an abelian category (for example $\otimes$ is the both the categorical coproduct and product), and indeed appropriate subcategories such as finite dimensional ones do form abelian categories.

We mention one important example of an isomorphism.

Example 2.10. Suppose that $(H, \varphi, \eta, \psi, \varepsilon, \chi)$ is a Hopf algebra whose antipode χ is bijective. Then its *opposite Hopf algebra* is $(H^{\mathrm{op}}, \varphi^{\mathrm{op}}, \eta^{\mathrm{op}}, \psi^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \chi^{\mathrm{op}})$ where we take the opposite algebra and coalgebra structures and as a function $\chi^{\mathrm{op}} = \chi$. Then the function

$$\widetilde{\chi}\colon H \to H^{\mathrm{op}}; \quad \widetilde{\chi}(h) = (\chi(h))^{\mathrm{op}}$$

is an isomorphism of Hopf algebras with inverse

$$\widetilde{\chi^{\mathrm{op}}}\colon H^{\mathrm{op}} \to H^{\mathrm{op}}; \quad \widetilde{\chi^{\mathrm{op}}}(h^{\mathrm{op}}) = \chi^{-1}(h).$$

A similar result applies if we interchange χ and χ^{-1}.

Later we will see that these isomorphisms allows us to interchange between left and right modules and comodules over H.

Proposition 2.11. *Suppose that* $(H, \varphi, \eta, \psi, \varepsilon, \chi)$ *is a* Hopf algebra.
(a) *If* (H, ψ, ε) *is cocommutative then* (H, φ, η, χ) *is a group object in* ${}^{\mathrm{co}}\mathbf{Coalg}_{\Bbbk}$*. In particular,* φ, η, χ *are coalgebra homomorphisms.*
(b) *If* (H, φ, η) *is commutative then* $(H, \psi, \varepsilon, \chi)$ *is a cogroup object in* ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$*. In particular,* ψ, ε, χ *are algebra homomorphisms.*

Proof. This follows from Proposition 2.2 since χ is the inverse map in each case. □

Definition 2.12. A Hopf algebra which is commutative or cocommutative is called a *classical Hopf algebra*. We have shown above that for such a Hopf algebra, $\chi \circ \chi = \mathrm{Id}$ and χ is an (co)algebra isomorphism $H \xrightarrow{\cong} H^{\mathrm{op}}$ to the opposite (co)algebra. A Hopf algebra for which $\chi \circ \chi = \mathrm{Id}$ is called *involutary* or *involutive*. Of course involutary Hopf algebras have bijective antipodes.

Remark 2.13. Although in general the antipode χ of a Hopf algebra H need not be either an algebra or a coalgebra homomorphism, its composition square $\chi^2 = \chi \circ \chi$ is by Lemma 2.3 and because χ^2 commutes with χ. This means that $\chi^2 H \subseteq H$ is a subHopf algebra; of course χ is not injective or surjective this might be a proper inclusion of a quotient Hopf algebra.

Definition 2.14. If H is a Hopf algebra then its set of *primitive elements* is

$$\mathrm{P}(H) = \{h \in H : \psi(h) = 1 \otimes h + h \otimes 1\}.$$

This is a vector subspace but also we have for $x, y \in \mathrm{P}(H)$,

$$\psi(xy - yx) = 1 \otimes (xy - yx) + (xy - yx) \otimes 1,$$

so $\mathrm{P}(H)$ is a Lie subalgebra of H with its commutator bracket. Notice also that if $x \in \mathrm{P}(H)$ then

$$x = \varepsilon(1)x + \varepsilon(x) = x + \varepsilon(x)$$

so $\varepsilon(x) = 0$.

Definition 2.15. If H is a Hopf algebra then its set of *group-like elements* is

$$\mathrm{G}(H) = \{g \in H : \psi(g) = g \otimes g\}.$$

If $g, h \in \mathrm{G}(H)$ then

$$\psi(gh) = gh \otimes gh$$

and since $\psi(1) = 1 \otimes 1$, $1 \in \mathrm{G}(H)$. This show that $\mathrm{G}(H)$ is a monoid under multiplication. If $g \in \mathrm{G}(H)$ then using the counit we get

$$\varepsilon(g)g = g = g\varepsilon(g)$$

so $\varepsilon(g) = 1$; now using the antipode we also find that

$$\chi(g)g = \varepsilon(g) = g\chi(g)$$

so g is a unit with inverse $g^{-1} = \chi(g)$. Therefore $\mathrm{G}(H) \leqslant H^{\times}$.

There is a more general notion that combines the group-like and the primitives. If $g \in \mathrm{G}(H)$ then the set of *g-primitives* is

$$\mathrm{G}_g(H) = \{h \in H : \psi(h) = g \otimes h + h \otimes g\}.$$

Lemma 2.16. *Let H be a Hopf algebra. Then the set of group-like elements $\mathrm{G}(H)$ is linearly independent. Hence the group-like elements span a cocommutative subHopf algebra isomorphic to the group algebra $\Bbbk\mathrm{G}(H)$.*

Proof. Suppose that $\mathrm{G}(H)$ is not linearly independent. Then there is a minimal $n \geqslant 1$ for which there is a subset $\{g_0, g_1, \ldots, g_n\} \subseteq \mathrm{G}(H)$ with $\{g_1, \ldots, g_n\}$ linearly independent and

$$g_0 = \sum_{1 \leqslant k \leqslant n} t_k g_k$$

for $t_k \in \Bbbk$. Applying ψ we obtain

$$g_0 \otimes g_0 = \sum_{1 \leqslant k \leqslant n} t_k g_k \otimes g_k \in H \otimes H$$

and so

$$\sum_{\substack{1 \leqslant k \leqslant n \\ 1 \leqslant \ell \leqslant n}} t_k t_\ell g_k \otimes g_\ell = \sum_{1 \leqslant k \leqslant n} t_k g_k \otimes g_k.$$

Since the basic tensors $g_k \otimes g_\ell \in H \otimes H$ are linearly independent we must have $t_k = 0$. This contradiction shows that no such minimal set exists.

The monoid $\mathrm{G}(H)$ spans a subspace with its elements as a basis, and which is closed under multiplication it forms a subalgebra visibly isomorphic to the group algebra $\Bbbk\mathrm{G}(H)$. Also the coproduct ψ restricts to it and agrees with the coproduct in the group algebra. Finally, it is closed under the action of the antipode. $\square$

In fact $\mathrm{G}(-)$ defines a functor $\mathrm{G}\colon \mathbf{HA}_\Bbbk \to \mathbf{Gp}$ and this has as its left adjoint the group algebra functor $\Bbbk(-)\colon \mathbf{Gp} \to \mathbf{HA}_\Bbbk$, so there is a natural isomorphism of bifunctors

$$\mathbf{HA}_\Bbbk(\Bbbk(-), -) \cong \mathbf{Gp}(-, \mathrm{G}(-)).$$

This will be discussed more in the examples.

Endomorphism algebras. For a vector space V, its endomorphism algebra is

$$\mathrm{End}_{\Bbbk}(V) = \hom(V, V)$$

with composition as its product. If V is finite dimensional then

$$\mathrm{End}_{\Bbbk}(V) \cong V \otimes V^*$$

as a vector space with the obvious pairing

$$(V \otimes V^*) \otimes (V \otimes V^*) \xrightarrow{\cong} V \otimes (V^* \otimes V) \otimes V^* \to V \otimes \Bbbk \otimes V^* \xrightarrow{\cong} V \otimes V^*$$

making this an isomorphism of algebras. Of course if we choose a basis for V and the corresponding dual basis for V^* we can find and isomorphism of algebras with the ring of $\dim_{\Bbbk} V$ by $\dim_{\Bbbk} V$ matrices

$$\mathrm{End}_{\Bbbk}(V) \cong \mathrm{M}_{\dim_{\Bbbk} V}(\Bbbk).$$

Polynomial rings and their duals.

Example 3.1. Let $\Bbbk[X]$ be the polynomial ring. We can give it a coproduct by making X primitive,

$$\psi(X) = X \otimes 1 + 1 \otimes X,$$

and the antipode is determined by

$$\chi(X) = -X.$$

So this Hopf algebra is bicommutative.

If $\Bbbk$ has characteristic 0 this has no ideals which are also coideals, but if the characteristic is $p > 0$ the for $k \geqslant 1$, (X^{p^k}) is a coideal and $\Bbbk[X]/(X^{p^k})$ is a quotient Hopf algebra.

This example can be generalised to a polynomial ring $\Bbbk[X_1, \ldots, X_n]$ and then there is an isomorphism of Hopf algebras

$$\Bbbk[X_1, \ldots, X_n] \cong \Bbbk[X_1] \otimes \cdots \otimes \Bbbk[X_n].$$

Example 3.2 (Divided power Hopf algebra)**.** Consider the $\Bbbk$-vector space $\Gamma_{\Bbbk}$ with basis γ_i $(i \geqslant 0)$. Make $\Gamma_{\Bbbk}$ into a commutative algebra with product

$$\gamma_i \gamma_j = \binom{i+j}{i} \gamma_{i+j}$$

and unity $1 = \gamma_0$. Make it a cocommutative coalgebra with product

$$\psi(\gamma_k) = \sum_{0 \leqslant i \leqslant k} \gamma_i \otimes \gamma_{k-j}$$

and counit

$$\varepsilon(\gamma_0) = 1, \qquad \varepsilon(\gamma_k) = 0 \quad (k > 0).$$

Then with this structure $\Gamma_{\Bbbk}$ is a bicommutative Hopf algebra with antipode defined recursively using $\overline{\gamma_1} = -\gamma_1$ and

$$\sum_{0 \leqslant i \leqslant k} \overline{\gamma_i} \gamma_{k-j} = 0.$$

If the characteristic of $\Bbbk$ is 0 then it is easy to show that there is an isomorphism of Hopf algebras $\Gamma_{\Bbbk} \cong \Bbbk[X]$ under which

$$\gamma_k \leftrightarrow \frac{1}{k!} X^k$$

where X is primitive. In this case $\mathrm{P}\Gamma_{\Bbbk} = \Bbbk\{\gamma_1\}$ and $\Gamma_{\Bbbk}$ is primitively generated.

If the characteristic of $\Bbbk$ is $p > 0$ we have relations such as $\gamma_k^p = 0$ when $k > 0$. As an algebra, $\Gamma_\Bbbk$ is generated by the elements γ_{p^r} with $r \geqslant 0$. Also $\mathrm{P}\Gamma_\Bbbk = \Bbbk\{\gamma_1\}$, and so $\Gamma_\Bbbk$ is not primitively generated.

The finite dual $\Gamma_\Bbbk^\circ$ is familiar: if we define

$$x\colon \Gamma_\Bbbk \to \Bbbk; \quad x(\gamma_k) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

then $x \in \Gamma_\Bbbk^\circ$ and

$$x^n(\gamma_k) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases}$$

so $x^n \in \Gamma_\Bbbk^\circ$ for every $n \geqslant 0$. Furthermore, x is primitive. Then there is an isomorphism of Hopf algebras

$$\Bbbk[X] \xrightarrow{\cong} \Gamma_\Bbbk^\circ; \quad X^k \mapsto x^k.$$

In fact this relationship is symmetric: $\Bbbk[X]^\circ \cong \Gamma_\Bbbk$.

The free vector space. Let X be a set and recall the free vector space on X, $\mathbb{F}(X)$.

For any non-empty set X, $\mathbb{F}(X \times X) \cong \mathbb{F}(X) \otimes \mathbb{F}(X)$ and the diagonal map $X \to X \times X$ induces a $\Bbbk$-linear map

$$\begin{tikzcd} \mathbb{F}(X) \arrow[r] \arrow[rr, bend left, "\psi"] & \mathbb{F}(X \times X) \arrow[r] & \mathbb{F}(X) \otimes \mathbb{F}(X) \end{tikzcd}$$

Since there is a bijection $X \times (X \times X) \cong (X \times X) \times X$ this is coassociative. If we take any set $\mathbf{1}$ with a single element it is a terminal object and there are bijections

$$\mathbf{1} \times X \cong X \cong X \times \mathbf{1}.$$

Also, $\mathbb{F}(\mathbf{1}) \cong \Bbbk$. Now the unique function $X \to \mathbf{1}$ induces a counit $\varepsilon\colon \mathbb{F}(X) \to \Bbbk$. Putting all this together we find that $(\mathbb{F}(X), \psi, \varepsilon)$ is a coalgebra. In fact the switch map gives a commutative diagram

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ X \times X \arrow[rr, leftrightarrow, "\mathrm{T}", "\cong"'] & & X \times X \end{tikzcd}$$

and using this we can show that $(\mathbb{F}(X), \psi, \varepsilon)$ is a cocommutative coalgebra.

If X is a monoid it has a product $X \times X \to X$ and a unit $\mathbf{1} \to X$. By functoriality, these induce maps

$$\varphi\colon \mathbb{F}(X) \otimes \mathbb{F}(X) \xrightarrow{\cong} \mathbb{F}(X \times X) \to \mathbb{F}(X), \quad \eta\colon \Bbbk \to \mathbb{F}(X),$$

so that $(\mathbb{F}(X), \varphi, \eta)$ is an algebra which is commutative if and only if the monoid X is commutative.

Now if X is a monoid we can put together the coalgebra and algebra structures to obtain a cocommutative bialgebra $(\mathbb{F}(X), \varphi, \eta, \psi, \varepsilon)$ which is commutative if and only if the monoid is commutative. With this structure, $\Bbbk X = \mathbb{F}(X)$ is called the *monoid algebra* of X.

There is also a dual object, namely

$$\Bbbk^X = \text{the set of all functions } X \to \Bbbk.$$

Then for two finite sets X, Y,

$$\Bbbk^{X \times Y} \cong \Bbbk^X \otimes \Bbbk^Y.$$

The diagonal map $X \to X \times X$ induces a multiplication

$$\Bbbk^X \otimes \Bbbk^X \xrightarrow{\cong} \Bbbk^{X\times X} \to \Bbbk^X$$

which is 'pointwise product' of functions. This makes into a commutative algebra. In fact

$$\Bbbk^X \cong \hom(\Bbbk X, \Bbbk).$$

When X is a finite monoid, there is a coproduct and $\Bbbk^X$ is then a commutative bialgebra.

If G is a group, the inverse map $G \to G$ induces a coalgebra map $\chi\colon \mathbb{F}(G) \to \mathbb{F}(G)$. Then $(\mathbb{F}(G), \varphi, \eta, \psi, \varepsilon, \chi)$ is a cocommutative Hopf algebra. The algebra $\Bbbk G = \mathbb{F}(G)$ is called the *group algebra* of G, and we know that it is also Hopf algebra. When G is finite, the dual $\Bbbk^G$ is also a commutative Hopf algebra, the *dual group algebra*.

This construction of an algebra and Hopf algebra for each group defines two left adjoints. Recall that every ring has a group of units and in particular every $\Bbbk$-algebra A has a group of units $A^\times$; we can think of this as defining a functor $(-)^\times\colon \mathbf{Alg}_\Bbbk \to \mathbf{Gp}$. Of course every Hopf algebra is also an algebra so there is a restriction to a functor $(-)^\times\colon \mathbf{HA}_\Bbbk \to \mathbf{Gp}$.

Proposition 3.3. *The functor* $\mathbb{F}\colon \mathbf{Gp} \to \mathbf{Alg}_\Bbbk$ *is a left adjoint to the unit functor, i.e., there is natural isomorphism of bifunctors*

$$\mathbf{Alg}_\Bbbk(\mathbb{F}(-), -) \cong \mathbf{Gp}(-, (-)^\times).$$

Similarly, the functor $\mathbb{F}\colon \mathbf{Gp} \to \mathbf{HA}_\Bbbk$ *is a left adjoint to the unit functor, i.e., there is natural isomorphism of bifunctors*

$$\mathbf{HA}_\Bbbk(\mathbb{F}(-), -) \cong \mathbf{Gp}(-, (-)^\times).$$

For finite groups, we can do something similar with $\mathbb{M}(G)$, this time obtaining a commutative Hopf algebra contravariantly functorial in G. It is common to set $\Bbbk^G = \mathbb{M}(G)$ and call this the dual group algebra of G.

