

MSp LOCALIZED AWAY FROM 2 AND ODD FORMAL GROUP LAWS

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ABSTRACT. We investigate the relationship between complex and symplectic cobordism localized away from the prime 2 and show that these theories are related much as a real Lie group is related to its complexification. This suggests that ideas from the theory of symmetric spaces might be used to illuminate these subjects. In particular, we give an explicit equivalence of ring spectra $\mathrm{MSp}[1/2] \wedge \mathrm{Sp}/\mathrm{U}_+ \simeq \mathrm{MU}[1/2]$ and deduce that $\mathrm{MU}[1/2]$ is a wedge of copies of $\mathrm{MSp}[1/2]$. We discuss the implications for the structure of the stable operation algebra $\mathrm{MSp}[1/2]^*\mathrm{MSp}[1/2]$ and the dual cooperation algebra $\mathrm{MSp}[1/2]_*\mathrm{MSp}[1/2]$. Finally we describe some realted Witt vector algebra and apply our results to the study of formal involutions on the category of formal group laws over a $\mathbb{Z}[1/2]$ -algebra.

INTRODUCTION

Cobordism theory is a part of topology in which geometry encounters homotopy theory in a particularly transparent way, and it is striking that the algebra which results is often remarkably interesting. This paper is concerned with complex and symplectic cobordism; it is well-known that the latter is extremely complicated at the prime 2, but we show that at odd primes, these two theories are related much as a real Lie group is related to its complexification. Ideas from the theory of symmetric spaces can thus be used as a guide to an area where the structure is otherwise perplexing. The following are our three main results.

Our main topological results are the following. Theorem 2.1 shows that there is an equivalence of ring spectra $\mathrm{MSp}[1/2] \longrightarrow \mathrm{odd}\mathrm{MU}$, where $\mathrm{odd}\mathrm{MU}$ is a certain ring spectrum constructed with the aid of an idempotent map of ring spectra $\mathrm{MU}[1/2] \longrightarrow \mathrm{MU}[1/2]$, produced using formal group theory; the proof is directly modelled on the approach to splitting the p -localization of MU used by Quillen [10, 1], and is consistent with it for all odd primes p . In Theorem 2.6, the homotopy ring $\mathrm{odd}\mathrm{MU}_*$ is characterized as universal for a class of formal group laws over $\mathbb{Z}[1/2]$ -algebras that we call *odd*. In Theorem 2.2 we show that there is an equivalence of the ring spectra $\mathrm{MU}[1/2]$ and $\mathrm{MSp}[1/2] \wedge \mathrm{Sp}/\mathrm{U}_+$.

In Theorem 3.2, we give an algebraic characterisation of $\mathrm{MSp}[1/2]_*\mathrm{MSp}[1/2] \cong \mathrm{odd}\mathrm{MU}_*\mathrm{odd}\mathrm{MU}$ using an identification of the second Hopf algebroid in terms of isomorphisms of odd formal group laws; this is related to work of Buchstaber [2], the precise connection being stated in Proposition 3.1. We obtain a related identification of the stable operation algebra in the cohomology theory $\mathrm{MSp}[1/2]^*(\)$ with a subalgebra of that for $\mathrm{MU}[1/2]^*(\)$; in keeping with work of Buchstaber & Shokurov and Morava [3, 9], this realizes an action of the even positive half of the formal vector fields Lie algebra associated to the diffeomorphism group of the circle. This seems to have connections with the KdV equation and Witten's topological field theory, as well as the work of Katsura, Shimizu & Ueno [5].

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We reformulate our topological results in scheme theoretic language and relate them to the scheme of formal involutions viewed as a sort of ‘symmetric space’. We also discuss some Witt vector algebra related to both the algebraic and topological aspects of our work.

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1. SOME CALCULATIONS WITH THE $[-1]$ -SERIES OF A FORMAL GROUP LAW

Throughout this section we will work with a graded commutative and unital ring $R = R_*$, which we assume is torsion free, hence can be embedded in its rationalization $R\mathbb{Q} = R \otimes \mathbb{Q}$. Let $F(X, Y) = \sum_{i,j} a_{i,j}^F X^i Y^j$ denote a commutative 1-dimensional formal group law over R , where $a_{i,j}^F \in R_{2i+2j-2}$. Over the ring $R\mathbb{Q}$, F admits unique logarithm $\log^F(X)$ and exponential $\exp^F(X)$, characterized by the three properties

$$\begin{aligned}\log^F(X) &\equiv X \pmod{(X^2)}, \\ \exp^F(\log^F(X)) &= X, \\ F(X, Y) &= \exp^F(\log^F(X) + \log^F(Y)).\end{aligned}$$

The $[-1]$ -series $[-1]_F(X) \in R[[X]]$ is characterized by the identity

$$F(X, [-1]_F(X)) = 0,$$

and also satisfies

$$[-1]_F(X) = \exp^F(-\log^F(X)) \equiv -X \pmod{(X^2)}.$$

We wish to understand the relationship between the two series $S^F(X) = X + [-1]_F(X)$ and $P^F(X) = -X[-1]_F(X)$, both of which lie in the ring $R[[X]] \subset R \otimes \mathbb{Q}[[X]]$. Consider the ring automorphism of $R[[X]]$ given by

$$\alpha: f(X) \longmapsto f([-1]_F X).$$

By induction on $n \geq 0$, the fixed subring of the quotient ring $R_n = R[[X]]/(X^{2n+1})$ is easily seen to be

$$R_n^\alpha = R[[P^F(X)]]/(X^{2n+1}),$$

where the inductive step uses the fact that

$$\alpha(X^{2n+1}) \equiv -X^{2n+1} \pmod{(P^F(X))^{n+1}}.$$

A straightforward argument using a limit in the X -adic topology now identifies the fixed subring of $R[[X]]$ as $R[[X]]^\alpha = R[[P^F(X)]]$. Thus there must be an expansion

$$S^F(X) = \sum_{r \geq 1} c_{2r-1}^F P^F(X)^r$$

with $c_{2r-1}^F \in R_{4r-2}$. These coefficients c_{2r-1}^F can be determined using Lagrange Inversion, which we next recall in a form suitable for our purposes.

Let $R((Z)) = R[[Z]][Z^{-1}]$, be the ring of finite tailed Laurent series in Z over R . For $f(Z) = \sum_{n_0 \leq n} a_n Z^n \in R((Z))$, we will write

$$[f(Z)]_{Z^n} = a_n = \text{coefficient of } Z^n$$

and let

$$\oint f(Z) dZ = [f(Z)]_{Z^{-1}}$$

be the *residue of $f(Z)$ with respect to Z* . Let $f'(Z) = \sum_{n_0-1 \leq n} (n+1)a_{n+1}Z^n$ denote the formal derivative of $f(Z)$ with respect to Z .

Theorem 1.1 (Lagrange Inversion). *Let $f(Z) = \sum_{n_0 \leq n} a_n Z^n \in R((Z))$ and let $h(Z) \in R[[Z]]$ with $h(Z) \equiv Z \pmod{(Z^2)}$. Then for the expansion $f(Z) = \sum_{n_0 \leq n} c_n h(Z)^n$, we have*

$$c_n = \oint \frac{f(Z)h'(Z) dZ}{h(Z)^{n+1}}, \quad \text{for } n \geq n_0.$$

To calculate such a residue, we can use change of variable, integration by parts and other standard techniques from elementary calculus. We list here some that we require, their proofs being straightforward formal versions of results from calculus.

