Fun with Dyer-Lashof operations

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Power operations and coactions

Recall the extended power construction for $n \geqslant 1$:

$$D_n X = E\Sigma_n \ltimes_{\Sigma_n} X^{(n)} = E\Sigma_n \ltimes_{\Sigma_n} \underbrace{X \wedge \cdots \wedge X}_{n}.$$

For a commutative S-algebra R and an R-module Y, this extends to a construction

$$D_n^R Y = E\Sigma_n \ltimes_{\Sigma_n} Y^{\wedge_n}.$$

Suppose that A, B, E are commutative S-algebras (it is enough to assume that E is H_∞). There are maps $D_n A \to A$, $D_n B \to B$ and $D_n E \to E$ which give rise to a diagram of A-module morphisms.

$$\begin{tikzcd}
A \wedge D_n E \arrow[rr, "I \wedge \mu_n"] \arrow[dr] & & A \wedge E \\
& D_n^A(A \wedge E) \arrow[ur] &
\end{tikzcd}$$

If $x\colon S^m \to A \wedge E$, then the composition of solid arrows in the commutative diagram

$$\begin{tikzcd}[column sep=small]
 & & S^k \arrow[dll, dotted, "e"'] \arrow[drr, dotted, "\Theta^e(x)"] & & \\
A \wedge D_n S^m \arrow[r, "\mathrm{id} \wedge D_n x"] \arrow[rrrr, bend right, "\widetilde{x}"'] & A \wedge D_n(A \wedge E) \arrow[rr, "I \wedge \mu_n"] & & A \wedge A \wedge E \arrow[r] & A \wedge E
\end{tikzcd}$$

defines a power operation

$$\Theta^e\colon A_m(E) \to A_k(E); \quad \Theta^e(x) = \widetilde{x}_* e$$

for each element $e \in A_k(D_n S^m) = \pi_k(A \wedge D_n S^m)$.
Remark: it is well-known that

$$D_2 S^m \sim \Sigma^m \mathbb{R}\mathrm{P}^\infty_m = \Sigma^m \mathbb{R}\mathrm{P}^\infty / \mathbb{R}\mathrm{P}^{m-1}.$$

Generalised coactions

Let A, B be two commutative ring spectra (later we will take them to be $\mathcal{E}_\infty$). For any spectrum X, the unit $S \to B$ induces

$$A \wedge X \cong A \wedge S \wedge X \to A \wedge B \wedge X$$

and so a homomorphism

$$A_*(X) \to A_*(B \wedge X) \cong B_*(X \wedge A).$$

If $B_*(A)$ is B_*-flat then

$$A_*(B) \boxtimes_{B_*} B_*(X) \cong A_*(B \wedge X) \cong B_*(X) \otimes_{B_*} B_*(A),$$

where $\boxtimes$ denotes bimodule tensor product and $\otimes$ denotes left module tensor product. If $A = B$, this gives the familiar left $A_*(A)$-comodule structure $\psi : A_*(X) \to A_*(A) \boxtimes_{A_*} A_*(X)$ and a right analogue $\widetilde{\psi} : A_*(X) \to A_*(X) \otimes_{A_*} A_*(A)$.

Power operations and coactions

Now assume that A, B, E are $\mathcal{E}_\infty$ and $B_*(A)$ is B_*-flat. The following commutative diagram is the source of formulae showing the interaction between power operations and right coactions.

$$
\begin{tikzcd}
A_*(S \wedge D_n(A \wedge E)) \arrow[r, "\cong"] & A_*(D_n(S \wedge A \wedge E)) \arrow[r] \arrow[ddd] & A_*(S \wedge A \wedge E) \arrow[r] \arrow[d, "\cong"'] & A_*(S \wedge E) \arrow[ddd, bend left] \\
A_*(S \wedge D_n S^m) \arrow[u, "(\mathrm{id} \wedge D_n x)_*"] \arrow[d] & & S_*((A \wedge A) \wedge E) \arrow[r] \arrow[d] & S_*(A \wedge E) \arrow[u, "\cong"'] \arrow[d] \\
A_*(B \wedge D_n S^m) \arrow[d, "(\mathrm{id} \wedge D_n x)_*"'] & & B_*((A \wedge A) \wedge E) \arrow[r] & B_*(A \wedge E) \arrow[d, "\cong"'] \\
A_*(B \wedge D_n(A \wedge E)) \arrow[r, "\cong"] & A_*(D_n^B(B \wedge (A \wedge E))) \arrow[r] & A_*(B \wedge (A \wedge E)) \arrow[r] \arrow[u, "\cong"'] & A_*(B \wedge E) \arrow[d, "\cong"] \\
 & & & B_*(E) \otimes_{B_*} B_*(A)
\end{tikzcd}
$$

