CHROMATIC CALCULATIONS FOR THE $S^1 \times S^1$ TRANSFER

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The material contained in these notes can be found in the references and we make no attempt to give details or precise citations for all of it.

Throughout, we let p be an odd prime. Later we will often need to assume $p \geqslant 5$ but most of the algebra works for $p = 3$ and some even apply when $p = 2$.

1. The Hopf algebra $BP_*(\mathbb{C}\mathrm{P}^\infty_+)$

See [1, 9, 10] for background details on this section.

We will write $\mathbb{C}\mathrm{P} = \mathbb{C}\mathrm{P}^\infty_+$. Then as a BP_*-module, $BP_*(\mathbb{C}\mathrm{P})$ has a canonical basis dual to powers of the canonical orientation $x = x^{BP} \in BP_*(\mathbb{C}\mathrm{P})$,

$$BP_*(\mathbb{C}\mathrm{P}) = BP_*\{\beta_0, \beta_1, \beta_2, \ldots\}.$$

where $\langle x^r, \beta_s \rangle = \delta_{r,s}$ and $\beta_0 = 1$. The digonal coproduct is given by

$$\Delta(\beta_n) = \sum_{0 \leqslant r \leqslant n} \beta_r \otimes \beta_{n-r}. \tag{1.1}$$

Tensor product of line bundles induces a commutative BP_*-algebra structure, making $BP_*(\mathbb{C}\mathrm{P})$ into a bicommutative BP_*-Hopf algebra. To calculate the product structure we use the formal group law

$$F(X, Y) = F^{BP}(X, Y) = X + Y + \sum_{r,s} a_{r,s} X^r Y^s.$$

Setting $\beta(X) = \sum_{r \geqslant 0} \beta_r X^r$, we obtain

$$\beta(X)\beta(Y) = \beta(F(X, Y)), \tag{1.2}$$

where on the left hand side the coefficient of $X^r Y^s$ is $\beta_r \beta_s$. If n is a natural number then

$$\beta(X)^n = \beta([n]X),$$

where $[n]X = [n]_F X$ is the n-series. We alse have

$$\beta(X)^{-1} = \beta([-1]X),$$

where $F([-1]X, X) = 0 = F(X, [-1]X)$. In fact, for any non-zero $q \in \mathbb{Z}_{(p)}$, we can make sense of $[q]X$ and

$$\beta(X)^q = \beta([q]X).$$

Now recall the Araki and Hazewinkel generators $v_n, w_n \in BP_{2p^n-2}$ for which $v_0 = w_0 = p$ and

$$v_n \equiv w_n \bmod I_n,$$

where $I_n = (v_0, \ldots, v_{n-1}) \lhd BP_*$. The Araki generators are defined by the identity

$$[p]X = \sum_{k \geqslant 0}^{F} v_k X^{p^k}. \tag{1.3}$$

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Notice that

$$[p]X \equiv \sum_{k\geqslant n}^{F} v_k X^{p^k} \bmod I_n. \tag{1.4}$$

Now we can read off some useful formulae for the β_r. We have the equation

$$\beta([p]X) = \prod_{k\geqslant 0} \beta(v_k X^{p^k}) \tag{1.5}$$

and the congruences

$$\beta(X)^p = \beta([p]X) \equiv \prod_{k\geqslant n} \beta(v_k X^{p^k}) \bmod I_n. \tag{1.6}$$

In particular we have

$$\beta_1^p \equiv v_1\beta_1 \bmod (p). \tag{1.7}$$

We can project off to $E(1)$-homology, and under the natural map

$$BP_* \longrightarrow E(1)_* = BP_*[v_1^{-1}]/(v_k : k \geqslant 2),$$

only the powers of v_1 survive.

