BRAVE NEW BOCKSTEIN OPERATIONS

ANDREW BAKER & ALAIN JEANNERET

Introduction

The picture of the stable homotopy category dramatically changed after the publication of [2] and improvements in [5]. These new technics lead to an easy and conceptual definition of the Bockstein operations associated to a quotient A of a commutative S-algebra R, see [5].

The aim of this paper is to discussed the algebra (and coalgebra) of operations associated to R-modules and their relationship with the stable operations. Most of the results of this paper are known to experts but the proofs are spread in various papers. Here we present elementary proofs and give various examples.

1. Background

In this paper we freely use the notation and terminology of [2, 5]. Let R be a commutative S-algebra, with R_* concentrated in even dimension, $\mathscr{M}_R$ be the category of R-modules and $\mathscr{D}_R$ its derived category.

A R-ring is a ring A in the category $\mathscr{D}_R$ and

$$\varphi_A\colon A \wedge_R A \longrightarrow A$$

denotes the product.

The functor $A^*_R(-) = \mathscr{D}_R(-, A)$ defines a cohomology theory while $A^R_*(-) = \pi_*(A\wedge_R-)$ defines a homology theory on either of the categories $\mathscr{M}_R$ or $\mathscr{D}_R$. Let $f\colon A \longrightarrow B$ be a map in $\mathscr{M}_R$. Then f induces homology and cohomology operations

$$f\colon A^R_*(-) \longrightarrow B^R_*(-), \quad f\colon A^*_R(-) \longrightarrow B^*_R(-).$$

Also for any homology theory $E^R_*(-)$ or cohomology theory $E^*_R(-)$ there are E_*-module homomorphisms

$$f_*\colon E^R_*(A) \longrightarrow E^R_*(B), \quad f^*\colon E^*_R(B) \longrightarrow E^*_R(A).$$

When $A = B = E$ it may very well happen that the operations f do not coincide with the morphisms f_* or f^*.

Recall some further definitions. Let X and Y be R-modules and $a \in A^R_*(X)$, $b \in A^R_*(Y)$. Then

$$a \otimes b = (\varphi \wedge 1 \wedge 1) \circ (1 \wedge \tau \wedge 1) \circ (a \wedge b) \in A^R_*(X \wedge_R Y),$$

where τ denotes the switch map, and if $x \in A^*_R(X)$, $y \in A^*_R(Y)$, then

$$x \otimes y = \varphi \circ (x \wedge y) \in A^*_R(X \wedge_R Y).$$

The Kronecker pairing

$$A^*_R(X) \longrightarrow \mathrm{Hom}_{A_*}(A^R_*(X), A_*)$$

is defined by

$$\langle x, a\rangle = \varphi \circ (1 \wedge x) \circ a.$$

If A is commutative (up to homotopy) then

$$\langle x \otimes y, a \otimes b\rangle = \langle x, a\rangle \, \langle y, b\rangle \,,$$

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if A is non commutative then the situation is much more delicate, see Lemma 3.4.

We now consider quotients of the commutative S-algebra R. Let $x \in R_d$ be an element which is not a zero divisor (recall that d is even). By construction R/x fits into the cofibre sequence

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho_x} R/x \xrightarrow{\beta_x} \Sigma^{d+1} R.$$

Let $\varphi\colon R/x \wedge_R R/x \longrightarrow R/x$ be a product, then any other product φ' on R/x is given by

$$\varphi' = \varphi + u \circ (\beta \wedge \beta)$$

for some $u \in \pi_{2d+2} R/x$, in particular for the opposite product

$$\varphi^{\mathrm{op}} = \varphi \circ \tau = \varphi + u \circ (\beta \wedge \beta).$$

The symbol A^{op} will denote the ring A endowed with the opposite product φ^{op}.

Let $S = \{x_1, x_2, \ldots\}$ be a regular sequence in R_* generating an ideal $I \triangleleft R_*$. We define the R-module $A = R/I$ or $A = R/S$ to be the smash product taken over R:

$$A = \bigwedge_{R}^{x_i \in S} R/x_i. \tag{1.1}$$

If we splice together all the products φ_i (on the various R/x_i) we obtain a product φ_A on A. The product φ_A as well as the φ_i are unital and associative up to homotopy; for details see [2, 5].

2. Brave new Bockstein operations

Let A be defined as in (1.1). We first fix some notation. By construction R/x_i fits into the cofibre sequence

$$\Sigma^{d_i} R \xrightarrow{x_i} R \xrightarrow{\rho_i} R/x_i \xrightarrow{\beta_i} \Sigma^{d_i+1} R \tag{2.1}$$

with $d_i = \deg(x_i)$. The composition

$$q_i = \rho_i \circ \beta_i\colon R/x_i \longrightarrow \Sigma^{d_i+1} R/x_i$$

is an R-module morphism known as a *Bockstein operation*. We then define $Q_i\colon A \longrightarrow A$ as q_i on the i-th smash factor and the identity on the others. With respect to the product φ_i on R/x_i and φ_A on A, the operations q_i and Q_i are derivations by [5], that is

$$q_i \circ \varphi_i = \varphi_i \circ (\mathrm{id} \wedge q_i \vee q_i \wedge \mathrm{id}), \tag{2.2}$$

$$Q_i \circ \varphi_A = \varphi_A \circ (\mathrm{id} \wedge Q_i \vee Q_i \wedge \mathrm{id}). \tag{2.3}$$

We will consider the map $Q_i \in A_R^*(A)$ more closely. A test spectrum for Q_i is R/x_i. From the cofibration (2.1) we easily deduce that

$$0 \to A_R^*(R) \xrightarrow{\beta_i^*} A_R^*(R/x_i) \xrightarrow{\rho_i^*} A_R^*(R) \to 0$$

and

$$0 \to A_*^R(R) \xrightarrow{(\rho_i)_*} A_*^R(R/x_i) \xrightarrow{(\beta_i)_*} A_*^R(R) \to 0$$

are split short exact sequences of $A_*^R(R) = A_*$-modules. Writing 1 for the unit in A_* (represented by the natural projection $R \longrightarrow A$) and recalling that $R_{\mathrm{odd}} = 0$, we find that there is a unique element $g_i \in A_R^0(R/x_i)$ for which $\rho_i^*(g_i) = 1$. Defining

$$e_i = \beta_i^*(1) \in A_R^{-d_i-1}(R/x_i),$$

observe that e_i is just the composite

$$R/x_i \xrightarrow{q_i} \Sigma^{d_i+1} R/x_i \xrightarrow{g_i} \Sigma^{d_i+1} A. \tag{2.4}$$

