Duals of P-algebras and their comodules

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Some references

- J. C. Moore & F. P. Peterson, Nearly Frobenius algebras, Poincaré algebras and their modules, J. Pure Appl. Algebra, **3** (1973), 83–93.
- H. R. Margolis, Spectra and the Steenrod Algebra: Modules over the Steenrod algebra and the stable homotopy category, North-Holland, (1983).

P-algebras

Let $\mathbb{k}$ be a field. A graded cocommutative Hopf algebra $A = A^*$ is a *P-algebra* if

- it is connected (i.e., $A^n = 0$ if $n < 0$ and $A^0 = \mathbb{k}$);
- finite type;
- a union of finite dimensional subHopf algebras $A(n) = A(n)^*$ where $A(n) \subset A(n+1)$.

Here each $A(n)$ is a Poincaré (duality) algebra and $A(n+1)$ is a free left/right $A(n)$-module.

Theorem

Let A be P-algebra.
(a) A is a free and injective left/right $A(n)$-module.
(b) A is a coherent $\mathbb{k}$-algebra.
(c) A is self-injective. More generally, every bounded below free module is injective.

More properties

Every coherent A-module M is finitely presented and there is a finite presentation

$$P_1 \to P_0 \to M \to 0$$

which is induced up from a finite presentation

$$P_1' \to P_0' \to M' \to 0$$

of a finitely generated $A(m)$-module for some m, so there is an exact sequence

$$A \otimes_{A(m)} P_1' \to A \otimes_{A(m)} P_0' \to M \to 0.$$

Similarly, every homomorphism $M \to N$ between coherent A-modules are induced up from homomorphisms between finitely generated $A(n)$-modules for some n.

Proposition

Let M be a coherent A-module. Then there is an embedding of M into a finitely generated free A-module which is also injective. Hence M has an injective resolution by finitely generated free A-modules.

Proof.

Let M_0 be a finitely generated $A(n)$-module with $M \cong A \otimes_{A(n)} M_0$. By a standard result $M_0 \hookrightarrow F$ where F is a finitely generated free $A(n)$-module, so $M \cong A \otimes_{A(n)} M_0 \hookrightarrow A \otimes_{A(n)} F$ is a monic since A is $A(n)$-flat. $\square$

An A-module is called *finite* if it is finite dimensional over $\Bbbk$.

Proposition

Let M be a finite A-module and N an A-module.

- *If N is bounded below and free, then*

$$\mathrm{Ext}_A^*(M, N) = \mathrm{Hom}_A(M, N) = 0;$$

- *If N is coherent, then*

$$\mathrm{Ext}_A^*(M, N) = 0.$$

Proof.

Since bounded below free modules are injective, $\mathrm{Ext}_A^s(M, N) = 0$ when $s > 0$, so it suffices to show that $\mathrm{Hom}_A(M, A) = 0$. For this, note that the image of a non-trivial homomorphism must lie in some $A(n)$ and have a highest degree element. But also $A(n) \subseteq A(n+1)$ and so by Poincaré duality there is a non-trivial product with an element of $A(n+1)$.
For the other part use an injective resolution of N consisting of finitely generated free modules. □

Note: This proof exploits a special case of the following general property of a P-algebra: for each non-zero $a \in A$ there are positive degree elements u, v for which ua, av are non-zero.

P_*-algebras

A *P_*-algebra* A_* is the degree-wise dual of a P-algebra A, i.e., $A_n = \mathrm{Hom}_{\Bbbk}(A^n, \Bbbk)$. It inherits the structure of a commutative Hopf algebra.

Every left A_*-comodule M_* is naturally a right A-module, but we can also dualise it to give a left A-module M^* where $M^n = \mathrm{Hom}_{\Bbbk}(M_n, \Bbbk)$.

A A_*-comodule M_* is *coherent* if M^* is a coherent A-module.

Because we have assumed A and A_* are finite type, A_* is a projective A_*-comodule. Furthermore, every coherent comodule admits a projective comodule resolution by finitely generated cofree comodules.