Proposition 1.2. *Let $f(Z), g(Z) \in R((Z))$, and let $h(Z) \in R[[Z]]$ satisfy $h(Z) \equiv Z \pmod{(Z^2)}$. Then we have*

(a): Change of Variable
$$\oint f(Z) dZ = \oint f(h(Z))h'(Z) dZ,$$

(b): Exactness
$$\oint \frac{df(Z)}{dZ} dZ = 0,$$

(c): Integration by Parts
$$\oint f(z) \frac{dg(Z)}{dZ} dZ = - \oint g(z) \frac{df(Z)}{dZ} dZ.$$

We can use Lagrange Inversion, Theorem 1.1, to calculate the coefficients c_{2r-1}^F . First make the changes of variable $X = \exp^F(Z)$ and $Y = Z^2$. Then

$$S^F(\exp^F(Z)) = \sum_{k \geq 1} 2e_{2k-1}^F Y^k,$$

where $\exp^F(Z) = \sum_{k \geq 0} e_k^F Z^{k+1}$ for $e_n^F \in R_{2n} \otimes \mathbb{Q}$. Setting $Q(Y) = P^F(\exp^F(Z))$ and $\bar{Q}(Y) = Q(Y)/Y = 1 + \dots$, we obtain

$$c_{2r-1}^F = \oint \frac{(\sum_{k \geq 1} 2e_{2k-1}^F Y^k) Q'(Y) dY}{Q(Y)^{r+1}}.$$

Integrating by parts and interchanging the summation and integral signs, we obtain

$$\begin{aligned} c_{2r-1}^F &= \sum_{r \geq k \geq 1} \frac{2k}{r} e_{2k-1}^F \oint \frac{Y^{k-1} dY}{Q(Y)^r} \\ &= \sum_{r \geq k \geq 1} \frac{2k}{r} e_{2k-1}^F [\bar{Q}(Y)^{-r}]_{Y^{r-k}}. \end{aligned}$$

Notice that the coefficient c_{2t-1}^F has the form

$$c_{2t-1}^F = 2e_{2t-1}^F + \text{decomposables in } R\mathbb{Q}.$$

In the cases $R_* = \text{MU}_*$ or $R_* = \text{BP}_*$ at the prime 2 (equipped with their canonical formal group laws) Milnor's criterion tells us that c_{2t-1}^F is a polynomial generator; indeed, for the case of BP_* , this gives a complete set of polynomial generators for BP_* over $\mathbb{Z}_{(2)}$.

The elements $e_n^F \in R_{2n} \otimes \mathbb{Q}$ may be interpreted as formal symmetric functions in infinitely many variables t_i . More precisely, we view e_n^F as the n -th elementary symmetric function

$\sigma_n(t) = \sum t_1 t_2 \cdots t_n$, obtained by symmetrising the monomial $t_1 t_2 \cdots t_n$. Thus we have the formal expansion

$$\sum_{n \geq 0} e_n^F Z^n = \prod_i (1 + t_i Z),$$

from which it follows that

$$\bar{Q}(Y) = \prod_i (1 - t_i^2 Y).$$

Hence we deduce that

$$\begin{aligned} \bar{Q}(Y) &= \sum_{n \geq 0} q_n^F Y^n \\ &= \sum_{n \geq 0} (-1)^n \sigma_n(t^2) Y^n, \end{aligned}$$

where $\sigma_n(t^2) = \sum t_1^2 t_2^2 \cdots t_n^2$ is the n -th elementary symmetric function in the t_i^2 . It is easily seen that

$$\sigma_n(t^2) = 2e_{2n}^F - 2e_{2n-1}^F e_1^F + \cdots + (-1)^n (e_n^F)^2.$$

Of course we actually want to know the first few coefficients in $\bar{Q}(Y)^{-n}$ for $n \geq 1$, and these will be complicated polynomials in the $\sigma_k(t^2)$ for $1 \leq k \leq n$. An alternative approach is to use the total symmetric functions rather than the elementary functions.

Recall that the n -th total symmetric function $\tau_n(t)$ in the variables t_i is obtained by the summing up all the monomials of the form $t_1^{r_1} t_2^{r_2} \cdots t_n^{r_n}$ where $0 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ and $r_1 + r_2 + \cdots + r_n = n$, and then symmetrising over all t_i 's. The generating function for the $\tau_n(t)$ is

$$\sum_{0 \leq n} \tau_n(t) Z^n = \prod_i (1 - t_i Z)^{-1}.$$

In the context above where we set $e_n^F = \sigma_n(t)$, thus we have

$$\bar{Q}(Y)^{-1} = \sum_{0 \leq n} \tau_n(t^2) Y^n$$

and we set $h_n^F = \tau_n(t^2)$ which is an integer polynomial in the elements q_k^F . Thus to evaluate c_{2r-1}^F we are reduced to a calculation with polynomials in the h_n^F .

We end this section with another technical result required later. The proof is a well known application of Newton iteration.

Theorem 1.3 (Hensel's Lemma). *Let R be a commutative, unital ring which is complete with respect to the I -adic topology for some ideal $I \triangleleft R$ and let $f(Z) \in R[[Z]]$ be a power series such that $f(z_0) \equiv 0 \pmod{I}$ for some $z_0 \in R$ and the formal derivative $f'(z_0)$ is a unit in R . Then the sequence $z_n \in R$ given by*

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_0)}$$

converges I -adically to a limit z satisfying $f(z) = 0$.

2. A UNIVERSAL FORMAL GROUP LAW FOR $\mathrm{MSp}[1/2]$

The main results of this section are the following. First we follow Quillen in using formal group theory to prove

Theorem 2.1. *There is an idempotent map of ring spectra $\varepsilon_2: \mathrm{MU}[1/2] \longrightarrow \mathrm{MU}[1/2]$ for which image ring spectrum ${}^{\mathrm{odd}}\mathrm{MU}$ and a canonical orientation $\mathrm{MSp} \longrightarrow {}^{\mathrm{odd}}\mathrm{MU}$ inducing an equivalence of ring spectra $\mathrm{MSp}[1/2] \longrightarrow {}^{\mathrm{odd}}\mathrm{MU}$.*

Next we deduce a topological result which in effect shows that $\mathrm{MU}[1/2]$ is a ‘Hopf algebra over MSp ’ in the sense that the natural product and diagonal maps of Sp/U give $\mathrm{MSp}[1/2]_*\mathrm{Sp}/\mathrm{U}$ the structure of a Hopf algebra over $\mathrm{MSp}[1/2]_*$.

Theorem 2.2. *There is an equivalence of ring spectra*

$$\mathrm{MSp}[1/2] \wedge \Sigma^\infty(\mathrm{Sp}/\mathrm{U}_+) \longrightarrow \mathrm{MU}[1/2].$$

Finally we obtain

Theorem 2.3. *There is an equivalence of ring spectra*

$$\mathrm{MU}[1/2] \simeq \bigvee_J \Sigma^{2|J|} \mathrm{MSp}[1/2]$$

where J ranges over all non-decreasing finite odd sequences

$$J = (1 \leq 2j_1 - 1 \leq 2j_2 - 1 \leq \cdots \leq 2j_\ell - 1),$$

including the empty sequence $()$, and $|J| = \sum_{1 \leq t \leq \ell} (2j_t - 1)$; moreover, these summands are indexed by a polynomial algebra $\mathbb{Z}[1/2][u_J : J \neq ()]$ with $|u_J| = 2|J|$ and have multiplication compatible with this product.

We begin by constructing an idempotent map of ring spectra $\varepsilon_2: \mathrm{MU}[1/2] \longrightarrow \mathrm{MU}[1/2]$ for which the cohomology theory associated to the image is representable by a ring spectrum equivalent to $\mathrm{MSp}[1/2]$. Our construction is modelled on that of Quillen’s idempotent for BP , as described by Adams [1].

Let $x = x^{\mathrm{MU}} \in \mathrm{MU}^2(\mathbb{C}\mathrm{P}^\infty) \subset \mathrm{MU}[1/2]^2(\mathbb{C}\mathrm{P}^\infty)$ denote the canonical orientation class of [1] which arises geometrically as the homotopy class of a map $\mathbb{C}\mathrm{P}^\infty \xrightarrow{\cong} \mathrm{MU}(1) \longrightarrow \Sigma^2\mathrm{MU}$. Let $[1/2]_{\mathrm{MU}}(X) \in \mathrm{MU}[1/2]_*[[X]]$ denote the series characterized by

$$[2]_{\mathrm{MU}}([1/2]_{\mathrm{MU}}(X)) = X.$$

The series

$$\varphi(X) = [1/2]_{\mathrm{MU}}(F^{\mathrm{MU}}(X, [-1]_{\mathrm{MU}}(-X))) \in \mathrm{MU}[1/2]_*[[X]]$$

may be interpreted as a strict isomorphism $\varphi: F' \cong F$, where F' is the formal group law defined by

$$F'(X, Y) = \varphi(F(\varphi^{-1}(X), \varphi^{-1}(Y))),$$

We choose a new orientation $y \in \mathrm{MU}[1/2]^2(\mathbb{C}\mathrm{P}^\infty)$ by requiring it to satisfy

$$x = [1/2]_{\mathrm{MU}}(F^{\mathrm{MU}}(y, [-1]_{\mathrm{MU}}(-y)))$$

and noting that in $\mathrm{MU}[1/2]^*(\mathbb{C}\mathrm{P}^\infty)$ we have

$$y \equiv x \pmod{(x^2)}.$$

Then y is an orientation whose associated formal group law is F' .

Definition 2.4. A formal group law G for which $[-1]_G(X) = -X$ is said to be *odd*.

