Fun and games with the Steenrod algebra

Online **A**lgebraic **T**opology **S**eminar, 27th April 2020

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Recollections on mod 2 (co)homology

Each of $H_*(-) = H_*(-;\mathbb{F}_2)$ and $H^*(-) = H^*(-;\mathbb{F}_2)$ is a homotopy functor from spaces to $\mathbb{Z}$-graded vector spaces. The reduced theories $\widetilde{H}_*(-)$ and $\widetilde{H}^*(-)$ gives functors from based spaces to $\mathbb{Z}$-graded vector spaces which extend to spectra. A *stable cohomology operation* θ *of degree* k is a sequence of natural transformations

$$\theta_n \colon H^n(-) \to H^{n+k}(-) \quad (n \in \mathbb{Z})$$

compatible with suspension isomorphisms, i.e., the following diagram commutes for all n and k.

$$\begin{array}{ccc} \widetilde{H}^n(-) & \xrightarrow{\theta_n} & \widetilde{H}^{n+k}(-) \\ \cong\downarrow & & \downarrow\cong \\ \widetilde{H}^{n+1}(\Sigma(-)) & \xrightarrow{\theta_{n+1}} & \widetilde{H}^{n+k+1}(\Sigma(-)) \end{array}$$

The set of all such operations $\mathcal{A}^k = H^k(H)$ is an $\mathbb{F}_2$-vector space, and these form the mod 2 *Steenrod algebra* $\mathcal{A} = \mathcal{A}^* = H^*(H)$, a non-commutative graded algebra with composition as product. The structure of $\mathcal{A}$ was determined by Serre, then Milnor showed that it was a cocommutative Hopf algebra and determined its dual Hopf algebra $\mathcal{A}_*$ where $\mathcal{A}_n = \mathrm{Hom}_{\mathbb{F}_2}(\mathcal{A}^n, \mathbb{F}_2)$. As an algebra, $\mathcal{A}$ is generated by the *Steenrod operations* $\mathrm{Sq}^n \in \mathcal{A}^n$ ($n \geqslant 1$) satisfying the *Adem relations* (here $\mathrm{Sq}^0 = 1$):

$$\text{For } 0 < r < 2s, \quad \mathrm{Sq}^r \mathrm{Sq}^s = \sum_{0 \leqslant j \leqslant \lfloor r/2 \rfloor} \binom{s-1-j}{r-2j} \mathrm{Sq}^{r+s-j} \mathrm{Sq}^j .$$

$$\mathrm{Sq}^1 \mathrm{Sq}^1 = \binom{0}{1} \mathrm{Sq}^2 = 0, \quad \mathrm{Sq}^1 \mathrm{Sq}^2 = \binom{1}{1} \mathrm{Sq}^3 = \mathrm{Sq}^3,$$

$$\mathrm{Sq}^2 \mathrm{Sq}^2 = \binom{1}{2} \mathrm{Sq}^4 + \binom{0}{0} \mathrm{Sq}^3 \mathrm{Sq}^1 = \mathrm{Sq}^3 \mathrm{Sq}^1,$$

$$\mathrm{Sq}^2 \mathrm{Sq}^3 = \binom{2}{2} \mathrm{Sq}^5 + \binom{1}{0} \mathrm{Sq}^4 \mathrm{Sq}^1 = \mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1 .$$

It can be shown that the algebra indecomposables are the Sq^{2^s}.
There is a basis of *admissible monomials*

$$\mathrm{Sq}^{(i_1,\ldots,i_\ell)} = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_\ell}$$

where $i_{r-1} \geqslant 2i_r$ for $2 \leqslant r \leqslant \ell$ and $i_\ell \geqslant 1$. Here ℓ is the *length* of the monomial and there is one length zero element, the identity operation $\mathrm{Sq}^0 = 1$.