As a consequence

$$A_R^*(R/x_i) \cong A_* g_i \oplus A_* e_i.$$

We write γ_i for $(\rho_i)_*(1)$. Let ε_i be the unique class in $A_*^R(R/x_i)$ for which $(\beta_i)_*(\varepsilon_i) = 1$ (uniqueness follows from the fact that $R_{\mathrm{odd}} = 0$), moreover $\deg(\varepsilon_i) = d_i + 1$. Therefore

$$A_*^R(R/x_i) \cong A_* \gamma_i \oplus A_* \varepsilon_i.$$

The Kronecker pairing

$$\langle -,-\rangle : A^*_R(R/x_i) \longrightarrow \mathrm{Hom}_{A_*}(A^R_*(R/x_i), A_*)$$

is easily seen to be an isomorphism. The basis $\{g_i, e_i\}$ is dual to $\{\gamma_i, \varepsilon_i\}$. This follows from the naturality of the pairing. For instance,

$$\langle e_i, \varepsilon_i\rangle = \langle \beta_i^*(1), \varepsilon_i\rangle = \langle 1, (\beta_i)_*(\varepsilon_i)\rangle = \langle 1, 1\rangle = 1.$$

The same shows that $\langle g_i, \gamma_i\rangle = 1$. The equality $\langle e_i, \gamma_i\rangle = 0$ holds for dimensional reasons.

We now determine the action of the Bockstein maps $Q_i\colon A \longrightarrow \Sigma^{d_i+1}A$ on the test spectra R/x_i and A. First we consider the case where we kill off a single element $x \in R_d$, that is $A = R/x$. In this situation

$$A^*_R(A) \cong A_* g \oplus A_* e$$

with $g = \mathrm{id}\colon A \longrightarrow A$ and $e = q\colon A \longrightarrow \Sigma^{d+1}A$. Here we find that $q^*(g) = e = q(g)$ and that $q^*(e) = 0 = q(e)$, that is

$$q^* = q\colon A^*_R(A) \longrightarrow A^*_R(A).$$

In the general case $A = R/I$ with $I = (x_1, x_2, \ldots)$ we observe that the map $R/x_i \longrightarrow A$ induces the vertical quotient maps in the following commutative diagram.

$$\begin{array}{ccc} (R/x_i)^*_R(R/x_i) & \xrightarrow{\cong} & R_*/(x_i)g_i \bigoplus R_*/(x_i)e_i \\ \big\downarrow & & \big\downarrow \\ A^*_R(R/x_i) & \xrightarrow{\cong} & A_* g_i \bigoplus A_* e_i \end{array}$$

The map Q_i is induced by q_i therefore, since $q_i = q_i^*$ in R/x_i-homology, we obtain

$$Q_i = q_i^*\colon A^*_R(R/x_i) \longrightarrow A^*_R(R/x_i).$$

Moreover $Q_i(g_i) = e_i$ and $Q_i(e_i) = 0$. Thus the operation Q_i is non-trivial in the homotopy category $\mathscr{D}_R$.

Dually in homology, we find that

$$Q_i = (q_i)_*\colon A^R_*(R/x_i) \longrightarrow A^R_*(R/x_i),$$

$Q_i(\varepsilon_i) = \gamma_i$ and $Q_i(\gamma_i) = 0$.

3. The operations and cooperations

In this section we describe $A^R_*(A)$ and $A^*_R(A)$ together with the various structures they carry. By the definition of A in (1.1) together with induction and passage to the limit, there is an isomorphism of A_*-modules,

$$A^R_*(A) \cong \bigotimes A^R_*(R/x_i) \cong \Lambda_{A_*}(\alpha_i : x_i \in S), \tag{3.1}$$

where α_i is the image of ε_i under the map $g_i\colon R/x_i \longrightarrow A$ which satisfies

$$\deg(\alpha_i) = \deg(\varepsilon_i) = d_i + 1.$$

Here $\Lambda_{A_*}(-)$ denotes an exterior algebra over A_*.

Dually we also have isomorphisms of A_*-modules:

$$A^*_R(A) \cong \widehat{\bigotimes} A^*_R(R/x_i) \cong \widehat{\Lambda}_{A_*}(Q_i : x_i \in S). \tag{3.2}$$

Here the completed tensor product is needed when the regular sequence $S = \{x_1, x_2, \ldots\}$ is infinite. The first isomorphism is induced by the maps $g_i\colon R/x_i \longrightarrow A$, therefore Q_i is mapped to e_i under g_i^*. Finally, $\widehat{\Lambda}_{A_*}(-)$ denotes the completed exterior algebra.

The duality morphism

$$\langle -,-\rangle : A^*_R(A) \longrightarrow \mathrm{Hom}_{A_*}(A^R_*(A), A_*) \tag{3.3}$$

is an isomorphism which satisfies

$$Q_i(\alpha_j) = \langle Q_i, \alpha_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The same arguments as above show that

$$A^R_*(A \wedge_R A) \cong A^R_*(A) \bigotimes A^R_*(A), \tag{3.4}$$

$$A^*_R(A \wedge_R A) \cong A^*_R(A) \bigotimes A^*_R(A). \tag{3.5}$$

Now we consider the various structures on $A^*_R(A)$ and $A^R_*(A)$. First define η as

$$\eta\colon A = R \wedge_R A \xrightarrow{1\wedge \mathrm{id}} A \wedge_R A$$

where $1\colon R \longrightarrow A$ (the projection) is the unit of A. The composition of R-module homomorphisms induces an algebra structure on $A^*_R(A)$. Thanks to the isomorphism of (3.5) the product of $x, y \in A^*_R(A)$ can be described as the composite

$$A \xrightarrow{\eta} A \wedge_R A \xrightarrow{x\wedge y} A \wedge_R A \xrightarrow{\varphi_A} A.$$

So the product is given by

$$A^*_R(A) \bigotimes A^*_R(A) \cong A^*_R(A \wedge_R A) \xrightarrow{\eta^*} A^*_R(A).$$

It is also possible to define a coalgebra structure on $A^*_R(A)$. The coproduct is induced by φ_A, that is

$$A^*_R(A) \xrightarrow{\varphi_A^*} A^*_R(A \wedge_R A) \cong A^*_R(A) \bigotimes A^*_R(A).$$

In homology the product is induced by φ_A, *i.e.*,

$$A^R_*(A) \bigotimes A^R_*(A) \cong A^R_*(A \wedge_R A) \xrightarrow{(\varphi_A)_*} A^R_*(A)$$

and the coproduct by η, *i.e.*,

$$A^R_*(A) \xrightarrow{\eta_*} A^R_*(A \wedge_R A) \cong A^R_*(A) \bigotimes A^R_*(A).$$

We can also define the same type of structures on $A^*_R(A^{\mathrm{op}})$ and $A^R_*(A^{\mathrm{op}})$.