Proposition

Let M_* be a finite A_*-comodule and N_* an A_*-comodule.

- *If N_* is bounded below and cofree, then*

$$\mathrm{Coext}^*_{A_*}(N_*, M_*) = \mathrm{Cohom}^*_{A_*}(N_*, M_*) = 0;$$

- *If N_* is coherent, then*

$$\mathrm{Coext}^*_{A_*}(N_*, M_*) = 0.$$

We will need some 'change of rings' spectral sequences for computing such Coext groups. It is well known that $\mathrm{Coext}^*_{A_*}(N_*, -)$ is computable using injective resolutions where a comodule is injective if it is a retract of a cofree comodule $A_* \otimes W$. Usually projective comodules are not available for infinite dimensional A_*, although we have seen that in certain situations they may exist. In general this means that $\mathrm{Coext}^*_{A_*}(-, -)$ is not a balanced functor.

Let $\mathbf{Mod}_{A}^{\natural,\mathrm{f.t.}}$ denote the category of finite type bounded below homologically graded A^*-modules (with A^* acting by decreasing degree), $\mathbf{Mod}_{A^*}^{\flat,\mathrm{f.t.}}$ denote the category of finite type bounded below cohomologically graded A^*-modules and $\mathbf{Mod}_{A^*}$ denoting the category of all A^*-modules. There is a commutative diagram of functors in which all functors are exact.

$$\begin{array}{ccc}
 & \nearrow & \mathbf{Mod}_{A^*}^{\sharp,\mathrm{f.t.}} \\
 & & (-)_* \uparrow\downarrow (-)^* \\
\mathbf{Comod}_{A_*}^{\flat,\mathrm{f.t.}} & \underset{(-)_*}{\overset{(-)^*}{\rightleftarrows}} & (\mathbf{Mod}_{A^*}^{\flat,\mathrm{f.t.}})^{\mathrm{op}} \\
 & & \downarrow \\
 & \searrow & \mathbf{Mod}_{A^*}^{\mathrm{op}}
\end{array}$$

So in the case of finite type bounded below comodules we can set

$$\mathrm{Coext}_{A_*}^{*}(N_*, M_*) = \mathrm{Ext}_{A^*}^{*}(M^*, N^*)$$

where $\mathrm{Ext}_{A^*}^{*}(-,-)$ is a balanced bifunctor on $\mathbf{Mod}_{A^*}$. There are four Cartan-Eilenberg spectral sequences for computing this, two depending on injective resolutions and two on projective ones.

Cartan-Eilenberg spectral sequences

There are two ways to set up Cartan-Eilenberg spectral sequences for computing $\mathrm{Ext}^*_S(M,N)$ where $R \to S \to S/\!/R = S \otimes_R \Bbbk$ is a normal sequence of Hopf algebras over a field $\Bbbk$ and S is a free R-module. For a left $S/\!/R$-module L and a left S-module M there is a spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^s_{S/\!/R}(L, \mathrm{Ext}^t_R(\Bbbk, M)) \Longrightarrow \mathrm{Ext}^{s+t}_S(L, M).$$

For left S-module M and left $S/\!/R$-module N there is a spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Ext}^s_{S/\!/R}(\mathrm{Tor}^t_R(\Bbbk, M), N) \Longrightarrow \mathrm{Ext}^{s+t}_S(M, N).$$

These can be defined as composite functor spectral sequences using injective and projective resolutions of M and standard adjunctions

$$\mathrm{Hom}_S(L, -) \cong \mathrm{Hom}_{S/\!/R}(L, \mathrm{Hom}_R(\Bbbk, -)),$$
$$\mathrm{Hom}_S(-, N) \cong \mathrm{Hom}_{S/\!/R}(\Bbbk \otimes_R (-), N).$$

