Fun with Dyer-Lashof operations

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Nordic Topology Meeting, Stockholm

(27th-28th August 2015)

based on arXiv:1309.2323

last updated 27/08/2015

Power operations and coactions

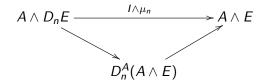
Recall the extended power construction for $n \ge 1$:

$$D_n X = E \Sigma_n \ltimes_{\Sigma_n} X^{(n)} = E \Sigma_n \ltimes_{\Sigma_n} \underbrace{X \land \cdots \land X}_n.$$

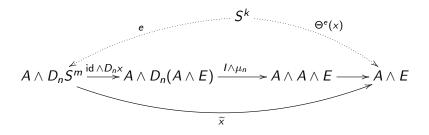
For a commutative S-algebra R and an R-module Y, this extends to a construction

$$D_n^R Y = E \Sigma_n \ltimes_{\Sigma_n} Y^{\wedge_n}.$$

Suppose that A, B, E are commutative S-algebras (it is enough to assume that E is H_{∞}). There are maps $D_nA \rightarrow A$, $D_nB \rightarrow B$ and $D_nE \rightarrow E$ which give rise to a diagram of A-module morphisms.



If $x: S^m \to A \land E$, then the composition of solid arrows in the commutative diagram



defines a power operation

$$\Theta^{e} \colon A_{m}(E) \to A_{k}(E); \quad \Theta^{e}(x) = \widetilde{x}_{*}e$$

for each element $e \in A_k(D_nS^m) = \pi_k(A \wedge D_nS^m)$. **Remark:** it is well-known that

$$D_2 S^m \sim \Sigma^m \mathbb{R} \mathrm{P}_m^\infty = \Sigma^m \mathbb{R} \mathrm{P}^\infty / \mathbb{R} \mathrm{P}^{m-1}$$

Generalised coactions

Let A, B be two commutative ring spectra (later we will take them to be \mathcal{E}_{∞}). For any spectrum X, the unit $S \to B$ induces

$$A \land X \cong A \land S \land X \to A \land B \land X$$

and so a homomorphism

$$A_*(X) o A_*(B \wedge X) \cong B_*(X \wedge A).$$

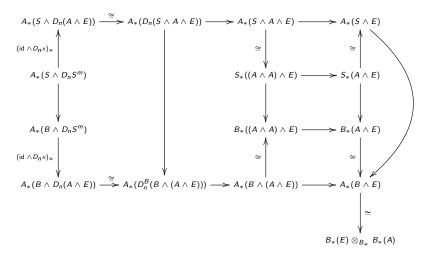
If $B_*(A)$ is B_* -flat then

$$A_*(B) oxtimes_{B_*} B_*(X) \cong A_*(B \wedge X) \cong B_*(X) \otimes_{B_*} B_*(A),$$

where \boxtimes denotes bimodule tensor product and \otimes denotes left module tensor product. If A = B, this gives the familiar left $A_*(A)$ -comodule structure $\psi : A_*(X) \to A_*(A) \boxtimes_{A_*} A_*(X)$ and a right analogue $\widetilde{\psi} : A_*(X) \to A_*(X) \otimes_{A_*} A_*(A)$.

Power operations and coactions

Now assume that A, B, E are \mathcal{E}_{∞} and $B_*(A)$ is B_* -flat. The following commutative diagram is the source of formulae showing the interaction between power operations and right coactions.



We will give explicit formulae in the case $A = B = H\mathbb{F}_2$. Here

$$\mathcal{A}_*(\mathcal{A}) = \mathcal{A}_* = \mathbb{F}_2[\xi_s:s \geqslant 1] = \mathbb{F}_2[\zeta_s:s \geqslant 1]$$

is the dual Steenrod algebra which is a Hopf algebra over \mathbb{F}_2 . The generators have degrees $|\xi_s| = |\zeta_s| = 2^s - 1$ and $\zeta_s = \chi(\xi_s)$ and we set $\zeta_0 = \xi_0 = 1$. The comultiplication is given by

$$\psi(\xi_s) = \sum_{0 \leqslant i \leqslant s} \xi_{s-i}^{2^i} \otimes \xi_i, \quad \psi(\zeta_s) = \sum_{0 \leqslant i \leqslant s} \zeta_i \otimes \zeta_{s-i}^{2^i}$$

and the right coaction is given by

$$\widetilde{\psi}(\xi_s) = \sum_{0 \leqslant i \leqslant s} \xi_i \otimes \zeta_{s-i}^{2^i}, \quad \widetilde{\psi}(\zeta_s) = \sum_{0 \leqslant i \leqslant s} \zeta_{s-i}^{2^i} \otimes \xi_i.$$

Here is the formula for the right coaction on a Dyer-Lashof operation applied to an element $x \in H_m(E)$ and $s \ge m$:

$$\widetilde{\psi} \mathbf{Q}^{s}(x) = \sum_{k=m}^{s} \mathbf{Q}^{k}(\widetilde{\psi}(x)) \left[\left(\frac{\zeta(t)}{t} \right)^{k} \right]_{t^{s-k}}$$

Here we use the Cartan formula to evaluate Dyer-Lashof operations on tensors and the right hand factor involves the generating function

$$\zeta(t)=\sum_{i\geqslant 0}\zeta_it^{2^i}.$$

For example, if $\psi(x) = \sum_i a_i \otimes x_i$ then $\widetilde{\psi}(x) = \sum_i x_i \otimes \chi(a_i)$ and

$$egin{aligned} \widetilde{\psi} \mathrm{Q}^{m+1}(x) &= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \mathrm{Q}^{m+1}(x_i \otimes \chi(a_i)) \ &= \sum_i x_i^2 \otimes \chi(a_i)^2 \zeta_1 + \sum_i \sum_j \mathrm{Q}^{m+1-j} x_i \otimes \mathrm{Q}^j \chi(a_i)). \end{aligned}$$