Eilenberg-Mac Lane spectra

We will give explicit formulae in the case $A = B = H\mathbb{F}_2$. Here

$$A_*(A) = \mathcal{A}_* = \mathbb{F}_2[\xi_s : s \geqslant 1] = \mathbb{F}_2[\zeta_s : s \geqslant 1]$$

is the dual Steenrod algebra which is a Hopf algebra over $\mathbb{F}_2$. The generators have degrees $|\xi_s| = |\zeta_s| = 2^s - 1$ and $\zeta_s = \chi(\xi_s)$ and we set $\zeta_0 = \xi_0 = 1$. The comultiplication is given by

$$\psi(\xi_s) = \sum_{0 \leqslant i \leqslant s} \xi_{s-i}^{2^i} \otimes \xi_i, \quad \psi(\zeta_s) = \sum_{0 \leqslant i \leqslant s} \zeta_i \otimes \zeta_{s-i}^{2^i}$$

and the right coaction is given by

$$\widetilde{\psi}(\xi_s) = \sum_{0 \leqslant i \leqslant s} \xi_i \otimes \zeta_{s-i}^{2^i}, \quad \widetilde{\psi}(\zeta_s) = \sum_{0 \leqslant i \leqslant s} \zeta_{s-i}^{2^i} \otimes \xi_i.$$

Here is the formula for the right coaction on a Dyer-Lashof operation applied to an element $x \in H_m(E)$ and $s \geqslant m$:

$$\widetilde{\psi}\mathrm{Q}^s(x) = \sum_{k=m}^{s} \mathrm{Q}^k(\widetilde{\psi}(x)) \left[\left(\frac{\zeta(t)}{t} \right)^k \right]_{t^{s-k}}.$$

Here we use the Cartan formula to evaluate Dyer-Lashof operations on tensors and the right hand factor involves the generating function

$$\zeta(t) = \sum_{i \geqslant 0} \zeta_i t^{2^i}.$$

For example, if $\psi(x) = \sum_i a_i \otimes x_i$ then $\widetilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i)$ and

$$\begin{aligned}\widetilde{\psi}\mathrm{Q}^{m+1}(x) &= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \mathrm{Q}^{m+1}(x_i \otimes \chi(a_i)) \\ &= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \sum_j \mathrm{Q}^{m+1-j} x_i \otimes \mathrm{Q}^j \chi(a_i)).\end{aligned}$$

Dyer-Lashof operations on $\mathcal{A}_*$

The Dyer-Lashof action on $\mathcal{A}_*$ was determined by Kochman (implicitly) and Steinberger. For example,

$$\mathrm{Q}^{2^s}\zeta_s = \zeta_{s+1}.$$

Other useful formula (which seem not to be well-known) are

$$\begin{aligned}
\mathrm{Q}^{2^s}\xi_s &= \xi_{s+1} + \xi_1\xi_s^2, \\
\mathrm{Q}^{2^s+1}\zeta_s &= \begin{cases} \zeta_1^4 & \text{if } s = 1, \\ 0 & \text{otherwise,} \end{cases} \\
\mathrm{Q}^{2^s+1}\xi_s &= \xi_1^2\xi_s^2.
\end{aligned}$$

Use of symmetric functions

There is an $\mathcal{E}_\infty$ orientation $M\mathrm{O} \to H\mathbb{F}_2$ which induces a ring epimorphism $H_*(M\mathrm{O}) \to \mathcal{A}_*$. The Thom isomorphism $H_*(B\mathrm{O}) \xrightarrow{\cong} H_*(M\mathrm{O})$ is also a ring isomorphism. In fact these are both compatible with the Dyer-Lashof actions, but the second is not an $\mathcal{A}_*$-comodule homomorphism.
The ring $H_*(B\mathrm{O}) = \mathbb{F}_2[a_k : k \geqslant 1]$ can be identified with the ring of symmetric functions where a_k is the k-th elementary function.
The k-th Newton polynomial is defined recursively by

$$\begin{aligned}\mathrm{N}_k(a) =& a_1\mathrm{N}_{k-1}(a) - a_2\mathrm{N}_{k-2}(a) + \cdots + \\ & \qquad (-1)^{k-2}a_{k-1}\mathrm{N}_{k-2}(a) + (-1)^{k-1}ka_k \\ =& a_1\mathrm{N}_{k-1}(a) + a_2\mathrm{N}_{k-2}(a) + \cdots + a_{k-1}\mathrm{N}_{k-2}(a) + ka_k.\end{aligned}$$

In general, $\mathrm{N}_{2k}(a) = \mathrm{N}_k(a)^2$.