The cocommutative coproduct $\psi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and antipode $\chi\colon \mathcal{A} \to \mathcal{A}$ are given by the formulae

$$\psi(\mathrm{Sq}^n) = \sum_{0\leqslant r\leqslant n} \mathrm{Sq}^r \otimes \mathrm{Sq}^{n-r}, \qquad \sum_{0\leqslant r\leqslant n} \chi(\mathrm{Sq}^r)\,\mathrm{Sq}^{n-r} = 0.$$

Note that the antipode is anti-commutative, i.e.,

$$\chi(\alpha\beta) = \chi(\beta)\chi(\alpha).$$

Here are the first few $\chi(\mathrm{Sq}^{2^s})$:

$$\chi(\mathrm{Sq}^1) = \mathrm{Sq}^1,\ \chi(\mathrm{Sq}^2) = \mathrm{Sq}^2,\ \chi(\mathrm{Sq}^4) = \mathrm{Sq}^4 + \mathrm{Sq}^1\mathrm{Sq}^4\mathrm{Sq}^1.$$

Theorem (Serre & Milnor)

The commutative Hopf algebra $\mathcal{A}_*$ is polynomial:

$$\mathcal{A}_* = \mathbb{F}_2[\xi_r : r \geqslant 1] = \mathbb{F}_2[\zeta_r : r \geqslant 1],$$

where $\xi_r, \zeta_r \in \mathcal{A}_{2^r-1}$ and $\zeta_r = \chi(\xi_r)$. The coproduct and antipode satisfy

$$\psi(\xi_n) = \sum_{0\leqslant j\leqslant n} \xi_{n-j}^{2^j} \otimes \xi_j, \quad \psi(\zeta_n) = \sum_{0\leqslant j\leqslant n} \zeta_j \otimes \zeta_{n-j}^{2^j},$$

$$\zeta_n = \sum_{1\leqslant k\leqslant n} \xi_k \zeta_{n-k}^{2^k}.$$

The non-zero primitives are the elements $\xi_1^{2^s} = \zeta_1^{2^s}$.

The Poincaré series for $\mathcal{A}$ and $\mathcal{A}_*$ is $\prod_{r\geqslant 1}(1-t^{2^r-1})^{-1}$.

Finite sub-Hopf algebras of $\mathcal{A}$

Important fact: $\mathcal{A} = \bigcup_{n\geqslant 0} \mathcal{A}(n)$, where $\mathcal{A}(n) \subseteq \mathcal{A}$ is the finite sub-Hopf algebra of dimension $2^{\binom{n+2}{2}}$ generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$ with dual quotient Hopf algebra

$$\begin{aligned}\mathcal{A}(n)_* &= \mathcal{A}_*/(\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots \zeta_{n+1}^2, \zeta_{n+2}, \ldots) \\ &= \mathcal{A}_*//\mathbb{F}_2[\zeta_1^{2^{n+1}}, \zeta_2^{2^n}, \zeta_3^{2^{n-1}}, \ldots \zeta_{n+1}^2, \zeta_{n+2}, \ldots].\end{aligned}$$

Here $\mathcal{A}(n)$ and $\mathcal{A}(n)_*$ have Poincaré series

$$\prod_{1\leqslant r\leqslant n+1} \frac{(1 - t^{2^{n+2-r}(2^r-1)})}{(1 - t^{2^r-1})}.$$

The highest degree element in $\mathcal{A}(n)_*$ is the residue class of

$$z_n = \zeta_1^{2^{n+1}-1}\zeta_2^{2^n-1}\zeta_3^{2^{n-1}-1}\cdots\zeta_{n+1}$$

and dual to this is a generator of the 1-dimensional socle $\mathrm{soc}\,\mathcal{A}(n)$. Under the dual pairing, z_n defines a Frobenius form making $\mathcal{A}(n)$ a *Poincaré duality algebra* and thus self-injective.