The logarithm of F' is

$$\begin{aligned} \log^{F'}(X) &= \frac{1}{2} (\log^{\mathrm{MU}}(X) - \log^{\mathrm{MU}}(-X)) \\ &= \sum_{n \geq 0} m_{2n} X^{2n+1}, \end{aligned}$$

where $\log^{\mathrm{MU}}(X) = \sum_{n \geq 0} m_n X^{n+1} \in \mathrm{MU}_* \otimes \mathbb{Q}[[X]]$. Clearly $\log^{F'}(X)$ is an odd function of X and a straightforward calculation shows that $[-1]_{F'}(X) = -X$, hence F' is odd.

The universality of MU_* implies that there is a unique homomorphism $\text{MU}_* \longrightarrow \text{MU}[1/2]_*$ which classifies F' and gives rise to a homomorphism $\varepsilon_2: \text{MU}[1/2]_* \longrightarrow \text{MU}[1/2]_*$. The latter is an idempotent, since it extends to a visibly idempotent homomorphism on $\text{MU}_* \otimes \mathbb{Q} = \mathbb{Q}[m_n : n \geq 1]$, given by

$$m_n \longmapsto \begin{cases} 0 & \text{if } n \text{ odd,} \\ m_n & \text{if } n \text{ even.} \end{cases}$$

Let ${}^{\text{odd}}\text{MU}_* = \text{im } \varepsilon_2 \subset \text{MU}[1/2]_*$ and let ${}^{\text{odd}}\text{MU}^*(\)$ denote the associated complex oriented multiplicative cohomology theory, which is a summand of complex cobordism with 2 inverted, $\text{MU}[1/2]^*(\)$.

Explicit polynomial generators for ${}^{\text{odd}}\text{MU}_*$ can be given using the method of I. Kozma [6]. For each prime ℓ , let \mathbf{f}_ℓ and \mathbf{v}_ℓ denote the *Frobenius* and *verschiebung* operators with respect to the formal group law F . Thus, using \sum^F to denote formal group summation, we have

$$\begin{aligned} \mathbf{f}_\ell(X) &= \sum_{\zeta^\ell=1}^F (\zeta X^{1/\ell}), \\ \mathbf{v}_\ell(X) &= X^\ell, \end{aligned}$$

and

$$\begin{aligned} \mathbf{v}_\ell \mathbf{f}_\ell(X) &= \sum_{\zeta^\ell=1}^F (\zeta X^\ell) \\ &= \sum_{k \geq 1}^F (\mathbb{T}_{\ell,k} X^{k\ell}) \end{aligned}$$

for some $\mathbb{T}_{\ell,k} \in \text{MU}_{2k\ell-2}$. In the ring $\text{MU}_* \otimes \mathbb{Q}$ these coefficients satisfy the identities

$$\mathbb{T}_{\ell,k} = \ell m_{k\ell-1} - \sum_{1 < s|k} m_{s-1} \mathbb{T}_{\ell,k/s}^s.$$

Using the well known criterion of Milnor (see [13, theorem, page 128], also [1, part II, Theorem 7.9]) we obtain the following. If $k = \ell^d$, then \mathbb{T}_{ℓ,ℓ^d} is a polynomial generator of MU_* . If n has two distinct prime factors ℓ_1 and ℓ_2 , then some integral linear combination of $\mathbb{T}_{\ell_1,n/\ell_1}$ and $\mathbb{T}_{\ell_2,n/\ell_2}$ will be a polynomial generator. We will assume that a choice of such an element $X_{n-1} \in \text{MU}_{2n-2}$ has been made for each n , with $X_{\ell^d+1-1} = \mathbb{T}_{\ell,\ell^d}$ for every prime ℓ . Then by induction we have

$$\varepsilon_2(\mathbb{T}_{\ell,k}) = \begin{cases} 0 & \text{if } \ell k \text{ is even,} \\ \mathbb{T}_{\ell,k} & \text{if } \ell k \text{ is odd.} \end{cases}$$

Theorem 2.5. *The elements X_{2^m} form a set of polynomial generators for ${}^{\text{odd}}\text{MU}_*$ as a $\mathbb{Z}[1/2]$ -algebra.*

Suppose (R, G) is a pair consisting of a $\mathbb{Z}[1/2]$ -algebra R together with an odd formal group law G on R classified by a ring homomorphism $\psi: \text{MU}[1/2]_* \longrightarrow R$. The pushed forward power series

$$\psi_* \varphi(X) = [1/2]_G (G(X, [-1]_G(-X))) \in R[[X]]$$

has the associated formal group law

$$G'(X, Y) = \psi_* \varphi (G(\psi_* \varphi^{-1}(X), \psi_* \varphi^{-1}(Y))),$$

classified by the homomorphism $\psi \circ \varepsilon_2$. But since G is odd, $\psi_* \varphi(X) = X$, hence $G' = G$ and so $\psi \circ \varepsilon_2 = \psi$. Hence ψ has a unique factorization

$$\psi: \text{MU}[1/2]_* \xrightarrow{\varepsilon_2} {}^{\text{odd}}\text{MU}_* \longrightarrow R.$$

We may now easily deduce

Theorem 2.6. *The pair $({}^{\text{odd}}\text{MU}_*, F')$ is universal for pairs (R, G) consisting of a $\mathbb{Z}[1/2]$ -algebra R together with an odd formal group law G on R .*

The theory ${}^{\text{odd}}\text{MU}^*(\)$ possesses canonical orientations for symplectic bundles, which can be regarded as complex bundles. In particular, the canonical quaternionic line bundle $\xi_1 \rightarrow \mathbb{H}\mathbb{P}^\infty$ has an orientation $w \in {}^{\text{odd}}\text{MU}^4(\mathbb{H}\mathbb{P}^\infty)$, and under the pullback induced from the canonical map $\mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$, this maps to

$$y(-[-1]_{F'}y) = y^2 \in {}^{\text{odd}}\text{MU}^4(\mathbb{C}\mathbb{P}^\infty).$$

For any commutative ring R , the image of the homomorphism $x_*: H_*(\mathbb{C}\mathbb{P}^\infty; R) \rightarrow H_*(\text{MU}; R)$ induced by the orientation x contains a set of algebra generators for $H_*(\text{MU}; R)$. Similarly, the map

$$y: \mathbb{C}\mathbb{P}^\infty \xrightarrow{\cong} \text{MU}(1) \xrightarrow{x} \Sigma^2\text{MU}[1/2] \rightarrow \Sigma^2{}^{\text{odd}}\text{MU}[1/2] \rightarrow \Sigma^2\text{MU}[1/2]$$

induces a map in homology which has generators in its image. Now it follows that the composite

$$\mathbb{C}\mathbb{P}^\infty \xrightarrow{\cong} \text{MU}(1) \xrightarrow{y} \Sigma^2{}^{\text{odd}}\text{MU}[1/2] \rightarrow \Sigma^2\text{MU}[1/2]$$

also provides algebra generators for $H_*({}^{\text{odd}}\text{MU}; \mathbb{Z}[1/2]) \subset H_*(\text{MU}; \mathbb{Z}[1/2])$. Then it follows that the map $w: \mathbb{H}\mathbb{P}^\infty \simeq \text{M}\text{Sp}(1) \rightarrow \Sigma^4\text{M}\text{Sp} \rightarrow {}^{\text{odd}}\text{MU}$ induces a map in homology whose image contains polynomial generators for $H_*({}^{\text{odd}}\text{MU}; \mathbb{Z}[1/2])$, since the natural map $\mathbb{C}\mathbb{P}^\infty \rightarrow \mathbb{H}\mathbb{P}^\infty$ induces a surjection in ordinary homology. Finally, we see that the orientation map $\text{M}\text{Sp} \rightarrow {}^{\text{odd}}\text{MU}$ induces an isomorphism in homology, since $H_*(\text{M}\text{Sp}; \mathbb{Z}[1/2])$ is generated by the image of the natural homomorphism $H_*(\mathbb{H}\mathbb{P}^\infty; \mathbb{Z}[1/2]) \rightarrow H_*(\text{M}\text{Sp}; \mathbb{Z}[1/2])$ and we know that $H_*(\text{M}\text{Sp}; \mathbb{Z}[1/2])$ maps surjectively to $H_*({}^{\text{odd}}\text{MU}; \mathbb{Z}[1/2])$.

