Fun with Dyer-Lashof operations

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Power operations and coactions

Recall the extended power construction for $n \geq 1$:

$$D_n X = E \Sigma_n \ltimes \Sigma_n X^{(n)} = E \Sigma_n \ltimes \Sigma_n X \wedge \cdots \wedge X.$$ 

For a commutative $S$-algebra $R$ and an $R$-module $Y$, this extends to a construction

$$D_n^R Y = E \Sigma_n \ltimes \Sigma_n Y \wedge^n.$$ 

Suppose that $A, B, E$ are commutative $S$-algebras (it is enough to assume that $E$ is $H_\infty$). There are maps $D_nA \to A$, $D_nB \to B$ and $D_nE \to E$ which give rise to a diagram of $A$-module morphisms.

\[ A \wedge D_n E \xrightarrow{I \wedge \mu_n} A \wedge E \]

\[ D_n^A(A \wedge E) \]

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If $x: S^m \to A \land E$, then the composition of solid arrows in the commutative diagram

$$
\begin{align*}
A \land D_n S^m & \xrightarrow{id \land D_n x} A \land D_n (A \land E) & \xrightarrow{I \land \mu_n} & A \land A \land E & \to & A \land E \\
A \land D_n S^m & \xrightarrow{id \land D_n x} A \land D_n (A \land E) & \xrightarrow{I \land \mu_n} & A \land A \land E & \to & A \land E \\
\end{align*}
$$

defines a power operation

$$
\Theta^e: A_m(E) \to A_k(E); \quad \Theta^e(x) = \tilde{x}_* e
$$

for each element $e \in A_k(D_n S^m) = \pi_k(A \land D_n S^m)$.

**Remark:** it is well-known that

$$
D_2 S^m \sim \Sigma^m \mathbb{RP}^\infty = \Sigma^m \mathbb{RP}^\infty / \mathbb{RP}^{m-1}.
$$
Generalised coactions

Let \( A, B \) be two commutative ring spectra (later we will take them to be \( \mathcal{E}_\infty \)). For any spectrum \( X \), the unit \( S \to B \) induces

\[
A \wedge X \cong A \wedge S \wedge X \to A \wedge B \wedge X
\]

and so a homomorphism

\[
A_*(X) \to A_*(B \wedge X) \cong B_*(X \wedge A).
\]

If \( B_*(A) \) is \( B_* \)-flat then

\[
A_*(B) \boxtimes_{B_*} B_*(X) \cong A_*(B \wedge X) \cong B_*(X) \otimes_{B_*} B_*(A),
\]

where \( \boxtimes \) denotes bimodule tensor product and \( \otimes \) denotes left module tensor product. If \( A = B \), this gives the familiar left \( A_*(A) \)-comodule structure \( \psi : A_*(X) \to A_*(A) \boxtimes_{A_*} A_*(X) \) and a right analogue \( \tilde{\psi} : A_*(X) \to A_*(X) \otimes_{A_*} A_*(A) \).
Now assume that $A, B, E$ are $\mathcal{E}_\infty$ and $B_*(A)$ is $B_*$-flat. The following commutative diagram is the source of formulae showing the interaction between power operations and right coactions.

\[
\begin{array}{ccccccccc}
A_* (S \wedge D_n(A \wedge E)) & \cong & A_* (D_n(S \wedge A \wedge E)) & \rightarrow & A_* (S \wedge A \wedge E) & \rightarrow & A_* (S \wedge E) \\
(id \wedge D_n^x)_* & & & & & & \\
A_* (S \wedge D_n S^m) & & & & & & \\
(id \wedge D_n^x)_* & & & & & & \\
A_* (B \wedge D_n S^m) & & & & & & \\
(id \wedge D_n^x)_* & & & & & & \\
A_* (B \wedge D_n(A \wedge E)) & \cong & A_* (D_n B (B \wedge (A \wedge E))) & \rightarrow & A_* (B \wedge (A \wedge E)) & \rightarrow & A_* (B \wedge E) \\
B_*(E) \otimes_{B_*} B_*(A) \\
\end{array}
\]
We will give explicit formulae in the case \( A = B = H\mathbb{F}_2 \). Here