By naturality of the Kronecker pairing, $A^*_R(A)$ is dual as algebra (resp. coalgebra) to the coalgebra (resp. algebra) $A^R_*(A)$. The same holds between $A^*_R(A^{\mathrm{op}})$ and $A^R_*(A^{\mathrm{op}})$. Observe however that none of $A^R_*(A)$, $A^R_*(A^{\mathrm{op}})$, $A^*_R(A)$ or $A^*_R(A^{\mathrm{op}})$ are Hopf algebras, for the following reason. It is easy to check that η is multiplicative, since

$$\begin{array}{ccc} A \wedge_R A & \xrightarrow{\varphi_A} & A \\ {\scriptstyle \eta\wedge\eta}\downarrow & & \downarrow{\scriptstyle \eta} \\ A \wedge_R A \wedge_R A \wedge_R A & \longrightarrow & A \wedge_R A \end{array}$$

commutes, where the bottom horizontal map is $(\varphi_A \wedge \varphi_A) \circ (1 \wedge \tau \wedge 1)$. The problem is that τ does not induced in cohomology and homology the usual switch map T on tensor product $T\colon V \otimes W \longrightarrow W \otimes V$. In fact the following diagram commutes.

$$\begin{array}{ccc} A^*_R(X) \bigotimes A^*_R(Y) & \xrightarrow{T} & (A^{\mathrm{op}})^*_R(Y) \bigotimes (A^{\mathrm{op}})^*_R(X) \\ \cong\downarrow & & \cong\downarrow \\ A^*_R(X \wedge_R Y) & \xrightarrow{\tau} & (A^{\mathrm{op}})^*_R(Y \wedge_R X) \end{array}$$

A similar result applies in homology.

If A is commutative (up to homotopy), then $A^*_R(A)$ is dual to $A^R_*(A)$ as Hopf algebras.

Theorem 3.1. *As algebras over A_*,*

$$A^*_R(A) \cong \widehat{\Lambda}_{A_*}(Q_i : x_i \in S).$$

The Q_i and the unit are primitives with respect to the coalgebra structure.

Proof. The algebra structure is easily established. By construction, the Q_i anti-commute,

$$Q_iQ_j = -Q_jQ_i.$$

Therefore $A^*_R(A)$ is an exterior algebra. For any i the product

$$\varphi_i \colon R/x_i \wedge_R R/x_i \longrightarrow R/x_i$$

induces a coalgebra structure on $A^*_R(R/x_i)$ satisfying

$$\begin{aligned}\varphi_i^*(g_i) &= g_i \otimes g_i,\\ \varphi_i^*(e_i) &= g_i \otimes e_i + e_i \otimes g_i.\end{aligned}$$

These equalities follow from the definition of g_i and e_i in (2.4) and the derivation rule (2.2). Then we paste all the $A^*_R(R/x_i)$ together to obtain the result. □

Recall that for any i there exists a well defined class $u_i \in \pi_* R/x_i$ such that $\varphi_i^{\mathrm{op}} = \varphi_i + u_i \circ (\beta_i \wedge \beta_i)$.

Theorem 3.2. *As algebras over* A_*,

$$A^*_R(A^{\mathrm{op}}) \cong \widehat{\Lambda}_{A_*}(Q_i : x_i \in S).$$

The Q_i *are primitives with respect to the coalgebra structure while the coproduct of* 1 *satisfies*

$$(\varphi_A^{\mathrm{op}})^*(1) = 1 \otimes 1 + \sum u_i \cdot Q_i \otimes Q_i.$$

Proof. The algebra structure of $A^*_R(A^{\mathrm{op}})$ is independent of φ_A^{op}, thus $A^*_R(A^{\mathrm{op}}) \cong A^*_R(A)$ as algebras.

The product $\varphi_i^{\mathrm{op}} \colon R/x_i \wedge_R R/x_i \longrightarrow R/x_i$ induces a coalgebra structure on $A^*_R(R/x_i^{\mathrm{op}})$, we easily check that

$$(\varphi_i^{\mathrm{op}})^*(e_i) = \varphi_i^*(e_i) + e_i \circ u_i \circ (\beta_i \wedge \beta_i)$$

but $e_i \circ u_i = 0$, by (2.4) together with the fact that q_i acts trivially on elements coming from R_*. Now

$$\begin{aligned}(\varphi_i^{\mathrm{op}})^*(g_i) &= \varphi_i^*(g_i) + g_i \circ u_i \circ (\beta_i \wedge \beta_i)\\ &= g_i \otimes g_i + u_i \cdot e_i \otimes e_i.\end{aligned}$$

The second equality follows from the definitions of e_i and g_i. Now we can paste all the $A^*_R(R/x_i^{\mathrm{op}})$ together to obtain the result. □

The case of homology seems much more interesting.

Theorem 3.3. *As algebras over* A_*,

$$A^R_*(A^{\mathrm{op}}) = \Lambda_{A_*}(\alpha_i : x_i \in S).$$

The α_i *and the unit are primitives with respect to the coalgebra structure.*

We first restrict our attention to the case where $A = R/x$ and $x \in R_d$. We have the cofibre sequence

$$\Sigma^d R \xrightarrow{x} R \xrightarrow{\rho} A \xrightarrow{\beta} \Sigma^{d+1} R.$$

Let $\varphi \colon A \wedge_R A \longrightarrow A$ be a product and

$$\varphi^{\mathrm{op}} = \varphi \circ \tau = \varphi + u \circ (\beta \wedge \beta)$$

for some well defined class $u \in A_{2d+2}$.

The following Lemma is the crucial step in the proof of Theorem 3.3.

Lemma 3.4. *For* $a \in A^R_*(X)$, $b \in A^R_*(Y)$, $x \in A^*_R(X)$ *and* $y \in A^*_R(Y)$, *the Kronecker pairing satisfies*

$$\langle x, a\rangle \langle y, b\rangle = \langle x \otimes y, a \otimes b\rangle - u\beta_*(x_*(a))\beta(y_*(b)).$$

Proof. The reader is encouraged to draw the diagrams corresponding to the following complicated equalities.

$$\begin{aligned}\langle x,a\rangle\,\langle y,b\rangle &= \varphi\circ(\varphi\wedge\varphi)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge\varphi\wedge 1)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge\varphi^{\mathrm{op}}\wedge 1)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &\qquad -\varphi\circ(1\wedge\varphi)\circ(1\wedge(u\circ(\beta\wedge\beta))\wedge 1)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b).\end{aligned}$$

The first equality follows from the definition, the second by associativity of the product and the third by definition of the opposite product.