Recall the existence of the fibrations

$$\text{Sp}(n)/\text{U}(n) \rightarrow \text{BU}(n) \rightarrow \text{BSp}(n),$$

induced from the ‘quaternionification map’ $\text{U}(n) \rightarrow \text{Sp}(n)$. For each m , there is a map

$$\text{BU}(m) \times \text{Sp}(n)/\text{U}(n) \rightarrow \text{BU}(m+n)$$

constructed using the external Whitney sum map. Hence there is a family of maps compatible under increasing values of n and k ,

$$\text{BSp}(k) \times \text{Sp}(n)/\text{U}(n) \rightarrow \text{BU}(2k) \times \text{Sp}(n)/\text{U}(n) \rightarrow \text{BU}(2k+n).$$

A straightforward calculation now shows that on passing to the limit with respect to n and k , there is a map of H-spaces $\text{BSp} \times \text{Sp}/\text{U} \rightarrow \text{BU}$ which induces an isomorphism of Hopf algebras over $\mathbb{Z}[1/2]$,

$$H_*(\text{BSp} \times \text{Sp}/\text{U}; \mathbb{Z}[1/2]) \rightarrow H_*(\text{BU}; \mathbb{Z}[1/2]).$$

We now pass to Thom spectra. Let $\chi \rightarrow \text{Sp}/\text{U}$ denote the pullback of the universal (virtual) bundle $\zeta \rightarrow \text{BU}$. The Thom complex $\text{M}\chi$ of ζ is a ring spectrum and possesses a canonical orientation $\text{M}\chi \rightarrow \text{MU}$. Smashing with MSp we obtain a morphism of ring spectra $\text{M}\text{Sp} \wedge \text{M}\chi \rightarrow \text{MU}$ and a commutative diagram of $\mathbb{Z}[1/2]$ -algebras,

$$\begin{array}{ccc} H_*(\text{BSp} \times \text{Sp}/\text{U}; \mathbb{Z}[1/2]) & \xrightarrow{\cong} & H_*(\text{BU}; \mathbb{Z}[1/2]) \\ \cong \downarrow & & \cong \downarrow \\ H_*(\text{M}\text{Sp} \wedge \text{M}\chi; \mathbb{Z}[1/2]) & \longrightarrow & H_*(\text{MU}; \mathbb{Z}[1/2]) \end{array}$$

in which the vertical maps are Thom isomorphisms in $H_*(\ ; \mathbb{Z}[1/2])$, and hence the bottom row is an isomorphism.

We would like to replace $\mathrm{MSp} \wedge \mathrm{M}\chi$ up to homotopy by $\mathrm{MSp} \wedge \Sigma^\infty(\mathrm{Sp}/\mathrm{U}_+)$, at least after inverting 2. We are able to do this using the following result from [7] provides a geometric realization of the Thom isomorphism.

Lemma 2.7. *Let $\xi \rightarrow B$ be a virtual bundle, orientable in the cohomology theory represented by the ring spectrum E , and let $u: \mathrm{M}\xi \rightarrow E$ be an orientation. Then there is an equivalence*

$$E \wedge \mathrm{M}\xi \simeq E \wedge \Sigma^\infty(B_+).$$

The equivalence of Theorem 2.2 is the composite

$$E \wedge \mathrm{M}\xi \xrightarrow{1 \wedge \delta} E \wedge \mathrm{M}\xi \wedge \Sigma^\infty(B_+) \xrightarrow{1 \wedge u \wedge 1} E \wedge E \wedge \Sigma^\infty(B_+) \xrightarrow{\mu \wedge 1} E \wedge \Sigma^\infty(B_+),$$

where $u: \mathrm{M}\xi \rightarrow E$ is the chosen orientation for ξ , $\delta: \mathrm{M}\xi \rightarrow \mathrm{M}\xi \wedge \Sigma^\infty(B_+)$ is the external diagonal used in defining the Thom isomorphism, and $\mu: E \wedge E \rightarrow E$ is the multiplication map.

Theorem 2.3 now follows by a standard argument from the fact that

$$\mathrm{MSp}[1/2]_*(\Sigma^\infty(\mathrm{Sp}/\mathrm{U}_+)) \cong {}^{\mathrm{odd}}\mathrm{MU}_*(\Sigma^\infty(\mathrm{Sp}/\mathrm{U}_+))$$

is a polynomial algebra over the ring $\mathrm{MSp}[1/2]_* \cong {}^{\mathrm{odd}}\mathrm{MU}_*$, hence is a free module.

3. CATEGORIES OF ODD FORMAL GROUP LAWS

In this section, let $R = R_*$ be a commutative unital graded $\mathbb{Z}[1/2]$ -algebra. We will also denote the category of commutative, 1-dimensional formal group laws over R by $\mathbf{FGL}(R)$, and the full category of odd formal group laws by ${}^{\mathrm{odd}}\mathbf{FGL}(R)$; here the morphisms are strict isomorphisms. Thus a morphism $F_1 \xrightarrow{\varphi} F_2$ between two formal group laws is a power series $\varphi(X) \in R[[X]]$ such that

$$\varphi(X) \equiv X \pmod{(X)^2}$$

and

$$\varphi(F_1(X, Y)) = F_2(\varphi(X), \varphi(Y)).$$

These small categories are groupoids, and are representable covariant functors of the algebra R . There are natural isomorphisms

$$(3.1) \quad \mathbf{FGL}(R) \cong \mathbf{Alg}_{\mathbb{Z}[1/2]}(\mathrm{MU}[1/2]_*\mathrm{MU}[1/2], R),$$

$$(3.2) \quad {}^{\mathrm{odd}}\mathbf{FGL}(R) \cong \mathbf{Alg}_{\mathbb{Z}[1/2]}({}^{\mathrm{odd}}\mathrm{MU}_*{}^{\mathrm{odd}}\mathrm{MU}, R),$$

where we define

$${}^{\mathrm{odd}}\mathrm{MU}_*{}^{\mathrm{odd}}\mathrm{MU} = {}^{\mathrm{odd}}\mathrm{MU}_* \otimes_{\mathrm{MU}[1/2]_*} \mathrm{MU}[1/2]_*\mathrm{MU}[1/2] \otimes_{\mathrm{MU}[1/2]_*} {}^{\mathrm{odd}}\mathrm{MU}_*,$$

making use of the fact that $\mathrm{MU}[1/2]_*\mathrm{MU}[1/2]$ is a bimodule over $\mathrm{MU}[1/2]_*$ and the idempotent ε_2 provides ${}^{\mathrm{odd}}\mathrm{MU}_*$ with the structure of a module over $\mathrm{MU}[1/2]_*$. The topological structure of $\mathrm{MU}_*\mathrm{MU}$ and ${}^{\mathrm{odd}}\mathrm{MU}_*{}^{\mathrm{odd}}\mathrm{MU}$ includes that of ‘bilateral Hopf algebras’ or ‘Hopf algebroids’, and the associated structure maps yield (natural with respect to the algebra R) groupoid structures on the right hand sides of Equations (3.1) and (3.2), and these isomorphisms are isomorphisms of groupoids. It also follows from standard results in this area [8, 11] that the idempotent ε_2 induces an equivalence of groupoids

$$\varepsilon_2: \mathbf{FGL}(R) \longrightarrow {}^{\mathrm{odd}}\mathbf{FGL}(R),$$

which is functorial in R . The fixed objects are precisely the odd formal group laws, while the fixed morphisms are the strict isomorphisms between odd formal group laws which are odd power series over R . The full subcategory of all such objects and morphisms is the image of ε_2 .

V. M. Buchstaber [2] introduced a different idempotent e_2 on $\text{MU}[1/2]$ with associated idempotent equivalence

$$\mathbf{e}_2: \mathbf{FGL}(R) \longrightarrow \text{odd}\mathbf{FGL}(R).$$

This is defined as follows. Associated to a formal group law F on R there is a strict isomorphism

$$\theta_F: F \longrightarrow \mathbf{e}_2(F)$$

given by the series

$$\theta_F(X) = X \sqrt{\frac{-[-1]_F(X)}{X}},$$

where for any series $h(X) \in R[[X]]$ satisfying $h(X) \equiv 1 \pmod{(X)}$, we define

$$\sqrt{h(X)} \equiv 1 \pmod{(X)}$$

to be the unique square root of $h(X)$ with this property (this of course depends crucially on the fact that R is an algebra over $\mathbb{Z}[1/2]$); this series can also be determined using the formal binomial expansion for $h(X)^{1/2}$. It is now straightforward to verify the following result.

