Topological André-Quillen homology as a cellular theory and some applications

Andrew Baker (Oslo & Glasgow)

(+ H. Gilmour & P. Reinhard)

Tbilisi May 2007 [25/05/2007]

References

[1] A. J. Baker & J. P. May, Minimal atomic complexes, Topology **43**.

[2] M. Basterra, André-Quillen cohomology of commutative S-algebras, J. Pure Appl. Algebra, **144**.

[3] M. Basterra & M. Mandell, Homology and cohomology of E_∞ ring spectra, Math. Z. **249**.

[4] P. Hu, I. Kriz & J. P. May, Cores of spaces, spectra and E_∞ ring spectra, Homology, Homotopy and App. **3**.

1 Topological André-Quillen homology

Let A be a commutative S-algebra (this is equivalent to A being an E_∞ ring spectrum). For a commutative A-algebra $A \longrightarrow B$, we write B/A. For such a pair B/A there is a B-module $\Omega_{B/A}$ which is well defined in the homotopy category $\overline{h}\mathscr{M}_B$ and characterised by the natural isomorphism

$$\overline{h}\mathscr{C}_A/B(B, B \vee M) \cong \overline{h}\mathscr{M}_B(\Omega_{B/A}, M).$$

Here $\overline{h}\mathscr{C}_A/B$ denotes the derived category of commutative A-algebras over B. If $M = \Omega_{B/A}$, the identity map corresponds to a morphism $B \longrightarrow B \vee \Omega_{B/A}$ which projects onto the *universal derivation* $\delta_{B/A} \in \overline{h}\mathscr{M}_A(B, \Omega_{B/A})$.

Associated to a sequence of morphisms of commutative S-algebras $A \longrightarrow B \longrightarrow C$ is a natural cofibre sequence of C-modules

$$\Omega_{B/A} \wedge_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B}.$$

$\Omega_{B/A}$ is defined in $\overline{h}\mathscr{M}_B$ by

$$\Omega_{B/A} = \mathrm{L}Q_B\mathrm{R}I_B(B^c \wedge_A B),$$

where $(-)^c$ is a cofibrant replacement functor, $\mathrm{R}I_B$ is the right derived functor of the augmentation ideal I_B of the category of B-algebras. The target of I_B and $\mathrm{R}I_B$ is the category of B-nucas (non-unital B-algebras). $\mathrm{L}Q_B$ is the left derived functor of Q_B which is defined by the following strict pushout diagram in the category of B-modules.

$$\begin{tikzcd}
N \wedge_B N \arrow[r] \arrow[d] & * \arrow[d] \\
N \arrow[r] & Q_B(N)
\end{tikzcd}$$

The *topological André-Quillen homology* of B/A with coefficients in a B-module M is

$$\mathrm{TAQ}_*(B/A; M) = \pi_* \Omega_{B/A} \wedge_B M.$$

Associated to maps $A \longrightarrow B \longrightarrow C$ as above, is a natural long exact *transitivity sequence*

$$\cdots \longrightarrow \mathrm{TAQ}_k(B/A; M) \longrightarrow \mathrm{TAQ}_k(C/A; M) \longrightarrow \mathrm{TAQ}_k(C/B; M) \longrightarrow \mathrm{TAQ}_{k-1}(B/A; M) \longrightarrow \cdots$$

We are interested in the situation where A and B are connective and the map $\varphi \colon A \longrightarrow B$ induces an isomorphism $A_0 \xrightarrow{\cong} B_0$ and we write $\Bbbk = A_0 = B_0$. There is an Eilenberg-Mac Lane object $H\Bbbk$, which can be taken to be a CW commutative A-algebra or B-algebra.