Poset coalgebras and algebras. Let $(\mathcal{P}, \preccurlyeq)$ be a *locally finite poset*, i.e., each interval

$$[x, y] = \{t \in \mathcal{P} : x \preccurlyeq t \preccurlyeq y\}$$

is finite. We define a vector space $C(\mathcal{P}, \preccurlyeq)$ with basis the symbols $[x, y]$ with $x \preccurlyeq y$. Then

$$\psi\colon C(\mathcal{P}, \preccurlyeq) \to C(\mathcal{P}, \preccurlyeq) \otimes C(\mathcal{P}, \preccurlyeq); \quad \psi([x, y]) = \sum_{t\in[x,y]} [x, t] \otimes [t, y]$$

is a coproduct and

$$\varepsilon\colon C \to \Bbbk; \quad \varepsilon([x, y]) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases}$$

is its counit. There is a dual *incidence algebra* $A(\mathcal{P}, \preccurlyeq)$ which consists of the finitely supported functions $f\colon \{[x, y] : x \preccurlyeq y\} \to \Bbbk$ with the product given by convolution,

$$(f * g)([x, y]) = \sum_{t\in[x,y]} f([x, t])g([t, y]),$$

and the unit is given by the constant functions.

Free algebras, bialgebras and Hopf algebras. The forgetful functor $\mathbf{Alg}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$ which forgets the multiplication has a left adjoint. Its construction involves the tensor powers of a vector space V: set $\mathrm{T}^0(V) = \Bbbk$ and for each $n \geqslant 1$,

$$\mathrm{T}^n(V) = V \otimes \mathrm{T}^{n-1}(V) = V^{\otimes n}.$$

Then

$$\mathrm{T}(V) = \bigoplus_{n \geqslant 0} \mathrm{T}^n(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}.$$

There are obvious linear mappings $\mathrm{T}^m(V) \otimes \mathrm{T}^n(V) \to \mathrm{T}^{n+n}(V)$ and these make $\mathrm{T}(V)$ into a $\Bbbk$-algebra. It is easy to see that for any $\Bbbk$-linear mapping $f \colon U \to V$ there is a unique algebra homomorphism $\mathrm{T}(f) \colon \mathrm{T}(U) \to \mathrm{T}(V)$ which extends $\mathrm{T}^1(f) = f \colon \mathrm{T}^1(U) \to \mathrm{T}^1(V)$. Then $\mathrm{T}(V)$ is called the *tensor algebra* or the *free algebra* on V.

Proposition 3.4. *The functor* $\mathrm{T} \colon \mathbf{Vect}_{\Bbbk} \to \mathbf{Alg}_{\Bbbk}$ *is left adjoint to the forgetful functor* $\mathbf{Alg}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$*, i.e., there is a natural isomorphism of bifunctors*

$$\mathbf{Alg}_{\Bbbk}(\mathrm{T}(-), (-)) \cong \mathbf{Vect}_{\Bbbk}((-), (-)).$$

We can modify this to the case of commutative algebras. The free algebra $\mathrm{T}(V)$ has a 2-sided ideal $\mathrm{I}(V)$ generated by all the elements of form

$$u \otimes v - v \otimes u \in \mathrm{T}^2(V) \quad (u, v \in V).$$

The quotient algebra

$$\mathrm{S}(V) = \mathrm{T}(V)/\mathrm{I}(V)$$

is commutative since we have implicitly killed all commutators (exercise!), and $\mathrm{S}(V)$ is called the *symmetric algebra* or the *free commutative algebra* on V.

Proposition 3.5. *The functor* $\mathrm{S} \colon \mathbf{Vect}_{\Bbbk} \to {}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$ *is left adjoint to the forgetful functor* ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk}$*, i.e., there is a natural isomorphism of bifunctors*

$${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}(\mathrm{S}(-), (-)) \cong \mathbf{Vect}_{\Bbbk}((-), (-)).$$

Notice that both $\mathrm{T}(V)$ and $\mathrm{S}(V)$ are naturally $\mathbb{N}$-graded algebras: the degree n part of $\mathrm{T}(V)$ is $\mathrm{T}^n(V)$ and its image in $\mathrm{S}(V)$ is $\mathrm{S}^n(V)$. As a vector space,

$$\mathrm{S}(V) = \bigoplus_{n \geqslant 0} \mathrm{S}^n(V).$$

3.1. **Free bialgebras and free Hopf algebras.** There is also a functor which forgets the algebra structure:

$$\mathbf{HA}_{\Bbbk} \to \mathbf{Coalg}_{\Bbbk}; \quad (H, \varphi, \eta, \psi, \varepsilon, \chi) \mapsto (H, \psi, \varepsilon, \chi).$$

This also has a left adjoint, but we have to construct it in stages.

We first form the composition

$$\mathbf{Coalg}_{\Bbbk} \to \mathbf{Vect}_{\Bbbk} \xrightarrow{\mathrm{T}} \mathbf{Bialg}_k$$

into the category of bialgebras, where the first map is the forgetful functor. Then for a coalgebra C, $\mathrm{T}(C)$ is the *free bialgebra* on C. Its elements are sums of monomials in elements of $C \cong \mathrm{T}^1(C)$ so the coproduct is obtained using

$$\psi(c_1 c_2 \cdots c_\ell) = \psi(c_1)\psi(c_2) \cdots \psi(c_\ell).$$

There is a similar construction forming the free *free commutative bialgebra* on C, $\mathrm{S}(C)$.

There are variants of these for (co)augmented coalgebras which form a category $\Bbbk/\mathbf{Coalg}_{\Bbbk}$ (i.e., coalgebras under $\Bbbk$). Given a coaugmented coalgebra $\eta\colon \Bbbk \to C$ we form $\mathrm{T}(C)$ then pass to the quotient bialgebra

$$\mathrm{T}(C)/(\eta(1) - 1).$$

This of course identifies $\eta(1) \in \mathrm{T}(C)$ with $1 \in \mathrm{T}^0(C)$. We can do a similar thing with the commutative version.

To get the free algebra functor into $\mathbf{HA}_{\Bbbk}$ we take the direct sum of coalgebras $C \oplus C^{\mathrm{op}}$, form the free algebra $\mathrm{T}(C \oplus C^{\mathrm{op}})$ and then impose relations to identify each element c^{op} with an antipode applied to c, i.e., quotient by the ideal generated by all the expressions

$$\sum c_{(1)} \otimes c^{\mathrm{op}}_{(2)} - \varepsilon(c) \otimes 1, \quad \sum c^{\mathrm{op}}_{(1)} \otimes c_{(2)} - \varepsilon(c) \otimes 1$$

where $c \in C$.

To get a free commutative Hopf algebra we can use S instead of T. In fact

$$\mathrm{S}(C \oplus C^{\mathrm{op}}) \cong \mathrm{S}(C) \otimes \mathrm{S}(C^{\mathrm{op}}).$$

Since a Hopf algebra is naturally a coaugmented coalgebra we can also do this by first applying the free bialgebra functors for coaugmented coalgebras.

Enveloping algebras of Lie algebras. Recall that a *Lie algebra over* $\Bbbk$ is a vector space L equipped with a linear mapping called the *Lie bracket*

$$[-,-]\colon L \otimes L \to L$$

which satisfies the following conditions for all $x, y, z \in L$:

(Jacobi identity) $\quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0;$

(Anticommutativity) $\quad [y, x] + [x, y] = 0;$

(Alternativity) $\quad [x, x] = 0.$

If the characteristic of $\Bbbk$ is not 2 then anticommutativity implies alternativity so then the last condition is redundant. Care is also required when the characteristic is 3 but we will ignore this subtlety.

A Lie algebra with trivial bracket $[x, y] = 0$ is called an *abelian Lie algebra*; abelian Lie algebras are essentially the same thing as vector spaces.

Lie algebras over $\Bbbk$ form an abelian category $\mathbf{Lie}_{\Bbbk}$ with homomorphisms preserving brackets.

For any algebra A, its elements for a Lie algebra with the usual commutator $[x, y] = xy - yx$ as its bracket. Of course this Lie algebra is abelian if and only if the algebra is commutative This construction defines a functor $\mathbf{Alg}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk}$. We will see that it has a left adjoint. But in fact there is another functor $\mathrm{P}\colon \mathbf{HA}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk}$ which also has a left adjoint.

To construct the adjoint in the algebra case we first recall the free algebra functor T. We can apply this to a Lie algebra L but the linear mapping $L = \mathrm{T}^1(L) \hookrightarrow \mathrm{T}(L)$ is not a homomorphism of Lie algebras if we make $\mathrm{T}(L)$ a Lie algebra using the commutator. To correct this we have to force relations by passing to a quotient algebra. We consider the 2-sided ideal $\mathrm{J}(L) \lhd \mathrm{T}(L)$ generated by all the elements

$$x \otimes y - y \otimes x - [x, y] \quad (x, y \in L).$$

Notice that $x \otimes y, y \otimes x \in \mathrm{T}^2(L)$ but $[x, y] \in \mathrm{T}^1(L)$. The resulting quotient algebra

$$\mathrm{U}(L) = \mathrm{T}(L)/\mathrm{J}(L)$$

is called the universal enveloping algebra of L. It can be verified that the mapping $L \to \mathrm{U}(L)$ is a Lie algebra homomorphism where $\mathrm{U}(L)$ is given the commutator as its Lie bracket (it is injective except possibly when the characteristic of $\Bbbk$ is 3).

Proposition 3.6. *The functor* $\mathrm{U}\colon \mathbf{Lie}_{\Bbbk} \to \mathbf{Alg}_{\Bbbk}$ *is left adjoint to the functor* $\mathbf{Alg}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk}$ *sending each algebra to its Lie algebra with the commutator bracket, i.e., there is a natural isomorphism of bifunctors*

$$\mathbf{Alg}_{\Bbbk}(\mathrm{U}(-),(-)) \cong \mathbf{Lie}_{\Bbbk}((-),(-)).$$

The Poincaré-Birkhoff-Witt Theorem is an important result which describes the vector space structure of $\mathrm{U}(L)$ at least given a certain kind of basis of L. Here is a version when L is of finite or countable dimension with a basis $x_1, x_2, \ldots$; we will denote the image of $x \in L$ in $\mathrm{U}(L)$ by $\widetilde{x}$.

Theorem 3.7 (Poincaré-Birkhoff-Witt Theorem)**.** *The distinct monomials*

$$\widetilde{x_1}^{k_1}\widetilde{x_2}^{k_2}\cdots\widetilde{x}_\ell^{k_\ell} \quad (k_i \geqslant 0)$$

form a basis for $\mathrm{U}(L)$*. In particular the linear map* $L \to \mathrm{U}(L)$ *sending* x *to* $\widetilde{x}$ *is injective.*

Since the map $L \to \mathrm{U}(L)$ is injective, it is usual to omit the tildes and write x for the image of $x \in L$ in $\mathrm{U}(L)$.

Of course we have chosen a particular ordering here; for example to express x_2x_1 we note that in $\mathrm{U}(L)$ we have

$$x_2x_1 = (x_1x_2 - x_2x_1) + x_1x_2 = [x_1, x_2] + x_1x_2$$

where $[x_1, x_2] \in L \subseteq \mathrm{U}(L)$ is a linear combination of the x_i.

For any Lie algebra L we can make $\mathrm{U}(L)$ into a Hopf algebra by defining $L \subseteq \mathrm{U}(L)$ to be contained in $\mathrm{PU}(L)$. Then $\mathrm{U}(L)$ is generated as an algebra by $\mathrm{PU}(L)$. Of course for any Hopf algebra H the inclusion $\mathrm{P}(H) \hookrightarrow H$ is a Lie homomorphism so it induces a Hopf algebra homomorphism $\mathrm{UP}(H) \to H$; if this is surjective then H is called *primitively generated.* Primitively generated Hopf algebras are cocommutative and in a sense the 'easy' ones to understand.

Proposition 3.8. *The functor* $\mathrm{U}\colon \mathbf{Lie}_{\Bbbk} \to \mathbf{HA}_{\Bbbk}$ *is left adjoint to the functor* $\mathrm{P}\colon \mathbf{HA}_{\Bbbk} \to \mathbf{Lie}_{\Bbbk}$*, i.e., there is a natural isomorphism of bifunctors*

$$\mathbf{HA}_{\Bbbk}(\mathrm{U}(-),(-)) \cong \mathbf{Lie}_{\Bbbk}((-),\mathrm{P}(-)).$$

Here are some examples.

Example 3.9. Let p be a prime number and $\Bbbk$ a field of characteristic p. Let

$$H = \Bbbk[X]/(X^p)$$

and write $x = X + (X^p) \in H$. Then the coproduct $\psi(x) = 1 \otimes x + x \otimes 1 + x \otimes x$ and counit $\varepsilon(x) = 0$ make H a bicommutative Hopf algebra.

It is easy to see that $\mathrm{P}H = \Bbbk\{x\}$ and so H is primitively generated.

This is a disguised version of the group algebra $\Bbbk C_p$. If the characteristic of $\Bbbk$ is not equal to p and $\Bbbk$ contains a primitive p-th root of unity then $\Bbbk C_p$ is not primitively generated.

Example 3.10. The polynomial ring $H = \Bbbk[X]$ given the coproduct

$$\psi(X^n) = \sum_{0 \leqslant i \leqslant n} \binom{n}{i} X^i \otimes X^{n-i}$$

is a commutative and cocommutative Hopf algebra which is primitively generated. If the characteristic of $\Bbbk$ is 0 then $\mathrm{P}H = \Bbbk\{x\}$, but if it is a prime number p then

$$\mathrm{P}H = \Bbbk\{x^{p^k} : k \geqslant 0\}.$$

Restricted Lie algebras. See Jacobson [Jac79, section V.7] or Milnor & Moore [MM65, section 6].

For any Hopf algebra H over a field of positive characteristic p, there is a Frobenius mapping

$$\mathrm{P}H \to \mathrm{P}H; \quad x \mapsto x^p.$$

Of course this is not linear over $\Bbbk$ but if $t \in \Bbbk$, then $(tx)^p = t^p x^p$. If $x, y \in \mathrm{P}H$ commute then $(x+y)^p = x^p + y^p$, but in general there is a more complicated formula.

A Lie algebra over a field of characteristic p is called a *restricted Lie algebra* if there is an additive homomorphism $(-)^{[p]} \colon L \to L$ (the *restriction*) such that

- for $x \in L$ and $t \in \Bbbk$, $(tx)^{[p]} = t^p x^{[p]}$;
- $\mathrm{ad}_{x^{[p]}} = \mathrm{ad}_x^p = \mathrm{ad}_x \circ \mathrm{ad}_x \circ \cdots \circ \mathrm{ad}_x$, where $\mathrm{ad}_x \colon L \to L$ is the linear mapping given by $\mathrm{ad}_x(y) = [x, y]$;
- for $x, y \in L$,
$$(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$$
where for an indeterminate Z, $is_i(x, y)$ is the coefficient of Z^{i-1} in $(\mathrm{ad}_{tx+y})^{p-1}(x)$.

Now for a Hopf algebra H over a field of characteristic p, its primitives form a restricted Lie algebra, so there is a functor from Hopf algebras to restricted Lie algebras and this has a left adjoint given on a restricted Lie algebra by

$$\mathrm{V}(L) = \mathrm{U}(L)/(\widetilde{x^{[p]}} - \widetilde{x}^p : x \in L),$$

a quotient Hopf algebra of the usual enveloping algebra; this is called the *restricted enveloping algebra* of L. When L is finite dimensional, so is $\mathrm{V}(L)$ whereas $\mathrm{U}(L)$ is infinite dimensional. There is also a version of the PBW Theorem for $\mathrm{V}(L)$.