We consider separately the two summands of the last term. The first one is

$$\begin{aligned}\varphi\circ(1\wedge\varphi)\circ&(1\wedge\varphi^{\mathrm{op}}\wedge 1)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge\varphi\wedge 1)\circ(1\wedge\tau\wedge 1)\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge\varphi\wedge 1)\circ(1\wedge 1\wedge x\wedge y)\circ(1\wedge\tau\wedge 1)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(\varphi\wedge 1\wedge 1)\circ(1\wedge 1\wedge x\wedge y)\circ(1\wedge\tau\wedge 1)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge x\wedge y)\circ(\varphi\wedge 1\wedge 1)\circ(1\wedge\tau\wedge 1)\circ(a\wedge b)\\ &= \varphi\circ(x\otimes y)\circ(a\otimes b)\\ &= \langle x\otimes y, a\otimes b\rangle\,.\end{aligned}$$

The first equality follows from the definition of the opposite product, the second is obvious, the third uses associativity of the product, the fourth is obvious, the fifth follows from the definition of the exterior product in homology and cohomology and the last one is the definition of the Kronecker pairing.

In the second summand,

$$\begin{aligned}\varphi\circ(1\wedge\varphi)\circ(1\wedge(u\circ(\beta\wedge\beta))\wedge 1)&\circ(1\wedge x\wedge 1\wedge y)\circ(a\wedge b)\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge(u\circ(\beta\wedge\beta))\wedge 1)\circ(x_*(a)\wedge y_*(b))\\ &= \varphi\circ(1\wedge\varphi)\circ(1\wedge u\wedge 1)\circ(\beta_*(x_*(a))\wedge\beta(y_*(b)))\\ &= u\beta_*(x_*(a))\beta(y_*(b)).\end{aligned}$$

The first equality follows from the definition of an induced morphism, the second from the definitions of induced morphism and induced operation while the last one is obvious. Now combining these results, the Lemma is proven. □

Proof of Theorem 3.3. We first restrict our attention to the special case $A = R/x$ with notation as above. We have already shown that

$$A^*_R(A)\cong\Lambda_{A_*}(q)$$

as A_*-algebras, where q stands for the composite $\rho\circ\beta$ and

$$A^R_*(A^{\mathrm{op}})\cong\Lambda_{A_*}(\alpha)$$

as A_*-modules with $\deg(\alpha)=d+1$. Moreover $\{1,q\}$ is the dual basis of $\{1,\alpha\}$. Now to show that $A^R_*(A^{\mathrm{op}})$ is an exterior algebra it is sufficient to prove that $\langle x,\alpha^2\rangle=0$ for any $x\in A^*_R(A)$. We only need to consider the cases $x=q$ and $x=1$. In general we have

$$\langle x,\alpha^2\rangle=\langle x,\varphi^{\mathrm{op}}_*(\alpha\otimes\alpha)\rangle=\langle(\varphi^{\mathrm{op}})^*(x),\alpha\otimes\alpha\rangle\,.$$

When $x=q$, since q is a derivation with respect to any product on A we find

$$\begin{aligned}(\varphi^{\mathrm{op}})^*(q)&=\varphi^*(q)+(u\circ(\beta\wedge\beta))^*(q)\\ &=q\otimes 1+1\otimes q+q\circ(u\circ(\beta\wedge\beta))\\ &=q\otimes 1+1\otimes q,\end{aligned}$$

since $A_{\mathrm{odd}}=0, q\circ u=0$. Applying Lemma 3.4 we obtain

$$\langle q\otimes 1,\alpha\otimes\alpha\rangle=u\beta_*(q_*(\alpha))\beta(q_*(\alpha))=0,$$

since $\beta \circ q = 0$. A similar calculation shows that $\langle 1 \otimes q, \alpha \otimes \alpha \rangle = 0$.

Next in the case $x = 1$, we have

$$\left\langle 1, \alpha^2 \right\rangle = \langle 1 \otimes 1, \alpha \otimes \alpha \rangle + \langle 1 \circ u \circ (\beta \wedge \beta), \alpha \otimes \alpha \rangle .$$

We will show that the first summand is equal to u. The element $1 \in A^*_R(A)$ is represented by the identity, hence because of Lemma 3.4 it suffices to show that $\beta_*(\alpha) = 1 = \beta(\alpha)$. As $A_* = R_*/(x)$, the algebra $A^R_*(A^{\mathrm{op}})$ has A_* in its center, thus the two homomorphisms $\rho, \rho_* \colon A_* \longrightarrow A^R_*(A^{\mathrm{op}})$ coincide. We have already shown that

$$q(\alpha) = q_*(\alpha) = 1,$$

so since ρ is injective it follows that

$$\beta(\alpha) = \beta_*(\alpha) = 1.$$

The second summand satisfies

$$\langle 1 \circ u \circ (\beta \wedge \beta), \alpha \otimes \alpha \rangle = -u.$$

This holds since $\beta_*(\alpha) = 1$, the sign comes from the fact that the twist map

$$\tau \colon \Sigma^{d+1} A \wedge A \longrightarrow A \wedge \Sigma^{d+1} A$$

is a map of degree -1 since d is even.

Now consider the general case $A = R/I$ and $I = (x_1, x_2, \ldots)$. For any i,

$$\varphi_i^{\mathrm{op}} = \varphi_i \circ \tau = \varphi_i + u_i \circ (\beta_i \wedge \beta_i).$$

The morphisms of R ring spectra $R/x_i \longrightarrow A$ induce ring homomorphisms

$$(R/x_i)^R_*(R/x_i^{\mathrm{op}}) \longrightarrow A^R_*(R/x_i^{\mathrm{op}})$$

and we now easily deduce that as algebras over A_*,

$$A^R_*(R/x_i^{\mathrm{op}}) \cong \Lambda_{A_*}(\alpha_i).$$

By construction of the product φ_A on A, the elements α_i commute to each other, therefore

$$A^R_*(A^{\mathrm{op}}) \cong \Lambda_{A_*}(\alpha_i : x_i \in S).$$

The coalgebra structure follows easily from the naturality of the pairing

$$\langle x \otimes y, \eta(\alpha_i) \rangle = \langle x \cdot y, \alpha_i \rangle$$

for $x, y \in A^*_R(A^{\mathrm{op}})$. Thus

$$\eta(\alpha_i) = \alpha_i \otimes 1 + 1 \otimes \alpha_i$$

and $\eta(1) = 1 \otimes 1$. □

A similar calculation establishes

Theorem 3.5. *As algebras over A_*,*

$$A^R_*(A) \cong A_*[\alpha_i : x_i \in S]/(\alpha_i^2 - u_i),$$

where the α_i and the unit are primitives with respect to the coalgebra structure.