2. Some coaction primitives

Now we will consider coaction primitives. The coaction

$$\rho\colon BP_*(\mathbb{C}\mathrm{P}) \longrightarrow BP_*(BP) \otimes_{BP_*} BP_*(\mathbb{C}\mathrm{P})$$

can be expressed neatly using the formlua

$$\rho\beta(X) = (1\otimes\beta)(\chi t(X)) = \sum_{k\geqslant 0} (\chi t(X))^k \otimes \beta_k, \tag{2.1}$$

where $\chi\colon BP_*(BP) \longrightarrow BP_*(BP)$ is the antipode and $t_k \in BP_{2p^k-2}(BP)$ are the Adams polynomial generators of

$$BP_*(BP) = BP_*[t_k : k \geqslant 1]$$

and $t_0 = 1$; the series $t(X)$ is defined by

$$t(X) = \sum_{k\geqslant 0}^{F} t_k X^{p^k}.$$

Theorem 2.1 (Hansen, Smith, et al)**.** *The coaction primitives are given by*

$$\mathrm{Ext}^{0,2k}_{BP_*(BP)}(BP_*, BP_*(\mathbb{C}\mathrm{P})) = \mathbb{Z}_{(p)}\{\beta_1^k\}$$

for each $k \geqslant 1$. Furthermore,

$$\beta_1^k = k!\beta_k + (\text{terms involving lower degree } \beta_r).$$

Proof. Since β_1 comes from the bottom cell $\mathbb{C}\mathrm{P}^1 = S^2$, it is unstably spherical and hence primitive, so its powers are also primitive. The fact that β_1^k is not divisible by p is most easily verified in K-theory or equivalently $E(1)$-theory. It is standard that

$$\mathrm{Ext}^{0,2k}_{BP_*(BP)}(BP_*, BP_*(\mathbb{C}\mathrm{P})) \otimes \mathbb{Q} \cong H_{2k}(\mathbb{C}\mathrm{P};\mathbb{Q}) \cong \mathbb{Q}. \qquad \square$$

Now let $\mathbb{C}\mathrm{P}^{(2)} = \mathbb{C}\mathrm{P}\wedge\mathbb{C}\mathrm{P} = \mathbb{C}\mathrm{P}^\infty \times \mathbb{C}\mathrm{P}^\infty_+$. In order to describe some primitives in $BP_*(\mathbb{C}\mathrm{P}^{(2)})$, we recall work of Knapp and others. In effect this can be done in K-theory by using the Hattori-Stong theorem which has an analogue for each $E(n)$ [3, theorem 3.1].

We define two families of primitives. From the second expression for it, the element

$$X_0 = \beta_1 \otimes \beta_p - \beta_p \otimes \beta_1 = \frac{1}{p!}(\beta_1 \otimes \beta_1^p - \beta_1^p \otimes \beta_1),$$

is clearly primitive. We define another primitive of degree $2(p+1)p^n$ by

$$H = \beta_1^{p^2-1} \otimes 1 - \beta_1^{(p^2-1)/2} \otimes \beta_1^{(p^2-1)/2} + 1 \otimes \beta_1^{p^2-1}$$

and then inductively define

$$X_n = \frac{1}{p}(X_{n-1}^p - H^{p^{n-1}} X_{n-1}) \in BP_*(\mathbb{C}\mathrm{P}^{(2)}) \otimes \mathbb{Q}.$$

Setting

$$\Sigma = \sum_{1 \leqslant r \leqslant p-1} r\beta_{(p-r)p+r} \otimes \beta_{rp+p-r} \in BP_*(\mathbb{C}\mathrm{P}^{(2)}),$$

we have the congruence

$$\Sigma^2 \equiv 0 \bmod (p, v_1). \tag{2.2}$$

In fact, each $X_n \in BP_*(\mathbb{C}\mathrm{P}^{(2)})$ and furthermore it satisfies the congruence

$$X_n \equiv v_2^{p^{n-1}+\cdots+p^2+p}(v_2 X_0 + \Sigma) \bmod (p, v_1). \tag{2.3}$$

Using these primitives, we follow Knapp [6] and Schwartz in defining yet another family.