\[
A_\ast (A) = A_\ast = \mathbb{F}_2[\xi_s : s \geq 1] = \mathbb{F}_2[\zeta_s : s \geq 1]
\]

is the dual Steenrod algebra which is a Hopf algebra over \( \mathbb{F}_2 \). The generators have degrees \( |\xi_s| = |\zeta_s| = 2^s - 1 \) and \( \zeta_s = \chi(\xi_s) \) and we set \( \zeta_0 = \xi_0 = 1 \). The comultiplication is given by

\[
\psi(\xi_s) = \sum_{0 \leq i \leq s} \xi_{s-i} \otimes \xi_i, \quad \psi(\zeta_s) = \sum_{0 \leq i \leq s} \zeta_i \otimes \zeta_{s-i}
\]

and the right coaction is given by

\[
\tilde{\psi}(\xi_s) = \sum_{0 \leq i \leq s} \xi_i \otimes \zeta_{s-i}^i, \quad \tilde{\psi}(\zeta_s) = \sum_{0 \leq i \leq s} \zeta_{s-i}^i \otimes \xi_i.
\]
Here is the formula for the right coaction on a Dyer-Lashof operation applied to an element \( x \in H_m(E) \) and \( s \geq m \):

\[
\tilde{\psi} Q^s(x) = \sum_{k=m}^{s} Q^k(\tilde{\psi}(x)) \left[ \left( \frac{\zeta(t)}{t} \right)^k \right] t^{s-k}.
\]

Here we use the Cartan formula to evaluate Dyer-Lashof operations on tensors and the right hand factor involves the generating function

\[
\zeta(t) = \sum_{i \geq 0} \zeta_i t^{2^i}.
\]

For example, if \( \psi(x) = \sum_i a_i \otimes x_i \) then \( \tilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i) \) and

\[
\tilde{\psi} Q^{m+1}(x) = \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i Q^{m+1}(x_i \otimes \chi(a_i))
\]

\[
= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \sum_j Q^{m+1-j} x_i \otimes Q^j \chi(a_i)).
\]
Dyer-Lashof operations on $A_*$

The Dyer-Lashof action on $A_*$ was determined by Kochman (implicitly) and Steinberger. For example,

$$Q^{2s} \zeta_s = \zeta_{s+1}.$$ 

Other useful formula (which seem not to be well-known) are

$$Q^{2s} \xi_s = \xi_{s+1} + \xi_1 \xi_s^2,$$

$$Q^{2s+1} \zeta_s = \begin{cases} 
\zeta_4 & \text{if } s = 1, \\
0 & \text{otherwise}, 
\end{cases}$$

$$Q^{2s+1} \xi_s = \xi_1^2 \xi_s^2.$$
There is an $\mathcal{E}_\infty$ orientation $MO \to H\mathbb{F}_2$ which induces a ring epimorphism $H_*(MO) \to \mathcal{A}_*$. The Thom isomorphism $H_*(BO) \xrightarrow{\cong} H_*(MO)$ is also a ring isomorphism. In fact these are both compatible with the Dyer-Lashof actions, but the second is not an $\mathcal{A}_*$-comodule homomorphism. The ring $H_*(BO) = \mathbb{F}_2[a_k : k \geq 1]$ can be identified with the ring of symmetric functions where $a_k$ is the $k$-th elementary function. The $k$-th Newton polynomial is defined recursively by

$$N_k(a) = a_1 N_{k-1}(a) - a_2 N_{k-2}(a) + \cdots + (-1)^{k-2} a_{k-1} N_{k-2}(a) + (-1)^{k-1} ka_k$$

$$= a_1 N_{k-1}(a) + a_2 N_{k-2}(a) + \cdots + a_{k-1} N_{k-2}(a) + ka_k.$$

In general, $N_{2k}(a) = N_k(a)^2$. 

