$M\mathrm{Sp}$: something old and something new

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$M\mathrm{Sp}$: some history

The compact symplectic groups $\mathrm{Sp}(n) \leqslant U(2n) \leqslant \mathrm{O}(4n)$ give rise to the classifying space of virtual symplectic ($=$ quaternionic) vector bundles which admits an $\mathcal{E}_\infty$ map $B\mathrm{Sp} \to BO$ and a Thom spectrum $M\mathrm{Sp}$ which an $\mathcal{E}_\infty$ ring spectrum.

Since the 1960s the homotopy ring $\pi_*(M\mathrm{Sp})$ has been studied intensively and Kochman determined its classical Adams spectral sequence for $p = 2$. The ring $\pi_*(M\mathrm{Sp})$ is complicated: there is 2-torsion of arbitrary 2-exponent and it is all nilpotent.

At an odd prime p, $M\mathrm{Sp}$ is a wedge of suspensions of BP, so we'll (mostly) work 2-locally from now on.

The image of $\pi_*(M\mathrm{Sp}) \to H_*(M\mathrm{Sp}; \mathbb{F}_2)$ is known by work of Floyd and Kochman; its first non-trivial positive part is in degree 32 so $M\mathrm{Sp}$ is not 'minimal atomic' in the sense of Baker & May. On the other hand, the TAQ Hurewicz homomorphism

$$\pi_*(M\mathrm{Sp}) \to \mathrm{TAQ}_*(M\mathrm{Sp}; \mathbb{F}_2) \cong H_*(k\mathrm{O}; \mathbb{F}_2)[4]$$

is trivial, so $M\mathrm{Sp}$ is minimal atomic as a 2-local $\mathcal{E}_\infty$ ring spectrum.

Kochman determined the image of $\pi_*(S) \to \pi_*(M\mathrm{Sp})$ modulo higher Adams filtration: its only non-zero positive degree elements are the images of the Adams $\mu_{8k+\varepsilon}$ family where $k \geqslant 0$, $\varepsilon = 1, 2$. Earlier, Ray showed that apart from $\eta \in \pi_1(S)$, the image of the classical J-homomorphism mapped trivially. We will see that this has interesting consequences for the K-localisation $M\mathrm{Sp}_K$.

Ray also produced an infinite family of indecomposable elements $\varphi_k \in \pi_{8k-3}(M\mathrm{Sp})$ of order 2 and much of the known structure of $\pi_*(M\mathrm{Sp})$ depends on these. They are detected by the Hurewicz homomorphism $\pi_*(M\mathrm{Sp}) \to K\mathrm{O}_*(M\mathrm{Sp})$.

We can lift φ_k to $\widetilde{\varphi}_k \in \pi_{8k-2}(M\mathrm{Sp} \wedge M(2))$ and consider its image in $BP_*(M\mathrm{Sp} \wedge M(2))$. For $s \geqslant 1$,

$$\widetilde{\varphi}_{2^{s-1}} \mapsto v_{s+1} + v_s N_{2^{s-1}} \quad \text{mod } (I_{s-1} + \text{decomposables}),$$

where $BP_*(M\mathrm{Sp}) = BP_*[N_k : k \geqslant 1]$ and $N_k \in BP_{4k}(M\mathrm{Sp})$. So these elements are in some sense 'transchromatic'.

Some recent results

An example left unresolved in Ravenel's localisation paper is the Bousfield class of $M\mathrm{Sp}$. It is known that globally

$$\langle S^0 \rangle \geqslant \langle M\mathrm{Sp} \rangle \geqslant \langle M\mathrm{U} \rangle$$

so 2-locally,

$$\langle S^0 \rangle \geqslant \langle M\mathrm{Sp} \rangle \geqslant \langle BP \rangle.$$

Theorem (B., `arXiv:2103.01253`)

Globally

$$\langle S^0 \rangle > \langle M\mathrm{Sp} \rangle > \langle M\mathrm{U} \rangle.$$

and 2-locally,

$$\langle S^0 \rangle > \langle M\mathrm{Sp} \rangle > \langle BP \rangle.$$

K and $K(1)$-localisations

We will consider two Bousfield localisations where $K = K\mathrm{U}_{(2)}$:

$$(-)_K = (-)_{K\mathrm{O}_{(2)}}, \quad (-)_{K(1)} = ((-)_K)\widehat{_2}.$$

We recall (here $s \in \mathbb{Z}$)

$$\pi_n(S_K) = \begin{cases} \mathbb{Z}_{(2)} \oplus \mathbb{Z}/(2) & \text{if } n = 0, \\ \mathbb{Z}/(2^\infty) = \mathbb{Q}/\mathbb{Z}_{(2)} & \text{if } n = -2, \\ \mathbb{Z}/(2^{\nu_2(s)+4}) & \text{if } n = 8s-1 \text{ and } s \neq 0, \\ \mathbb{Z}/(2) & \text{if } n = 8s \text{ and } s \neq 0, \\ \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) & \text{if } n = 8s+1, \\ \mathbb{Z}/(2) & \text{if } n = 8s+2, \\ \mathbb{Z}/(8) & \text{if } n = 8s+3, \\ 0 & \text{otherwise.} \end{cases}$$

In positive degrees these groups detect the image of the classical J-homomorphism under the localisation map $\pi_*(S) \to \pi_*(S_K)$.