For $k \geqslant 1$, let

$$k(p-1) - 1 = r_0 + r_1 p \cdots + r_d p^d$$

be the p-adic expansion of $k(p-1)-1$. We define the primitive

$$Y_k = X_0^{r_0} X_1^{r_1} \cdots X_d^{r_d}.$$

Now we need a condition on k. Writing the p-adic expansion of k as

$$k = k_0 + k_1 p + \cdots + k_e p^e,$$

we can consider the system of inequalities

$$k_0 - 1 \geqslant k_1 \geqslant k_2 \geqslant \cdots \geqslant k_{e-1} \geqslant k_e. \tag{2.4}$$

Kummer's formula for the p-adic valuation of a binomial coefficient shows that (2.4) is equivalent to

$$\binom{pk-1}{k} \not\equiv 0 \bmod p. \tag{2.5}$$

Theorem 2.2. *For $k \geqslant 1$, the primitive $Y_k \in BP_*(\mathbb{C}\mathrm{P}^{(2)})$ satisfies the following congruence modulo (p, v_1):*

$$Y_k \equiv \begin{cases} v_2^{k-2}(v_2 X_0^{p-2} + (k-1)X_0^{p-3}\Sigma) & \text{if (2.4) is true,} \\ 0 & \text{if (2.4) is false.} \end{cases}$$

3. Stably spherical primitives

Some of the Y_k are known to be stably spherical, *i.e.*, they are in the image of the BP-Hurewicz homomorphism

$$\underline{bp}\colon \pi_*^S(\mathbb{C}\mathrm{P}^{(2)}) \longrightarrow BP_*(\mathbb{C}\mathrm{P}^{(2)}).$$

Theorem 3.1 (see [4, proposition 3.4]). *Each of X_0, X_1, X_2 is stably spherical, hence so are all monomials in them.*

Of course this implies that certain of the Y_k are stably spherical, such as

$$Y_1 = X_0^{p-2},\ Y_2 = X_0^{p-3}X_1,\ \cdots,\ Y_{(p-1)/2} = X_0^{(p-1)/2}X_1^{(p-3)/2},$$

all of which are non-zero $\mathrm{mod}(p, v_1)$, while

$$Y_{(p+1)/2} = X_0^{(p+1)/2}X_1^{(p-5)/2} \equiv 0 \bmod (p, v_1).$$

4. The S^1 and $S^1 \times S^1$ transfers

The S^1-transfer is a map of spectra $\mathrm{tr}\colon \Sigma^\infty\mathbb{C}\mathrm{P} \longrightarrow S^{-1}$ whose domain we will denote by $\mathbb{C}\mathrm{P}$. Smashing two copies together we obtainthe double transfer

$$\mathrm{tr}_2 = \mathrm{tr}\wedge\mathrm{tr}\colon \mathbb{C}\mathrm{P}^{(2)} = \mathbb{C}\mathrm{P}\wedge\mathbb{C}\mathrm{P} \longrightarrow S^{-1}\wedge S^{-1} \sim S^{-2}.$$

Note that tr_2 factors through tr in two obvious ways.

$$\begin{tikzcd}
 & S^{-1}\wedge\mathbb{C}\mathrm{P} \arrow[dr, "\mathrm{tr}"] & \\
\mathbb{C}\mathrm{P}\wedge\mathbb{C}\mathrm{P} \arrow[ur, "\mathrm{tr}\wedge\mathrm{Id}"] \arrow[rr, "\mathrm{tr}_2"] \arrow[dr, "\mathrm{Id}\wedge\mathrm{tr}"'] & & S^{-2} \\
 & \mathbb{C}\mathrm{P}\wedge S^{-1} \arrow[ur, "\mathrm{tr}"'] &
\end{tikzcd}$$

It is an important fact that raises Adams-Novikov filtration by 1 since there is a factorisation up to the first stage of the Adams tower. This means that in the ANSS E_2-term, tr induces maps $\mathrm{E}_2^{s,t}(\mathbb{C}\mathrm{P}) \longrightarrow \mathrm{E}_2^{s+1,t+1}(S)$. Similarly, tr_2 induces maps $\mathrm{E}_2^{s,t}(\mathbb{C}\mathrm{P}^{(2)}) \longrightarrow \mathrm{E}_2^{s+2,t+2}(S)$.