Remark 3.6. The R ring spectra A and A^{op} are usually not isomorphic because the rings $A^R_*(A)$ and $A^R_*(A^{\mathrm{op}})$ are not isomorphic. As illustrated in the examples below, in some cases A and A^{op} may be isomorphic as S ring spectra, *i.e.*, there is a morphism of S ring spectra $A \longrightarrow A^{\mathrm{op}}$ that is not a morphism of R ring spectra.

4. Spectral sequences

In this section we consider the two spectral sequences converging to $A_R^*(A)$ and $A_*^R(A)$ respectively. Here, as usual, A satisfies (1.1).

From [2], there is a multiplicative spectral sequence

$$\mathrm{E}_2^{**} = \mathrm{Ext}^*_{R_*}(A_*, A_*) \Rightarrow A_R^*(A).$$

We determine the E_2^{**}-term with the aid of the Koszul resolution of A_*:

$$\Lambda_{R_*}(\omega_i : x_i \in S) \longrightarrow A_* \to 0, \tag{4.1}$$

with differentials $d(\omega_i) = x_i$. The Ext module is the homology of the complex

$$\mathrm{Hom}_{R_*}(\Lambda_{R_*}(\omega_i : x_i \in S)) \to 0.$$

Then we easily obtain

$$\mathrm{Ext}^*_{R_*}(A_*, A_*) \cong \widehat{\Lambda}_{A_*}(\tau_i : x_i \in S)$$

as A_*-algebras. For dimensional reasons, the differentials act trivially on the τ_i, and so by multiplicativity the spectral sequence collapses. It is not hard to identify the Bockstein Q_i in this spectral sequence

Theorem 4.1. *The cofibre E of Q_i satisfies*

$$\pi_*(E) \cong R_*/(x_1, x_2, \ldots, x_i^2, \ldots)$$

and the extension

$$0 \to A_* \longrightarrow E_* \longrightarrow A_* \to 0$$

represents Q_i in $\mathrm{Ext}^*_{R_*}(A_*, A_*)$.

Proof. By construction of the Q_i the following diagram commutes.

$$\begin{array}{ccccccc} \Sigma^{-d_i-1}R/x_i & \xrightarrow{\beta_i} & R & \xrightarrow{x_i} & R & \xrightarrow{\rho_i} & R/x_i \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-d_i-1}A & \xrightarrow{Q_i} & A & \longrightarrow & E & \longrightarrow & A \end{array}$$

In homotopy it induces the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R_* & \xrightarrow{x_i} & R_* & \longrightarrow & R_*/(x_i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_* & \longrightarrow & E_* & \longrightarrow & A_* & \longrightarrow & 0 \end{array}$$

which shows that E_* is the push-out of the left-handed square, because the left vertical morphism is the natural projection, hence E_* has the desired form.

Observe that the extension

$$0 \to A_* \longrightarrow E_* \longrightarrow A_* \to 0$$

is classified by $\tau_i \in \mathrm{Ext}^1_{R_*}(A_*, A_*)$, where by construction, τ_i is represented by the composition

$$\bigoplus_i R_*\omega_i \longrightarrow R_*\omega_i \longrightarrow A_*$$

of the projection on the i-th factor and of the quotient map.

It remains to prove that Q_i is represented by τ_i in the spectral sequence. First we consider the spectral sequence for $A_R^*(R/x_i)$:

$$\mathrm{E}_2^{**} = \mathrm{Ext}^*_{R_*}(R_*/(x_i), A_*) \Rightarrow A_R^*(R/x_i).$$

We have $\mathrm{Ext}^*_{R_*}(R_*/(x_i), A_*) \cong \Lambda_{A_*}(\tau_i)$ and $A_R^*(R/x_i) \cong \Lambda_{A_*}(e_i)$. The spectral sequence collapses for dimensional reason and e_i is represented by τ_i because we do not have elements of filtration greater than one and obviously it is not of filtration zero.

For any quotient A of R, the Koszul resolution of (4.1) can be realized geometrically

$$\cdots \longrightarrow \bigvee_{i,j} R\omega_i\omega_j \longrightarrow \bigvee_{i} R\omega_i \longrightarrow A.$$

This is the main ingredient in the construction of the spectral sequence in our cases. To show that Q_i is represented by $\tau_i \in \mathrm{Ext}^1_{R_*}(A_*, A_*)$, it suffices to compare the two geometrical resolutions

$$\begin{array}{ccccccc} & & R & \longrightarrow & R & \longrightarrow & R/x_i \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \bigvee R\omega_i & \longrightarrow & R & \longrightarrow & A \end{array}$$

in which the left vertical morphism is the inclusion on the i-th factor. Now we easily deduce the result. □

We now turn to homology and consider the Künneth spectral sequence of [2],

$$\mathrm{E}^2_{**} \cong \mathrm{Tor}^{R_*}_{**}(A_*, A_*) \Longrightarrow A^R_*(A). \tag{4.2}$$

As in the cohomological case, we use a Koszul resolution to compute the E^2-term. We obtain

$$\mathrm{Tor}^{R_*}_*(A_*, A_*) \cong \Lambda_{A_*}(t_i : x_i \in S).$$

In this situation the spectral sequence is known to be multiplicative by [1], and we easily deduce that it collapses. Observe however that we do not need the multiplicative structure to show that it collapses. We may proceed as follows.

Observe first that it is sufficient to consider the case where we kill off only a finite number of elements in R_* (this is legitimate because we are working in homology). In this case, the Koszul resolution (4.1) is free and of finite type, this implies that the exact couples used to construct the spectral sequences (in homology and in cohomology) are dual to each other and the modules involved are free of finite rank. The collapse of the cohomology spectral sequence then implies the collapse of the homology spectral sequence.

We can also consider the spectral sequence

$$\mathrm{E}^2_{**} \cong \mathrm{Tor}^{R_*}_{**}(A_*, A^{\mathrm{op}}_*) \Longrightarrow A^R_*(A^{\mathrm{op}}). \tag{4.3}$$

Similar arguments show that it collapses. Because $A_* = A^{\mathrm{op}}_*$, the spectral sequences (4.2) and (4.3) coincide. In this example we see that even though the spectral sequence is multiplicative we cannot recover $A^R_*(A)$ and $A^R_*(A^{\mathrm{op}})$ from it.