Proposition 3.1. *The idempotents ε_2 and e_2 satisfy the following.*

- a) *For any formal group law over R , $\mathbf{e}_2(F)$ is odd.*
- b) *For any strict isomorphism $\varphi: F_1 \longrightarrow F_2$, $\mathbf{e}_2(\varphi)$ is odd.*
- c) *There are equations*

$$\mathbf{e}_2 \varepsilon_2 = \mathbf{e}_2,$$

$$\varepsilon_2 \mathbf{e}_2 = \varepsilon_2.$$

- d) *The restrictions of the idempotents ε_2 and e_2 induce inverse isomorphisms of rings*

$$\text{im } e_2 \begin{matrix} \xrightarrow{\varepsilon_2} \\ \xleftarrow{e_2} \end{matrix} \text{im } \varepsilon_2.$$

It is worth noting that Buchstaber's idempotent $e_2: \text{MU}[1/2]_* \longrightarrow \text{MU}[1/2]_*$ fixes the image of the canonical map $\text{MSp}_* \longrightarrow \text{MU}[1/2]_*$. Indeed there is a diagram of ring spectra

$$\begin{array}{ccc} \text{MSp}[1/2] & \longrightarrow & \text{MU}[1/2] \\ \text{id} \downarrow & & e_2 \downarrow \\ \text{MSp}[1/2] & \longrightarrow & \text{MU}[1/2] \end{array}$$

from which we can deduce that the image of the idempotent $e_2: \text{MU}[1/2]^*() \longrightarrow \text{MU}[1/2]^*()$ is a cohomology theory which agrees with the image of the associated natural transformation of cohomology theories $\text{MSp}[1/2]^*() \longrightarrow \text{MU}[1/2]^*()$. However, the same is *not* true for our idempotent ε_2 .

From [14] we obtain the following facts about the spectrum MSp , which do not use any detailed knowledge of the structure of the still mysterious coefficient ring MSp_* . The object

$$\text{MSp}_* \text{MSp} = \text{MSp}_*(\text{MSp}) = \text{MSp}_*[Q_n : n \geq 1]$$

is endowed with the structure of a Hopf algebroid, where the elements $Q_n \in \text{MSp}_{4n} \text{MSp}$ may be chosen so that the coproduct on the Q_n is given by the composition law

$$Q(T) \longmapsto (1 \otimes Q)(Q \otimes 1(T)),$$

where

$$\begin{aligned} Q(T) &= \sum_{n \geq 0} Q_n T^{n+1}, \\ (1 \otimes Q)(T) &= \sum_{n \geq 0} 1 \otimes Q_n T^{n+1}, \\ (Q \otimes 1)(T) &= \sum_{n \geq 0} Q_n \otimes 1 T^{n+1}. \end{aligned}$$

Furthermore, the natural ring homomorphism $\mathrm{MSp}_* \mathrm{MSp} \longrightarrow \mathrm{MU}_* \mathrm{MU}$ is a morphism of Hopf algebroids over \mathbb{Z} . After inverting 2 and applying the idempotent ε_2 on the factors of $\mathrm{MU}[1/2]$, we obtain the following result.

Theorem 3.2. *The evident natural homomorphism $\mathrm{MSp}[1/2]_* \mathrm{MSp}[1/2] \longrightarrow {}^{\mathrm{odd}}\mathrm{MU}_* {}^{\mathrm{odd}}\mathrm{MU}$ is an isomorphism of Hopf algebroids over $\mathbb{Z}[1/2]$.*

Notice that the natural orientation $p^{\mathrm{MSp}} \in \mathrm{MSp}^4(\mathbb{HP}^\infty)$ maps to an orientation $p^{\mathrm{odd}} \in {}^{\mathrm{odd}}\mathrm{MU}^4(\mathbb{HP}^\infty)$, and on pulling back to \mathbb{CP}^∞ , becomes the square of the complex orientation $y \in {}^{\mathrm{odd}}\mathrm{MU}^2(\mathbb{CP}^\infty)$.

The morphism of Hopf algebroids $\mathrm{MU}[1/2]_* \mathrm{MU}[1/2] \longrightarrow {}^{\mathrm{odd}}\mathrm{MU}_* {}^{\mathrm{odd}}\mathrm{MU}$ induced by the idempotent ε_2 is an equivalence, in the sense that there is a commutative diagram of groupoids which is natural in R

$$\begin{array}{ccc} \mathbf{Alg}_{\mathbb{Z}[1/2]}(\mathrm{MU}[1/2]_* \mathrm{MU}[1/2], R) & \longrightarrow & \mathbf{Alg}_{\mathbb{Z}[1/2]}({}^{\mathrm{odd}}\mathrm{MU}_* {}^{\mathrm{odd}}\mathrm{MU}, R) \\ \cong \downarrow & & \cong \downarrow \\ \mathbf{FGL}(R) & \longrightarrow & {}^{\mathrm{odd}}\mathbf{FGL}(R) \end{array}$$

and in which the rows are equivalences of groupoids induced by ε_2 . For example, this implies that the associated Ext groups arising as the Adams-Novikov E_2 -terms for the homology theories defined by $\mathrm{MU}[1/2]$ and $\mathrm{MSp}[1/2]$ are naturally isomorphic.

There is a well known decomposition of \mathbb{Z} -algebras,

$$\mathrm{MU}_* \mathrm{MU} = \mathrm{MU}_* \otimes_{\mathbb{Z}} S_*,$$

where $S_* = \mathbb{Z}[B_n : n \geq 1]$ with B_n the usual generator of $\mathrm{MU}_* \mathrm{MU}$ over MU_* . This splitting is such that the subalgebra MU_* is invariant with respect to the coaction map over S_* . Dually, we have a semi-tensor decomposition of \mathbb{Z} -algebras,

$$\mathrm{MU}^* \mathrm{MU} = \mathrm{MU}_* \otimes_{\mathbb{Z}} S^*,$$

where $S^* = \mathrm{Hom}_{\mathbb{Z}}(S_*, \mathbb{Z})$. Here the subalgebra MU_* is invariant under the action of S^* . The idempotent $\varepsilon_{2*} \varepsilon_2$, obtained by applying ε_2 to both the factors of MU in $\mathrm{MU}_* \mathrm{MU}$, has image

$$\varepsilon_{2*} \varepsilon_2(S_*) = \mathbb{Z}[1/2][B_{2n} : n \geq 1] = {}^{\mathrm{odd}}S_*$$

when restricted to $S_*[1/2]$.

These decompositions are compatible with the action of the idempotent ε_2 , and there is an isomorphism

$$\mathrm{MSp}[1/2]^* \mathrm{MSp}[1/2] \cong {}^{\mathrm{odd}}\mathrm{MU}_* \otimes {}^{\mathrm{odd}}S^*,$$

where ${}^{\mathrm{odd}}S^* = \mathrm{Hom}_{\mathbb{Z}[1/2]}({}^{\mathrm{odd}}S_*, \mathbb{Z}[1/2])$. There is a well known semi-tensor product decomposition

$$\mathrm{MSp}^* \mathrm{MSp} \cong \mathrm{MSp}_* \otimes_{\mathbb{Z}} Q^*,$$

where is dual to $\mathbb{Q}_* = \mathbb{Z}[Q_n : n \geq 1]$; similarly, the $\mathbb{Z}[1/2]$ -Hopf algebra oddS^* is isomorphic to the dual of $\mathbb{Z}[1/2][Q_n : n \geq 1] \subset \text{MSp}[1/2]_* \text{MSp}[1/2]$.

The work of [3, 9] suggests that we should be able to relate the algebra oddS^* to the Virasoro algebra. In [9], the operations $s_{e_n} \in \text{MU}^{2n} \text{MU}$ of [1] are shown to realize the action of the operators $v_k = z^{k+1} d/dz$ in the rational Lie algebra V^+ with basis $\{v_k : k \geq 1\}$; it follows that the operations $\varepsilon_2 \circ s_{e_n} \circ \varepsilon_2$ realize the action of the even half of V^+ , namely the subalgebra V^{2+} with basis $\{v_{2k} : k \geq 1\}$. In this way, we can interpret the Hopf algebra $\mathbb{Q} \otimes \mathbb{Q}^*$ as the universal enveloping algebra of V^{2+} . A perhaps more interesting point involves the action of this even part of the Virasoro algebra on the generators of $\text{MSp}[1/2]_*(\text{Sp}/U_+)$ generating the augmentation ideal.

4. SOME WITT VECTOR-LIKE HOPF ALGEBRAS

The constructions in this section are reminiscent of, and influenced by, the algebra of Witt vectors associated to a formal group law as described in [4]. However, they differ in ways that appear novel. Throughout the section, let $\mathbb{k} = \mathbb{k}_*$ be any graded commutative unital ring.