The Wall relations

The Adem relations are not minimal, and also do not restrict to the $\mathcal{A}(n)$ subalgebras: for example, the identities

$$\mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^5 = \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^1\,\mathrm{Sq}^4$$

are not meaningful in $\mathcal{A}(1)$ since $\mathrm{Sq}^4 \notin \mathcal{A}(1)$. Wall found a minimal set of relations amongst the generators Sq^{2^s} which do restrict to the $\mathcal{A}(n)$.
For $0 \leqslant s \leqslant r-2$ and $1 \leqslant t$, let

$$\Theta(r,s) = \mathrm{Sq}^{2^r}\,\mathrm{Sq}^{2^s} + \mathrm{Sq}^{2^s}\,\mathrm{Sq}^{2^r},$$
$$\Phi(t) = \mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^t} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^{t-1}} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t}.$$

Then $\Theta(r,s) \in \mathcal{A}(r-1)$ and $\Phi(r) \in \mathcal{A}(r-1)$ so these can be expressed as polynomial expressions in the Sq^{2^k} for $0 \leqslant k \leqslant r-1$.

The elements

$$\mathrm{Sq}^{2^r}\,\mathrm{Sq}^{2^s} + \mathrm{Sq}^{2^s}\,\mathrm{Sq}^{2^r} + \Theta(r,s),$$
$$\mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^t} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t}\,\mathrm{Sq}^{2^{t-1}} + \mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^{t-1}}\,\mathrm{Sq}^{2^t} + \Phi(t)$$

give a minimal set of relations for $\mathcal{A}$. In particular, such elements with $r, t \leqslant n$ form a minimal set of relations for $\mathcal{A}(n)$.

In the first few cases the Wall relations are

$$\begin{aligned}
\mathcal{A}(0): \quad & \mathrm{Sq}^1\,\mathrm{Sq}^1 = 0,\\
\mathcal{A}(1): \quad & \mathrm{Sq}^1\,\mathrm{Sq}^1 = \mathrm{Sq}^2\,\mathrm{Sq}^2 + \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1 = 0\\
\mathcal{A}(2): \quad & \mathrm{Sq}^1\,\mathrm{Sq}^1 = \mathrm{Sq}^2\,\mathrm{Sq}^2 + \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\\
& \qquad = \mathrm{Sq}^4\,\mathrm{Sq}^4 + \mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^2\,\mathrm{Sq}^4\\
& \qquad = \mathrm{Sq}^1\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^1 + \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2 = 0.
\end{aligned}$$

Using these it is possible to produce explicit bases for the $\mathcal{A}(n)$s. For example, there is a formula for the top dimensional element, here are the cases $n = 0, 1, 2$:

$$\mathrm{Sq}^1,\ \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2,\ \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4\,.$$

Generalisations of the Steenrod algebra

Modern categories of spectra are symmetric monoidal with respect to smash products before passing to homotopy. The category of S-modules $\mathscr{M}_S$ is an important example and provides a good model for the category of spectra.

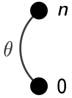
A commutative monoid in this category is equivalent to an $\mathcal{E}_\infty$ ring spectrum and is called a *commutative S-algebra*. Examples include S, $H\mathbb{Z}$, $H\mathbb{F}_p$, $k\mathrm{O}$, $k\mathrm{U}$, $M\mathrm{U}$ and so on. Every commutative S-algebra R has a module category $\mathscr{M}_R$ which is also closed symmetric monoidal with respect to a relative smash product $\wedge_R$ and function object $F_R(-,-)$; it also has a model structure and homotopy category $\mathscr{D}_R$ in which to do homotopy theory.

If R is connective and $\pi_0 R = \mathbb{Z}$ or $\pi_0 R = \mathbb{Z}_{(p)}$ there is a morphism of commutative S-algebras $R \to H = H\mathbb{F}_p$ so H is a commutative R-algebra, and then there are relative homology and cohomology theories

$$H^R_*(-) = \pi_*(H \wedge_R -), \quad H^*_R(-) = \pi_{-*}(F_R(-, H)).$$

The relative Steenrod algebra $H_R^*(H)$ is the algebra of stable operations in $H_R^*(-)$. When $R = S$ it is $\mathcal{A}$.
When $p = 2$, $H_{k\mathrm{O}}^*(H) = \mathcal{A}(1)$ and $H_{\mathrm{tmf}}^*(H) = \mathcal{A}(2)$.