The *ordinary topological André-Quillen homology* of B/A is

$$\begin{aligned}\mathrm{HAQ}_*(B/A) &= \mathrm{TAQ}_*(B/A; H\Bbbk) \\ &= \pi_*\Omega_{B/A} \wedge_B H\Bbbk.\end{aligned}$$

We introduce coefficients in a $\Bbbk$-module M by

$$\begin{aligned}\mathrm{HAQ}_*(B/A; M) &= \mathrm{TAQ}_*(B/A; HM) \\ &= \pi_*\Omega_{B/A} \wedge_B HM.\end{aligned}$$

When $C_0 = \Bbbk$, the transitivity sequence gives

$$\cdots \longrightarrow \mathrm{HAQ}_k(B/A) \longrightarrow \mathrm{HAQ}_k(C/A)$$
$$\longrightarrow \mathrm{HAQ}_k(C/B) \longrightarrow \mathrm{HAQ}_{k-1}(B/A) \longrightarrow \cdots$$

Two fundamental results are due to Maria Basterra [2].

Lemma 1.1. *Let $\varphi\colon A \longrightarrow B$ be an n-equivalence, where $n \geqslant 1$. Then $\Omega_{B/A}$ is n-connected and there is a map of A-modules $\tau\colon \mathrm{C}_\varphi \longrightarrow \Omega_{B/A}$ for which*

$$\tau_*\colon \pi_{n+1}\,\mathrm{C}_\varphi \xrightarrow{\cong} \pi_{n+1}\Omega_{B/A}.$$

Corollary 1.2 (Hurewicz theorem). *The map τ induces isomorphisms*

$$\tau_*\colon \pi_k\,\mathrm{C}_\varphi \xrightarrow{\cong} \mathrm{HAQ}_k(B/A) \quad (k \leqslant n+1).$$

Using $\delta_{B/A}$ we can define a Hurewicz homomorphism

$$\theta\colon \pi_* B \longrightarrow \mathrm{HAQ}_*(B/A)$$

which factors through the usual Hurewicz homomorphism. There are versions of the Hurewicz theorem for θ. Also, for a morphism of connective S-algebras $\varphi\colon A \longrightarrow B$ with $A_0 = B_0 = \mathbb{Z}$, φ is a weak equivalence if and only if $\varphi_*\colon \mathrm{HAQ}_*(A/S) \longrightarrow \mathrm{HAQ}_*(B/S)$ is an isomorphism.

To calculate HAQ we need to know about its values on certain basic objects. For any A-module X, there is a free commutative A-algebra on X, $\mathbb{P}_A X = \bigvee_{i \geqslant 0} X^{(i)}/\Sigma_i$. If $A \longrightarrow A'$ is a morphism of commutative S-algebras, then

$$\mathbb{P}_{A'}(A' \wedge_A X) \cong A' \wedge_A \mathbb{P}_A X.$$

The A-algebra map $\mathbb{P}_A X \longrightarrow \mathbb{P}_A * = A$ induced by collapsing X to a point makes A into an $\mathbb{P}_A X$-algebra and there is a cofibration sequence of $\mathbb{P}_A X$-modules

$$\mathbb{P}_A^+ X \longrightarrow \mathbb{P}_A X \longrightarrow \mathbb{P}_A * = A,$$

where $\mathbb{P}_A^+ X = \bigvee_{i \geqslant 1} X^{(i)}/\Sigma_i$. For the A-sphere $S^n = S_A^n$ $(n > 0)$ we get the commutative A-algebra $\mathbb{P}_A S^n$ with augmentation $\mathbb{P}_A S^n \longrightarrow A$, we may view an A-module or algebra as a $\mathbb{P}_A S^n$-module or algebra.

Proposition 1.3. *Let X be a cell A-module, so $\mathbb{P}_A X$ is a q-cofibrant A-algebra. Then*

$$\Omega_{\mathbb{P}_A X/A} = \mathbb{P}_A X \wedge_A X.$$

Hence

$$\mathrm{TAQ}_*(\mathbb{P}_A X/A; M) = \pi_* X \wedge_A M.$$

In particular, when A is connective and $\Bbbk = A_0$,

$$\mathrm{HAQ}_r(\mathbb{P}_A S^n/A) = \begin{cases} \Bbbk & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