The algebra $H_* = \mathbb{k}[b_n : n \geq 1]$ with $b_n \in H_{2n}$ and $b_0 = 1$ admits the cocommutative coproduct

$$\psi(b_n) = \sum_{0 \leq k \leq n} b_k \otimes b_{n-k}$$

and antipode χ for which

$$\sum_{0 \leq k \leq n} \chi(b_k) b_{n-k} = 0.$$

In terms of generating functions we have for the series $b(T) = \sum_{0 \leq n} b_n T^n$,

$$\psi b(T) = \sum_{0 \leq k \leq n} \psi(b_n) T^n = b(T) \otimes b(T),$$

and

$$\chi b(T) = \sum_{0 \leq k \leq n} \chi(b_n) T^n = b(T)^{-1}.$$

This Hopf algebra represents the group scheme of sequences \mathbf{W} on graded \mathbb{k} -algebras for which

$$\mathbf{W}(R) = \mathbf{Alg}_{\mathbb{k}}(H_*, R),$$

where $\varphi \in \mathbf{Alg}_{\mathbb{k}}(H_*, R)$ is identified with the sequence $\{\varphi(b_n)\}_{0 \leq n}$; ‘addition’ is given on sequences by

$$\{c_n\} * \{d_n\} = \left\{ \sum_{0 \leq k \leq n} c_k d_{n-k} \right\}.$$

Writing $c = \{c_n\}$ and $d = \{d_n\}$, we will also write

$$c * d = \{(c * d)_n\} = \{c_n\} * \{d_n\}.$$

Now suppose that $1/2 \in \mathbb{k}$. There is an endomorphism of \mathbf{W} determined by

$$w \longmapsto (1/2)w$$

which is induced by the Hopf algebra homomorphism satisfying

$$b(T) \longmapsto b(T)^{1/2},$$

which can be calculated using the formal binomial expansion. There is also an involution τ given by

$$\tau \cdot b(T) = b^\tau(T) = b(-T).$$

There are two idempotent endomorphisms on \mathbf{W} induced by the Hopf algebra endomorphisms

$$\begin{aligned} b(T) &\longmapsto b^+(T) = b(T)^{1/2}b(-T)^{1/2}, \\ b(T) &\longmapsto b^-(T) = b(T)^{1/2}b(-T)^{-1/2}. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} b_{2n}^+ &\equiv b_{2n} \pmod{\text{(decomposables)}}, \\ b_{2n-1}^- &\equiv b_{2n-1} \pmod{\text{(decomposables)}}, \end{aligned}$$

hence these elements form a set of polynomial generators for H_* .

Remark 4.1. Although $b_{2n-1}^+ = 0$, the elements $b_{2n}^- \in H_*^-$ need not be zero since $b^-(T)$ is not an odd series.

We will denote the images of these by \mathbf{W}^+ and \mathbf{W}^- , where $\mathbf{W}^+(R) = \mathbf{W}(R)^\tau$. Denoting the corresponding representing Hopf algebras by H_*^+ and H_*^- , we have

$$H_*^+ = \mathbb{k}[b_{2n}^+ : n \geq 1], \quad H_*^- = \mathbb{k}[b_{2n-1}^- : n \geq 1].$$

The following result is now immediate.

Theorem 4.2. *There is a decomposition of group schemes*

$$\mathbf{W} = \mathbf{W}^+ \times \mathbf{W}^-,$$

or equivalently of Hopf algebras

$$H_* = H_*^+ \otimes_{\mathbb{k}} H_*^-.$$

Let $c = \{c_n\}_{0 \leq n}$ be a sequence in the \mathbb{k} -algebra R with $c_0 = 1$. The power series $c(X)X = \sum_{0 \leq n} c_n X^{n+1}$ has a composition inverse $\tilde{c}(X)X = \sum_{0 \leq n} \tilde{c}_n X^{n+1}$; the associated sequence $\tilde{c} = \{\tilde{c}_n\}_{0 \leq n}$ is called the *reverted series* or the *reversion* of c . When R has no \mathbb{Z} -torsion, we may determine \tilde{c}_n by using Lagrange Inversion to obtain the well known formula

$$\tilde{c}_n = \frac{1}{n+1} [c(T)^{-n-1}]_{T^n},$$

i.e., the coefficient of T^n in $\frac{1}{n+1}c(T)^{-n-1}$. Notice that $\tilde{\tilde{c}} = c$. Following Adams [1] and working in the algebra H_* , we denote the reversion of $b = \{b_n\}$ by $m = \{m_n\} = \{\tilde{b}_n\}$. Since

$$m_n \equiv -b_n \pmod{\text{(decomposables)}},$$

we have $H_* = \mathbb{k}[m_n : n \geq 1]$. We can interpret this as defining yet another group scheme $\widetilde{\mathbf{W}}$, for which

$$\widetilde{\mathbf{W}}(R) = \mathbf{Alg}_{\mathbb{k}}(H_*, R)$$

is identified with the set of all sequences in R , but this time $\varphi \in \mathbf{Alg}_{\mathbb{k}}(H_*, R)$ corresponds to $\{\varphi(m_n)\}$ rather than $\{\varphi(b_n)\}$. The addition law here is given by

$$c \diamond d = \widetilde{\widetilde{c * d}}$$

or equivalently,

$$\{c_n\} \diamond \{d_n\} = \left\{ \widetilde{\widetilde{(c * d)_n}} \right\}.$$

The coproduct in H_* gives rise to this group structure. Of course, there is an isomorphism of group schemes $\mathbf{W} \cong \widetilde{\mathbf{W}}$ for which on R ,

$$\{c_n\} \longleftrightarrow \{\tilde{c}_n\}.$$

This is induced by the algebra automorphism of H_* given by

$$b_n \longleftrightarrow m_n.$$

Now let F be a formal group law over the \mathbb{k} -algebra R . For simplicity, we assume that F is *odd*, however it is possible to rework our discussion using an arbitrary formal group law and replacing certain occurrences of $-T$ by the $[-1]_F$ -series.

Given a sequence $\{c_n\}_{0 \leq n}$ in R , we can consider the series $\sum_{0 \leq n} c_n X^{n+1}$. A standard calculation in formal group theory shows that there are unique elements c_n^F in R for which

$$\sum_{0 \leq n} c_n X^{n+1} = \sum_{0 \leq n} c_n^F X^{n+1}.$$

Moreover, each $c_n^F - c_n$ is a polynomial over \mathbb{k} in the coefficients of F together with the c_i^F and c_i for $i < n$. Given two such sequences $c = \{c_n\}$ and $d = \{d_n\}$, we have

$$(c \diamond d)(X) = \sum_{0 \leq n} (c \diamond_F d)_n X^{n+1}$$

where $(c \diamond_F d)_n^F - c_n^F - d_n^F$ is a polynomial over \mathbb{k} in the c_i^F and d_i^F for $i < n$ together with the coefficients of F . This gives a group scheme \mathbf{W}^F (depending on F) for which the underlying set of $\mathbf{W}^F(R)$ again consists of sequences in R , but this time we have the addition law

$$\{c_n\} \diamond_F \{d_n\} = \{(c \diamond_F d)_n\}.$$

As a representing object for \mathbf{W}^F we have

$$H_*^F = \mathbb{k}[m_n^F : n \geq 1],$$

where we view the generators m_n^F as the coefficients of the logarithm of the formal group law F . Thus

$$\mathbf{W}^F(R) \cong \mathbf{Alg}_{\mathbb{k}}(H_*^F, R),$$

where we identify $\varphi \in \mathbf{Alg}_{\mathbb{k}}(H_*^F, R)$ with the sequence $\{\varphi(m_n^F)\}$. H_*^F admits a coproduct ψ_F and antipode χ^F giving rise to the above group structure.

Theorem 4.3. *There is an isomorphism of group schemes $\widetilde{\mathbf{W}} \cong \mathbf{W}^F$, or equivalently of Hopf algebras $H_* \cong H_*^F$.*

Proof. We identify a series $\{c_n\} \in \widetilde{\mathbf{W}}(R)$ with a power series

$$\sum_{1 \leq n} c_n X^n = \sum_{1 \leq n} (c_n^F X^n)$$

and hence with an element of $\{c_n^F\}$ of $\mathbf{W}^F(R)$. Moreover, the group structures match up by definition of $(c \diamond_F d)$ □

We may also use the decomposition $\mathbf{W} \cong \mathbf{W}^+ \times \mathbf{W}^-$ to induce a similar decomposition of \mathbf{W}^F .

Theorem 4.4. *There is an decomposition of group schemes*

$$\mathbf{W}^F = \mathbf{W}^{F+} \times \mathbf{W}^{F-},$$

or equivalently of Hopf algebras

$$H_*^F = H_*^{F+} \otimes_{\mathbb{k}} H_*^{F-}.$$

It is worth remarking that the Hopf algebras H_*^{F+} and H_*^{F-} are polynomial,

$$H_*^{F+} = \mathbb{k}[m_{2n}^{F+} : n \geq 1], \quad H_*^{F-} = \mathbb{k}[m_{2n-1}^{F-} : n \geq 1],$$

but that an element of $c \in \mathbf{W}^{F+}(R)$ has the form $\{c_n\}$ in which not all of the c_{2n-1} need be 0. Similarly, for $c \in \mathbf{W}^{F-}(R)$, the terms c_{2n} need not vanish.