Ravenel showed that for $s \in \mathbb{N}$, the multiplication pairing

$$\mu_s \colon \pi_{8s-1}(S_K) \otimes \pi_{-8s-1}(S_K) \to \pi_{-2}(S_K) \cong \mathbb{Z}/(2^\infty)$$

is injective and it follows that the images of the maps μ_s exhaust $\pi_{-2}(S_K)$,

$$\pi_{-2}(S_K) = \bigcup_{s \geqslant 1} \operatorname{im} \mu_s.$$

On passing to $\pi_*(S_{K(1)})$ we get the same groups except that

$$\pi_n(S_{K(1)}) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}/(2) & \text{if } n = 0, \\ \mathbb{Z}_2 = \mathbb{Z}_2\{\zeta_2\} = \mathbb{Z}_2\{\zeta\} & \text{if } n = -1, \\ 0 & \text{if } n = -2, \end{cases}$$

where $\mathbb{Z}_2$ denotes the 2-adic integers.
Here

$$\pi_{-1}(S_{K(1)}) = \mathrm{Hom}(\mathbb{Z}/(2^\infty), \pi_{-2}(S_K))$$

and more generally, for any spectrum X, there is a functorial short exact sequence (which splits):

$$0 \to \mathrm{Ext}(\mathbb{Z}/(2^\infty), \pi_*(X_K)) \to \pi_*(X_{K(1)}) \\ \to \mathrm{Hom}(\mathbb{Z}/(2^\infty), \pi_{*-1}(X_K)) \to 0.$$

The $\mathcal{E}_\infty$ morphism $S_K \to M\mathrm{Sp}_K$ induces a commutative diagram

$$\begin{tikzcd}
\pi_{8s-1}(\mathrm{SO}) \otimes \pi_{-8s-1}(S_K) \arrow[r, "J \otimes \mathrm{Id}"] \arrow[dr, dashed, "0"'] & \pi_{8s-1}(S) \otimes \pi_{-8s-1}(S_K) \arrow[d] \arrow[r, "\cong"] & \operatorname{im} \mu_s \arrow[d] \\
& \pi_{8s-1}(M\mathrm{Sp}) \otimes \pi_{-8s-1}(M\mathrm{Sp}_K) \arrow[r] & \operatorname{im} \widetilde{\mu}_s
\end{tikzcd}$$

in which the dashed arrow is trivial by results of Ray and

$$\widetilde{\mu}_s : \pi_{8s-1}(M\mathrm{Sp}_K) \otimes \pi_{-8s-1}(M\mathrm{Sp}_K) \to \pi_{-2}(M\mathrm{Sp}_K)$$

is the multiplication.
It follows that $\pi_{-2}(S_K) \to \pi_{-2}(M\mathrm{Sp}_K)$ is trivial.

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(2)}, \pi_{-2}(S_K)) \arrow[r, "\cong"] \arrow[d, "0"] & \pi_{-1}(S_{K(1)}) \arrow[d] \\
\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}_{(2)}, \pi_{-2}(M\mathrm{Sp}_K)) \arrow[r] & \pi_{-1}(M\mathrm{Sp}_{K(1)})
\end{tikzcd}$$

which shows that $\pi_{-1}(S_{K(1)}) \to \pi_{-1}(M\mathrm{Sp}_{K(1)})$ is also trivial, so the generator $\zeta \in \pi_{-1}(S_{K(1)})$ maps to zero and $M\mathrm{Sp}_{K(1)}$ has characteristic ζ in the sense of Szymik, i.e., there is a morphism of $\mathcal{E}_\infty$ ring spectra $S_{K(1)}//(\zeta) \to M\mathrm{Sp}_{K(1)}$.

The $\mathcal{E}_\infty$ cone for ν and σ

The following results are joint work with Gerd Laures and Jan Holz.

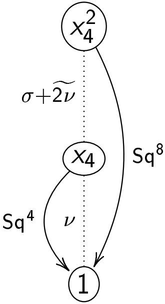
Since ν and σ both map to zero in $M\mathrm{Sp}$ there is an $\mathcal{E}_\infty$ morphism $S//(\nu,\sigma) \to M\mathrm{Sp}$ which is a 9-equivalence in two different ways.