5. Some examples

In this section we will be mostly interested in the morphism

$$F^*\colon A^*_R(A) = \mathscr{D}_R(A, A) \longrightarrow A^*(A) = \mathscr{D}_S(A, A)$$

induced by the forgetful functor $\mathscr{D}_R \longrightarrow \mathscr{D}_S$. The problem is to determine under which conditions the maps Q_i are non-trivial in the category $\mathscr{D}_S$. We will also consider the dual map $F_*\colon A_*(A) \longrightarrow A^R_*(A)$ induced from the evident natural transformation between the smash product bifunctors

$$(-)\underset{S}{\wedge}(-) \longrightarrow (-)\underset{R}{\wedge}(-).$$

Throughout the section we will make use of the following remark.

Remark 5.1. Let $\widetilde{S} \subset S$ be regular sequences in R_*. Then

$$R/S = R/\widetilde{S}\underset{R}{\wedge} \bigwedge_{R}^{x_i \in S - \widetilde{S}} R/x_i.$$

If $\varphi\colon R/\widetilde{S} \longrightarrow R/\widetilde{S}$ is a morphism of R-modules then $\varphi \wedge \mathrm{id}\colon R/S \longrightarrow R/S$ is also a morphism of R-modules, therefore we obtain a ring map

$$\mathscr{D}_R(R/\widetilde{S}, R/\widetilde{S}) = R/\widetilde{S}^*_R(R/\widetilde{S}) \longrightarrow \mathscr{D}_R(R/S, R/S) = R/S^*_R(R/S).$$

If $\bar{X} \subset X$ are any sequences in R_* then the inclusion $\bar{X} \subset X$ induces a morphism of R-modules $\bar{X}^{-1}R \longrightarrow X^{-1}R$. Thus we obtain the following lattice of ring maps.

$$\begin{array}{ccc} \mathscr{D}_R(\bar{X}^{-1}R/\widetilde{S}, \bar{X}^{-1}R/\widetilde{S}) & \longrightarrow & \mathscr{D}_R(\bar{X}^{-1}R/S, \bar{X}^{-1}R/S) \\ \downarrow & & \downarrow \\ \mathscr{D}_R(X^{-1}R/\widetilde{S}, X^{-1}R/\widetilde{S}) & \longrightarrow & \mathscr{D}_R(X^{-1}R/S, X^{-1}R/S) \end{array}$$

We begin with the commutative S-algebra $R = MU$, the spectrum of the complex cobordism [2]. Let p be a prime number and $MU_{(p)}$ be the p-localization of MU; $MU_{(p)}$ is again a commutative S-algebra satisfying

$$\pi_*(MU_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots],$$

with $deg(x_i) = 2i$. We can choose the generators x_i such that x_{p^i-1} is the i-th Hazewinkel generator of BP_*, we write v_i rather than x_{p^i-1}. Here we recall that BP is a summand of $MU_{(p)}$ for which

$$BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots] \subset MU_{(p)*}.$$

We consider the regular sequence $S = \{x_i : i \neq p^k - 1,\ i \in \mathbb{N}\}$. The quotient $MU_{(p)}/S$ is a model for BP. Then we compute

$$BP^*_{MU_{(p)}}(BP) \cong \widehat{\Lambda}_{BP_*}(K_i : x_i \in S).$$

We denote the Bockstein operations by K_i rather than Q_i. Observe that the K_i are trivial in the stable category $\mathscr{D}_S$ since the ring homomorphism

$$F^* \colon BP^*_{MU_{(p)}}(BP) \longrightarrow BP^*(BP)$$

is trivial (in positive dimension) because $BP^{\mathrm{odd}}(BP) = 0$. Therefore we will not be interested in the generators K_i. If BP were a commutative S-algebra (this is not known to be true at the time of writing) we could start with $R = BP$ and the situation would be cleaner.

Dually in homology and because BP is commutative

$$BP_*^{MU_{(p)}}(BP) \cong \Lambda_{BP_*}(\kappa_i : x_i \in S) \tag{5.1}$$

and the the morphism

$$F_* \colon BP^*(BP) \longrightarrow BP_*^{MU_{(p)}}(BP)$$

is trivial in positive dimension.

Let $\widetilde{S} = \{t_1, t_2, \ldots\}$ be a regular sequence in BP_* and let A be the quotient

$$A = MU_{(p)}/\{S \cup \widetilde{S}\} = BP \bigwedge_{MU_{(p)}} MU_{(p)}/\widetilde{S}.$$

We have

$$A^*_{MU_{(p)}}(A) \cong \widehat{\Lambda}_{A_*}(K_i : x_i \in S) \widehat{\otimes} \widehat{\Lambda}_{A_*}(Q_i : x_i \in \widetilde{S}).$$

Theorem 5.2. *Let $F^* \colon A^*_{MU_{(p)}}(A) \longrightarrow A^*(A)$ be the forgetful map. Then with the above notation, the K_i are in the kernel of F^* while the Q_i are not.*

Proof. Let H denote the mod p Eilenberg-Mac Lane spectrum

$$H = K(\mathbb{F}_p; 0) = MU_{(p)}/(p, x_1, x_2, \ldots) = BP \bigwedge_{MU_{(p)}} MU_{(p)}/(p, v_1, v_2, \ldots).$$

We write BP/t_i for $BP \bigwedge_{MU_{(p)}} MU_{(p)}/t_i$. Since the mod p homology Hurewicz homomorphism for BP is trivial, the cofibration

$$\Sigma^{|t_i|} MU_{(p)} \xrightarrow{t_i} MU_{(p)} \xrightarrow{\rho_i} MU_{(p)} \xrightarrow{\beta_i} \cdots$$

induces for any i

$$H_*(BP/t_i) \cong H_*(BP) \otimes \Lambda_{\mathbb{F}_p}(\varepsilon_i) \tag{5.2}$$

with $(\beta_i)_*(\varepsilon_i) = 1$ and

$$H_*^{MU_{(p)}}(BP/t_i) \cong H_*^{MU_{(p)}}(BP) \otimes \Lambda_{\mathbb{F}_p}(\varepsilon_i). \tag{5.3}$$

A similar calculation to that in the previous sections shows that

$$H_*^{MU_{(p)}}(BP) \cong \Lambda_{\mathbb{F}_p}(\kappa_i : x_i \in S). \tag{5.4}$$