5. APPLICATIONS TO SOME HOPF ALGEBRAS FROM ALGEBRAIC TOPOLOGY

If $\mathbb{k} = E_*$ is the coefficient ring of a complex oriented cohomology theory, then $E_*(\mathbf{BU}_+) \cong E_*[b_n : n \geq 1]$, where we may identify b_n with the standard generator β_n^E of [1].

A particular case of interest is the universal one, $E = \mathbf{MU}$. This is made more interesting by the existence of a map of ring spectra

$$\Delta: \mathbf{MU} \longrightarrow \mathbf{MU} \wedge \mathbf{BU}_+ = \mathbf{MU} \wedge \Sigma^\infty(\mathbf{BU}_+),$$

which is the ‘external Thom diagonal’ (see the discussion after Theorem 2.2). This gives rise to a multiplicative cohomology operation

$$\bar{\Delta}: \mathbf{MU}^*(\) \longrightarrow (\mathbf{MU} \wedge \mathbf{BU}_+)^*(\) \cong \mathbf{MU}_*(\mathbf{BU}_+) \otimes_{\mathbf{MU}_*} \mathbf{MU}^*(\)$$

whose effect on the orientation class $x \in \mathbf{MU}^2(\mathbb{C}\mathbf{P}^\infty)$ is given by

$$\bar{\Delta}(x) = b(x)x = \sum_{n \geq 0} b_n x^{n+1}.$$

By Quillen’s result [10, 1] identifying \mathbf{MU}_* with Lazard’s ring, we can view the \mathbf{MU}_* -algebra

$$\mathbf{MU}_*(\mathbf{BU}_+) = \mathbf{MU}_*[b_n^{\mathbf{MU}} : n \geq 1] = \mathbf{MU}_*[m_n^{\mathbf{MU}} : n \geq 1]$$

(where $m_n^F = \widetilde{b}_n^F$) as giving rise to the scheme (on the category of commutative rings) which evaluated at R gives all pairs $(F, f(X))$ consisting of a formal group law F over R together with a power series

$$f(X) = X + \sum_{1 \leq n} c_n X^{n+1}.$$

Interpreting this as $\mathbf{Alg}_{\mathbb{Z}}(\mathbf{MU}_*(\mathbf{BU}_+), R)$, we identify the element φ of the latter with the sequence $\{\varphi(b_n^{\mathbf{MU}})\}$.

We may apply the ideas of Section 4 to describe the $\text{odd}\mathbf{MU}_*$ -Hopf algebra $\text{odd}\mathbf{MU}_*(\mathbf{BU}_+)$ equipped with the canonical odd formal group law $\text{odd}\mathbf{FGL}$. By the discussion following Theorem 2.1 together with the Atiyah-Hirzebruch spectral sequence, in this case we have

$$\text{odd}\mathbf{MU}_*(\mathbf{BU}_+)^+ = \text{odd}\mathbf{MU}_*(\mathbf{BSp}_+),$$

$$\text{odd}\mathbf{MU}_*(\mathbf{BU}_+)^- = \text{odd}\mathbf{MU}_*(\mathbf{Sp}/\mathbf{U}_+)$$

as Hopf algebras over the $\mathbb{Z}[1/2]$ -algebra $\text{odd}\mathbf{MU}_*$. This gives the following interpretation of the scheme (on $\mathbb{Z}[1/2]$ -algebras) represented by $\text{odd}\mathbf{MU}_*(\mathbf{BU}_+)^-$: points in R are pairs (F, φ) consisting of an odd formal group law F over R together with a strict isomorphism $\varphi: F_\varphi \longrightarrow F$, where

$$\varphi(X) = u(X)X$$

for some series $u \in \mathbf{W}^{F^-}(R)$. Given two such pairs $(F, \varphi), (F, \theta)$, having the same formal group law F , their ‘composition’ is given by the operation of ‘addition’ defined by \diamond . So if $\theta(X) = v(X)X$, we have another such pair

$$(F, \varphi) \underset{F}{\diamond} (F, \theta) = (F, (u \underset{F}{\diamond} v)(X)X).$$

Thus we see that ${}^{\text{odd}}\text{MU}_*(\text{Sp}/\text{U})$ represents a groupoid scheme, with objects over a $\mathbb{Z}[1/2]$ -algebra R being odd formal group laws over R , and morphisms being such strict isomorphisms.

6. AN INTERPRETATION IN TERMS OF HOMOGENEOUS SPACES

In this section we sketch an interpretation of some of our preceding constructions in terms of the geometry of homogeneous spaces. For a graded $\mathbb{Z}[1/2]$ -algebra $R = R_*$ we will write $G(R)$ for the group of invertible power series with coefficients in R , i.e.,

$$G(R) = \left\{ g(T) = \sum_{0 \leq k} g_k T^{k+1} : g_k \in R_{2k}, g_0 = 1 \right\}$$

with composition as its operation. We also set

$$L(R) = \mathbf{Alg}_{\mathbb{Z}}(\text{MU}_*, R),$$

which by [10] can be identified with the set of (graded) formal group laws defined over R . There is a right action

$$\begin{aligned} L(R) \times G(R) &\longrightarrow L(R); \\ (F, g) &\longmapsto F^g \end{aligned}$$

defined by changing coordinates via

$$F^g(X, Y) = g^{-1}(F(g(X), g(Y))).$$

The subgroup

$$G_{\text{odd}}(R) = \left\{ g(T) \in G(R) : g(T) = \sum_{0 \leq k} g_{2k} T^{2k+1} \right\}$$

of odd power series, and the subset $L_{\text{odd}}(R)$ of odd formal group laws, are defined analogously; thus $\mathbf{FGL}(R)$ is the category associated to the transformation group (G, L) , while ${}^{\text{odd}}\mathbf{FGL}(R)$ is defined by the action of G_{odd} on L_{odd} . Finally, we define $G_{\text{ev}}(R)$ to be the *set* of all even power series over R with zero constant term.

It is now easy to see that the action of G on the subobject L_{odd} of L factors through a map

$$L_{\text{odd}} \times_{G_{\text{odd}}} G \longrightarrow L$$

which is an equivalence over $\mathbb{Z}[1/2]$. To see this, first recall that we have seen that under these circumstances any formal group law is isomorphic to some odd formal group law, implying that the map is surjective. On the other hand, if two pairs (F_1, g_1) and (F_2, g_2) (with F_1 and F_2 both odd) map to the same group law $F_1^{g_1} = F_2^{g_2}$, then $h = g_2 \circ g_1^{-1}$ is an isomorphism from the odd group law F_1 with the odd group law F_2 , and therefore commutes with the automorphisms

$$[-1]_{F_1}(T) = [-1]_{F_2}(T) = -T,$$

implying that h is odd, and that the equivalence classes $[F_1, g_1]$ and $[F_2, g_2]$ are equal. Hence this map is also injective.

This description of L yields a diagram of functors

$$\begin{array}{ccc} L_{\text{odd}} & \longrightarrow & L = L_{\text{odd}} \times_{G_{\text{odd}}} G \\ & & \downarrow \\ & & G \times_{G_{\text{odd}}} \text{pt} = G/G_{\text{odd}} \end{array}$$

for L as a G_{odd} -equivariant fibre bundle; but the map $L_{\text{odd}} \longrightarrow L$ of Section 2 induced from the ring homomorphism $\text{MU}_* \longrightarrow {}^{\text{odd}}\text{MU}_*$ is a retraction of the total space to the fibre, which can be interpreted as a product splitting

$$L \longrightarrow L_{\text{odd}} \times G/G_{\text{odd}}.$$

If we now define the set of *involutive power series over R* by

$$\text{Invol}(R) = \left\{ e(T) = \sum_{i \geq 0} e_i T^{i+1} : e_0 = -1, e_i \in R_{2i}, e(e(T)) = T \right\},$$

then a similar construction to the above defines the G -equivariant map

$$\begin{aligned} G/G_{\text{odd}} &\longrightarrow \text{Invol}; \\ g &\longmapsto g^{-1}(-g(T)) \end{aligned}$$

of representable functors. The inversion formula derived in Section 1 for the equation

$$T + e(T) = \sum_{i \geq 1} c_{2i-1} (Te(T))^i$$

expresses the coefficients c_{2i-1} as functions on the space of involutions; conversely, Hensel's Lemma, Theorem 1.3, applied to the equation

$$H(E) = \sum_{i \geq 1} c_{2i-1} (TE)^i - T - E = 0$$

over the ring $R[[T]]$ and the ideal $I = (T)$, yields an expression for the involution $e(T)$ as a formal power series in terms of the coefficients c_{2i-1} . Hence these coefficients generate the ring of homogeneous functions on the scheme of all such formal involutions.