- If we realise both spectra with minimal CW structures then it induces an equivalence of 8-skeleta; i.e., $H_k(S//(\nu,\sigma);\mathbb{F}_2) \xrightarrow{\cong} H_k(M\mathrm{Sp};\mathbb{F}_2)$ if $0 \leqslant k \leqslant 8$.
- If we realise both $\mathcal{E}_\infty$ ring spectra with minimal CW structures then it induces an equivalence of 8-skeleta; i.e., $\mathrm{TAQ}_k(S//(\nu,\sigma);\mathbb{F}_2) \xrightarrow{\cong} \mathrm{TAQ}_k(M\mathrm{Sp};\mathbb{F}_2)$ if $0 \leqslant k \leqslant 8$.

Here Dyer-Lashof monomials only start appearing in degree 9 and

$$\begin{aligned}
H_*(S//(\nu,\sigma);\mathbb{F}_2) &= \mathbb{F}_2\{1, x_4, x_4^2, x_8, \ldots\},\\
H_*(M\mathrm{Sp};\mathbb{F}_2) &= \mathbb{F}_2\{1, q_1, q_1^2, q_2, \ldots\},\\
\mathrm{TAQ}_*(S//(\nu,\sigma);\mathbb{F}_2) &= \mathbb{F}_2\{x_4, x_8\},\\
\mathrm{TAQ}_*(M\mathrm{Sp};\mathbb{F}_2) &= \mathbb{F}_2\{1, \xi_1^4, \ldots\}[4].
\end{aligned}$$

To see what is going on, the (usual) 8-skeleton of $S//(\nu)$ has the cell structure shown with Steenrod operations in homology.



So here σ on the bottom cell is not dead but it has order 4 and Adams filtration 2. So the extra 8-cell in $S//(\nu, \sigma)$ and $M\mathrm{Sp}$ has to be attached to kill it.

Theorem (Holz, PhD thesis)

The $\mathcal{E}_\infty$ ring spectrum $(S//(\nu,\sigma))_{K(1)}$ has characteristic ζ.

The proof involves a geometric construction of an *Artin-Schreier class* $a \in \pi_0((KO \wedge S//(\nu,\sigma))_{K(1)}$ for which

$$\psi^9 a = a + 1.$$

Corollary (Holz & Laures)

There are equivalences of $\mathcal{E}_\infty$ ring spectra

$$(S//(\nu,\sigma))_{K(1)} \sim S_{K(1)}//(\zeta) \wedge (?), \quad M\mathrm{Sp}_{K(1)} \sim S_{K(1)}//(\zeta) \wedge (??).$$

Independently I used a different approach to these results by using the θ-algebra structure of the 2-completed K-homology to show that there is a $K(1)$-equivalence

$$S//(\nu) \sim \prod_j \sigma^{4\rho(j)} KO.$$

Here are some recent results which provide a very general context for understanding what is happening.

Theorem (B. & Laures)

Let p be a prime and work p-locally.

- *If $p = 2$ and $j_{4m-1} \in \pi_{4m-1}(S)$ is a generator of* $\operatorname{im} J_{4m-1} \subseteq \pi_{4m-1}(S)$, *then $K_0^\vee(S//(j_{4m-1}))$ contains an Artin-Schreier element so $(S//(j_{4m-1}))_{K(1)}$ has characteristic ζ.*
- *If p is odd and $j_{2(p-1)n-1} \in \pi_{2(p-1)n-1}(S)$ is a generator of* $\operatorname{im} J_{2(p-1)n-1} \subseteq \pi_{2(p-1)n-1}(S)$, *then $K_0^\vee(S//(j_{2(p-1)n-1}))$ contains an Artin-Schreier element so $(S//(j_{2(p-1)n-1}))_{K(1)}$ has characteristic ζ.*

For $p = 2$, it follows that $(S//(\nu))_{K(1)}$, $(S//(\sigma))_{K(1)}$, $(S//(\sigma))_{K(1)}$ have characteristic ζ. There is a map $S//(\nu) \to S//(\eta)$, so $(S//(\eta))_{K(1)}$ also has characteristic ζ.

Thanks for listening!

Some references can be found on the next slide.

References

- N. Ray, The symplectic J-homomorphism, Invent. Math., **12** (1971), 237–248.
- D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math., **106** (1984), 351–414.
- A. Baker, Some chromatic phenomena in the homotopy of $M\mathrm{Sp}$, Adams Memorial Symposium on Algebraic Topology, Vol. 2, Ed. N. Ray and G. Walker, London Mathematical Society Lecture Note Series, **175** (1992), 263–280.
- G. Laures, An E_∞ splitting of spin bordism, Amer. J. Math., **125** (2003), 977–1027.
- A. J. Baker & J. P. May, Minimal atomic complexes, Topology, **43** (2004), 645–665.
- A. Baker, H. Gilmour & P. Reinhard, Topological André-Quillen homology for cellular commutative S-algebras, Abhand. Math. Sem. Univ. Hamburg, **78** (2008), 27–50.