A CW S-algebra A is a colimit of S-algebras $A^{[n]}$, where $A^{[0]} = S$ and $A^{[n+1]}$ is the pushout of a diagram

$$\begin{tikzcd} & \mathbb{P}_S K_n \arrow[dl] \arrow[dr] & \\ A^{[n]} & & \mathbb{P}_S \mathrm{C} K_n \end{tikzcd}$$

where K_n is a wedge of n-spheres. Since $\mathrm{C}K_n \sim *$, we also have $\mathbb{P}_S \mathrm{C}K_n \sim S$. In fact,

$$A^{[n+1]} = A^{[n]} \wedge_{\mathbb{P}_S K_n} \mathbb{P}_S \mathrm{C}K_n.$$

Properties of the transitivity sequence now give a long exact sequence of the form

$$\cdots \longrightarrow H_{k+1}(\Sigma K_n) \longrightarrow \mathrm{HAQ}_k(A^{[n]}/S) \\ \longrightarrow \mathrm{HAQ}_k(A^{[n+1]}/S) \longrightarrow H_k(\Sigma K_n) \longrightarrow$$

where $H_k(\Sigma K_n)$ is only nonzero if $k = n + 1$.

So for a CW S-algebra, $\mathrm{HAQ}_*(A/S)$ behaves like cellular homology for CW complexes. Of course, we can take any coefficient group in place of $\mathbb{Z}$.

Bousfield localisations can be carried out on S-algebras and their modules. In particular, we can also localise at a prime p. So we could work with the p-local sphere in place of S and with p-local CW algebras.

2 Minimal atomic S-algebras

Assumptions *From now on, we work p-locally. S denotes the p-local sphere. All S-algebras A are commutative and connective with $A_0 = \mathbb{Z}_{(p)}$ and all homotopy groups f.g. over $\mathbb{Z}_{(p)}$.*

A is *atomic* if every S-algebra self map $A \longrightarrow A$ is a weak equivalence.

A is *irreducible* if every S-algebra map $B \longrightarrow A$ inducing a mono on $\pi_*(-)$ is a weak equivalence.

An atomic A is *minimal atomic* if every S-algebra map $B \longrightarrow A$ inducing a mono on $\pi_*(-)$ and with B atomic is a weak equivalence.

A CW S-algebra is *minimal* if for every n,

$$\mathrm{HAQ}_n(A^{[n]}/S;\mathbb{F}_p) \longrightarrow \mathrm{HAQ}_n(A^{[n+1]}/S;\mathbb{F}_p)$$

is an isomorphism.

We use an important general fact (remember our assumptions above).

Lemma 2.1. *For every S-algebra A, there is a weak equivalence $B \longrightarrow A$ with B a minimal CW S-algebra.*

Theorem 2.2. *Let A be an S-algebra. Then the following are equivalent.*

- *A is minimal atomic.*
- *A is irreducible.*
- *For all $k > 0$, the Hurewicz homomorphism $\theta\colon A_k \longrightarrow \mathrm{HAQ}(A/S; \mathbb{F}_p)$ is trivial.*

The proofs are described in Helen Gilmour's thesis and are parallel to those of [1] for spectra and simply connected spaces, but using HAQ in place of ordinary homology.

3 Some examples

If A is a commutative S-algebra that is minimal atomic as an S-module, the usual Hurewicz homomorphism $\pi_k A \longrightarrow H_k(A;\mathbb{F}_p)$ is trivial for $k > 0$. So ku, ko, $H\mathbb{Z}$, $H\mathbb{Z}/p^n$ are all minimal atomic p-locally. If BP were a commutative S-algebra it would be too but this is still not known.

Many Thom spectra are amenable to study using a result of Basterra & Mandell [3].

