# Spectra

## settings for stable homotopy theory and Adams spectral sequences

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# Historical and mathematical motivations

- Freudenthal suspension theorem: For a based space $X$ and finite CW complex $Z$, there is a $k \gg 0$ such that

$$\{Z, X\} = \operatorname*{colim}_{\ell} [\Sigma^{\ell} Z, \Sigma^{\ell} X] \cong [\Sigma^{k} Z, \Sigma^{k} X]$$

  In particular, for $n \in \mathbb{Z}$ and $k \gg 0$,

$$\pi_n^{\mathrm{S}}(X) = \{S^n, X\} = \operatorname*{colim}_{\ell} \pi_{n+\ell}(\Sigma^{\ell} X) \cong \pi_{n+k}(\Sigma^{k} X).$$

  Suggests forming a homotopy category with objects formed from sequences $\underline{X} = (\Sigma^{\ell} X)_{\ell \in \mathbb{Z}}$ and at least when $Z$ is a finite CW complex, morphisms $\underline{Z} \to \underline{X}$ being elements of $\{Z, X\}$.
- Spanier-Whitehead duality: For a finite CW complex $X$, an embedding $j \colon X \hookrightarrow S^{n+1}$ has a complementary subspace $D_j X$ which depends on $j$ up to homotopy. However, for $k \gg 0$, up to homotopy equivalence $\Sigma^{k-n} D_j X$ is independent of $j$, so the Spanier-Whitehead dual $DX = \underline{\Sigma^{-n} D_j X}$ would be a well-defined object in the above homotopy category.

- Brown representability: Every reasonable cohomology theory $h^*$ defined on based CW complexes is representable. This means that there is a sequence of spaces $(E_n)_{n\in\mathbb{Z}}$ for which there is a natural isomorphism $h^n(-) \cong [-, E_n]$ and furthermore the suspension isomorphisms $h^n(-) \cong h^{n+1}(\Sigma -)$ correspond to weak equivalences $E_n \sim \Omega E_{n+1}$. The sequence $(E_n)_{n\in\mathbb{Z}}$ should be a spectrum and a morphism $(E_n)_{n\in\mathbb{Z}} \to (F_n)_{n\in\mathbb{Z}}$ between two such objects should correspond to a natural transformation of cohomology theories.
- Products in (co)homology theories: These should arise from 'smash products' $(E_n)_{n\in\mathbb{Z}} \wedge (F_n)_{n\in\mathbb{Z}}$ extending the usual space level smash products $E_m \wedge F_n$. So a good category of spectra should have such smash products.

**Early history of spectra ($<$ 1975):** Lima/Spanier, Kan, G. Whitehead, Atiyah, Boardman/Vogt/Adams.

# Modern categories of spectra ($> 1990$)

- EKMM: first categories of spectra with strictly monoidal smash product ($S$-modules).
- Symmetric spectra.
- Orthogonal spectra.
- Diagram spectra.
- Equivariant spectra.

**Properties**

- Stable monoidal model category structures allowing passage to homotopy categories and triangulated structures.
- Monoids and commutative monoids equivalent to $\mathcal{A}_\infty$ and $\mathcal{E}_\infty$ ring spectra giving relative module categories. Multiplicative (co)homology theories have spectral sequences for calculations based on homological algebra of homotopy rings.
- Bousfield localisation works well.

# Some examples

Let $\mathscr{M}_S$ be the category of (left) $S$-modules. There is a symmetric monoidal smash product $\wedge = \wedge_S$ on this, with unit $S$. The homotopy/derived category $\mathscr{D}_S$ inherits a derived smash product $\wedge$. There is also a function object $F(-,-) = F_S(-,-)$.
A commutative monoid (aka commutative ring spectrum) $R \wedge R \to R$ has left $R$-modules $M$ which are equipped with suitably associative and unital products $\mu_M \colon M \wedge R \to M$. For two $R$-modules $M$ and $N$ there is a coequaliser diagram

$$M \wedge R \wedge N \underset{\mu_N}{\overset{\mu_M}{\rightrightarrows}} M \wedge N \longrightarrow M \wedge_R N$$

which defines the relative smash product. The category of $R$-modules $\mathscr{M}_R$ is also a symmetric monoidal model category with homotopy category $\mathscr{D}_R$. There is also a relative function object $F_R(-,-)$ defined by an equaliser diagram.