Affine group schemes. Algebraic Geometry since Grothendieck has been centred around representable functors on the category of algebras over a base ring. A commutative algebra $R \in {}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$ defines a functor

$$\mathrm{Spec}(R) \colon {}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk} \to \mathbf{Set}; \quad \mathrm{Spec}(R)(A) = {}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}(R, A).$$

This is called an *affine scheme*. Its space of *geometric points* is given by its value on an algebraic closure $\overline{\Bbbk}$, $\mathrm{Spec}(R)(\overline{\Bbbk})$.

In practise such a functor often has a factorisation through a functor into a concrete category $\mathbf{C}$ such as the category of groups; in this case we say that it is a $\mathbf{C}$-*scheme*.

$$\begin{tikzcd} & \mathbf{C} \arrow[d] \\ {}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk} \arrow[ur, "\mathrm{Spec}(R)"] \arrow[r] & \mathbf{Set} \end{tikzcd}$$

Let's suppose that $\mathrm{Spec}(R)$ takes values in $\mathbf{Gp}$ so it is *group scheme*. Now the coproduct in ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$ is given by $\otimes$ and $\Bbbk$ is an initial object. Therefore for any A,

$$\mathrm{Spec}(R \otimes R)(A) \cong \mathrm{Spec}(R)(A) \times \mathrm{Spec}(R)(A)$$

and $\mathrm{Spec}(\Bbbk)(A)$ contains only the unit homomorphism $\Bbbk \to A$. The multiplication is a natural transformation

$$\mathrm{Spec}(R \otimes R) \cong \mathrm{Spec}(R) \times \mathrm{Spec}(R) \to \mathrm{Spec}(R)$$

so if we evaluate on $R \otimes R$ we get

$$\mathrm{Spec}(R \otimes R)(R \otimes R) \to \mathrm{Spec}(R)(R \otimes R)$$

which sends $\mathrm{Id}_{R\otimes R}$ to a homomorphism $\psi\colon R \to R\otimes R$. Similarly the identity evaluated on $\Bbbk$ gives

$$\mathrm{Spec}(\Bbbk)(\Bbbk) \to \mathrm{Spec}(R)(\Bbbk)$$

which sends $\mathrm{Id}_{\Bbbk}$ to an element $\varepsilon\colon R \to \Bbbk$. Final the inverse map gives a natural transformation $\mathrm{Spec}(R) \to \mathrm{Spec}(R)$ which when evaluated on R sends Id_R to $\chi\colon R \to \Bbbk$. All of these structure maps are algebra homomorphisms by definition and make $(R,\psi,\varepsilon,\chi)$ a cogroup object in ${}^{\mathrm{co}}\mathbf{Alg}_{\Bbbk}$, in other words we have a commutative Hopf algebra; if the group scheme takes values in abelian groups then it will be cocommutative. Here are some examples.

Each commutative algebra A has a group of units $A^\times$. To specify a unit means to pick an element and another element which is its inverse. We can do this with the affine scheme $\mathrm{Spec}(\Bbbk[U,V]/(UV-1))$ where

$$\psi(U) = U\otimes U, \quad \psi(V) = V\otimes V, \quad \varepsilon(U) = 1 = \varepsilon(V), \quad \chi(U) = V, \quad \chi(V) = U.$$

It is usual to set $V = U^{-1}$ and write $\Bbbk[U,U^{-1}] = \Bbbk[U,V]/(UV-1)$. This is called the *multiplicative group scheme* and denote $\mathbb{G}_{\mathrm{m}}$.

For each natural number $n \geqslant 1$, there is a natural transformation $[n]\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ induce by the Hopf algebra homomorphism $\Bbbk[U,U^{-1}] \to \Bbbk[U,U^{-1}]$ which maps U to U^n. This corresponds to the n-th power map when evaluated on an algebra A.

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}}(A) \arrow[r, "{[n]}"] \arrow[d, equal] & \mathbb{G}_{\mathrm{m}}(A) \arrow[d, equal] \\
A^\times \arrow[r, "(-)^n"] & A^\times
\end{tikzcd}$$

In fact $\mathbb{G}_{\mathrm{m}}[n] = \ker[n]$ is also a scheme, given by

$$\mathbb{G}_{\mathrm{m}}[n] = \mathrm{Spec}(\Bbbk[U]/(U^n-1)),$$

represented by the quotient Hopf algebra $\Bbbk[U]/(U^n-1) = \Bbbk[U,U^{-1}]/(U^n-1)$.

This can be generalised to a non-abelian group scheme $\mathbb{GL}_n$ for $n \geqslant 2$. When $n=2$ this is

$$\mathbb{GL}_2 = \mathrm{Spec}\big(\Bbbk[A,B,C,D,E]/((AD-BC)E-1)\big)$$

with coproduct induced by matrix mutiplication

$$\begin{aligned}
\psi(A) &= A\otimes A + B\otimes C, & \psi(B) &= A\otimes B + B\otimes D,\\
\psi(C) &= C\otimes A + D\otimes C, & \psi(D) &= C\otimes B + D\otimes D,\\
\psi(E) &= E\otimes E.
\end{aligned}$$

The antipode is induced by the formula for finding the inverse of a 2 by 2 matrix (Cramer's Rule).

There is a normal subgroup scheme $\mathbb{SL}_2 \lhd \mathbb{GL}_2$ given by

$$\mathbb{SL}_2 = \mathrm{Spec}\big(\Bbbk[A,B,C,D]/((AD-BC)-1)\big)$$

where $\Bbbk[A,B,C,D]/((AD-BC)-1)$ is a quotient Hopf algebra of $\Bbbk[A,B,C,D,E]/((AD-BC)E-1)$.

Combinatorial Hopf algebras. The symmetric function Hopf algebra can be defined over any commutative ring $\Bbbk$. It is bicommutative and

$$\mathrm{Symm}(\Bbbk) = \Bbbk[e_n : n \geqslant 1]$$

with coproduct given by

$$\psi(e_n) = \sum_{0\leqslant i\leqslant n} e_i \otimes e_{n-i}$$

where $e_0 = 1$. Its vector space of primitives is spanned by the elements s_n defined by $s_1 = e_1$ and the Newton recursion formula

$$s_n = e_1 s_{n-1} - e_2 s_{n-2} + e_3 s_{n-3} - \cdots + (-1)^{n-2} e_{n-1} s_1 + (-1)^{n-1} n e_n.$$

If the characteristic of $\Bbbk$ is zero then

$$\mathrm{Symm}(\Bbbk) = \Bbbk[s_n : n \geqslant 1]$$

but if it is a prime $p > 0$ then for any k,

$$s_{pk} = s_k^p.$$

The e_n are essentially the elementary symmetric functions in infinitely many indeterminates while the s_n are the power sums. The antipode is given by

$$\chi(e_n) = h_n$$

where the h_n are the total symmetric functions. There is another set of polynomial generators that occurs, namely the w_n defined recursively by

$$p_n = \sum_{k|n} k w_k^{n/k}.$$

If the characteristic of $\Bbbk$ is $p > 0$ then for each m with $p \nmid m$, there is a subHopf algebra

$$B[m] = \Bbbk[w_{mp^r} : r \geqslant 0] \subseteq \mathrm{Symm}(\Bbbk)$$

and a Hopf algebra splitting

$$\mathrm{Symm}(\Bbbk) = \bigotimes_{p \nmid m} B[m].$$

This is related to Witt vectors and also the Necklace Algebra of Rota and Metropolis [MR83].

Frobenius algebras. Frobenius algebras are commonly encountered, and we will see later that every finite dimensional Hopf algebra is a Frobenius algebra.

Definition 3.11. A finite dimensional $\Bbbk$-algebra A is a *Frobenius algebra* if it has a *Frobenius form* $\lambda \in A^* = \hom(A, \Bbbk)$ which is non-trivial on every simple left submodule.

A left submodule is of course a left ideal; it is *simple* if it has no non-trivial proper submodules. A given Frobenius algebra can have many different Frobenius forms.

A Frobenius form λ has an associated non-degenerate $\Bbbk$-*bilinear Frobenius form*

$$\beta \colon A \times A \to \Bbbk; \quad \beta(x, y) = \lambda(xy)$$

which satisfies

$$\beta(xy, z) = \beta(x, yz).$$

This can be used to show that λ is non-trivial on every simple right submodule.

The Frobenius form induces two $\Bbbk$-linear mappings

$$A \to A^*; \quad a \mapsto a \cdot \lambda, \quad a \mapsto \lambda \cdot a$$

where

$$a \cdot \lambda(x) = \lambda(xa), \quad \lambda \cdot a(x) = \lambda(ax).$$

If we make A^* a left or right A-module by premultiplying on the right or the left these become left and right A-module isomorphisms. In particular this means that A is injective as a left or right A-module, i.e., it is *self-injective.* This has lots of implications: for example, A is a *Kasch algebra*, i.e., every simple left or right module is isomorphic to a submodule of A.

As well as the algebra (A, φ, η) structure, λ also induces a coalgebra structure $(A, \varphi^\dagger, \eta^\dagger)$ which make the following diagrams commute.

$$
\begin{tikzcd}
A \arrow[rr, "\varphi^\dagger"] \arrow[d, leftrightarrow, "\cong"'] & & A \otimes A \arrow[d, leftrightarrow, "\cong"] \\
A^* \arrow[r, "\varphi^*"] & (A \otimes A)^* \arrow[r, leftrightarrow, "\cong"] & A^* \otimes A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta^\dagger"] \arrow[d, leftrightarrow, "\cong"'] & \Bbbk \arrow[d, leftrightarrow, "\cong"] \\
A^* \arrow[r, "\eta^*"] & \Bbbk^*
\end{tikzcd}
$$

Note that (A, φ, η) and $(A, \varphi^\dagger, \eta^\dagger)$ do not interact appropriately to form a bialgebra. We will see later that finite dimensional Hopf algebras are Frobenius algebras but the coproduct associated to the Frobenius form is not the same as that of the Hopf coalgebra structure.

Taft Hopf algebras. For $n \geqslant 1$ and $\zeta \in \Bbbk$ a primitive n-th root of unity. As an algebra let

$$H_{n,\zeta} = \Bbbk\langle u, v\rangle/(u^n - 1,\, v^n,\, vu - \zeta uv).$$

The coproduct is given by

$$\psi(u) = u \otimes u, \quad \psi(v) = v \otimes u + 1 \otimes v$$

and the antipode by

$$\chi(u) = u^{-1}, \quad \chi(v) = -vu^{-1}.$$

For $n > 1$ this Hopf algebra $H_{n,\zeta}$ is neither commutative nor cocommutative. In fact

$$\chi^2(u) = u, \quad \chi^2(v) = uvu^{-1} = \zeta^{-1}v.$$

Then $H_{n,\zeta}$ has as a basis the elements $u^i v^j$ $(0 \leqslant i, j < n)$ and the linear mapping χ^2 is diagonalised with respect to it. Notice that $\dim_{\Bbbk} H_{n,\zeta} = n^2$.

4. SubHopf algebras, adjoint actions and normal subalgebras

A Hopf algebra H can contain subalgebras, subcoalgebras and subbialgebras. A subbialgebra $K \subseteq H$ where the antipode χ restricts to give an antipode for K is called a *subHopf algebra*; of course K is a Hopf algebra in its own right.

$$\begin{tikzcd} K \arrow[r, dotted, "\chi|_K"] \arrow[d, hook] & K \arrow[d, hook] \\ H \arrow[r, "\chi"] & H \end{tikzcd}$$

This is analogous to the notion of a subgroup of a group.

Proposition 4.1. *The image of the antipode $\chi H \subseteq H$ is a subHopf algebra. More generally, for $n \geqslant 2$, $\chi^n H \subseteq H$ is a subHopf algebra.*

Proof. We know that for H the identities

$$\psi \circ \chi = (\chi \otimes \chi) \circ \mathrm{T} \circ \psi, \quad \chi \circ \varphi = \varphi \circ \mathrm{T} \circ (\chi \otimes \chi),$$

which imply

$$\varphi(\chi H \otimes \chi H) \subseteq \chi H, \quad \psi \chi H \subseteq \chi H \otimes \chi H,$$

hence χH is a subbialgebra of H.

We also have

$$\begin{aligned} (\chi * \mathrm{Id}) \circ \chi &= \varphi \circ (\chi \otimes \mathrm{Id}) \circ \psi \circ \chi \\ &= \varphi \circ (\chi \otimes \mathrm{Id}) \circ (\chi \otimes \chi) \circ \mathrm{T} \circ \psi \\ &= \varphi \circ (\chi \otimes \chi) \circ (\chi \otimes \mathrm{Id}) \circ \mathrm{T} \circ \psi \\ &= \varphi \circ (\chi \otimes \chi) \circ \mathrm{T} \circ (\chi \otimes \mathrm{Id}) \circ \psi \\ &= \chi \circ \varphi \circ (\chi \otimes \mathrm{Id}) \circ \psi \\ &= \chi \circ \varphi \circ (\chi * \mathrm{Id}) \\ &= \chi \circ \eta \circ \varepsilon = \eta \circ \varepsilon, \end{aligned}$$

and a similar calculation shows that $(\mathrm{Id} * \chi) \circ \chi = \eta \circ \varepsilon$. These identities show that the restriction of χ to χH is an antipode for it, therefore χH is a subHopf algebra of H. □

Now we will consider analogue of a *normal* subgroup. There are two approaches which roughly correspond to the two ways of thinking about when a subgroup is normal (i.e., requiring left and right cosets being equal, or being closed under conjugation).

Let $A \subseteq H$ be a subalgebra and let $A^+ = \ker \varepsilon_A$, the kernel of the counit restricted to A. Then $HA^+ \subseteq H$ is a left ideal and $A^+H \subseteq H$ is a right ideal. If $HA^+ = A^+H$ we can form the quotient algebra H/HA^+, but this won't always be a Hopf algebra. If $K \subseteq H$ is a subHopf algebra and if $HK^+ = K^+H$, this is also a coideal and H/HK^+ is a quotient Hopf algebra. So this looks like a reasonable way to define a 'normal' subHopf algebra.

The alternative approach requires the two *adjoint actions*.

Definition 4.2. For $h \in H$, the left and right *adjoint actions* $\mathrm{ad}^{\mathrm{l}}_h \colon H \to H$ and $\mathrm{ad}^{\mathrm{r}}_h \colon H \to H$ are given by

$$\mathrm{ad}^{\mathrm{l}}_h(x) = \sum h_{(1)} x \chi(h_{(2)}), \quad \mathrm{ad}^{\mathrm{r}}_h(x) = \sum \chi(h_{(1)}) x h_{(2)}.$$

Lemma 4.3. *The adjoint actions are left and right actions of H on itself, i.e., for $h', h'' \in H$,*

$$\mathrm{ad}^{\mathrm{l}}_{h'h''} = \mathrm{ad}^{\mathrm{l}}_{h'} \circ \mathrm{ad}^{\mathrm{l}}_{h''}, \quad \mathrm{ad}^{\mathrm{r}}_{h'h''} = \mathrm{ad}^{\mathrm{r}}_{h''} \circ \mathrm{ad}^{\mathrm{r}}_{h'}.$$

Furthermore, for $h, x, y \in H$,

$$\mathrm{ad}^{\mathrm{l}}_h(xy) = \sum \mathrm{ad}^{\mathrm{l}}_{h_{(1)}}(x)\mathrm{ad}^{\mathrm{l}}_{h_{(2)}}(y), \qquad \varepsilon(\mathrm{ad}^{\mathrm{l}}_h(x)) = \varepsilon(h)\varepsilon(x), \qquad \mathrm{ad}^{\mathrm{l}}_h(1) = 1,$$
$$\mathrm{ad}^{\mathrm{r}}_h(xy) = \sum \mathrm{ad}^{\mathrm{r}}_{h_{(1)}}(x)\mathrm{ad}^{\mathrm{r}}_{h_{(2)}}(y), \qquad \varepsilon(\mathrm{ad}^{\mathrm{r}}_h(x)) = \varepsilon(h)\varepsilon(x), \qquad \mathrm{ad}^{\mathrm{r}}_h(1) = 1.$$

The left/right adjoint actions makes H into a left/right module over itself.