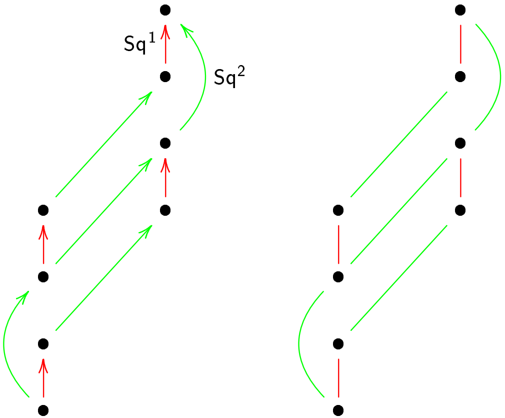
Realisation question: When working with spectra (or equivalently S-modules) we can ask whether an $\mathcal{A}$-module M is realisable as $H^*(X)$ for some S-module X. Similarly, for an $\mathcal{A}(1)$-module we can ask if it is $H_{k\mathrm{O}}^*(Y)$ for a $k\mathrm{O}$-module Y and for an $\mathcal{A}(2)$-module we can ask if it is $H_{\mathrm{tmf}}^*(Z)$ for a tmf-module Z.
Example: When can we realise an $\mathcal{A}$-module of the following form with $0 \neq \theta \in \mathcal{A}$?



Algebraic observation: Module only exists if θ is indecomposable, i.e., $n = 2^s$ and $\theta = \mathrm{Sq}^{2^s}$ +decomposables.
Hopf invariant 1 Theorem (Adams): Only realisable if $s = 0, 1, 2, 3$.

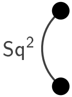
Realisability of $\mathcal{A}(1)$-modules

We will work with left modules $M = M^*$ involving multiplication maps $\mathcal{A}(1)^r \otimes M^n \to M^{n+r}$. Here some pictures of $\mathcal{A}(1)$ which is a free cyclic module realisable as $H^*_{k\mathrm{O}}(H)$.

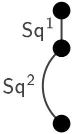


Here are some more realisable $\mathcal{A}(1)$-modules. In each case we can form a finite CW spectrum W then take $k\mathrm{O} \wedge W$ to get $H^*_{k\mathrm{O}}(k\mathrm{O} \wedge W) \cong H^*(W)$ with its $\mathcal{A}$-action restricted to an action of the subalgebra $\mathcal{A}(1) \subseteq \mathcal{A}$.

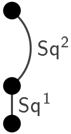
$\mathbb{F}_2 = H^*(S^0)$



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^2\} = H^*(C_\eta)$

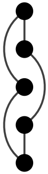


$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1, \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\}$



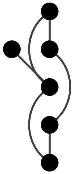
$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\}$

The Joker



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\,\mathrm{Sq}^2\}$

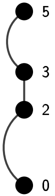
The whiskered Joker



$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\}$

The construction of the Joker example uses the Toda bracket $\langle 2,\eta,2\rangle = \{\eta^2\} \subseteq \pi_2(S^0)$. Later we'll see other examples of Toda brackets playing a rôle.

What about this one?



$$\mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\}$$

Let's first think about whether the above diagram can be realised as an $\mathcal{A}$-module. Notice that the top class is $\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2$. Using Adem relations we have

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2 = \mathrm{Sq}^2\,\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1 = \mathrm{Sq}^1\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^1$$

which is not possible.

Despite this, there is a $k\mathrm{O}$-module realising this module, namely $H\mathbb{Z}$ for which $H^*_{k\mathrm{O}}(H\mathbb{Z}) \cong \mathcal{A}(1)/\mathcal{A}(1)\{\mathrm{Sq}^1\}$.