**Algebraic Bousfield localisation:** For such a commutative ring spectrum $R$, given $u \in R_d = \pi_d(R)$, there is a commutative ring spectrum $R[u^{-1}]$ with $\pi_*(R[u^{-1}]) = R_*[u^{-1}]$. This extends to an $R$-module $M$ by setting $M[u^{-1}] = R[u^{-1}] \wedge_R M$ so that

$$\pi_*(M[u^{-1}]) = \pi_*(M)[u^{-1}].$$

Homology and cohomology theories can be defined on $\mathscr{D}_R$ by setting

$$E^R_*(-) = \pi_*(E \wedge_R -), \quad E^*_R(-) = \pi_{-*}(F_R(-, E)).$$

Versions of Brown representability are true.
Bousfield localisations exist for such homology theories and preserve commutative ring spectra. However, the resulting localised module categories may not have the obvious smash products. For example, localisation with respect to $H\mathbb{F}_p$ for $p$ a prime is essentially $p$-adic completion $(-)\widehat{_p}$. The category of $p$-complete $S$-modules has smash product given by

$$(M \wedge_{S\widehat{_p}} N)\widehat{_p} = (M \wedge N)\widehat{_p}.$$

# Cohomology operations and homology cooperations

A cohomology theory $E^*_R(-)$ has a ring of operations $E^*_R(E)$. For example, when $R = S$ and $E = H = H\mathbb{F}_p$ this is the Steenrod algebra $\mathcal{A} = \mathcal{A}(p)^*$. This acts on the left of $E^*_R(X)$ for any $R$-module $X$, and also on the left of $E^R_*(X)$. Warning: in general, these actions are not $E_* = \pi_*(E)$-linear.
Usually $E^*_R(E)$ has to be thought of as a topological ring and $E^*_R(X)$ as a topological $E_*$-module; then the action of $E^*_R(E)$ is continuous. This led to the dual viewpoint becoming standard.
Assuming that $E^R_*(E)$ is $E_*$-flat, $E^R_*(X)$ is naturally a right $E^R_*(E)$-comodule. Under suitable finiteness conditions, there is a strong relationship between the left action of $E^*_R(E)$ and the $E^R_*(E)$-coaction. Here $E^R_*(E)$ is a Hopf algebroid since there are two $E_*$-module structures which need not agree but are interchanged by the antipode. Similarly, $E^*_R(E)$ is an $R_*$-algebra which has two $E_*$-module structures.

When $R = S$ and $E = H = H\mathbb{F}_p$ everything is as simple as possible and $H_*(H) = \mathcal{A}_*$ is a commutative Hopf algebra over $E_* = \mathbb{F}_p$; of course $H^*(H) = \mathcal{A}$ is a cocommutative Hopf algebra.
Consider the case of $H_*(\mathbb{RP}^\infty) = H_*(\mathbb{RP}^\infty; \mathbb{F}_2)$. The action of $\mathcal{A}$ on $H^*(\mathbb{RP}^\infty) = H^*(\mathbb{RP}^\infty; \mathbb{F}_2) = \mathbb{F}_2[z]$ is given by

$$\mathrm{Sq}^r z^s = \binom{s}{r} z^{r+s}.$$

If $a_n \in H_n(\mathbb{RP}^\infty)$ is dual to $z^n$ then the left action is determined by

$$\chi(\mathrm{Sq}^r) a_t = \binom{t-r}{r} a_{t-r}.$$

Here $\mathcal{A}_* = \mathbb{F}_2[\zeta_1, \zeta_2, \ldots]$ with $\xi_k \in \mathcal{A}_{2^k-1}$ and the coproduct is

$$\psi\zeta_n = \zeta_n \otimes 1 + \sum_{i=1}^{n-1} \zeta_i \otimes \zeta_{n-i}^{2^i} + 1 \otimes \zeta_n.$$

The coaction $H_*(\mathbb{RP}^\infty) \to H_*(\mathbb{RP}^\infty) \otimes \mathcal{A}_*$ is given by

$$\psi a_n = a_n \otimes 1 + \sum_{i=1}^{n-1} a_i \otimes [\zeta(T)^i]_{T^n}, \quad \zeta(T) = \sum_{i=0}^{\infty} \zeta_i T^{2^i}.$$

## $BP$ for a prime $p$

The Thom spectrum $MU$ turns out to be very important not least because $MU_* = \pi_*(MU)$ has an algebraic universality property connected with formal group laws. For a prime $p$ there is an idempotent self map of $MU_{(p)}$ which defines the Brown-Peterson spectrum $BP$ and gives a good hold on $BP_*BP$. Here are some important facts.

$$\begin{aligned} BP_* &= \mathbb{Z}_{(p)}[v_i : i \geqslant 1], \qquad & |v_i| = 2(p^i - 1),\ v_0 = p. \\ BP_*BP &= BP_*[t_i : i \geqslant 1], \qquad & |t_i| = 2(p^t - 1),\ t_0 = p; \end{aligned}$$