For the comodule version, suppose $K\backslash\backslash H \rightarrowtail H \twoheadrightarrow K$, is a sequence of commutative Hopf algebras over $\Bbbk$, where

$$K\backslash\backslash H = \Bbbk \square_K H = H \square_K \Bbbk \subseteq H.$$

There are adjunctions

$$\mathrm{Cohom}_H(M, -) \cong \mathrm{Hom}_{K\backslash\backslash H}(M, \Bbbk\square_K(-)),$$
$$\mathrm{Cohom}_H(-, N) \cong \mathrm{Hom}_S(\Bbbk\square_K(-), N).$$

Let M be a left $K\backslash\backslash H$-comodule and N a left H-comodule. Then there is a spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^s_{K\backslash\backslash H}(M, \mathrm{Cotor}^t_K(\Bbbk, N)) \Longrightarrow \mathrm{Coext}^{s+t}_H(M, N).$$

If N is a trivial K-comodule then

$$\mathrm{E}_2^{s,t} \cong \mathrm{Coext}^s_{K\backslash\backslash H}(M, \mathrm{Cotor}^t_K(\Bbbk, \Bbbk) \overset{K\backslash\backslash H}{\wedge} N).$$

Here $U \overset{C}{\wedge} V$ indicates the tensor product of two comodules over a commutative Hopf algebra C with diagonal coaction

$$U \otimes V \to (C \otimes U) \otimes (C \otimes V) \xrightarrow{\cong} C \otimes C \otimes U \otimes V \to C \otimes U \otimes V.$$

Let M be a left H-comodule which admits a projective resolution and let N be a left $K\backslash\backslash H$-comodule. There is a spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^s_{K\backslash\backslash H}(\mathrm{Cotor}^t_K(\mathbb{k}, M), N) \Longrightarrow \mathrm{Coext}^{s+t}_H(M, N).$$

If M is a trivial K-comodule then

$$\mathrm{E}_2^{s,t} \cong \mathrm{Coext}^s_{K\backslash\backslash H}(\mathrm{Cotor}^t_K(\mathbb{k}, \mathbb{k}) \overset{K\backslash\backslash H}{\wedge} M, N).$$

The condition that M admits a projective resolution is crucial; when H is a P_*-algebra it is satisfied by a coherent comodule M.

Topological examples

For each prime p, the mod p Steenrod algebra is a P-algebra. When $p = 2$,

$$\mathcal{A} = \bigcup_{n \geqslant 0} \mathcal{A}(n)$$

where $\mathcal{A}(n)$ is the finite dimensional subHopf algebra generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n}$, with

$$\dim \mathcal{A}(0) = 2,\ \dim \mathcal{A}(1) = 8,\ \dim \mathcal{A}(2) = 64, \ldots$$

Many subHopf algebras and quotient Hopf algebras of $\mathcal{A}$ are P-algebras; for example, the primitively generated subHopf algebra $\mathcal{E} \subseteq \mathcal{A}$ generated by the Milnor primitives.

The dual Steenrod algebra

The commutative Hopf algebra $\mathcal{A}_*$ is polynomial:

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r : r \geqslant 1] = \mathbb{F}_2[\zeta_r : r \geqslant 1],$$

where $\xi_r, \zeta_r \in \mathcal{A}_{2^r-1}$ and $\zeta_r = \chi(\xi_r)$. The coproduct and antipode satisfy

$$\psi(\xi_n) = \sum_{0\leqslant j\leqslant n} \xi_{n-j}^{2^j} \otimes \xi_j, \quad \psi(\zeta_n) = \sum_{0\leqslant j\leqslant n} \zeta_j \otimes \zeta_{n-j}^{2^j},$$

$$\zeta_n = \sum_{1\leqslant k\leqslant n} \xi_k \zeta_{n-k}^{2^k}.$$

The non-zero primitives are the elements $\xi_1^{2^s} = \zeta_1^{2^s}$.
The dual of $\mathcal{A}(n)$ is the quotient Hopf algebra

$$\begin{aligned}\mathcal{A}(n)_* &= \mathcal{A}_*/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \ldots) \\ &= \mathcal{A}_*//\mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \ldots, \zeta_{n+1}^2, \zeta_{n+2}, \ldots].\end{aligned}$$