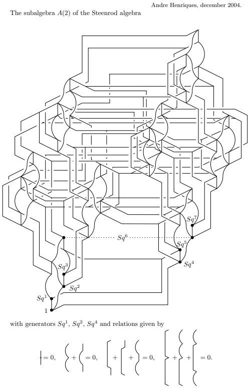
Another approach using a Toda bracket

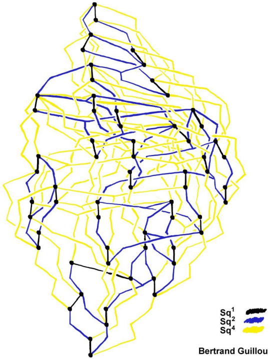
The existence of a CW spectrum $W = S^0 \cup e^2 \cup e^3 \cup e^5$ whose cohomology is $\mathcal{A}/\mathcal{A}\{\mathrm{Sq}^1\}$ is equivalent to the Toda bracket $\langle \eta, 2, \eta \rangle \subseteq \pi_3(S^0)$ containing 0. But $\langle \eta, 2, \eta \rangle = \{\pm 2\nu\} \not\ni 0$.
If we interpret the Toda bracket as being in $\pi_3(k\mathrm{O})$, since the image of ν is 0, we can build a CW $k\mathrm{O}$-module of this form using $k\mathrm{O}$ cells; the result is equivalent to $H\mathbb{Z}$ as a $k\mathrm{O}$-module.

There are many other examples of realisable cyclic $\mathcal{A}(1)$-modules! Of course there are also non-cyclic examples which can be realised by various methods such as by attaching cells or forming mapping cones of maps between $k\mathrm{O}$-modules.

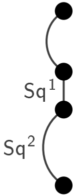
Realisability of $\mathcal{A}(2)$-modules with tmf-modules

Here are some pictures of $\mathcal{A}(2)$.





All of the above examples for $k\mathrm{O}$ of the form $k\mathrm{O} \wedge W$ can be replaced by $\mathrm{tmf} \wedge W$ so that $H^*_{\mathrm{tmf}}(\mathrm{tmf} \wedge W) \cong H^*(W)$ as $\mathcal{A}(2)$-modules. The Sq^4 argument works to show there is no $\mathcal{A}(2)$-module of the form shown; the Toda bracket argument also applies as the image of ν in $\pi_3(\mathrm{tmf})$ is non-zero.



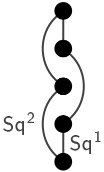
A good source of $\mathcal{A}(2)$-modules is by using *doubling* which exploits the fact that there is a degree halving surjective homomorphism of Hopf algebras $\mathcal{A}(2) \twoheadrightarrow \mathcal{A}(1)$ under which

$$\mathrm{Sq}^n \mapsto \begin{cases} \mathrm{Sq}^{n/2} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

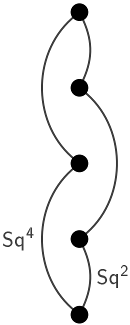
By restricting and doubling degrees, every $\mathcal{A}(1)$-module M induces an $\mathcal{A}(2)$-module ${}^{(1)}M$.

Doubling the Joker

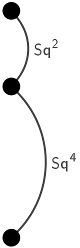
Joker



$^{(1)}$Joker



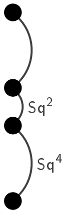
The following examples are of the form $H^*(W)$. Their constructions depending on $\eta\nu \in \pi_4(S^0) = 0$. The two CW spectra are stably Spanier-Whitehead dual.



$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^7\}$ $\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^1\,\mathrm{Sq}^2, \mathrm{Sq}^4\}$

It is also possible to realise the double of the (whiskered) Joker using the Toda bracket $\langle \eta, \nu, \eta \rangle = \{\nu^2\} \subseteq \pi_6(S^0)$. The double of $\mathcal{A}(1)$ is a also realisable as a spectrum so we can smash it with tmf to realise this $\mathcal{A}(2)$-module.