$$\begin{tikzcd}
 & & \Sigma^2\mathbb{C}\mathrm{P}\wedge\mathbb{C}\mathrm{P} \arrow[dl, "\mathrm{tr}"] \arrow[ddll, dotted, bend right, "\mathrm{tr}_2"'] \arrow[ddrr, dotted, "\widetilde{\mathrm{tr}}_2"] & & & & \\
 & \Sigma\mathbb{C}\mathrm{P} \arrow[dl, "\mathrm{tr}"] \arrow[dr, dotted, "\widetilde{\mathrm{tr}}"] & & & & & \\
S \arrow[dr] & & \Sigma^{-1}\overline{BP} \arrow[ll] \arrow[dr] & & \Sigma^{-2}\overline{BP}^{(2)} \arrow[ll] \arrow[dr] & & {} \arrow[ll] \\
 & BP & & BP\wedge\overline{BP} & & BP\wedge\overline{BP}^{(2)} &
\end{tikzcd}$$

A less obvious fact is that there is also lifting up the Chromatic tower, so there is a corresponding shift in the Chromatic SS. Both of these statements apply to tr_2, although the proof of the Chromatic statement is more involved.

5. The $S^1 \times S^1$ transfer in Morava $K(2)$-homology

We can now discuss how the map induced by tr_2 behaves on when composed with projection ρ onto Morava $K(2)$-theory, thus we determine some of the effect of the composition

$$\begin{aligned}\mathrm{Tr}_2\colon \mathrm{Ext}^{0,*}_{BP_*(BP)}(BP_*, BP_*(\mathbb{C}\mathrm{P}^{(2)})) &\xrightarrow{(\widetilde{\mathrm{tr}}_2)_*} \mathrm{Ext}^{2,*}_{BP_*(BP)}(BP_*, BP_*) \\ &\xrightarrow{\rho} \mathrm{Ext}^{2,*}_{K(2)_*K(2)}(K(2)_*, K(2)_*).\end{aligned}$$

Theorem 5.1. *For $k \geqslant 1$,*

$$\mathrm{Tr}_2(Y_k) = \begin{cases} \dfrac{(k+1)}{2} g_1 + \dfrac{(1-k)}{2}\zeta_2 h_{1,1} & \text{if (2.4) is true,} \\ 0 & \text{if (2.4) is false.} \end{cases}$$

Corollary 5.2. *If $p \mid k$, then*

$$\mathrm{Tr}_2(Y_k) = 0.$$

If k satisfies (2.4)*, then*

$$\mathrm{Tr}_2(Y_k) = \frac{1}{k}\rho(\beta_{k/1}).$$

If $p \nmid k$ and k does not satisfy (2.4)*, then*

$$\mathrm{Tr}_2(Y_k) \neq \rho(\beta_{k/1}).$$

These results use the following formula which can be deduced from [7]:

$$\rho(\beta_{k/1}) = -\binom{k+1}{2} g_1 + \frac{k}{2}\zeta_2 h_{1,1}.$$

We are also using Ravenel's calculation [8] of the algebra

$$\mathrm{Ext}^{**}_{K(2)_*K(2)}(K(2)_*, K(2)_*) = K(2)_*(\zeta_2, h_{1,0}, h_{1,1}, g_0, g_1),$$

where $\zeta_2, h_{1,0}, h_{1,1} \in \mathrm{Ext}^1$ and $g_0, g_1 \in \mathrm{Ext}^2$ satisfy the relations

$$h_{1,0}g_1 = g_0 h_{1,1}, \quad h_{1,0}g_0 = h_{1,1}g_1 = h_{1,0}h_{1,1} = h_{1,0}^2 = h_{1,1}^2 = \zeta_2^2 = 0.$$

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