Now recall the group schemes \mathbf{W}^- and \mathbf{W}^{F^-} of Section 4.

Theorem 6.1. *There are equivalences of schemes*

$$\mathbf{W}^- \xrightarrow[\Phi]{\cong} G_{\text{ev}} \xleftarrow[\Phi^F]{\cong} \mathbf{W}^{F^-},$$

where F is an odd formal group law.

Proof. The first equivalence is given by assigning to the series $c = \{c_n\} \in \mathbf{W}^-(R)$ the even series

$$\sum_{k \geq 1} c_{2k-1} X^{2k},$$

this gives an bijection $\mathbf{W}^-(R) \cong G_{\text{ev}}(R)$ since the c_{2k-1} determine a unique element $\{c_n\}$ of $\mathbf{W}^-(R)$. The second equivalence follows from the equivalences

$$\mathbf{W} \cong \widetilde{\mathbf{W}} \cong \mathbf{W}^F$$

of Theorem 4.3, together with that of Theorem 4.4. □

We next show that the decomposition of L into a product can be given the structure of a family of abelian group objects parameterized by L_{odd} . Suppose that F is an odd formal group law defined over a $\mathbb{Z}[1/2]$ -algebra R , and that $u \in \mathbf{W}^{F^-}(R)$; then there is a corresponding element $\Phi^F(u) \in G_{\text{ev}}(R)$ and we also have the series $u(T) = \sum_{n \geq 0}^F c_n T^{n+1}$. Define a new formal group law F_u over R by requiring the series

$$\varphi_u(T) = T \underset{F}{+} u(T)$$

to be a strict isomorphism $\varphi_u: F_u \longrightarrow F$.

Theorem 6.2.

a) *The function*

$$G_{\text{ev}}(R) \longrightarrow \text{Invol}(R); \quad u(T) \longmapsto [-1]_{F_u}(T),$$

is a bijection.

b) *The map*

$$G(R)/G_{\text{odd}}(R) \longrightarrow \text{Invol}(R); \quad g \longmapsto g^{-1}(-g(T)),$$

is an isomorphism. Equivalently, $G(R)$ acts transitively on the set of all involutions $\text{Invol}(R)$.

Proof.

a) We may write

$$[-1]_{F_u}(T) = (-T) \underset{F}{+} w(T)$$

where $w(T) = \sum_{k \geq 2} w_k T^k$ for some $w_k \in R_{2k-2}$. On applying φ_u and remembering that F is odd, we obtain

$$\begin{aligned} (-T) \underset{F}{+} (-u(T)) &= [-1]_F(\varphi_u(T)) \\ &= (-T) \underset{F}{+} w(T) \underset{F}{+} u((-T) \underset{F}{+} w(T)), \end{aligned}$$

since F is odd, and this yields

$$w(T) = - \left(u(T) \underset{F}{+} u((-T) \underset{F}{+} w(T)) \right).$$

Comparing coefficients of T^2 and T^3 gives $w_2 = -2u_2$ and expresses w_3 as a polynomial over $\mathbb{Z}[1/2]$ in w_2 and u_2 together with the coefficients of the formal group law F . By induction and comparison of the coefficients of T^{2k} and T^{2k+1} , we find that

$$\begin{aligned} w_{2k} &= -2u_{2k} + P_{2k}, \\ w_{2k+1} &= P_{2k+1}, \end{aligned}$$

where P_n is a polynomial over $\mathbb{Z}[1/2]$ in the u_{2i} ($2i < n$) and w_j ($j < n$) together with the coefficients of the formal group law F . Now these equations may be inverted to express the u_{2k} in terms of the w_ℓ ,

$$u_{2k} = (-1/2)(w_{2k} - P_{2k}).$$

Hence, given the series $w(T)$ there is exactly one even power series $u(T)$ for which

$$[-1]_{F_u}(T) = (-T) \underset{F}{+} w(T).$$

b) Notice that

$$g^{-1}(-g(T)) = -T \iff g(-T) = -g(T) \iff g(T) \text{ is odd.}$$

Hence, this map is injective.

To prove surjectivity, we use an idea from the proof of part (a). Let $e(T) \in \text{Invol}(R)$. Taking F to be any odd formal group law over R (e.g., $F = \hat{G}_a$, the additive group law), we can find an even series $u(T)$ with no constant term which satisfies

$$[-1]_{F_u}(T) = e(T).$$

Since

$$[-1]_{F_u}(T) = \varphi_u^{-1}(-\varphi_u(T)),$$

we see that for $g = \varphi_u$,

$$e(T) = g^{-1}(-g(T)).$$

This completes the proof of Theorem 6.2. \square

7. FORMAL INVOLUTIONS AND SYMMETRIC SPACES

We continue to assume that $R = R_*$ is a graded commutative $\mathbb{Z}[1/2]$ -algebra. Let $f(T) \in R[[T]]$ with

$$f(T) = T + (\text{higher order terms});$$

we call such series *strictly invertible* over R . We define the series

$$e_f = f^{-1} \circ [-1] \circ f, \quad \text{i.e.,} \quad e_f(T) = f^{-1}(-f(T))$$

where $[-1](T) = -T$. Notice that the $e_f \circ e_f = \text{id}$ in the sense that $e_f(e_f(T)) = T$.

Theorem 7.1. *Suppose f and g are strictly invertible over R , and that the involutions e_f and e_g commute under composition. Then $e_f = e_g$.*

The proof will require the following Lemma. We will say that a series $e(T) \in R[[T]]$ is a *formal involution* if $e \circ e = \text{id}$, i.e., $e(e(T)) = T$.

Lemma 7.2. *Suppose $e(T)$ is a formal involution which strictly invertible over R . Then $e = \text{id}$, i.e., $e(T) = T$.*

Proof. By Lagrange Inversion of Theorem 1.1, the coefficients of the series $e(T) = T + \sum_{n \geq 1} e_n T^{n+1}$ satisfy

$$e_n = -e_n + E_n$$

for some polynomial E_n over $\mathbb{Z}[1/2]$ in the coefficients e_1, \dots, e_{n-1} . Now by induction on n (and remembering that $1/2 \in R$) we get $e_n = 0$ for $n \geq 1$, since the initial case follows from the equation $e_1 = -e_1$. \square

Now suppose that $e_f \circ e_g = e_g \circ e_f$ for two strictly invertible series f and g . Expanding this equation and suppressing the composition symbols gives

$$f^{-1}[-1]fg^{-1}g = g^{-1}[-1]gf^{-1}[-1]g.$$

On setting $h = fg^{-1}$, this becomes

$$(h[-1]h^{-1}[-1])(h[-1]h^{-1}[-1]) = \text{id},$$

which implies that $h[-1]h^{-1}[-1]$ is a formal involution. Since $h[-1]h^{-1}[-1](T)$ is clearly strictly invertible, Lemma 7.2 now gives $h[-1]h^{-1}[-1] = \text{id}$ and so $h[-1] = [-1]h$, implying that $h = fg^{-1}$ is odd. Thus $f = h \circ g$ and g lie in the same left coset of $G_{\text{odd}}(R)$ in $G(R)$, i.e., $fG_{\text{odd}}(R) = gG_{\text{odd}}(R)$ in $G(R)/G_{\text{odd}}(R)$. Hence $e_f = e_g$, which proves Theorem 7.1. Our next corollary shows that $G(R)/G_{\text{odd}}(R)$ is a kind of symmetric space.

Corollary 7.3. *At every point $e \in G(R)/G_{\text{odd}}(R)$ there is an involution*

$$G(R)/G_{\text{odd}}(R) \longrightarrow G(R)/G_{\text{odd}}(R); \quad c \longmapsto e^{-1} \circ c \circ e,$$

which has e as the unique fixed point.

There is thus some similarity of the quotient G/G_{odd} with the Riemannian symmetric spaces defined by the quotient of the group of complex points of a simple Lie group by a maximal compact subgroup. This fits in with the observation that the Lie algebra of G is twice as big as the Lie algebra of G_{odd} , so that one can try to think of G_{odd} as the ‘real’ points of some ‘Lie’ group, with G as the ‘complex’ points.

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