Theorem 3.1. *Let $f\colon X \longrightarrow BF$ be an infinite loop map with associated Thom spectrum Mf. Then $\Omega_{Mf/S} = Mf \wedge \underline{X}$, where $\underline{X}$ is the spectrum with zeroth space X. Hence*

$$\mathrm{HAQ}_*(Mf/S) = H_*(\underline{X})$$

and the Hurewicz homomorphism θ (with any coefficients) is the composition

$$\pi_*Mf \longrightarrow H_*(Mf) \xrightarrow[\cong]{\text{Thom}} H_*(X) \xrightarrow{\text{eval}} H_*(\underline{X}),$$

where eval *annihilates decomposables in $H_*(X)$.*

MU p-locally: $H_*(MU;\mathbb{F}_p) = \mathbb{F}_p[b_r : r \geqslant 1]$ and
$\mathrm{HAQ}_*(MU/S;\mathbb{F}_p) = H_*(\Sigma^2 ku;\mathbb{F}_p) \subseteq A(p)_{*-2}$.
For p odd,

$$\theta(b_r) = \begin{cases} \xi_s & \text{if } r = p^s, \\ 0 & \text{otherwise,} \end{cases}$$

while if $p = 2$

$$\theta(b_r) = \begin{cases} \xi_s^2 & \text{if } r = 2^s, \\ 0 & \text{otherwise.} \end{cases}$$

So MU is never minimal atomic.

MSp/U 2-locally: The fibration $Sp/U \longrightarrow BU \longrightarrow BSp$ has an associated map of Thom spectra $MSp/U \longrightarrow MU$ and the induced maps in homology and homotopy are injective. In fact,

$$H_*(MSp/U) = \mathbb{Z}_{(2)}[y_{2r-1} : r \geqslant 1] \subseteq H_*(MU)$$

where $y_{2r-1} \equiv b_{2r-1}$ (mod decomp). This time, $\underline{Sp/U} = \Sigma^2 ko$ and θ is trivial here. So MSp/U is minimal atomic and is a core of MU.

MSp 2-locally: By a result of Floyd,

$$\mathrm{im}[MSp_* \longrightarrow MO_*] \subseteq (MO_*)^{(2)},$$

so it follows that

$$\mathrm{im}[MSp_* \longrightarrow H_*(MSp;\mathbb{F}_2)] \subseteq H_*(MSp;\mathbb{F}_2)^{(2)},$$

and so θ is trivial and therefore MSp is minimal atomic.

4 TAQ for some periodic S-algebras

We can identify $\Omega_{KU/S}$ using Snaith's result that the localization of $\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+$ with respect to the generator $\beta_1 \in \pi_2 \Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+$ is equivalent to the periodic K-theory spectrum KU. This result can be rigidified to give an equivalence of commutative S-algebras $\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+ \simeq KU$.

Proposition 4.1. *We have*

$$\Omega_{KU/S} = KU \wedge \Sigma^2 H\mathbb{Z} \simeq KU\mathbb{Q}.$$

Proof. By [3],

$$\Omega_{\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+/S} = \Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+ \wedge \Sigma^2 H\mathbb{Z}.$$

Since the functor $\Omega_{(-)/A}$ commutes with smashing localizations,

$$\Omega_{\Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+[\beta^{-1}]/S} = \Sigma^\infty \mathbb{C}\mathrm{P}^\infty_+[\beta^{-1}] \wedge \Sigma^2 H\mathbb{Z}. \quad \square$$

We have the following consequence.

Corollary 4.2. *Let p be a prime and let $K(1)$ be the first Morava K-theory at p. In the category of $K(1)$-local KU-modules, $\Omega_{KU/S} \simeq *$.*

Proof. Since $\Omega_{KU/S}$ is already KU-local, its $K(1)$-localization agrees with its p-completion, and this is trivial since $\Omega_{KU/S}$ is rationally a wedge of suspensions of $H\mathbb{Q}$. □

For the Lubin-Tate spectrum E_n, we have

Proposition 4.3. *In the category of $K(n)$-local E_n-modules, $\Omega_{E_n/S} \simeq *$.*

Proof. This uses an argument of Rognes to show that (even after Bousfield localization)

$$B \simeq \mathrm{THH}(B/A) \Longrightarrow \Omega_{B/A} \simeq *.$$

We can use a spectral sequence to show that $K(n)_* \mathrm{THH}(E_n/S) = K(n)_* E_n$, hence $E_n \longrightarrow \mathrm{THH}(E_n/S)$ is a $K(n)$-equivalence. □