Kochman showed that

$$\mathrm{Q}^r \mathrm{N}_k(a) = \binom{r-1}{k-1} \mathrm{N}_{k+r}(a).$$

The elements $q_k \in H_*(M\mathrm{O})$ corresponding to the $\mathrm{N}_k(a)$ satisfy simple formulae for the $\mathcal{A}_*$-coaction, in particular

$$\widetilde{\psi}(q_{2^s-1}) = \sum_i q_{2^{s-1}-1}^{2^i} \otimes \xi_i.$$

Furthermore, under the orientation homomorphism, $q_{2^s-1} \mapsto \zeta_s$. If we set $\mathrm{N}_k(\xi) = \mathrm{N}_k(\xi_1, 0, \ldots, 0, \xi_2, 0, \ldots)$ where we replace a_r by 0 except when $r = 2^2 - 1$ when we replace it by ξ_s, this gives

$$\mathrm{N}_k(\xi) = \xi_1 \mathrm{N}_{k-1}(\xi) + \xi_2 \mathrm{N}_{k-2^2+1}(\xi) + \cdots .$$

Then we get

$$\mathrm{Q}^r \zeta_s = \mathrm{Q}^r \mathrm{N}_{2^s-1}(\xi) = \binom{r-1}{2^s-2} \mathrm{N}_{2^s-1+r}(\xi)$$

which is non-zero precisely when $r \equiv 0, -1 \bmod 2^s$.

Proving a splitting result

Consider the factorisation of the bottom cell of $B\mathrm{O}$ through an infinite loop map $S^1 \to \mathrm{Q}S^1 \xrightarrow{j} B\mathrm{O}$. The associated Thom spectrum Mj is an $\mathcal{E}_\infty$ ring spectrum which is weakly equivalent to the reduced free spectrum $\widetilde{\mathbb{P}}(S^0 \cup_2 e^1)$, hence for any $\mathcal{E}_\infty$ ring spectrum E with $\operatorname{char} \pi_0(E) = 2$ there is an $\mathcal{E}_\infty$ morphism $\widetilde{\mathbb{P}}(S^0 \cup_2 e^1) \to E$ and we denote this spectrum by $S/\!/2$. The homology of $S/\!/2$ is

$$H_*(S/\!/2) = \mathbb{F}_2[\mathrm{Q}^I x_1 : I \text{ admissible, } \operatorname{exc}(I) > 1],$$

where $x_1 \in H_1(S/\!/2)$ satisfies $\operatorname{Sq}^1_* x_1 = 1$. There is a morphism $S/\!/2 \to H\mathbb{F}_2$ which induces $H_*(S/\!/2) \to \mathcal{A}_*$ so that $x_1 \mapsto \zeta_1$.

Define a sequence of elements $x_s \in H_{2^s-1}(S/\!/2)$ by

$$x_s = \mathrm{Q}^{2^{s-1}} x_{s-1} = \mathrm{Q}^{2^{s-1}} \mathrm{Q}^{2^{s-2}} \cdots \mathrm{Q}^{2^2} x_1.$$

These form a regular sequence in the polynomial ring $H_*(S/\!/2)$ and generate an ideal $I \lhd H_*(S/\!/2)$ whose quotient ring $H_*(S/\!/2)/I$ is polynomial.
The right coaction on the x_s is

$$\widetilde{\psi}(x_s) = \sum_{0 \leqslant i \leqslant s} x_{s-i}^{2^i} \otimes \xi_i$$

where $x_0 = 1$. Induction shows that $x_s \mapsto \zeta_s \in \mathcal{A}_{2^2-1}$, so $H_*(S/\!/2) \to \mathcal{A}_*$ is an epimorphism.

Theorem

The composition

$$\begin{tikzcd}
H_*(S//2) \arrow[rr] \arrow[dr, "\widetilde{\psi}"'] & & H_*(S//2)/I \otimes \mathcal{A}_* \\
& H_*(S//2) \otimes \mathcal{A}_* \arrow[ur] &
\end{tikzcd}$$

is an isomorphism of $\mathcal{A}_*$-comodule algebras. Hence $S//2$ is a wedge of suspensions of $H\mathbb{F}_2$.

A similar conclusion holds for any $\mathcal{E}_\infty$ ring spectrum E with $\operatorname{char} \pi_0(E) = 2$; this is a result of Steinberger.