Now we can define a subalgebra $A \subseteq H$ to be ad-*invariant* if for every $h \in H$, $\mathrm{ad}^{\mathrm{l}}_h A \subseteq A$ and $\mathrm{ad}^{\mathrm{r}}_h A \subseteq A$. Although in general this notion involves two independent conditions, for some Hopf algebras such as group algebras the left and right adjoint actions give equivalent information.

Lemma 4.4. *Suppose that the coproduct ψ is cocommutative. Then the following conditions are equivalent:*

- *A is* ad-*invariant;*
- *for every $h \in H$, $\mathrm{ad}^{\mathrm{l}}_h A \subseteq A$;*
- *for every $h \in H$, $\mathrm{ad}^{\mathrm{r}}_h A \subseteq A$.*

Proof. By Lemma 2.4, $\chi\colon H \to H$ is a bijection and indeed $\chi^{-1} = \chi$.

Suppose that for every $h \in H$, $\mathrm{ad}^{\mathrm{l}}_h A \subseteq A$. Then for every $a \in A$ and $h \in H$, let $h' = \chi(h)$ so that $h = \chi(h')$ and

$$\begin{aligned}
\mathrm{ad}^{\mathrm{r}}_h(a) &= \sum \chi(h_{(1)})ah_{(2)} \\
&= \sum \chi(\chi(h')_{(1)})a\chi(h')_{(2)} \\
&= \sum \chi(\chi(h'_{(2)}))a\chi(h'_{(1)}) \\
&= \sum h'_{(2)}a\chi(h'_{(1)}) \\
&= \sum h'_{(1)}a\chi(h'_{(2)}) = \mathrm{ad}^{\mathrm{l}}_{h'}(a) \in A,
\end{aligned}$$

where we have used cocommutativity in the last step. Therefore

$$\forall\, h \in H,\ \mathrm{ad}^{\mathrm{l}}_h A \subseteq A \quad \Longrightarrow \quad \forall\, h \in H,\ \mathrm{ad}^{\mathrm{r}}_h A \subseteq A.$$

Similarly,

$$\forall\, h \in H,\ \mathrm{ad}^{\mathrm{r}}_h A \subseteq A \quad \Longrightarrow \quad \forall\, h \in H,\ \mathrm{ad}^{\mathrm{l}}_h A \subseteq A.$$ □

Proposition 4.5. *Let $K \subseteq H$ be a subHopf algebra.*
(a) *If K is* ad-*invariant then $HK^+ = K^+H$ and this is a Hopf ideal. Futhermore the quotient mapping $H \to H/HK^+$ is a homomorphism of Hopf algebras.*
(b) *If $HK^+ = K^+H$ and H is faithfully flat as a left or right K-module then K is* ad-*invariant.*
(c) *If H is finite dimensional then K is* ad-*invariant if and only if $HK^+ = K^+H$.*

Proof. Proofs can be found in [Mon93, Rad12].

Since free modules are faithfully flat, part (c) follows from the Nichols-Zoeller Theorem 6.1 that we will meet later. □

This result leads us to define a subHopf algebra $K \subseteq H$ to be *normal* if it is ad-invariant and therefore $HK^+ = K^+H$ is a Hopf ideal in H and $H \to H/HK^+$ is a homomorphism of Hopf algebras. Following Milnor & Moore [MM65] it is common to write

$$H/\!/K = H/HK^+ \cong H \otimes_K \Bbbk,$$

where the right hand term is defined using the right K-module structure on H and the counit $K \to \Bbbk$ to define the trivial K-module, and thie isomorphism is one of left H-modules.

Example 4.6. If G is a group then the adjoint actions in $\Bbbk G$ are given by $\mathrm{ad}^{\mathrm{l}}_g = g(-)g^{-1}$ and $\mathrm{ad}^{\mathrm{r}}_g = g^{-1}(-)g$ for $g \in G \subseteq \Bbbk G$, so $\mathrm{ad}^{\mathrm{r}}_g = \mathrm{ad}^{\mathrm{l}}_{g^{-1}}$. Hence a subalgebra $A \subseteq \Bbbk G$ is ad-invariant if and only if for all $g \in G$, $\mathrm{ad}^{\mathrm{l}}_g A = A$.

If $N \lhd G$, then $\Bbbk N \subseteq \Bbbk G$ is a normal subHopf algebra and $\Bbbk G/\!/\Bbbk N \cong \Bbbk G/N$, the group algebra of the quotient group G/N.

Modules over an algebra. Algebras are rings with additional structure, so they have modules; in fact a module over a $\Bbbk$-algebra is automatically a $\Bbbk$-vector space.

Definition 5.1. Given a $\Bbbk$-algebra (A, φ, η), a *left A-module* (M, μ) is a $\Bbbk$-vector space M and a $\Bbbk$-linear map $\mu\colon A \otimes M \to M$ for which the following diagrams commute.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "\mathrm{Id}\otimes\mu"] \arrow[d, "\varphi\otimes\mathrm{Id}"'] & A \otimes M \arrow[d, "\mu"] \\ A \otimes M \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} \Bbbk \otimes M \arrow[r, "\cong"] \arrow[d, "\eta\otimes\mathrm{Id}"'] & M \\ A \otimes M \arrow[ur, "\mu"'] & \end{tikzcd}$$

A similar definition applies to a *right A-module*, but we can also view it as a *left* module over the opposite algebra A^{op}. The action of the algebra for a right module can be thought of either as a map $A^{\mathrm{op}} \otimes M \to M$ or as a map $M \otimes A \to M$.

An *A-module homomorphism* $\theta\colon (M, \mu) \to (M', \mu')$ is a $\Bbbk$-linear mapping $\theta : M \to M'$ that makes the following diagram commute.

$$\begin{tikzcd} A \otimes M \arrow[r, "\mathrm{Id}\otimes\theta"] \arrow[d, "\mu"'] & A \otimes M' \arrow[d, "\mu'"] \\ M \arrow[r, "\theta"] & M' \end{tikzcd}$$

Of course a homomorphism has a kernel, an image and a cokernel, all of which are A-modules. Furthermore, the set of all homomorphisms $M \to N$ between two A-modules is a subspace $\mathrm{Hom}_A(M, M') \subseteq \hom(M, M')$.

Example 5.2. Recall Example 1.1. A left module M over the enveloping algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ is sometimes called a *A-A-bimodule* because it is simultaneously a left and a right A-module and the two actions commute: if $a', a'' \in A$ and $m \in M$, then

$$(a'm)a'' = a'(ma'').$$

An important example of such a module is A itself acted on by A through left and right multiplication. This gives rise to an algebra homomorphism $A^{\mathrm{e}} \to \mathrm{End}_{\Bbbk}(A)$. When A is finite dimensional this need not be injective, but if $A^{\mathrm{e}} \cong \mathrm{End}_{\Bbbk}(A)$ then A is called an *Azumaya algebra*. Examples include matrix rings of central simple algebras over $\Bbbk$ and they give rise to the *Brauer group* of the field which appears in Galois Theory and Class Field Theory.

The multiplication map

$$A^{e} = A \otimes A^{\mathrm{op}} \to A; \quad x \otimes y^{\mathrm{op}} \mapsto xy$$

is a surjective homomorphism of A^{e}-modules. If A is a projective A^{e}-module (or equivalently if this is a split surjection) then A is called *separable*. For the case where A is a field extension of $\Bbbk$ this is equivalent to the notion of separability met in Galois Theory.

For a vector space W, the tensor product $A \otimes W$ becomes a left A-module where the composition

$$\begin{tikzcd} A \otimes (A \otimes W) \arrow[r, leftrightarrow, "\cong"'] \arrow[rr, bend left] & (A \otimes A) \otimes W \arrow[r, "\varphi"'] & A \otimes W \end{tikzcd}$$

is the multiplication; $A \otimes W$ is called an *extended* A-module. The set of A-module homomorphisms $\theta\colon (M, \mu) \to (M', \mu')$ is a vector subspace $\mathrm{Hom}_A(M, M') \subseteq \hom(M, M')$. There is an adjunction isomorphism

$$\mathrm{Hom}_A(A \otimes W, M) \cong \hom(W, M) \tag{5.1}$$

under which $\theta \in \mathrm{Hom}_A(A \otimes W, M)$ corresponds to

$$\begin{tikzcd} W \arrow[r, leftrightarrow, "\cong"'] \arrow[rrr, bend left] & \Bbbk \otimes W \arrow[r, "\eta\otimes\mathrm{Id}"'] & A \otimes W \arrow[r, "\theta"'] & M \end{tikzcd}$$

and $f \in \mathrm{hom}(W, M)$ corresponds to $A \otimes W \to M$ given on basic tensors by

$$a \otimes w \mapsto af(w).$$

For any $\Bbbk$-vector space, the vector space $\mathrm{hom}(A, W)$ becomes a left A-module with the multiplication of $a \in A$ and $f \in \mathrm{hom}(A, W)$ given by

$$(af)(x) = f(xa).$$

Notice that if $b \in A$,

$$(a(bf))(x) = (bf)(xa) = f((xa)b) = f(x(ab)) = ((ab)f)(x),$$

so $a(bf) = (ab)f$ as required.

This A-module fits into another important adjunction. For any A-module L, there is an isomorphism

$$\mathrm{hom}(L, W) \xrightarrow{\cong} \mathrm{Hom}_A(L, \mathrm{hom}(A, W)); \quad f \longmapsto (a \mapsto af(-)). \tag{5.2}$$

The inverse sends $g \in \mathrm{Hom}_A(L, \mathrm{hom}(A, W))$ to the composition

$$L \xrightarrow{g} \mathrm{hom}(A, W) \xrightarrow{\eta^*} \mathrm{hom}(\Bbbk, W) \xrightarrow{\cong} W$$

induced by the unit $\eta\colon \Bbbk \to A$.

Lemma 5.3. *Let A be a $\Bbbk$-algebra.*
(a) *For any $\Bbbk$-vector space W, the extended A-module $A \otimes W$ is a free module.*
(b) *For any A-module M, let M_0 denote its underlying vector space. Then there is a surjective A-module homomorphism $A \otimes M_0 \to M$.*
(c) *If P is a projective A-module, then there is an isomorphism of A-modules $A \otimes M_0 \cong P \oplus Q$ where Q is another projective module.*
(d) *For any $\Bbbk$-vector space W, $\mathrm{hom}(A, W)$ is an injective A-module.*
(e) *If I is an injective A-module then there is an isomorphism of A-modules $\mathrm{hom}(A, I_0) \cong I \oplus J$ where J is also an injective module.*

Proof. (a) Choose a basis of W and use it to give a basis for the A-module $A \otimes W$.
(b) Use the isomorphism (5.1).
(c) This is a standard argument: use (b) and projectivity.
(d) Suppose that we have a diagram of A-modules with exact row

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] \arrow[d] & V \\ & \mathrm{hom}(A, W) & \end{tikzcd}$$

Now apply $\mathrm{Hom}_A(-, \mathrm{hom}(A, W))$ to the row to obtain a commutative diagram where we use (5.2) to get the vertical isomorphisms.

$$\begin{tikzcd} \mathrm{Hom}_A(U, \mathrm{hom}(A, W)) & \mathrm{Hom}_A(V, \mathrm{hom}(A, W)) \arrow[l] \\ \mathrm{hom}(U, W) \arrow[u, leftrightarrow, "\cong"'] & \mathrm{hom}(V, W) \arrow[l] \arrow[u, leftrightarrow, "\cong"'] \end{tikzcd}$$

But the original $\Bbbk$-linear map $U \to V$ is split injection, so the linear map in the bottom row is surjective, hence so is the one in the top. It follows that the original diagram of A-modules can be extended with the dotted arrow to

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r] \arrow[d] & V \arrow[dl, dotted] \\
& \hom(A, W) &
\end{tikzcd}$$

and so $\hom(A, W)$ is injective.
(e) This is proved in a similar way to (c) using (5.2). $\square$

Now we can summarise all of this in categorical language.

Theorem 5.4. *There is an abelian category* $\mathbf{Mod}_A$ *whose objects are the left* A*-modules and whose morphisms are given by* $\mathbf{Mod}_A(M, N) = \mathrm{Hom}_A(M, N)$*. The usual* $\oplus = \times$ *is the coproduct and product; more generally, in this category arbitrary coproducts and coproducts exist. This category has enough projectives and injectives.*

Comodules over a coalgebra. Dually, a coalgebra has comodules.

Definition 5.5. Given a $\Bbbk$-coalgebra (C, ψ, ε), a *left comodule* (N, ν) is a $\Bbbk$-vector space N and a k-linear map $\nu \colon N \to C \otimes N$ called the *coaction* or *comultiplication* which makes the following diagrams commute.

$$\begin{tikzcd}
C \otimes C \otimes N & C \otimes N \arrow[l, "\mathrm{Id} \otimes \nu"'] \\
C \otimes N \arrow[u, "\psi \otimes \mathrm{Id}"] & N \arrow[l, "\nu"] \arrow[u, "\nu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes N & N \arrow[l, "\cong"'] \arrow[dl, "\nu"] \\
C \otimes N \arrow[u, "\varepsilon \otimes \mathrm{Id}"] &
\end{tikzcd}$$

A *right* C*-comodule* is the same thing as a left comodule over the opposite coalgebra C^{op}.

Sweedler notation is often used for the coproduct of a comodule, one version is

$$\nu(n) = \sum n_{(1)} \otimes n_{(0)}$$

where $n_{(1)} \in C$ and $n_{(0)} \in N$, so the index (0) is reserved for elements in the comodule.

A C*-comodule homomorphism* $\rho \colon (N, \nu) \to (N', \nu')$ is a $\Bbbk$-linear mapping $\theta : N \to N'$ that makes the following diagram commute.

$$\begin{tikzcd}
C \otimes N \arrow[r, "\mathrm{Id} \otimes \rho"] & C \otimes N' \\
N \arrow[u, "\nu"] \arrow[r, "\rho"] & N' \arrow[u, "\nu'"']
\end{tikzcd}$$

It is easy to see that the image and the cokernel of a homomorphism ρ are comodules. To see that kernels exist, let $\rho \colon N \to N'$ be a C-comodule homomorphism. As a linear mapping ρ has a kernel and there is an exact sequence of linear mappings

$$0 \longrightarrow \ker \rho \longrightarrow N \xrightarrow{\rho} N'$$

and we can extend this to a commutative diagram of solid arrows

$$\begin{tikzcd}
0 \arrow[r] & \ker \rho \arrow[r, "\mathrm{inc}"] \arrow[d, dotted] & N \arrow[r, "\rho"] \arrow[d, "\nu"'] & N' \arrow[d, "\nu'"'] \\
0 \arrow[r] & C \otimes \ker \rho \arrow[r, "\mathrm{Id} \otimes \mathrm{inc}"] & C \otimes N \arrow[r, "\rho"] & C \otimes N'
\end{tikzcd}$$

in which the bottom row is exact because tensoring over a field is an exact functor. Now a diagram chase shows that $\nu \circ \mathrm{inc}$ factors through $C \otimes \ker \rho$ hence we can fill in the dotted arrow and more diagram chasing shows that it is a comultiplication making $\ker \rho$ a comodule and a kernel for ρ.