Here the κ_i of (5.1) and (5.4) correspond under the natural map

$$BP_*^{MU_{(p)}}(BP) \longrightarrow H_*^{MU_{(p)}}(BP).$$

The products φ_i and φ_i^{op} on BP/t_i induce the same algebra structure on either $H_*(BP/t_i)$ or $H_*^{BP}(BP/t_i)$, therefore the isomorphisms of (5.2) and (5.3) are isomorphisms of $\mathbb{F}_p$-algebras. By induction we obtain the isomorphisms of $\mathbb{F}_p$-algebras

$$H_*(A) \cong H_*(BP) \otimes \Lambda_{\mathbb{F}_p}(\alpha_i : t_i \in \widetilde{S}),$$
$$H_*^{MU_{(p)}}(A) \cong \Lambda_{\mathbb{F}_p}(\kappa_i : x_i \in S) \otimes \Lambda_{\mathbb{F}_p}(\alpha_i : t_i \in \widetilde{S}),$$

where α_i is the image of ε_i and the map

$$F_*\colon H_*(A) \longrightarrow H_*^{MU_{(p)}}(A)$$

is the natural projection onto $\Lambda_{\mathbb{F}_p}(\alpha_i : t_i \in \widetilde{S})$.

By definition of the Bockstein $q_i\colon BP/t_i \longrightarrow BP/t_i$, we have $(q_i)_*(\varepsilon_1) = 1$. Therefore $Q_i\colon A \longrightarrow A$ satisfies $Q_i(\alpha_i) = 1$ and Q_i is non-trivial in $\mathscr{D}_S$. Thus we have proved that the Bocksteins Q_i are not in the kernel of the forgetful map $F^*\colon A^*_{MU_{(p)}}(A) \longrightarrow A^*(A)$. □

When $A = H$ we have the following identification. The forgetful map

$$F^*\colon H^*_{MU_{(p)}}(H) \longrightarrow H^*(H)$$

is a morphism of Hopf algebras and the element $Q_i \in H^{2p^i-1}_{MU_{(p)}}(H)$ is mapped to a non-trivial primitive element in degree $2p^i - 1$ in the Steenrod algebra $H^*(H)$, so it is the Milnor's basis element written Q_i and the image of F^* is the exterior algebra on such elements.

In homology we have

$$H_*(H) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \ldots)$$

and τ_i is dual to Milnor basis Q_i. Therefore the map

$$F_*\colon H_*(H) \longrightarrow H_*^{MU_{(p)}}(H) \cong \Lambda_{\mathbb{F}_p}(\alpha_0, \alpha_1, \ldots) \otimes \Lambda_{\mathbb{F}_p}(\kappa_i : x_i \in S)$$

sends ξ_i to 0 and τ_i to α_i.

So far we have that the Bockstein operations Q_i are not in the kernel of the forgetful map

$$F^*\colon A^*_{MU_{(p)}}(A) \longrightarrow A^*(A),$$

but we are not claiming that the latter is injective (this is false in general). For instance, consider the p-local Eilenberg-Mac Lane spectrum

$$H_{(p)} = K(\mathbb{Z}_{(p)}; 0) = MU_{(p)}/(x_1, x_2, \ldots) = BP \bigwedge_{MU_{(p)}} MU_{(p)}/(v_1, v_2, \ldots).$$

The map $H_{(p)} \longrightarrow H$ induces the commutative diagram

$$\begin{array}{ccccc}
\Lambda_{\mathbb{Z}_{(p)}}(K_i : x_i \in S) \otimes \Lambda_{\mathbb{Z}_{(p)}}(Q_i : i > 0) & \xrightarrow{\cong} & (H_{(p)})^*_{MU_{(p)}}(H_{(p)}) & \xrightarrow{F^*} & (H_{(p)})^*(H_{(p)}) \\
 & & \downarrow & & \downarrow \\
\Lambda_{\mathbb{F}_p}(K_i : x_i \in S) \otimes \Lambda_{\mathbb{F}_p}(Q_i : i \geqslant 0) & \xrightarrow{\cong} & H^*_{MU_{(p)}}(H) & \xrightarrow{F^*} & H^*(H)
\end{array}$$

and by construction the left vertical map identifies the corresponding K_i and Q_i for $i > 0$. The bottom horizontal map is the injection into the exterior algebra generated by Milnor's elements. In degree 0 the diagram is just the mod p reduction. In positive degrees the image of

$Q_i \in (H_{(p)})^*_{MU_{(p)}}(H_{(p)})$ in $(H_{(p)})^*(H_{(p)})$ is not trivial. By [4], in positive degrees $(H_{(p)})^*(H_{(p)})$ is torsion of order exactly p, so the image of pQ_i is trivial and the map

$$F^*\colon (H_{(p)})^*_{MU_{(p)}}(H_{(p)}) \longrightarrow (H_{(p)})^*(H_{(p)})$$

is not injective.

The dual situation in homology is quite interesting. In this case the map $H_{(p)} \longrightarrow H$ induces the following commutative diagram.

$$\begin{array}{ccccc}
(H_{(p)})_*(H_{(p)}) & \xrightarrow{F_*} & (H_{(p)})_*^{MU_{(p)}}(H_{(p)}) & \xrightarrow{\cong} & \Lambda_{\mathbb{Z}_{(p)}}(\alpha_i : i > 0) \otimes \Lambda_{\mathbb{Z}_{(p)}}(\kappa_i : x_i \in S) \\
{\scriptstyle \pi_*}\downarrow & & \downarrow & & \\
H_*(H) & \xrightarrow{F_*} & H_*^{MU_{(p)}}(H) & \xrightarrow{\cong} & \Lambda_{\mathbb{F}_p}(\alpha_i : i \geqslant 0) \otimes \Lambda_{\mathbb{F}_p}(\kappa_i : x_i \in S)
\end{array}$$

In degree 0 the diagram is just the mod p reduction. In positive degrees the top horizontal map is zero since by [4], $(H_{(p)})_*(H_{(p)})$ is torsion of order p and $(H_{(p)})_*^{MU_{(p)}}(H_{(p)})$ is $\mathbb{Z}_{(p)}$ free. The right vertical map identifies the κ_i and α_i. The map F_* is the canonical surjection described above. Since the composite $\pi_* \circ F_*$ is trivial, the image of π_* has trivial intersection with the exterior algebra generated by the α_i. This is consistent with the calculations of [4].