There are lots of recursive formulae for the coproduct and antipode and explicit choices for the generators $v_i$ and $t_i$.
There is a map of homotopy ring spectra $BP \to H = H\mathbb{F}_p$ which induces a ring homomorphism $BP_*BP \to H_* = \mathcal{A}_*$ with

$$v_i \mapsto 0, \quad t_i \mapsto \zeta_i.$$

# Lubin-Tate and Morava $K$-theories

For each prime $p$ and $n \geqslant 1$ there is a commutative ring spectrum $E_n$ (the $n$-th Lubin-Tate spectrum for $p$) with homotopy ring

$$(E_n)_* = \pi_*(E_n) = \mathrm{W}(\mathbb{F}_{p^n})[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}]$$

where $u_i \in (E_n)_0$ and $u \in (E_n)_2$. There is also a ring spectrum $K_n$ (not even homotopy commutative when $p = 2$) which is an algebra over $E_n$ and has homotopy ring

$$(K_n)_* = \pi_*(K_n) = \mathbb{F}_{p^n}[u, u^{-1}].$$

So $K_n$ is a kind of residue (skew) field for $E_n$. In fact $E_n$ is $K_n$-local. The operation algebra $E_n^*(E_n)$ is a kind of 'twisted pro-group ring' for a certain $p$-adic Lie group $\mathbb{G}_n$ which acts continuously on $(E_n)_*$. The correct dual object is

$$E_{n*}^{\vee} E_n = \pi_*(\mathrm{L}_{K_n}(E_n \wedge E_n)).$$

An easier object to think about is the original 'Morava stabiliser algebra' whose degree 0 part is a Hopf algebra over $\mathbb{F}_{p^n}$:

$$(K_n)_0(E_n) = \mathbb{F}_{p^n}[t_i : i \geqslant 1]/(t_i^{p^n} - t_i : i \geqslant 1).$$

# Adams(-Novikov) spectral sequences

This construction works for any $R$ so we'll leave it out of notation or take $R = S$.

For a spectrum $X$ want to calculate $\pi_*(X)$ or more generally $\mathscr{D}(Y, X)^*$. Let $E$ be a homotopy commutative ring spectrum and let $\overline{E}$ be the cofibre of the unit $S \to E$.

Form a resolution

$$\begin{tikzcd}
X_0 & & X_1 \arrow[ll, "\circ"'] \arrow[dr] & & X_2 \arrow[ll, "\circ"'] \arrow[dr] & & X_3 \arrow[ll, "\circ"'] & \cdots \arrow[l] \\
& W_0 \arrow[ur] & & W_1 \arrow[ur] & & W_2 \arrow[ur] &
\end{tikzcd}$$

where $X_0 = X$, $W_s = E \wedge X_s$ and so $X_{s+1} = \overline{E} \wedge X_s$. Notice that for each $s$, the unit map is split by multiplication.

$$\begin{tikzcd}
\pi_*(E \wedge X_s) \arrow[r, "\mathrm{unit}\wedge 1"] \arrow[dr, equal] & \pi_*(E \wedge E \wedge X_s) \arrow[d] \\
& \pi_*(E \wedge X_s)
\end{tikzcd}$$

The maps

$$X \longleftarrow \Sigma^{-1}X_1 \longleftarrow \Sigma^{-2}X_2 \longleftarrow \Sigma^{-3}X_3 \longleftarrow \cdots$$

induce a filtration on $\pi_*(X)$ and there is an associated spectral sequence.

If we assume that $E_*E = E_*(E)$ is $E_*$-flat then the $\mathrm{E}_2$-term has an algebraic form, namely

$$\mathrm{E}_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{E_*E}(E_*, E_*X),$$

where the Ext is calculated in the category of right $E_*E$-comodules using weakly injective resolutions.

Under enough assumptions we can replace $E_*E$-comodules with $E^*E$-modules and even replace homology with cohomology. For example, when $E = H = H\mathbb{F}_p$ we can take

$$\mathrm{E}_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{\mathcal{A}}(H^*X, \mathbb{F}_p).$$

In general convergence is a complicated issue, but in good situations the target is $\pi_*(X_E)$, the homotopy of the Bousfield $E$-localisation of $X$, $X_E$.

# Lubin-Tate spectra

For the Lubin-Tate spectrum $E_n$ and related ring spectra, the Adams $\mathrm{E}_2^{*,*}(X)$ can be interpreted as continuous group cohomology. Here the target is $\pi_*(X_{K_n})$ and the input is

$$E_n^\vee X = \pi_*((E_n \wedge X)_{K_n})$$

which for finite $X$ is a topological $(E_n)_*$-module on which the Morava stabiliser group $\mathbb{G}_n$ acts continuously. Then

$$\mathrm{E}_2^{s,t}(X) \cong \mathrm{H}_{\mathrm{c}}^s(\mathbb{G}_n; E_t^\vee X) \Longrightarrow \pi_{t-s}(X_{K_n}),$$

where for a profinte group $G$ and a topological $G$-module $M$, $\mathrm{H}_{\mathrm{c}}^*(G; M)$ is continuous cohomology of a topological $G$-module; when $M$ is discrete this agrees with Galois cohomology

$$\mathrm{H}_{\mathrm{c}}^*(G; M) = \operatorname*{colim}_{\substack{N \lhd G \\ |G:N| < \infty}} \mathrm{H}_{\mathrm{c}}^*(G/N; M^N).$$