For a vector space W, the tensor product $C \otimes W$ becomes a left C-comodule where the composition

$$\begin{tikzcd} C \otimes W \arrow[r, "\psi"'] \arrow[rr, bend left] & (C \otimes C) \otimes W \arrow[r, leftrightarrow, "\cong"'] & C \otimes (C \otimes W) \end{tikzcd}$$

is the comultiplication; $C \otimes W$ is called an *extended* C-comodule. The set of C-module homomorphisms $\rho\colon (N,\nu) \to (N',\nu')$ is a vector subspace $\mathrm{Cohom}_C(N,N') \subseteq \hom(N,N')$. There is an adjunction isomorphism

$$\mathrm{Cohom}_C(N, C \otimes W) \cong \hom(N,W) \tag{5.3}$$

under which $\rho \in \mathrm{Cohom}_C(N, C \otimes W)$ corresponds to

$$\begin{tikzcd} N \arrow[r, "\rho"'] \arrow[rrr, bend left] & C \otimes W \arrow[r, "\varepsilon \otimes \mathrm{Id}"'] & \Bbbk \otimes W \arrow[r, leftrightarrow, "\cong"'] & W \end{tikzcd}$$

and $g \in \hom(N,W)$ corresponds to the following composition.

$$\begin{tikzcd} N \arrow[r, "\nu"'] \arrow[rr, bend left] & C \otimes N \arrow[r, "\mathrm{Id} \otimes g"'] & C \otimes W \end{tikzcd}$$

Now we will show that the extended comodule $C \otimes W$ is an injective comodule. Suppose given the following commutative diagram of comodule homomorphisms with an exact row.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\rho"] \arrow[d] & N' \\ & C \otimes W & \end{tikzcd}$$

Applying $\mathrm{Cohom}_C(-, C \otimes W) \cong \hom(-,W)$ to the row we get a diagram of vector spaces

$$\begin{tikzcd} 0 & \mathrm{Cohom}_C(N, C \otimes W) \arrow[l] \arrow[d, leftrightarrow, "\cong"'] & \mathrm{Cohom}_C(N', C \otimes W) \arrow[l, "\rho^*"'] \arrow[d, leftrightarrow, "\cong"'] \\ 0 & \hom(N,W) \arrow[l] & \hom(N',W) \arrow[l, "\rho^*"'] \end{tikzcd}$$

with exact bottom row. Therefore the top row is exact so we can fill in the dotted arrow with a comodule homomorphism.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\rho"] \arrow[d] & N' \arrow[dl, dotted] \\ & C \otimes W & \end{tikzcd}$$

This shows that $C \otimes W$ is injective and any summand of such a comodule is as well.

Now for any comodule N, we can also view N as just a vector space. Using the isomorphism (5.3) we obtain $\mathrm{Cohom}_C(N, C \otimes N) \cong \hom(N,N)$ and $\mathrm{Id}_N \in \hom(N,N)$ corresponds to a comodule homomorphism $N \to C \otimes N$ and the commutative diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[ddr, equal, bend right] & C \otimes N \arrow[d, "\varepsilon\otimes\mathrm{Id}"] \\
& \Bbbk \otimes N \arrow[d, leftrightarrow, "\cong"] \\
& N
\end{tikzcd}$$

shows that it is injective, so N embeds into the extended comodule $C \otimes N$ which is an injective comodule. It follows that every injective comodule J is a summand of the extended comodule $C \otimes J$.

We can summarise this information in a statement about the category of comodules.

Theorem 5.6. *For a $\Bbbk$-coalgebra C, its comodules and comodule homomorphisms form an abelian category $\mathbf{Comod}_C$ with enough injectives. This category has $\oplus$ as coproduct and product.*

If C is finite dimensional then $\mathbf{Comod}_C$ also has enough projectives.

In general the comodule category of a coalgebra may not have enough projectives, although in many cases it does. This asymmetry leads to slight differences in their homological algebra compared to that of algebras. The finite dimensional case can be verified using ideas in the discussion that follows, see Proposition 5.13.

Now recall that a coalgebra C has an associated algebra C^*. A left C-module has an action $\nu^\dagger \colon C^* \otimes N \to N$ defined by

$$\gamma n = \nu^\dagger(\gamma \otimes n) = \sum \gamma(n_{(1)}) n_{(0)},$$

where of course $\gamma(n_{(1)}) \in \Bbbk$. If $\alpha, \beta \in C^*$,

$$\begin{aligned}
\alpha(\beta n) &= \sum \alpha(\beta(n_{(1)})n_{(0)}) \\
&= \sum \beta(n_{(1)})\alpha\big((n_{(0)})_{(1)}\big)(n_{(0)})_{(0)}
\end{aligned}$$

while

$$\begin{aligned}
(\alpha\beta)n &= \sum (\alpha\beta)(n_{(1)})n_{(0)}) \\
&= \sum \alpha\big((n_{(1)})_{(0)}\big)\beta\big((n_{(1)})_{(1)}\big)n_{(0)} = \sum \beta\big((n_{(1)})_{(1)}\big)\alpha\big((n_{(1)})_{(0)}\big)n_{(0)} \\
&= \sum \beta(n_{(1)}\alpha\big((n_{(0)})_{(1)}\big)(n_{(0)})_{(1)},
\end{aligned}$$

so $\alpha(\beta n) = (\alpha\beta)n$. Another argument shows that $\varepsilon^* n = n$. So with this multiplication, N becomes a left C^*-module.

Definition 5.7. Let A be a $\Bbbk$-algebra and M a left A-module. Then M is *locally finite* if every element $m \in M$ is contained in a submodule which is a finite dimensional subspace.

In particular this means that for each $m \in M$, the *cyclic submodule*

$$Am = \{am : a \in A\} \subseteq M$$

is a finite dimensional subspace. The locally finite A-modules form a full abelian subcategory $\mathbf{Mod}_A^{\mathrm{l.f.}}$ of the full category $\mathbf{Mod}_A$ of all A-modules.

Definition 5.8. Let C be a $\Bbbk$-coalgebra and N a left C-comodule. Then N is *locally finite* if every element $m \in M$ is contained in a subcomodule which is a finite dimensional subspace.

In fact this notion is redundant!

Lemma 5.9. *Let C be a coalgebra. Then every C-comodule is locally finite.*

Proof. Let N be a C-comodule. The idea of the proof is that for $n \in N$, the coproduct

$$\nu(n) = \sum n_{(1)} \otimes n_{(0)}$$

gives rise to a finite dimensional subspace spanned by the elements $n_{(0)} \in N$. Now using coassociativity of ν, this can be shown to be a subcomodule. □

Lemma 5.10. *Let C be a coalgebra and C^* its dual algebra. Let N be a left C-comodule which we also view as a left C^*-module. Then N is a locally finite C^*-module.*

Proof. It is sufficient to show that for $n \in N$, the cyclic submodule $C^*n \subseteq N$ is finite dimensional. Lemma 5.9 tells us that n is contained in a finite dimensional subcomodule $W \subseteq N$ and by definition of the action of C^* on n, $C^*n \subseteq W$. □

Dualising from an algebra to a coalgebra is more problematic unless the finite dual is used. Details can be found in Montgomery [Mon93] or Radford [Rad12]. We summarise the main results.

Lemma 5.11. *Let A be an algebra and A° its finite dual coalgebra. Let M be a locally finite left A-comodule. Then M can be given the structure of a left A°-comodule.*

Proposition 5.12. *There is an isomorphism of abelian categories*

$$\mathbf{Mod}_A^{\mathrm{l.f.}} \xrightarrow{\sim} \mathbf{Comod}_{A^\circ}.$$

Of course when A is finite dimensional, $A^\circ = A^*$, and locally finite is equivalent to every element being in a finitely generated submodule. If we restrict attention to finite dimensional modules and comodules we obtain an important related result.

Proposition 5.13. *There is an isomorphism of abelian categories*

$$\mathbf{Mod}_A^{\mathrm{f.d.}} \xrightarrow{\sim} \mathbf{Comod}_{A^*}^{\mathrm{f.d.}}.$$

In particular, projective/injective modules correspond to projective/injective comodules.

Of course the finite dimensional projective A-modules are summands of direct sums of copies of A. Also A is an A^*-comodule through the adjunction

$$\hom(A \otimes A, A) \cong \hom(A, A^* \otimes A).$$

under which the product correspond to a coaction $A \to A^* \otimes A$ making it an A^*-comodule, and in fact it is projective.

Tensor and cotensor products. Suppose that A is an algebra, (M, μ) is a right A-module and (N, ν) is a left A-module. One definition of the *tensor product* $M \otimes_A N$ makes it the cokernel of the $\Bbbk$-linear mapping $(\mu \otimes \mathrm{Id} - \mathrm{Id} \otimes \nu)\colon M \otimes A \otimes N \to M \otimes N$. In other words there is an exact sequence

$$M \otimes A \otimes N \xrightarrow{\mu\otimes\mathrm{Id} - \mathrm{Id}\otimes\nu} M \otimes N \longrightarrow M \otimes_A N \longrightarrow 0 \tag{5.4}$$

of $\Bbbk$-linear mappings. Unless A is commutative this is not an A-module. We also have formulae such as

$$A \otimes_A N \cong N, \quad M \otimes_A A \cong M.$$

Notice also that if $B \subseteq A$ is a subalgebra we can also define $M \otimes_B N$ and there is a linear surjection $M \otimes_B N \to M \otimes_A N$.

Proposition 5.14. *For a fixed right A-module M, the functor $M \otimes_A (-)\colon \mathbf{Mod}_A \to \mathbf{Vect}_{\Bbbk}$ is right exact, i.e., it sends every short exact sequence of left A-modules*

$$0 \to N' \to N \to N'' \to 0$$

to an exact sequence

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

The left derived functors of $M \otimes_A (-)$ are denoted $\mathrm{Tor}^A_*(M, -)$; these can be computed using projective resolutions.

Remark 5.15. A left A-module P is called *flat* if for every right A-module M, $\mathrm{Tor}^A_s(M, P) = 0$ for $s > 0$. In fact we can calculate $\mathrm{Tor}^A_*(M, N)$ by using any *flat resolution* $P_\bullet \to N \to 0$, i.e., a resolution consisting of flat modules P_s. Then $\mathrm{Tor}^A_*(M, N)$ is the homology of the chain complex $N \otimes_A P_\bullet$. Free and projective modules are flat, and so are colimits of flat modules.

Now let's dualise to comodules. Suppose that C is a coalgebra, (M, μ) is a right C-comodule and (N, ν) is a left C-comodule. We define the cotensor product $M \square_C N$ as the kernel of $(\mu \otimes \mathrm{Id} - \mathrm{Id} \otimes \nu)\colon M \otimes N \to M \otimes C \otimes N$, so there is an exact sequence

$$0 \longrightarrow M \square_C N \longrightarrow M \otimes N \xrightarrow{\mu\otimes\mathrm{Id} - \mathrm{Id}\otimes\nu} M \otimes C \otimes N \tag{5.5}$$

and $M \square_C N$ is only a C-comodule if C is cocommutative. We have

$$C \square_C N \cong N, \quad M \square_C C \cong M.$$

A surjection of coalgebras $C \to D$ induces an injective linear mapping $M \square_C N \to M \square_D N$.

Proposition 5.16. *For a fixed right C-comodule M, the functor $M \square_C (-)\colon \mathbf{Comod}_A \to \mathbf{Vect}_{\Bbbk}$ is left exact, i.e., it sends every short exact sequence of left C-comodules*

$$0 \to N' \to N \to N'' \to 0$$

to an exact sequence

$$0 \to M \square_C N' \to M \square_C N \to M \square_C N''.$$

The right derived functors of $M \square_C (-)$ are denoted $\mathrm{Cotor}^*_C(M, -)$.

Modules over a Hopf algebra. For a Hopf algebra we have both modules and comodules. We will focus on (left) modules but similar things apply to comodules.

From now on, let $(H, \varphi, \eta, \psi, \varepsilon, \chi)$ be a Hopf algebra which we will assume has an invertible antipode; this condition holds if the Hopf algebra is classical since then $\chi \circ \chi = \mathrm{Id}$. We will often indicate the multiplication in a module (M, μ) by writing $hx = \mu(h \otimes x)$.

There are two obvious left modules. First we can let H act on itself by left multiplication, so for μ we just take φ. This is sometimes called the *left regular representation* of H; this module is free of rank 1. At the other extreme we can let H act on $\Bbbk$ using the counit ε, so μ is the map

$$H \otimes \Bbbk \to \Bbbk; \quad h \otimes 1 \mapsto \varepsilon(h).$$

In fact for any vector space W we can let H act on W by

$$H \otimes W \to W; \quad h \otimes w \mapsto \varepsilon(h)w.$$

Such representations are called *trivial representations*, and the one with $W = \Bbbk$ is often called *the* trivial representation and it is *simple* or *irreducible*.

Now we come to an important property of the category of modules over a Hopf algebra: it forms a closed monoidal category. Let (M_1, μ_1) and (M_2, μ_2) be two left H-modules. Their tensor product $M_1 \otimes M_2$ is a $\Bbbk$-vector space which also admits a multiplication $\widetilde{\mu}$ which is defined to make the diagram

$$\begin{tikzcd}
H \otimes (M_1 \otimes M_2) \arrow[rr, "\widetilde{\mu}"] \arrow[d, "\psi\otimes\mathrm{Id}\otimes\mathrm{Id}"'] & & M_1 \otimes M_2 \\
(H \otimes H) \otimes (M_1 \otimes M_2) \arrow[rr, "\mathrm{Id}\otimes\mathrm{T}\otimes\mathrm{Id}", "\cong"'] & & (H \otimes M_1) \otimes (H \otimes M_2) \arrow[u, "\mu_1\otimes\mu_2"']
\end{tikzcd}$$

commute and making it an H-module $(M_1 \otimes M_2, \widetilde{\mu})$. Using Sweedler notation we can write this explicitly as

$$\widetilde{\mu}(h \otimes m_1 \otimes m_2) = \sum h_{(1)} m_1 \otimes h_{(2)} m_2.$$

If H is cocommutative then the switch map

$$M_1 \otimes M_2 \xrightarrow[\cong]{\mathrm{T}} M_2 \otimes M_1.$$

is an isomorphism of H-modules, but when H is *not* cocommutative this is not usually true.

If W is any vector space with the trivial H-module structure, there is an isomorphism of H-modules

$$W \otimes M \cong M \otimes W.$$

In particular,

$$\Bbbk \otimes M \cong M \cong M \otimes \Bbbk.$$

For any H-module M we can consider the subspace of *H-invariants*

$$M^H = \{x \in M : \forall h \in H,\ hx = \varepsilon(h)x\} \subseteq M.$$

What about $M^* = \hom(M, \Bbbk)$? There is a natural *right* H-module structure on this given by taking for $h \in H$ and $f \in M^*$,

$$(f \cdot h)(x) = f(hx).$$

We can twist this into a *left* action by defining

$$(h \cdot f)(x) = f(\chi(h)x).$$

More generally, for two H-modules M, N, $\hom(M, N)$ becomes a module with the action given in Sweedler notation by

$$(h \cdot g)(x) = \sum h_{(1)} g(\chi(h_{(2)})x).$$

By an interesting calculation, the subspace of H-invariants of $\hom(M, N)$ turns out to be

$$\hom(M, N)^H = \mathrm{Hom}_H(M, N). \tag{5.6}$$

In particular,

$$M^H \cong \{f(1) \in M : f \in \hom(\Bbbk, M)^H\} \cong \mathrm{Hom}_H(\Bbbk, M).$$

In fact taking invariants gives a functor $(-)^H \colon \mathbf{Mod}_H \to \mathbf{Vect}_\Bbbk$ which is *left exact*, i.e., it sends every short exact sequence

$$0 \to L \to M \to N \to 0$$

to an exact sequence

$$0 \to L^H \to M^H \to N^H.$$

This means it has right derived functors denoted by $\mathrm{Ext}^*_H(\Bbbk, -)$ and also called the *cohomology of H with coefficients in M*. When $H = \Bbbk G$ is a group algebra this is the cohomology of G.