Now we consider the case of $P(n) = BP \bigwedge_{MU_{(p)}} MU_{(p)}/I_n$ with $I_n = (p, v_1, \ldots, v_{n-1})$ and let $p = 2$ (when p is odd $P(n)$, is homotopy commutative and the situation is much simpler to understand). Let φ_n be a product on $P(n)$ compatible with the $MU_{(p)}$ structure. Then from [3], the opposite product φ_n^{op} satisfies

$$\varphi_n^{\mathrm{op}} = \varphi_n + v_n \varphi_n \circ (Q_{n-1} \wedge Q_{n-1}).$$

Also recall from [3] that $P(n)_*(P(n))$ is the quotient algebra

$$P(n)_*(P(n)) = P(n)_*[\alpha_0, \ldots, \alpha_{n-1}, t_i : i > n]/(\alpha_i^2 + t_{i+1} + v_{i+1} : i = 0, \ldots, n-1),$$

where t_{i+1} and v_i are to be interpreted as 0 for $i < n$. Similarly,

$$P(n)_*(P(n)^{\mathrm{op}}) = P(n)_*[\alpha_0, \ldots, \alpha_{n-1}, t_i : i > n]/(\alpha_i^2 + t_{i+1} : i = 0, \ldots, n-1).$$

We have determined that

$$P(n)_*^{MU_{(p)}}(P(n)) \cong P(n)_*[\alpha_0, \ldots, \alpha_{n-1}]/((\alpha_i^2 + v_{i+1}) \otimes \Lambda_{P(n)_*}(\kappa_i : x_i \in S)$$

and

$$P(n)_*^{MU_{(p)}}(P(n)^{\mathrm{op}}) \cong \Lambda_{P(n)_*}(\alpha_0, \ldots, \alpha_{n-1}) \otimes \Lambda_{P(n)_*}(\kappa_i : x_i \in S),$$

where the maps

$$F_*\colon P(n)_*(P(n)) \longrightarrow P(n)_*^{MU_{(p)}}(P(n)), \quad F_*\colon P(n)_*(P(n)^{\mathrm{op}}) \longrightarrow P(n)_*^{MU_{(p)}}(P(n)^{\mathrm{op}})$$

are the natural projections.

Observe that since $P(n)_*^{MU_{(p)}}(P(n))$ and $P(n)_*^{MU_{(p)}}(P(n)^{\mathrm{op}})$ are not isomorphic as $P(n)_*$ algebras, $P(n)$ and $P(n)^{\mathrm{op}}$ cannot be isomorphic as $MU_{(p)}$ ring spectra. However, from [3] they are isomorphic as ring spectrum.

In cohomology, the map

$$F^*\colon P(n)^*_{MU_{(p)}}(P(n)) \cong \Lambda_{P(n)_*}(Q_0, \ldots, Q_{n-1}) \otimes \widehat{\Lambda}_{P(n)_*}(K_i : x_i \in S) \longrightarrow P(n)^*(P(n))$$

is trivial on the second factor and is the inclusion on the subalgebra generated by the Bockstein on the first one.

As further example we consider the case where $R = X^{-1}MU_{(p)}$ with $X \neq \emptyset$. It may very well happen that the R-maps Q_i become trivial when regarded as maps in the stable category $\mathscr{D}_S$ (we have already proven that the Q_i are non-trivial in the category $\mathscr{D}_R$). For instance, let $R = v_n^{-1}MU_{(p)}$ and $\widetilde{S} = \{v_0 = p, v_1, \ldots, \hat{v}_n, \ldots\}$. Then $A = K(n)$ is the n-th Morava K-theory for which

$$K(n)_* \cong \mathbb{F}_p[v_n, v_n^{-1}].$$

According to Wolbert [6],

$$K(n)^*_{v_n^{-1}MU_{(p)}}(K(n)) \cong K(n)^*_{MU_{(p)}}(K(n))$$

where the latter is

$$\widehat{\Lambda}_{K(n)_*}(Q_i : i \neq n)\widehat{\otimes}\widehat{\Lambda}_{K(n)_*}(K_i : x_i \in S).$$

We already know that the Bockstein Q_i for $i < n$ are non-trivial in $K(n)^*(K(n))$, in which they generate an exterior algebra). We now prove that the map $F^* \colon K(n)^*_{MU_{(p)}}(K(n)) \longrightarrow K(n)^*(K(n))$ sends Q_i to 0 when $i > n$.

For any spectrum X there is a natural isomorphism of $K(n)_*$-modules

$$K(n)^*(X) \cong \varprojlim_{\alpha} K(n)^*(X^\alpha), \tag{5.5}$$

where the limit is taken over finite subcomplexes $X^\alpha \subset X$. Let $i > n$. From the natural isomorphism of (5.5), it suffices to show that $Q_i(x) = 0$ for any $x \in K(n)^*(X)$ with X finite.

Consider the spectrum

$$C_{n,i} = v_n^{-1}BP \bigwedge_{v_n^{-1}MU_{(p)}}^{j \neq n,i} v_n^{-1}MU_{(p)}/v_j$$

which satisfies

$$\pi_*(C_{n,i}) \cong \mathbb{F}_p[v_n, v_n^{-1}, v_i].$$

The spectrum $C_{n,i}$ is equipped with a natural morphism of ring spectra $\alpha \colon P(n) \longrightarrow C_{n,i}$. If X is a finite spectrum, then by Landweber exactness, the map α induces an isomorphism

$$C^*_{n,i}(X) \cong P(n)^*(X) \bigotimes_{P(n)^*} C^*_{n,i}.$$

It is well known that an element of v_i torsion in $P(n)^*(X)$ is also of v_n torsion. Therefore if $y \in C^*_{n,i}(X)$ with $v_i y = 0$, then $y = 0$.

Let $x \in K(n)^*(X)$ with X finite, we will show that $Q_i(x) = 0$. By construction, $Q_i = \rho_i \circ \beta_i$, where

$$\Sigma^{2(p^i-1)}C_{n,i} \xrightarrow{v_i} C_{n,i} \xrightarrow{\rho_i} K(n) \xrightarrow{\beta_i} C_{n,i}.$$

The class $Q_i(x)$ is the composite

$$X \xrightarrow{x} K(n) \xrightarrow{\beta_i} C_{n,i} \xrightarrow{\rho_i} C_{n,i},$$

but by definition

$$X \xrightarrow{x} K(n) \xrightarrow{\beta_i} C_{n,i} \xrightarrow{v_i} C_{n,i}$$

is trivial, that is $\beta_i(x) \in C^*_{n,i}(X)$ is a v_i-torsion element. By the remark above $\beta_i(x) = 0$ and so $Q_i(x) = 0$.

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Department of Mathematics, University of Glasgow, Glasgow G12 8QW, Scotland.
E-mail address: `a.baker@maths.gla.ac.uk`
URL: `http://www.maths.gla.ac.uk/∼ajb`

Mathematisches Institut, Universität Bern, Bern CH 3012, Switzerland.
E-mail address: `alain.jeanneret@math-stat.unibe.ch`