The Dyer-Lashof action on A_* was determined by Kochman (implicitly) and Steinberger. For example,

$$\mathbf{Q}^{2^{s}}\zeta_{s} = \zeta_{s+1}.$$

Other useful formula (which seem not to be well-known) are

$$Q^{2^{s}}\xi_{s} = \xi_{s+1} + \xi_{1}\xi_{s}^{2},$$

$$Q^{2^{s}+1}\zeta_{s} = \begin{cases} \zeta_{1}^{4} & \text{if } s = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$Q^{2^{s}+1}\xi_{s} = \xi_{1}^{2}\xi_{s}^{2}.$$

Use of symmetric functions

There is an \mathcal{E}_{∞} orientation $MO \to H\mathbb{F}_2$ which induces a ring epimorphism $H_*(MO) \to \mathcal{A}_*$. The Thom isomorphism $H_*(BO) \xrightarrow{\cong} H_*(MO)$ is also a ring isomorphism. In fact these are both compatible with the Dyer-Lashof actions, but the second is not an \mathcal{A}_* -comodule homomorphism. The ring $H_*(BO) = \mathbb{F}_2[a_k : k \ge 1]$ can be identified with the ring of symmetric functions where a_k is the k-th elementary function.

The k-th Newton polynomial is defined recursively by

$$N_{k}(a) = a_{1}N_{k-1}(a) - a_{2}N_{k-2}(a) + \dots + (-1)^{k-2}a_{k-1}N_{k-2}(a) + (-1)^{k-1}ka_{k} = a_{1}N_{k-1}(a) + a_{2}N_{k-2}(a) + \dots + a_{k-1}N_{k-2}(a) + ka_{k}.$$

In general, $N_{2k}(a) = N_k(a)^2$.

Kochman showed that

$$\mathbf{Q}^{r}\mathbf{N}_{k}(\mathbf{a}) = \binom{r-1}{k-1}\mathbf{N}_{k+r}(\mathbf{a}).$$

The elements $q_k \in H_*(MO)$ corresponding to the $N_k(a)$ satisfy simple formulae for the A_* -coaction, in particular

$$\widetilde{\psi}(q_{2^s-1}) = \sum_i q_{2^{s-1}-1}^{2^i} \otimes \xi_i.$$

Furthermore, under the orientation homomorphism, $q_{2^s-1} \mapsto \zeta_s$. If we set $N_k(\xi) = N_k(\xi_1, 0, \dots, 0, \xi_2, 0, \dots)$ where we replace a_r by 0 except when $r = 2^2 - 1$ when we replace it by ξ_s , this gives

$$N_k(\xi) = \xi_1 N_{k-1}(\xi) + \xi_2 N_{k-2^2+1}(\xi) + \cdots$$

Then we get

$$\mathbf{Q}^{r}\zeta_{s} = \mathbf{Q}^{r}\mathbf{N}_{2^{s}-1}(\xi) = \binom{r-1}{2^{s}-2}\mathbf{N}_{2^{s}-1+r}(\xi)$$

which is non-zero precisely when $r \equiv 0, -1 \mod 2^{s}$.

Consider the factorisation of the bottom cell of *BO* through an infinite loop map $S^1 \to QS^1 \xrightarrow{j} BO$. The associated Thom spectrum *Mj* is an \mathcal{E}_{∞} ring spectrum which is weakly equivalent to the reduced free spectrum $\widetilde{\mathbb{P}}(S^0 \cup_2 e^1)$, hence for any \mathcal{E}_{∞} ring spectrum *E* with char $\pi_0(E) = 2$ there is an \mathcal{E}_{∞} morphism $\widetilde{\mathbb{P}}(S^0 \cup_2 e^1) \to E$ and we denote this spectrum by S//2. The homology of S//2 is

$$H_*(S//2) = \mathbb{F}_2[Q'x_1 : I \text{ admissible, } exc(I) > 1],$$

where $x_1 \in H_1(S//2)$ satisfies $Sq_*^1 x_1 = 1$. There is a morphism $S//2 \to H\mathbb{F}_2$ which induces $H_*(S//2) \to A_*$ so that $x_1 \mapsto \zeta_1$.

Define a sequence of elements $x_s \in H_{2^s-1}(S//2)$ by

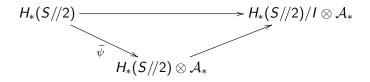
$$x_s = Q^{2^{s-1}} x_{s-1} = Q^{2^{s-1}} Q^{2^{s-2}} \cdots Q^{2^2} x_1$$

These form a regular sequence in the polynomial ring $H_*(S//2)$ and generate an ideal $I \lhd H_*(S//2)$ whose quotient ring $H_*(S//2)/I$ is polynomial. The right coaction on the x_s is

$$\widetilde{\psi}(x_s) = \sum_{0 \leqslant i \leqslant s} x_{s-i}^{2^i} \otimes \xi_i$$

where $x_0 = 1$. Induction shows that $x_s \mapsto \zeta_s \in \mathcal{A}_{2^2-1}$, so $H_*(S//2) \to \mathcal{A}_*$ is an epimorphism.

Theorem The composition



is an isomorphism of A_* -comodule algebras. Hence S//2 is a wedge of suspensions of $H\mathbb{F}_2$.

A similar conclusion holds for any \mathcal{E}_{∞} ring spectrum E with char $\pi_0(E) = 2$; this is a result of Steinberger.