We can also define the *H-coinvariants* of an H-module M to be

$$M_H = M/\{hm - \varepsilon(h)m : h \in H,\ m \in M\}$$

This can be shown to be isomorphic to the tensor product $\Bbbk \otimes_H M$ where we view $\Bbbk$ as a right H-module. Taking coinvariants gives a functor $(-)_H\colon \mathbf{Mod}_H \to \mathbf{Vect}_{\Bbbk}$ which is *right exact*, i.e., it sends every short exact sequence

$$0 \to L \to M \to N \to 0$$

to an exact sequence

$$L_H \to M_H \to N_H \to 0.$$

The left derived functors are $\operatorname{Tor}^H_*(\Bbbk, -)$ and $\operatorname{Tor}^H_*(\Bbbk, M)$ is also known as the *homology of H with coefficients in M*. When $H = \Bbbk G$ for a group G, this is the homology of G.

When M and N are two left H-modules,

$$(M \otimes N)_H \cong \Bbbk \otimes_H (M \otimes N)$$

is also isomorphic to the quotient $(M \otimes N)/T$ where T is the subspace spanned by the elements

$$hm \otimes n - m \otimes hn \quad (h \in H,\, m \in M,\, n \in N).$$

As a special case of this, suppose that L is a right H-module; we can make this into a left H-module by defining the action to be

$$h \cdot \ell = \ell\chi(h).$$

Then with this left H-module L and a left H-module N,

$$(L \otimes N)_H \cong L \otimes_H N$$

where the latter is the right-left tensor product over H.

We can assemble all of these ideas into an important categorical result which we will make use of later.

Theorem 5.17. *The category of left H-modules $\mathbf{Mod}_H$ under $\otimes$ and* $\hom(-,-)$ *is closed monoidal. So for H-modules L, M, N there is a functorial adjunction isomorphism*

$$\mathbf{Mod}_H(L \otimes M, N) \xleftrightarrow{\cong} \mathbf{Mod}_H(L, \hom(M, N)). \tag{5.7}$$

If H is cocommutative $\mathbf{Mod}_H$ is symmetric monoidal.

If M is finite dimensional then (5.7) *gives rise to a functorial isomorphism*

$$\mathbf{Mod}_H(L \otimes M, N) \xleftrightarrow{\cong} \mathbf{Mod}_H(L, N \otimes M^*). \tag{5.8}$$

We will return to the issue of the lack of symmetry for non-cocommutative Hopf algebras when we discuss *quantum groups*. We mention one general observation that shows care is need in such situations.

Suppose that M is a finite dimensional H-module. Then the dual space $M^* = \hom(M, \Bbbk)$ and the double dual space $M^{**} = (M^*)^* = \hom(M^*, \Bbbk)$ admit left H-modules structures as described above.

Lemma 5.18. *The canonical linear isomorphism $M \to M^{**}$ need not be an isomorphism of H-modules, but does induce an isomorphism of H-modules*

$$(\chi^2)^*M \xrightarrow{\cong} M^{**},$$

where $(\chi^2)^*M$ is the vector space M given the H-module structure with

$$h \cdot m = \chi^2(h)m.$$

Of course if H is involutary $(\chi^2)^*M = M$ but in general these need not even be isomorphic H-modules.

If W is a vector space which we view as a trivial H-module, then $H \otimes W$ is a left H-module with action on basic tensors

$$h(k \otimes w) = (hk) \otimes w;$$

this is often called an *extended module on* W. More generally, if $K \subseteq H$ is a subalgebra then for a left K-module N there is an *induced* H*-module* $H \otimes_K N$ where the tensor product is formed using right K-module structure on H. There is also the coinduced module $\mathrm{Hom}_K(H, N)$ where the left H-multiplication is induced by *right* multiplication on the codomain.

If M is an H-module it is useful to forget its module structure and take its underlying vector space with the trivial H-module structure which we will denote ${}_\varepsilon M$. The next result is really important and useful when doing homological algebra over a Hopf algebra.

Proposition 5.19. *Suppose that H is a Hopf algebra. For a left H-module M there are isomorphisms of left H-modules*

$$H \otimes M \xleftrightarrow{\cong} H \otimes {}_\varepsilon M \xleftrightarrow{\cong} M \otimes H,$$

where $H \otimes {}_\varepsilon M$ is the extended module for the vector space M. Hence $H \otimes M$ is a free H-module.

Proof. The following $\Bbbk$-linear maps are inverse H-module maps:

$$H \otimes M \to H \otimes M; \quad h \otimes x \mapsto \sum h_{(1)} \otimes \chi(h_{(2)})x,$$
$$H \otimes {}_\varepsilon M \to H \otimes M; \quad h \otimes x \mapsto \sum h_{(1)} \otimes h_{(2)}x.$$

A similar argument works for $M \otimes H$. □

If H is finite dimensional it is also true that for any H-module M, there is an isomorphism of H-modules

$$\hom(H, M) \cong H^* \otimes {}_\varepsilon M$$

where $H^* = \hom(H, \Bbbk)$ is injective; later we will see that $H^* \cong H$ so $\hom(H, M)$ is also a free H-module.

Representations of finite groups. A representation of a finite group G over $\Bbbk$ is equivalent to a $\Bbbk G$-module. For a vector space V, the induced module $V \uparrow_1^G = \Bbbk G \otimes V$ is free and for a $\Bbbk G$-module M it is well known that $\Bbbk G \otimes M \cong M \uparrow_1^G$.

If M, N are two $\Bbbk G$-modules then so is $M \otimes N$ with $g \in G$ acting on basic tensors by

$$g \cdot (m \otimes n) = gm \otimes gn.$$

Similarly, $M^* = \hom(M, \Bbbk)$ is a $\Bbbk G$-module with action of $g \in G$ on $f \in M^*$ given by

$$(g \cdot f)(m) = f(g^{-1}m) \quad (m \in M).$$

This is sometimes called the *dual* or *contragredient* module of M.

If $H \leqslant G$ then $\Bbbk H \subseteq \Bbbk G$ is a subHopf algebra and for any $\Bbbk H$-module L, there is an induced module $L \uparrow_H^G$; in particular,

$$\Bbbk G/H \cong \Bbbk G \otimes_{\Bbbk H} \Bbbk.$$

If M is a $\Bbbk G$-module we can view it as a $\Bbbk H$-module and then as $\Bbbk G$-modules,

$$\Bbbk G \otimes_{\Bbbk H} M \cong \Bbbk G/H \otimes M.$$

In fact the only subHopf algebras of $\Bbbk G$ are the $\Bbbk H$. If $N \lhd G$ the $\Bbbk G/N$ is a quotient Hopf algebra of $\Bbbk G$.

In the representation theory of a finite group it is well known that the tensor product of two G-modules M and N is a G-module $M \otimes N$ with the action of $g \in G$ on basic tensors given by

$$g(x \otimes y) = gx \otimes gy;$$

this of course is equivalent to the Hopf algebra definition since in $\Bbbk G$ the coproduct on an element $g \in G \subseteq \Bbbk G$ is given by $\psi(g) = g \otimes g$.

Hopf module algebras and coalgebras. Hopf algebras often act or coact on other things such as algebras and coalgebras. Let $(H, \varphi, \eta, \psi, \varepsilon)$ be a Hopf algebra.

Definition 5.20. An *H-module algebra* is a $\Bbbk$-algebra (A, φ_A, η_A) which is an H-module with multiplication denoted by $h \cdot a$ for $h \in H$ and $a \in A$, which satisfies

$$h \cdot (ab) = \sum_i (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad h \cdot 1 = \varepsilon(h) \qquad (h \in H,\ a, b \in A).$$

An *H-module coalgebra* is a $\Bbbk$-coalgebra $(C, \psi_C, \varepsilon_C)$ which is an H-module with multiplication denoted by $h \cdot a$ for $h \in H$ and $a \in A$, which satisfies

$$\psi_C(h \cdot c) = \sum \sum (h_{(1)} \cdot c_{(1)}) \otimes (h_{(2)} \cdot c_{(2)}), \quad \varepsilon(h \cdot 1) = \varepsilon(h), \qquad (h \in H,\ c \in C).$$

An *H-module bialgebra/Hopf algebra* is a bialgebra/Hopf algebra that is both an H-module algebra and a H-module coalgebra.

Example 5.21. An important example is provided by the left adjoint action of H on itself: for $h, x \in H$,

$$h \cdot x = \mathrm{ad}^{\mathrm{l}}_h(x) = \sum h_{(1)} x \chi(h_{(2)}).$$

This makes H into an H-module Hopf algebra. To see that the product formula holds, let $a, b, h \in H$. Using a modified version of Sweedler notation where $h_{(ij)} = (h_{(i)})_{(j)}$, we have

$$\begin{aligned}
h \cdot (ab) &= \sum h_{(1)} ab \chi(h_{(2)}) \\
&= \sum h_{(11)} (\varepsilon(h_{(12)})1) ab \chi(h_{(2)}) \\
&= \sum h_{(11)} a (\varepsilon(h_{(12)})1) b \chi(h_{(2)}) \\
&= \sum h_{(11)} a \chi(h_{(121)}) h_{(122)} b \chi(h_{(2)}) \\
&= \sum h_{(11)} a \chi(h_{(12)}) h_{(21)} b \chi(h_{(22)}) \\
&= \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b),
\end{aligned}$$

where we have used coassociativity to rewrite the penultimate sum. Verifying the formula $h \cdot 1 = \varepsilon(h)1$ requires a simpler calculation.

If H is commutative then

$$h \cdot x = \sum h_{(1)} \chi(h_{(2)}) x = \varepsilon(h) x$$

so in this case the action is trivial.

If $A \subseteq H$ is a subalgebra which is closed under the left adjoint action (i.e., for all $h \in H$, $\mathrm{ad}^{\mathrm{l}}_h A \subseteq A$) then the adjoint action restricted to A makes it into an H-module subalgebra.

Example 5.22. Let $\Bbbk G$ be the group algebra of a group G and $N \lhd G$. Then for $g \in G \subseteq \Bbbk G$ and $n \in N \subseteq \Bbbk N \subseteq \Bbbk G$, the adjoint action is given by

$$g \cdot n = gng^{-1}$$

so $\Bbbk N$ is a $\Bbbk G$-module Hopf algebra. This case is very important in representation theory and cohomology of finite groups.

6. Finite dimensional Hopf algebras

Finite dimensional Hopf algebras have a rich theory, some aspects of which are generalisations of the special case of group algebras of finite groups.

The Nichols-Zoeller Theorem. Here is an important result about finite dimensional Hopf algebras. Earlier versions of this for arbitrary graded connected Hopf algebras were due to Milnor & Moore [MM65]. More general results are known, for example when K is finite dimensional.

Theorem 6.1 (Nichols & Zoeller)**.** *Let H be a finite dimensional Hopf algebra and let K be a subHopf algebra. Then when viewed as a left or right K-module, H is free. Hence*

$$\dim_{\Bbbk} H = (\dim_{\Bbbk} K)(\operatorname{rank}_K H).$$

Proof. The proof seems to require some module theory that requires work to develop. Proofs can be found in [Rad12, theorem 9.3.3] or [Mon93, chapter 3]. □

The dimension formulae is of course a generalisation of Lagrange's Theorem. For a finite group G and $H \leqslant G$, $\Bbbk G$ is a finite dimensional Hopf algebra and $\Bbbk H$ is a subHopf algebra, with $\dim_{\Bbbk} \Bbbk G = |G|$ and $\dim_{\Bbbk} \Bbbk H = |H|$. Here is a nice generalisation.

Corollary 6.2. *Let H be a finite dimensional Hopf algebra. Then the grouplike elements form a finite subgroup $\mathrm{G}(H) \leqslant H^{\times}$ and $|\mathrm{G}(H)|$ divides $\dim_{\Bbbk} H$.*

Proof. By Lemma 2.16, $\mathrm{G}(H)$ is linearly independent so it must be finite with $|\mathrm{G}(H)| \leqslant \dim_{\Bbbk} H$. In fact it spans the subHopf algebra $\Bbbk\mathrm{G}(H) \subseteq H$, so by Theorem 6.1, $|\mathrm{G}(H)| \mid \dim_{\Bbbk} H$. □

Antipodes and finite dimensionality. In general the antipode of a Hopf algebra need not be bijective. But it often is, for example when the Hopf algebra is commutative or cocommutative. Here is another important case.

Theorem 6.3. *Let H be a finite dimensional Hopf algebra. Then its antipode $\chi\colon H \to H$ is bijective.*

The proof will require a lemma which does not require H to be finite dimensional.

Lemma 6.4. *Let H be a Hopf algebra and suppose that $K = \chi H \subseteq H$. If the restriction $\chi_{|K}\colon K \to K$ is a bijection then χ is a bijection.*

Proof. The linear mapping $H \to K$ given by χ is surjective and $\chi_{|K}\colon K \to K$ is injective, so $H = \ker\chi \oplus K$ as vector spaces. Let $\pi\colon H \to H$ be projection onto the second factor; then $\ker\chi = \ker\pi$ and $\pi_{|K} = \mathrm{Id}_K$.

By Proposition 4.1, $K \subseteq H$ is a subHopf algebra and $\ker\chi$ is a Hopf ideal of H. Hence $\varepsilon \ker\pi = \{0\}$ and

$$\psi \ker\chi \subseteq \ker\chi \otimes H + H \otimes \ker\chi = \ker\pi \otimes H + H \otimes \ker\chi,$$

so for $h \in \ker\chi$ and working with the convolution in $\hom(H,H)$,

$$(\pi * \chi)(h) = 0 = \varepsilon(h)1.$$

For $k \in K$ we have

$$(\pi * \chi)(k) = \sum \pi(k_{(1)})\chi(k_{(2)}) = \sum k_{(1)}\chi(k_{(2)}) = \varepsilon(k)1.$$

It follows that π is the convolution inverse of χ, but this is Id_H. So in fact $\pi = \mathrm{Id}_H$ and χ is surjective with $\ker\chi = 0$, hence χ is a bijection. □

Proof of Theorem 6.3. We will prove this is stages using a 'downward induction' argument. See Radford [Rad12, theorem 7.1.14] for more on this.

Since $\chi\colon H \to H$ is a linear mapping and H is finite dimensional, *Fitting's Lemma* implies that for some large enough n,

$$H = \operatorname{im}\chi^n \oplus \ker\chi^n.$$

where $K = \operatorname{im}\chi^n = \chi^n H \subseteq H$ is a subHopf algebra on which the restriction of χ is injective. Since $K = \chi K = \chi(\chi^{n-1}H)$, we can apply Lemma 6.4 to the subHopf algebra $\chi^{n-1}H \subseteq H$ to deduce that χ is bijective on $\chi^{n-1}H$. Now we can repeat this argument to show that χ is bijective on each $\chi^k H$ with $1 \leqslant k \leqslant n-1$ and then show that it is bijective on H itself. □

Recall that a finite submonoid of a group is always a subgroup. A similar result holds for Hopf algebras.

