Power operations in completed K-theory

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K-theory completed at a prime

Let p be a prime, and let $K = KU_{(p)}$ be the p-localisation of KU. This gives rise to a homology theory $K_*(-) = \pi_*(K \wedge -)$. Then *$p$-completed K-theory* is the covariant homotopy functor defined by

$$K^\vee_*(-) = \pi_*\left((K \wedge -)\widehat{_p}\right).$$

In practise we replace the $\mathbb{Z}$-grading by a $\mathbb{Z}/2$-grading, to obtain $K^\vee_\bullet(-)$. This takes values in $\mathbb{Z}$-graded *L-complete* $\mathbb{Z}_p$-modules. Here the covariant endofunctors L_s ($s \geqslant 0$) on $\mathbb{Z}_p$-modules are the left derived functors of p-adic completion, and the $\mathbb{Z}_p$-module M is said to be *L-complete* if the natural homomorphism $M \to L_0 M$ is an isomorphism. The symmetric monoidal abelian category of all L-complete $\mathbb{Z}/2$-graded $\mathbb{Z}_p$-modules will be denoted by $\mathscr{M}$.

Continuous actions and coactions

The cooperation algebra $K_*(K)$ is known to satisfy $K_{\mathrm{odd}}(K) = 0$, and $K_0(K)$ is free as a left or right $K_0 = \mathbb{Z}_{(p)}$-module, hence $K^\vee_\bullet(K) = K_\bullet(K)^\wedge_p$ is *pro-free*.
Because pro-free L-complete $\mathbb{Z}_p$-modules are flat on $\mathscr{M}$, the left $K_*(K)$-coaction on $K_*(-)$ extends to a coaction

$$K^\vee_\bullet(-) \to K^\vee_\bullet(K) \widehat{\otimes} K^\vee_\bullet(-).$$

This is dual to a continuous action of the pro-group ring $\mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ via Adams operations indexed on the p-adic units $\mathbb{Z}_p^\times$.

$$
\begin{aligned}
K_0(K) &= \{f(w) \in \mathbb{Q}[w, w^{-1}] : f\mathbb{Z}_{(p)}^{\times} \subseteq \mathbb{Z}_{(p)}\} \\
&= \{f(w) \in \mathbb{Q}[w, w^{-1}] : f\mathbb{Z}_{p}^{\times} \subseteq \mathbb{Z}_{p}\},
\end{aligned}
$$

and

$$
K_0^{\vee}(K) = \mathrm{Cont}(\mathbb{Z}_p^{\times}, \mathbb{Z}_p).
$$

This is complete with respect to the p-adic sup-norm. Here we can view w as the inclusion function $\mathbb{Z}_p^{\times} \to \mathbb{Z}_p$.
There is an continuous operation

$$
\Theta\colon K_0^{\vee}(K) \to K_0^{\vee}(K); \qquad \Theta(f) = \frac{f - f^p}{p}.
$$

We define a sequence of elements θ_n $(n \geqslant 0)$ by $\theta_0 = w$ and for $n \geqslant 1$,

$$
\theta_n = \Theta(\theta_{n-1}).
$$

Then the monomials $\theta_0^{r_0}\theta_1^{r_1}\cdots\theta_\ell^{r_\ell}$ with $r_j = 0, 1, \ldots, p-1$ form a topological basis for $K_0^{\vee}(K)$.

Power operations

The spectra $KU, K, K_p^{\widehat{}}$ are all E_∞ ring spectra and so their associated homology theories admit power operations.

McClure/Barthels-Frankland: There is a power operation $\mathrm{Q}\colon K^\vee_\bullet(-) \to K^\vee_\bullet(-)$ defined on the functor $K^\vee_\bullet(-)$ on E_∞ (or H_∞) ring spectra and enjoying various properties including the following.

If $|x| = |y| = 0$,

$$\mathrm{Q}(x+y) = \mathrm{Q}\,x + \mathrm{Q}\,y - \sum_{1\leqslant r\leqslant p-1} \frac{1}{p}\binom{p}{r} x^r y^{p-r},$$

$$\mathrm{Q}(xy) = y^p\,\mathrm{Q}\,x + x^p\,\mathrm{Q}\,y + \mathrm{Q}\,x\,\mathrm{Q}\,y.$$

If $a \in \mathbb{Z}_p^\times$ and $|x| = 0$,

$$\psi^a\,\mathrm{Q}\,x = \mathrm{Q}(\psi^a x),$$

$$\mathrm{Q}(ax) = a\,\mathrm{Q}(x) + \frac{(a-a^p)}{p}x^p.$$

Power operations on the cooperation algebra

For $a \in \mathbb{Z}_p^\times$, the action of ψ^a on $K_0^\vee(K) = \mathrm{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ satisfies,

$$\psi^a f = f(a^{-1}-),$$

so for example,

$$\psi^a w^r = a^{-r} w^r.$$

Theorem

The action of Q *on* $K_0^\vee(K)$ *is given by*

$$\mathrm{Q}\, f = \frac{(f - f^p)}{p} = \Theta(f).$$

This gives the iteration $\theta_0 = w$, and for $n \geqslant 0$,

$$\mathrm{Q}\, \theta_n = \theta_{n+1}.$$

The operator Q makes $K_0^\vee(K)$ into a *p-complete* $\mathbb{Z}_p$*-*θ*-algebra*.