For each $s \geqslant 0$ there is a subHopf algebra

$$\mathcal{A}_*^{(s)} = \mathbb{F}_2[\zeta_1^{2^s}, \zeta_2^{2^s}, \ldots, \zeta_n^{2^s}, \ldots] \subseteq \mathcal{A}_*$$

with dual $\mathcal{A}_{(s)} = \mathcal{A}_{(s)}^*$ a quotient Hopf algebra of $\mathcal{A}$. Each $\mathcal{A}_{(s)}$ is a P-algebra and $\mathcal{A}_*^{(s)}$ is a P_*-algebra.
There is also a family of finitely subHopf algebras

$$\mathcal{P}(n)_*^{(s)} = \mathbb{F}_2[\zeta_1^{2^s}, \zeta_2^{2^s}, \ldots, \zeta_n^{2^s}] \subseteq \mathcal{P}(\infty)_*^{(s)} = \mathcal{A}_*^{(s)}.$$

All of these are P_*-algebras as are the quotients $\mathcal{A}_*^{(s)}//\mathcal{P}(n)_*^{(s)}$.

Some sample calculations

The Adams spectral sequence for calculating homotopy classes of maps has the form

$$\mathrm{E}_2^{s,t}(X,Y) = \mathrm{Coext}_{\mathcal{A}_*}^{s,t}(H_*(X), H_*(Y)) \Longrightarrow Y^{s-t}(X) = [X,Y]^{s-t}.$$

Here are some examples.
Take $X = BP$ (the Brown-Peterson spectrum) and $Y = S^0$ where $H_*(BP) = \mathcal{A}_*^{(1)}$ and $H_*(S^0) = \mathbb{F}_2$. To calculate the E_2-term we use a Cartan-Eilenberg spectral sequence

$$\mathrm{E}_2^{s,t} = \mathrm{Coext}^s_{\mathcal{A}_*^{(1)}}(\mathcal{A}_*^{(1)}, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \mathrm{Coext}^{s+t}_{\mathcal{A}_*}(\mathcal{A}_*^{(1)}, \mathbb{F}_2).$$

Here we have suppressed the internal grading on $\mathrm{Cotor}^{t,*}(\mathbb{F}_2, \mathbb{F}_2)$ which is concentrated in $* \leqslant 0$. Since $\mathcal{A}_*^{(1)}$ is projective over itself $\mathrm{E}_2^{s,*} = 0$ when $s > 0$. Later we will show that $\mathrm{E}_2^{0,t} = 0$.

Comparing some Bousfield classes

In his seminal paper on localization of spectra, Ravenel introduced a family of ring spectra and maps

$$S^0 = X_0 \to X_1 \to X_2 \to \cdots \to BP$$

and showed that their Bousfield classes satisfied

$$\langle S^0 \rangle = \langle X_0 \rangle > \langle X_1 \rangle > \langle X_2 \rangle > \cdots > \langle X_s \rangle > \langle X_{s+1} \rangle > \cdots > \langle BP \rangle.$$

The proof requires showing that for example $X_n^*(BP) = 0$. Again we can reduce this to showing that

$$\mathrm{Cohom}_{\mathcal{A}_*^{(1)}//\mathcal{P}(n)_*^{(1)}}(\mathcal{A}_*^{(1)}, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2)) = 0.$$

To do this we need to know more about $\mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2)$ as a $\mathcal{A}_*^{(1)}//\mathcal{P}(n)_*^{(1)}$-comodule where $0 \leqslant n$.