Proposition 6.5. *Let H be a Hopf algebra and let $B \subseteq H$ be a subbialgebra that is finite dimensional. Then the antipode of H restricts to an antipode B, therefore B is a subHopf algebra.*

Proof. Consider the convolution monoids $\hom(B,B)$, $\hom(B,H)$ and $\hom(H,H)$; since B is a subbialgebra of H, $\hom(B,B) \subseteq \hom(B,H)$ is a submonoid and the inclusion $\mathrm{inc}_B\colon B \to H$ induces a monoid homomorphism $\mathrm{inc}_B\colon \hom(H,H) \to \hom(B,H)$. Let $\chi' = \chi \circ \mathrm{inc}_B \in \hom(B,H)$ be the restriction of the antipode $\chi\colon H \to H$ to B. The restriction of the identity Id_H to B is just the inclusion $\mathrm{inc}_B\colon B \to H$, and

$$\chi' * \mathrm{inc}_B = \mathrm{inc}_B^*(\chi * \mathrm{Id}_H) = \mathrm{inc}_B^*(1_H)$$

which is the identity in $\hom(B,H)$. So χ' is the $*$-inverse of $\mathrm{inc}_B \in \hom(B,H)$.

Now $\hom(B,B) \subseteq \hom(B,H)$ is a submonoid and $(\mathrm{inc}_B)_*(\mathrm{Id}_B) = \mathrm{inc}_B \circ \mathrm{Id}_B = \mathrm{inc}_B$. But $\hom(B,B)$ and $\hom(B,H)$ are also algebras with $\hom(B,B) \subseteq \hom(B,H)$ a subalgebra. Let $\Lambda\colon \hom(B,B) \to \hom(B,B)$ be the $\Bbbk$-linear endomorphism given by left multiplication by Id_B. Since $(\mathrm{inc}_B)_*(\mathrm{Id}_B) = \mathrm{inc}_B \in \hom(B,H)$ has a left inverse, it is injective, hence so is Λ. As B is finite dimensional so is $\hom(B,B)$ and therefore Λ must be invertible. It follows that Id_B is invertible in $\hom(B,B)$ under $*$, hence B has antipode χ_B making it a Hopf algebra. By construction, $\chi_B = \chi \circ \mathrm{inc}_B$ so B is a subHopf algebra. □

Hopf modules. Let $(H, \varphi, \eta, \psi, \varepsilon, \chi)$ be a Hopf algebra (not necessarily finite dimensional). Then H and $H \otimes H$ are both left H-modules and the coproduct $\psi\colon H \to H \otimes H$ is a module homomorphism since H is a bialgebra. Similarly, if M is a left H-module $H \otimes M$ is also an H-module.

Definition 6.6. Suppose that M is a left H-module which is also a left H-comodule (M,μ). Then (M,μ) is a (*left*) *H-Hopf module* if $\mu\colon M \to H \otimes M$ is an H-module homomorphism.

A *homomorphism of H-Hopf modules* $\theta\colon (M,\mu) \to (N,\nu)$ is a $\Bbbk$-linear mapping $\theta\colon M \to N$ which is both an H-module homomorphism and an H-comodule homomorphism.

If W is any vector space then $H \otimes W$ is both a left H-module and a left H-comodule and it is easy to check it is a Hopf module.

For a Hopf module (M,μ) we define its subspace of *coinvariants* to be

$$M_{\mathrm{coinv}} = \{m \in M : \mu(m) = 1 \otimes m\} \subseteq M.$$

This vector subspace of M can be identified with the cotensor product $\Bbbk\Box_H M$ where we view $\Bbbk$ as a right H-comodule.

Here is the main result about Hopf modules, again we do not assume finite dimensionality.

Theorem 6.7 (Fundamental Theorem of Hopf Modules)**.** *Let M be an H-Hopf module. Then there is an isomorphism of Hopf modules*

$$H \otimes M_{\mathrm{coinv}} \xrightarrow{\cong} M.$$

Hence every H-Hopf module is a free H-module.

Proof. We start by defining the linear mapping

$$\Theta\colon H \otimes M_{\mathrm{coinv}} \xrightarrow{\cong} M; \quad \Theta(h \otimes m) = hm.$$

Since M_{coinv} is just a vector space, this is a homomorphism of H-modules.

Let $h \in H$ and $m \in M_{\mathrm{coinv}}$. The coaction applied to the element $hm \in M$ gives

$$\mu(hm) = \sum h_{(1)}1 \otimes h_{(2)}m = \big(\sum h_{(1)} \otimes h_{(2)}\big)(1 \otimes m) = h\mu(m),$$

so this is a homomorphism of H-comodules.

Now for $m \in M$, let

$$\mu(m) = \sum m_{(1)} \otimes m_{(2)} \in H \otimes M.$$

Then

$$\begin{aligned}
\mu\big(\sum \chi(m_{(1)})m_{(2)}\big) &= \sum \chi(m_{(1)})\mu(m_{(2)}) \\
&= \sum \chi(m_{(1)})_{(1)}(m_{(2)})_{(1)} \otimes \chi(m_{(1)})_{(2)}(m_{(2)})_{(2)} \\
&= \sum \chi(m_{(12)})(m_{(21)}) \otimes \chi(m_{(11)})(m_{(22)}) \\
&= \sum \chi(m_{(121)})(m_{(122)}) \otimes \chi(m_{(11)})(m_{(2)}) \\
&= \sum \varepsilon(m_{(12)}) \otimes \chi(m_{(11)})(m_{(2)}) \\
&= \sum 1 \otimes \varepsilon(m_{(12)})\chi(m_{(11)})(m_{(2)}) \\
&= \sum 1 \otimes \chi(m_{(1)})(m_{(2)}),
\end{aligned}$$

hence $\sum \chi(m_{(1)})m_{(2)} \in M_{\mathrm{coinv}}$.

Now consider the $\Bbbk$-linear map $\Psi\colon M \to H \otimes M_{\mathrm{coinv}}$ given by

$$\Psi(m) = \sum m_{(1)} \otimes \chi((m_{(2)})_{(1)})(m_{(2)})_{(2)}.$$

Since H acts trivially on M_{coinv}, this is an H-module homomorphism and an H-comodule homomorphism by coassociativity. Also,

$$\begin{aligned}
\Theta\Psi(m) &= \Theta\big(\sum m_{(1)} \otimes \chi((m_{(2)})_{(1)})(m_{(2)})_{(2)}\big) \\
&= \Theta\big(\sum (m_{(1)})_{(1)} \otimes \chi((m_{(1)})_{(2)})m_{(2)}\big) \\
&= \sum (m_{(1)})_{(1)}\chi((m_{(1)})_{(2)})m_{(2)} \\
&= \sum \varepsilon(m_{(1)})m_{(2)} \\
&= m,
\end{aligned}$$

and when $\mu(m) = 1 \otimes m$,

$$\begin{aligned}
\Psi\Theta(h \otimes m) = \Psi(hm) &= \sum h_{(1)} \otimes \chi((h_{(2)})_{(1)})(h_{(2)})_{(2)}m \\
&= \sum h_{(1)} \otimes \varepsilon(h_{(2)})m \\
&= \sum \varepsilon(h_{(2)})h_{(1)} \otimes m \\
&= h \otimes m.
\end{aligned}$$

Therefore Ψ and Θ are inverse functions.

If we choose a $\Bbbk$-basis for M_{coinv} then we get an H basis for $H \otimes M_{\text{coinv}}$, hence $M \cong H \otimes M_{\text{coinv}}$ is a free H-module. □

This result tells us that for a non-trivial Hopf module M, M_{coinv} is also non-trivial. But we can say more when we have appropriate finiteness conditions.

Corollary 6.8. *If M_{coinv} is finite dimensional then*

$$\operatorname{rank}_H M = \dim_{\Bbbk} M_{\text{coinv}},$$

and if H is also finite dimensional then

$$\dim_{\Bbbk} M = \dim_{\Bbbk} H \dim_{\Bbbk} M_{\text{coinv}}.$$

Here is another interesting application. For a algebra H, a subspace $L \subseteq H$ is a *left coideal* if the image of the coproduct applied to L satisfies $\psi L \subseteq H \otimes L$, so L is a subcomodule of H.

Corollary 6.9. *If H is finite dimensional and a non-zero left ideal $I \subseteq H$ is also a left coideal, then $I = H$.*

Proof. The conditions imply that I is a Hopf module which is a subHopf module of H. By the Fundamental Theorem,

$$I \cong H \otimes I_{\text{coinv}}$$

as H-modules, so $\dim_{\Bbbk} I \geqslant \dim_{\Bbbk} H$ which is only possible if $I = H$. □

Applications to finite dimensional Hopf algebras. Now let H be a finite dimensional Hopf algebra.

Theorem 6.10. *If H is a finite dimensional Hopf algebra then its dual H^* is an H-Hopf module which is free of rank 1 as an H-module, i.e., $H^* \cong H$ as left H-modules.*

Proof. The dual $H^* = \hom(H, \Bbbk)$ is both a left H-module where for $h \in H$ and $f \in H^*$,

$$h \cdot f = f(\chi(h)-).$$

It is also an algebra where the product is obtained by dualising the coproduct of H, i.e., it is the composition

$$\begin{tikzcd} H^* \otimes H^* \arrow[r, leftrightarrow, "\cong"'] \arrow[rr, bend left, "\psi^\dagger"] & (H \otimes H)^* \arrow[r, "\psi^*"'] & H^* \end{tikzcd}$$

In fact this is a homomorphism of left H-modules where we use the antipode and the left multiplication on the domains of $H^* = \hom(H, \Bbbk)$ and $(H \otimes H)^* = \hom(H \otimes H, \Bbbk)$ to define their module structures.

Now we make H^* into a Hopf module over H by defining the coaction $\mu\colon H^* \to H \otimes H^*$ as follows: for $f \in H^*$,

$$\mu(f) = \sum f_{(1)} \otimes f_{(2)} \in H \otimes H^*$$

where the terms $f_{(1)}$ are characterised by requiring that for all $g \in H^*$, the product $fg \in H^*$ satisfies

$$fg = \sum g(f_{(1)}) f_{(2)}.$$

A verification that this is an H-module homomorphism can be found in the proof of [Lor18, theorem 10.9].

The Fundamental Theorem tells us that H^* is a free module and since $\dim_{\Bbbk} H^* = \dim_{\Bbbk} H$ it must have rank 1, i.e., $H^* \cong H$ as H-modules. □

Since H^* is an injective H-module this result says that H is also injective as well as projective, i.e., it is *self-injective*.

We can also give another proof of Theorem 6.3. For if $z \in \ker \chi \subseteq H$ then for any $f \in H^*$,

$$z \cdot f = f(\chi(z)-) = 0,$$

but since $H^* \cong H$, this is only possible if $z = 0$.

We now have an important result on finite dimensional Hopf algebras. A graded analogue of this was proved by Browder & Spanier [BS62], then the ungraded case was proved by Larson & Sweedler [LS69].

Theorem 6.11 (Larson & Sweedler). *If H is a finite dimensional Hopf algebra then it is a Frobenius algebra.*

Proof. The existence of a left H-module isomorphism $H \xrightarrow{\cong} H^*$ gives us an element $\lambda \in H^*$ which is the image of $1 \in H$. By definition of the module structure on H^*, the image of $h \in H$ is then $h\lambda \in H^*$ where

$$(h\lambda)(x) = \lambda(xh).$$

If this λ is trivial on some simple left submodule $S \subseteq H$ then for any non-zero element $s \in S$, $s\lambda = 0$, contradicting the definition of λ. It follows that λ is a Frobenius form and so H is a Frobenius algebra. □

This result has many interesting consequences. An algebra A is called a *Kasch algebra* if every left or right simple A-module is isomorphic to a minimal left or right ideal (these are its simple submodules).

If S is a (non-trivial) simple left or right A-module, then by Schur's Lemma its endomorphism algebra $\mathrm{End}_A(S)$ is a division algebra central over $\Bbbk$. It is easy to see that the sum of all the submodules of R isomorphic to S is actually a finite direct sum

$$S_1 \oplus S_2 \oplus \cdots \oplus S_m,$$

where for each i, $S_i \cong S$. The number m is well-defined and is called the *multiplicity* of S in R.

Proposition 6.12. *Every Frobenius algebra A is a Kasch algebra. In particular, if S is a simple A-module then its multiplicity in R is equal to $\dim_{\mathrm{End}_A(S)} S$.*

Proof. If S is a non-trivial simple left A-module, then the opposite division algebra $\mathrm{E} = \mathrm{End}_A(S)^{\mathrm{op}}$ acts on $\mathrm{Hom}_A(S, A)$ by precomposition making it a left E-vector space (i.e., a left E-module).

There are isomorphisms of E-vector spaces

$$\begin{aligned} \mathrm{Hom}_A(S, A) &\cong \mathrm{Hom}_A(S, A^*) = \mathrm{Hom}_A(S, \hom(A, \Bbbk)) \\ &\cong \mathrm{Hom}_{\Bbbk}(A \otimes_A S, \Bbbk) \\ &\cong \mathrm{Hom}_{\Bbbk}(S, \Bbbk). \end{aligned}$$

Every non-trivial A-module homomorphism $S \to A$ must be injective by simplicity, so the multiplicity of S is

$$\dim_{\mathrm{E}} \mathrm{Hom}_A(S, A) = \dim_{\mathrm{End}_A(S)} S \neq 0.$$ □

If $\dim_{\Bbbk} S = 1$, then $\mathrm{End}_A(S) = \Bbbk$ and $\dim_{\Bbbk} \mathrm{Hom}_A(S, A) = 1$, so S occurs with multiplicity 1, i.e., there is a unique submodule of R isomorphic to S.

Of course this result applies to any finite dimensional Hopf algebra. In particular the counit $\varepsilon \colon H \to \Bbbk$ gives us a 1-dimensional simple left or right module and each of these occurs as a unique submodule. These 1-dimensional subspaces are called the spaces of left or right *integrals* of H:

$$\textstyle\int^{\mathrm{l}}_H = \{z \in H : \forall h \in H,\ hz = \varepsilon(h)z\}, \quad \int^{\mathrm{r}}_H = \{z \in H : \forall h \in H,\ zh = \varepsilon(y)z\}.$$

In general, $\int^{\mathrm{l}}_H \neq \int^{\mathrm{r}}_H$, but if $\int^{\mathrm{l}}_H = \int^{\mathrm{r}}_H$ then H is called *unimodular* and we set $\int_H = \int^{\mathrm{l}}_H = \int^{\mathrm{r}}_H$. In general,

$$\textstyle\chi\int^{\mathrm{l}}_H = \int^{\mathrm{r}}_H, \quad \chi\int^{\mathrm{r}}_H = \int^{\mathrm{l}}_H,$$

so when H is unimodular,

$$\textstyle\chi\int_H = \int_H.$$

Although we know that a finite dimensional Hopf algebra H has a Frobenius form we have not yet explained how to find a suitable element of H^*.

Lemma 6.13. *Let H a finite dimensional Hopf algebra. Then in the dual Hopf algebra H^*, any non-zero right integral $\lambda \in \int^{\mathrm{r}}_{H^*}$ is a Frobenius form for H.*

Example 6.14. Let $\Bbbk G$ be the group algebra of a finite group. Every element can be uniquely written as $\sum_{g\in G} t_g g$ where t_g. The element $z_0 = \sum_{g\in G} g$ satisfies

$$(\sum_{g\in G} t_g g)z_0 = (\sum_{g\in G} t_g)z_0 = \varepsilon(\sum_{g\in G} t_g g)z_0 = z_0(\sum_{g\in G} t_g g),$$

so $\Bbbk G$ is unimodular and

$$\textstyle\int_{\Bbbk G} = \{tz_0 : t \in \Bbbk\}.$$

Define the form $\lambda \in (\Bbbk G)^*$ by

$$\lambda\big(\sum_{g\in G} t_g g\big) = t_1.$$

Then λ is a Frobenius form for $\Bbbk G$.

For a finite dimensional algebra it is useful to know whether it is semisimple, and therefore its modules are completely reducible (i.e., direct sums of simple modules). Semisimplicity is equivalent to the triviality of the Jacobson radical of the algebra.