Theorem

The p-complete $\mathbb{Z}_p$*-*θ*-algebra* $K_0^\vee(K)$ *is generated by the element* w*. Hence* $K_0^\vee(K)$ *is a quotient of a monogenic free p-complete* $\mathbb{Z}_p$*-*θ*-algebra,*

$$K_0^\vee(K) \cong$$

$$\mathbb{Z}_p[\theta^s(x) : s \geqslant 0]\widehat{_p} \Big/ (\!((\theta^s(x))^p - \theta^s(x) + p\theta^{s+1}(x) : s \geqslant 0)\!).$$

Completed K-theory of free algebras

For a spectrum X, the free algebra on X is

$$\mathbb{P}X = \bigvee_{r \geqslant 0} X^{(r)}/\Sigma_r.$$

The functor $\mathbb{P}\colon \mathscr{M}_S \to \mathscr{M}_S$ preserves pushouts so sends cell S-modules to cell commutative S-algebras/E_∞ ring spectra. In particular the two notions of skeleta correspond,

$$\mathbb{P}(X^{[n]}) = (\mathbb{P}X)^{\langle n \rangle}.$$

Theorem

If X has finitely many even dimensional cells, then $K^\vee_\bullet(X)$ is free on a finite basis of even degree elements x_i say, and

$$K^\vee_\bullet(\mathbb{P}X) = \mathbb{Z}_p[\mathrm{Q}^s x_i : s \geqslant 0,\ i]^{\widehat{}}_p.$$

There is also a reduced free algebra functor $\widetilde{\mathbb{P}}\colon S^0/\mathscr{M}_S \to S^0/\mathscr{M}_S$ on the comma category under the cofibrant 0-sphere.

Mapping cones on image of J elements

Suppose that $f\colon S^{2n-1} \to S^0$ is a map. We can form the classical mapping cone C_f and also the free algebras $\mathbb{P}C_f, \widetilde{\mathbb{P}}C_f$.
We have

$$K^\vee_\bullet(C_f) = \mathbb{Z}_p\{x_0, x_{2n}\},$$

and then

$$K^\vee_\bullet(\mathbb{P}C_f) = \mathbb{Z}_p[\mathrm{Q}^s x_0, \mathrm{Q}^s x_{2n} : s \geqslant 0]\widehat{_p},$$
$$K^\vee_\bullet(\widetilde{\mathbb{P}}C_f) = \mathbb{Z}_p[\mathrm{Q}^s x_{2n} : s \geqslant 0]\widehat{_p}.$$

If f has non-trivial e-invariant then there is a non-trivial coaction on x_{2n}.

Example

Take $p = 2$ and let $f\colon S^1 \to S^0$ be the Hopf map η. Then in

$$K_0^\vee(C_f) = \mathbb{Z}_p\{x_0, x_2\},$$

the coaction is

$$\Psi x_2 = \frac{1-w}{2} \otimes x_0 + w \otimes x_2.$$

If we introduce the elements

$$\Theta_0 = \frac{1-w}{2}, \quad \Theta_n = \Theta(\Theta_{n-1}),$$

then the monomials $\Theta_0^{r_0} \cdots \Theta_\ell^{r_\ell}$ with $r_j = 0, 1$ form a topological basis for $K_0^\vee(K)$.

There is a map $C_\eta \to K$ which extends to a unique morphism of algebras $\widetilde{\mathbb{P}}C_\eta \to K$. In $K_0^\vee(-)$ we have $x_2 \mapsto \Theta_0$, hence

$$\mathrm{Q}^s x_2 \mapsto \mathrm{Q}^s \Theta_0 = \Theta_s.$$

It follows that this map of θ-algebras is surjective, and realises the quotient homomorphism from the free θ-algebra,

$$\mathbb{Z}_2[\theta^s(x) : s \geqslant 0]_2^{\widehat{}} \longrightarrow$$
$$\mathbb{Z}_2[\theta^s(x) : s \geqslant 0]_2^{\widehat{}} \Big/ (((\theta^s(x))^2 - \theta^s(x) + 2\theta^{s+1}(x) : s \geqslant 0)).$$

Similar results apply on replacing η by ν, and for p odd, by α_1.