What about this one?



$$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2\}$$

We can't rule this out with Steenrod operations. What about a Toda bracket argument? Constructing a suitable CW complex requires the Toda bracket $\langle \nu, \eta, \nu \rangle \subseteq \pi_8(S^0)$ to contain 0. But

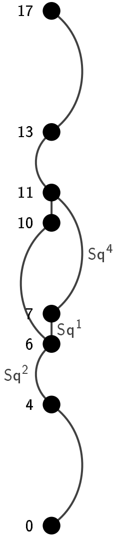
$$\langle \nu, \eta, \nu \rangle = \{\overline{\nu}\} = \{\eta\sigma + \varepsilon\} \not\ni 0.$$

Here the image of σ in $\pi_7(\mathrm{tmf})$ is 0 but the image of ε is not. This means that there is no tmf-module with this cohomology! If it did exist its homotopy would be $\pi_*(k\mathrm{O})[v_2]$.

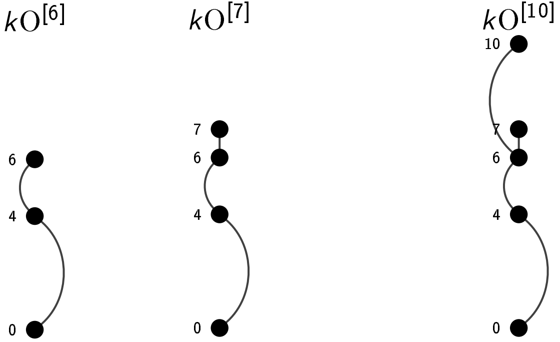
Some tmf-modules related to $k\mathrm{O}$

The cohomology of the tmf-module $k\mathrm{O}$ is shown below.

$$H^*_{\mathrm{tmf}}(k\mathrm{O}) \cong \mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2\}$$



We can realise kO as a CW tmf-module with cells corresponding to the basis shown. Each skeleton gives a tmf-module with corresponding $\mathcal{A}(2)$-module as its cohomology.

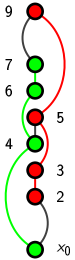


$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4\}$ $\quad$ $\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^1\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4\}$

There are many other $\mathcal{A}(2)$-modules including many cyclic ones. Here is an interesting example that is realisable as the cohomology of a tmf-module.

$$\mathcal{A}(2)/\mathcal{A}(2)\{\mathrm{Sq}^1, A, B\}$$

$$A = \mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1,$$
$$B = \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4 + \mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,.$$



It doesn't come from an $\mathcal{A}$-module since Adem relations imply

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2\,\mathrm{Sq}^4\,x_0 = (\mathrm{Sq}^8\,\mathrm{Sq}^1 + \mathrm{Sq}^1\,\mathrm{Sq}^8)x_0.$$

Exploiting dualisation

Many of the examples are 'self-dual'. For a left module M over a Hopf algebra H over a field $\Bbbk$, the dual $\mathrm{D}M = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$ is naturally a right H-module but using the antipode this can be made into a left module. In the graded case, $(\mathrm{D}M)^n = \mathrm{Hom}_{\Bbbk}(M^{-n}, \Bbbk)$, and M is *stably self-dual* if for some k, $\mathrm{D}M \cong M[k]$.

Every finite dimensional Hopf algebra is a Frobenius algebra or in the graded case a Poincaré duality algebra, hence stably self-dual.

Natural question: Which (cyclic) modules are stably self-dual?

Partial answer: Any $\mathcal{A}(n)$-module of form $\mathcal{A}(n) \otimes_K \mathbb{F}_2$ where $K \subseteq \mathcal{A}(n)$ is a subHopf algebra. If K is normal then $\mathcal{A}(n)//K = \mathcal{A}(n) \otimes_K \mathbb{F}_2$ is a quotient Hopf algebra.

There is a version of Spanier-Whitehead duality for finite CW R-modules and $H^*_R(D_R X) \cong \mathrm{D}(H^*_R(X))$ as left $H^*_R(H)$-modules. In particular, for a dualisable S-module W, $D_R(R \wedge W) \sim R \wedge D_S W$. This allows us to realise many examples, however dualising a cyclic module may not give a cyclic module.

Thanks for listening, stay safe and well!