Theorem 6.15. *Let H be a finite dimensional Hopf algebra. The following conditions are equivalent:*
(a) *Every H-module is completely reducible.*
(b) *For any non-zero left integral $z \in \int^{\mathrm{l}}_H$, $\varepsilon(z) \neq 0$.*
(c) *For any non-zero right integral $z \in \int^{\mathrm{r}}_H$, $\varepsilon(z) \neq 0$.*
(d) *Every right H-module is completely reducible.*
If these conditions hold then H is semisimple and unimodular.

A finite dimensional Hopf algebra which is semisimple has a representation theory very similar to that of a finite group over a field whose characteristic does not divide its order. In the setting of Example 6.14,

$$\varepsilon(z_0) = \varepsilon\big(\sum_{g\in G} g\big) = |G| \in \Bbbk,$$

so $\Bbbk G$ is semisimple if and only if the characteristic of $\Bbbk$ does not divide $|G|$. Of course this is a well-known fact in the representation theory of finite groups!

q-combinatorics. Suppose that we are working in a non-commutative ring R where $q, x, y \in R$ with $q \neq 1$ in the centre of R and the other elements satisfy

$$yx = qxy.$$

What is the analogue of the usual binomial expansion of $(x + y)^n$ when $n \geqslant 1$?

To describe the answer we introduce analogues of standard combinatorial expressions. We will set

$$[n]_q = \frac{q^n - 1}{q - 1} = q^{n-1} + q^{n-2} + \cdots + q + 1,$$

so

$$\lim_{q \to 1} [n]_q = n.$$

First we have the *q-factorials*; these are defined recursively for $n \geqslant 0$:

$$[0]_q! = 1, \quad [n]_q! = [n]_q([n-1]_q!) = \frac{q^n - 1}{q - 1}[n-1]_q!,$$

so

$$[n]_q! = \prod_{1 \leqslant k \leqslant n} (q^{k-1} + q^{k-2} + \cdots + q + 1).$$

Notice that

$$\lim_{q \to 1} [n]_q! = n!.$$

Next we have the q-binomial coefficients for $0 \leqslant k \leqslant n$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{(q^n - 1)(q - 1)}{(q^k - 1)(q^{n-k} - 1)}.$$

These satisfy two generalisations of Pascal's Triangle which are easily verified:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q, \tag{7.1}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{7.2}$$

Proposition 7.1. *For $n \geqslant 1$,*

$$(x + y)^n = \sum_{0 \leqslant k \leqslant n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k y^{n-k}.$$

Proof. This can be proved by induction on n using one of the identities

$$(x + y)(x + y)^{n-1} = (x + y)^n = (x + y)^{n-1}(x + y)$$

and one of the identities (7.1) or (7.2). $\square$

Here is an application. Suppose that $z \in R$ is nilpotent; the *q-exponential* of z is

$$\exp_q(z) = \sum_{k \geqslant 0} \frac{1}{[k]_q!} z^k$$

which is of course a finite sum. Notice that

$$\lim_{q \to 1} \exp_q(z) = \exp(z).$$

Now suppose that $x, y \in R$ as above are also nilpotent; then for some large enough m, for any $0 \leqslant i \leqslant m$ we have $x^i y^{m-i}$. So we have

$$\exp_q(x + y) = \exp_q(x) \exp_q(y). \tag{7.3}$$

To see this, expand out the left hand side to obtain

$$\begin{aligned}
\exp_q(x+y) &= \sum_{k\geqslant 0} \frac{1}{[k]_q!}(x+y)^k \\
&= \sum_{k\geqslant 0} \frac{1}{[k]_q!} \left(\sum_{0\leqslant i\leqslant k} \begin{bmatrix} k \\ i \end{bmatrix}_q x^i y^{k-i} \right) \\
&= \sum_{k\geqslant 0} \sum_{0\leqslant i\leqslant k} \frac{1}{[i]_q!} x^i \frac{1}{[k-i]_q!} y^{k-i} \\
&= \sum_{i\geqslant 0} \sum_{j\geqslant 0} \frac{1}{[i]_q!} x^i \frac{1}{[j]_q!} y^j \\
&= \exp_q(x) \exp_q(y),
\end{aligned}$$

where the sums are really finite.

For polynomials in a variable X, there is a *q-derivative* ∂_q given by

$$\partial_q f(X) = \frac{f(qX) - f(X)}{(q-1)X},$$

so for example,

$$\partial_q X^n = [n]_q X^{n-1}.$$

For nilpotent z, we have

$$\partial_q \exp_q(z) = \exp_q(z).$$

The Quantum Plane. Recall the Quantum Plane of Example 2.1, the non-commutative bialgebra $\mathcal{O}_q(\Bbbk^2)$ where $q \neq 1$, generated by two elements x, y satisfying $yx = qxy$. The coproduct ψ and counit ε are given by

$$\psi(x) = x \otimes x, \quad \psi(y) = y \otimes 1 + x \otimes y, \quad \varepsilon(x) = 1, \quad \varepsilon(y) = 0.$$

The quantum version of the general linear group for $1 \neq q \in \Bbbk^\times$ is a Hopf algebra $\mathcal{GL}_q(2)$ which we will now define. As an algebra, $\mathcal{GL}_q(2)$ is generated by a, b, c, d, e satisfying the relations

$$\begin{gathered}
ca = qac, \qquad ba = qab, \qquad db = qbd, \\
dc = qcd, \qquad cb = bc \qquad da - ad = (q - q^{-1})bc, \\
(ad - q^{-1}bc)e = 1.
\end{gathered}$$

It turns out that $(ad - q^{-1}bc)$ is in the centre of $\mathcal{GL}_q(2)$, hence so is

$$e = (ad - q^{-1}bc)^{-1}.$$

We can also define the quotient Hopf algebra $\mathcal{SL}_q(2)$ where we have the additional relations

$$ad - q^{-1}bc = 1 = e.$$

This is called the *quantum special linear group.* The coproduct, counit and antipode are given by

$$\begin{gathered}
\psi(a) = a \otimes a + b \otimes c, \quad \psi(b) = b \otimes d + a \otimes b, \quad \psi(c) = c \otimes a + d \otimes b, \quad \psi(d) = d \otimes d + c \otimes b, \\
\varepsilon(a) = 1 = \varepsilon(d), \qquad \varepsilon(b) = 0 = \varepsilon(c), \\
\chi(a) = d, \qquad \chi(d) = a, \qquad \chi(b) = c, \qquad \chi(c) = b.
\end{gathered}$$

Just as the special linear group acts linearly on the plane, so the quantum special linear group coacts on the quantum plane, i.e., there is a coaction $\rho\colon \mathcal{O}_q(\Bbbk^2) \to \mathcal{SL}_q(2) \otimes \mathcal{O}_q(\Bbbk^2)$. This is given on the generators by

$$\rho(x) = a \otimes x + b \otimes y, \quad \rho(y) = c \otimes x + d \otimes y.$$

Quasitriangular Hopf algebras. For a non-cocommutative Hopf algebra H, its module category $\mathbf{Mod}_H$ is monoidal under tensor product but not always *symmetric* monoidal since in general $M \otimes N$ need not be isomorphic to $N \otimes M$. One way to 'correct' this is to impose extra structure.

Definition 7.2. A *quasitriangular Hopf algebra* $(H, \mathcal{R})$ is a Hopf algebra H with an element $\mathcal{R} \in H \otimes H$ satisfying

- $\mathcal{R}$ is a unit in the algebra $H \otimes H$ and for all $h \in H$,

  $$\mathrm{T} \circ \psi(h) = \mathcal{R}(\psi(h))\mathcal{R}^{-1};$$

- In the algebra $H \otimes H \otimes H$ we have the identities

  $$(\psi \otimes \mathrm{Id}_H)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\mathrm{Id}_H \otimes \psi)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$

  where $\mathcal{R}_{ij} \in H \otimes H \otimes H$ means the image of $\mathcal{R}$ under the algebra homomorphism $H \otimes H \to H \otimes H \otimes H$ obtained by including the i and j factors (so $\mathcal{R}_{12} = \mathcal{R} \otimes 1$ and $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$ for example).

Lemma 7.3. *Suppose that $(H, \mathcal{R})$ is a quasitriangular Hopf algebra.*
(a) *We have*

$$\begin{aligned}
(\varepsilon \otimes \mathrm{Id}_H)(\mathcal{R}) &= 1 \otimes 1 = (\mathrm{Id}_H \otimes \varepsilon)(\mathcal{R}), \\
(\chi \otimes \mathrm{Id}_H)(\mathcal{R}) &= \mathcal{R}^{-1}, \\
(\mathrm{Id}_H \otimes \chi)(\mathcal{R}^{-1}) &= \mathcal{R},
\end{aligned}$$

and therefore

$$(\chi \otimes \chi)(\mathcal{R}) = \mathcal{R}.$$

(b) $(H, \mathcal{R}_{21}^{-1})$ *is also a quasitriangular Hopf algebra where*

$$\mathcal{R}_{21}^{-1} = \mathrm{T}(\mathcal{R}^{-1}).$$

(c) *The* Yang-Baxter identity *holds in $H \otimes H \otimes H$:*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Notice that the Yang-Baxter equation is similar to the following identity in the symmetric group S_3:

$$(1\,2)(1\,3)(2\,3) = (1\,3) = (2\,3)(1\,3)(1\,2).$$

It is also a relation in the 3-rd braid group so is sometimes called the *braid relation.*

Theorem 7.4. *Suppose that $(H, \mathcal{R})$ is a quasitriangular Hopf algebra.*
(a) *The antipode of H is a bijectiion.*
(b) *There is a unit $u \in H^\times$ such that $\chi^2 = u(-)u^{-1}$, and moreover*

$$\psi(u) = (\mathrm{T}(\mathcal{R})\mathcal{R})^{-1}(u \otimes u).$$

Proof. See Majid [Maj02, chapter 5]. □

To illustrate the impact of a quasitriangular on a Hopf algebra, recall from Lemma 5.18 that for a finite dimensional H-module M, $M^{**} \cong (\chi^2)^* M$. Using (b) it is easy to see that for a quasitriangular Hopf algebra we have $(\chi^2)^* M \cong M$ and therefore $M^{**} \cong M$. We will see far more is true.

Braidings on module categories. In the following we assume that $(H, \mathcal{R})$ is a quasitriangular Hopf algebra. We will often write $\mathcal{R}$ using Sweedler-style notation as a sum

$$\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2.$$

The module category $\mathbf{Mod}_H$ is monoidal under $\otimes$. We define

$$M \overset{\mathrm{op}}{\otimes} N = N \otimes M$$

with the usual H-action given by multiplication by $\mathrm{T} \circ \psi(h)$:

$$h(m \overset{\mathrm{op}}{\otimes} n) = h(n \otimes m) = \sum h_{(1)} n \otimes h_{(2)} m = \sum h_{(2)} m \overset{\mathrm{op}}{\otimes} h_{(1)} n.$$

Lemma 7.5. *For two left H-modules M and N,*

$$\Psi_{M,N} \colon M \otimes N \to M \overset{\mathrm{op}}{\otimes} N = N \otimes M; \quad \Psi_{M,N}(m \otimes n) = \sum \mathcal{R}_1 m \overset{\mathrm{op}}{\otimes} \mathcal{R}_2 n = \sum \mathcal{R}_2 n \otimes \mathcal{R}_1 m$$

defines an isomorphism of H-modules.

Proof. Notice that

$$\Psi_{M,N} = \mathrm{T} \circ \mathcal{R}$$

where $\mathcal{R}$ means the multiplication by $\mathcal{R}$ function on $H \otimes H$. By the first part of Definition 7.2, for $h \in H$,

$$\psi(h) \circ \mathcal{R} = \mathrm{T} \circ \mathcal{R} \circ \psi(h).$$

We have for $h \in H$, $m \in M$ and $n \in N$,

$$\begin{aligned}
\Psi_{M,N}(h(m \otimes n)) &= \mathrm{T}(\mathcal{R}\psi(h)(m \otimes n)) \\
&= \mathrm{T} \circ \mathcal{R} \circ \psi(h)(m \otimes n) \\
&= \psi(h) \circ \mathcal{R}(m \otimes n) \\
&= \psi(h) \circ \mathrm{T} \circ \mathrm{T} \circ \mathcal{R}(m \otimes n) \\
&= \mathrm{T} \circ \psi(h) \circ \Psi_{M,N}(m \otimes n) \\
&= h\Psi_{M,N}(m \otimes n).
\end{aligned}$$

It is clear that $\Psi_{M,N}$ does has an inverse, namely $\mathcal{R}^{-1} \circ \mathrm{T}$. $\square$

The existence of $\mathcal{R}$ makes the monoidal category $(\mathbf{Mod}_H, \otimes)$ into a *braided monoidal category.* This involves $\Psi_{-,-}$ as well as functorial isomorphisms

$$\Phi_{U,V,W} \colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

which obey the *Pentagon Condition* making the following diagram commute for all U, V, W, Z.

$$\begin{tikzcd}
& (U \otimes V) \otimes (W \otimes Z) \arrow[dr, "\Phi"] & \\
((U \otimes V) \otimes W) \otimes Z \arrow[ur, "\Phi"] \arrow[d, "\Phi \otimes \mathrm{Id}"'] & & U \otimes (V \otimes (W \otimes Z)) \\
(U \otimes (V \otimes W)) \otimes Z \arrow[rr, "\Phi"] & & U \otimes ((V \otimes W) \otimes Z) \arrow[u, "\mathrm{Id} \otimes \Phi"']
\end{tikzcd}$$

Furthermore, Φ and Ψ must obey the *Hexagon Conditions* making the following diagrams commute.

$$
\begin{tikzcd}
 & U \otimes (V \otimes W) \arrow[dl, "\mathrm{id} \otimes \Psi"'] \arrow[dr, "\Phi^{-1}"] & \\
U \otimes (W \otimes V) \arrow[d, "\Phi^{-1}"'] & & (U \otimes V) \otimes W \arrow[d, "\Psi"] \\
(U \otimes W) \otimes V \arrow[dr, "\Psi \otimes \mathrm{Id}"'] & & W \otimes (U \otimes V) \arrow[dl, "\Phi^{-1}"] \\
 & (W \otimes U) \otimes V &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (U \otimes V) \otimes W \arrow[dl, "\Phi"'] \arrow[dr, "\Psi \otimes \mathrm{Id}"] & \\
U \otimes (V \otimes W) \arrow[d, "\Psi"'] & & (V \otimes U) \otimes W \arrow[d, "\Phi"] \\
(V \otimes W) \otimes U \arrow[dr, "\Phi"'] & & V \otimes (U \otimes W) \arrow[dl, "\mathrm{Id} \otimes \Psi"] \\
 & V \otimes (W \otimes U) &
\end{tikzcd}
$$

Notice that we do not assume that $\Psi_{V,U} = \Psi_{U,V}^{-1}$ as it would if the tensor product were symmetric. This is related to the fact that $\mathcal{R}^2$ may not be $1 \otimes 1$. This means that the group of functorial isomorphisms acting on a tensor product of H-modules $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ is not the symmetric group S_n but rather the n-th *braid group* Br_n which admits an epimorphism $\pi_n \colon \mathrm{Br}_n \to \mathrm{S}_n$ with infinite kernel.

The group Br_n has a presentation with generators $b_1, b_2, \ldots, b_{n-1}$ and relations

$$b_i b_j = b_j b_i \quad (|i-j| \geqslant 2),$$

and the Yang-Baxter equation

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}.$$

Similarly, S_n has a presentation with generators $s_1, s_2, \ldots, s_{n-1}$ and relations

$$s_i s_j = s_j s_i \quad (|i-j| \geqslant 2),$$

and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},$$

as well as

$$s_i^2 = 1.$$

Here $s_i = (i\, i+1)$ and $\pi_n(b_i) = s_i$.

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