We will focus on the case $n = 0$, the general case is similar. First we recall that

$$\mathrm{Cotor}^{*,*}_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[q_0, q_1, q_2, \ldots]$$

where $q_k \in \mathrm{Cotor}^{1,2^{k+1}-1}_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}$. Next we can determine the induced $\mathcal{A}^{(1)}_*//\mathcal{A}^{(2)}_*$-coaction: q_0 is primitive and for $k \geqslant 1$,

$$\mu(q_k) = \xi_k^2 \otimes q_0 + 1 \otimes q_k.$$

This means that we can filter each $\mathrm{Cotor}^{t,*}_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}(\mathbb{F}_2, \mathbb{F}_2)$ by a finite increasing sequence of subcomodules

$$\mathrm{F}^{t,k} = \mathbb{F}_2\{q_0^{r_0} q_1^{r_1} \cdots q_\ell^{r_\ell} : r_0 \geqslant t - k, \sum_{0 \leqslant i \leqslant \ell} r_i = t\}.$$

where the coaction on $\mathrm{F}^{t,k}/\mathrm{F}^{t,k-1}$ is trivial.

Now any non-zero $\mathcal{A}_*^{(1)}//\mathcal{A}^{(2)}$-comodule homomorphism $\mathcal{A}_*^{(1)} \to \mathrm{Cotor}^{t,*}_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2)$ has to factor through a filtration F^{t,k_0} where k_0 is minimal. Hence we can compose with the quotient homomorphism to find a non-trivial homomorphism $\mathcal{A}_*^{(1)} \to \mathrm{F}^{t,k_0}/\mathrm{F}^{t,k_0-1}$ and the project onto a suspension of $\mathbb{F}_2$. But since $\mathcal{A}_*^{(1)}$ is a cofree $\mathcal{A}_*^{(1)}//\mathcal{A}^{(2)}$-comodule over a P_*-algebra, this contradicts earlier results.

Some new results

It is known that locally at 2, $\langle M\mathrm{Sp}\rangle \geqslant \langle BP\rangle$.

Theorem

$$\langle S^0\rangle > \langle M\mathrm{Sp}\rangle > \langle BP\rangle.$$

The proof that $\langle M\mathrm{Sp}\rangle > \langle BP\rangle$ involves showing that

$$\mathrm{Coext}^{*,*}_{\mathcal{A}_*}(H_*(BP), H_*(M\mathrm{Sp})) = 0$$

and this reduces to showing that

$$\mathrm{Cohom}_{\mathcal{A}^{(1)}_*}(\mathcal{A}^{(1)}_*, \mathcal{A}^{(2)}_* \overset{\mathcal{A}^{(1)}_*}{\wedge} \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}(\mathbb{F}_2, \mathbb{F}_2)) = 0$$

and this can be reduced to the vanishing of

$$\mathrm{Cohom}_{\mathcal{A}^{(1)}_*}(\mathcal{A}^{(1)}_*, \mathcal{A}^{(1)}_* \Box_{\mathcal{A}^{(1)}_*//\mathcal{A}^{(2)}_*} \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}(\mathbb{F}_2, \mathbb{F}_2)) \cong \mathrm{Cohom}_{\mathcal{A}^{(1)}_*//\mathcal{A}^{(2)}_*}(\mathcal{A}^{(1)}_*, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}^{(1)}_*}(\mathbb{F}_2, \mathbb{F}_2)).$$

The proof that $\langle S^0 \rangle > \langle M\mathrm{Sp} \rangle$ is harder because it involves the vanishing of

$$\mathrm{Cohom}_{\mathcal{A}_*^{(2)}}(\mathcal{A}_*^{(2)}, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}(\mathbb{F}_2, \mathbb{F}_2))$$

and this can be done by reducing to

$$\mathrm{Cohom}_{\mathcal{A}_*^{(2)}//\mathcal{A}_*^{(3)}}(\mathcal{A}_*^{(2)}, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}(\mathbb{F}_2, \mathbb{F}_2))$$

and defining a suitable filtration on the $\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(3)}$-comodule $\mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(2)}}(\mathbb{F}_2, \mathbb{F}_2)$. This requires analysis of the $\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(3)}$-comodules $\mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2)$ and $\mathrm{Cotor}^t_{\mathcal{A}_*^{(1)}//\mathcal{A}_*^{(2)}}(\mathbb{F}_2, \mathrm{Cotor}^t_{\mathcal{A}_*//\mathcal{A}_*^{(1)}}(\mathbb{F}_2, \mathbb{F}_2))$.

Dinlediğiniz için teşekkürler!