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Kochman showed that

$$Q^r N_k(a) = \binom{r - 1}{k - 1} N_{k+r}(a).$$

The elements $$q_k \in H_*(MO)$$ corresponding to the $$N_k(a)$$ satisfy simple formulae for the $$A_\ast$$-coaction, in particular

$$\widetilde{\psi}(q_{2s-1}) = \sum_i q_{2s-1}^{2^i} \otimes \xi_i.$$

Furthermore, under the orientation homomorphism, $$q_{2s-1} \mapsto \zeta_s$$. If we set $$N_k(\xi) = N_k(\xi_1, 0, \ldots, 0, \xi_2, 0, \ldots)$$ where we replace $$a_r$$ by 0 except when $$r = 2^2 - 1$$ when we replace it by $$\xi_s$$, this gives

$$N_k(\xi) = \xi_1 N_{k-1}(\xi) + \xi_2 N_{k-2^2+1}(\xi) + \cdots.$$

Then we get

$$Q^r \zeta_s = Q^r N_{2^2-1}(\xi) = \binom{r - 1}{2^s - 2} N_{2^s-1+r}(\xi)$$

which is non-zero precisely when $$r \equiv 0, -1 \mod 2^s$$. 
Consider the factorisation of the bottom cell of $BO$ through an infinite loop map $S^1 \to QS^1 \xrightarrow{j} BO$. The associated Thom spectrum $Mj$ is an $E_{\infty}$ ring spectrum which is weakly equivalent to the reduced free spectrum $\tilde{P}(S^0 \cup_2 e^1)$, hence for any $E_{\infty}$ ring spectrum $E$ with $\text{char } \pi_0(E) = 2$ there is an $E_{\infty}$ morphism $\tilde{P}(S^0 \cup_2 e^1) \to E$ and we denote this spectrum by $S//2$. The homology of $S//2$ is

$$H_\ast(S//2) = \mathbb{F}_2[Q^lx_1 : l \text{ admissible, } \text{exc}(l) > 1],$$

where $x_1 \in H_1(S//2)$ satisfies $Sq^1 x_1 = 1$. There is a morphism $S//2 \to H\mathbb{F}_2$ which induces $H_\ast(S//2) \to \mathcal{A}_\ast$ so that $x_1 \mapsto \zeta_1$. 
Define a sequence of elements $x_s \in H_{2s-1}(S//2)$ by

$$x_s = Q^{2^{s-1}} x_{s-1} = Q^{2^{s-1}} Q^{2^{s-2}} \cdots Q^{2^2} x_1.$$ 

These form a regular sequence in the polynomial ring $H_*(S//2)$ and generate an ideal $I \triangleleft H_*(S//2)$ whose quotient ring $H_*(S//2)/I$ is polynomial.

The right coaction on the $x_s$ is

$$\tilde{\psi}(x_s) = \sum_{0 \leq i \leq s} x_{s-i}^{2^i} \otimes \xi_i$$

where $x_0 = 1$. Induction shows that $x_s \mapsto \zeta_s \in A_{2^{s-1}}$, so $H_*(S//2) \to A_*$ is an epimorphism.
Theorem

The composition

\[ H_*(S//2) \xrightarrow{\tilde{\psi}} H_*(S//2) \otimes A_* \xrightarrow{\sim} H_*(S//2)/I \otimes A_* \]

is an isomorphism of \( A_* \)-comodule algebras. Hence \( S//2 \) is a wedge of suspensions of \( HF_2 \).

A similar conclusion holds for any \( \mathcal{E}_\infty \) ring spectrum \( E \) with \( \text{char} \ \pi_0(E) = 2 \); this is a result of Steinberger.