PROFINITE GROUPOIDS AND THEIR COHOMOLOGY

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Abstract. We define a notion of *profinite groupoid* and a suitable class of modules over these. We describe two functors which in the case of a connected profinite groupoid agree; the derived functors of these are the analogues of continuous cohomology of profinite groupoids. The resulting 'change of groupoid' formulæ reduce its computation to continuous cohomology of profinite groups.

Introduction

Although groupoids have been well studied in many branches of mathematics, the literature on their cohomology is somewhat hard to find. Motivated by their importance in algebraic topology, in this paper we outline a theory of *profinite groupoids* and the cohomology of *proper modules* which is intended to generalize standard ideas in Galois cohomology as described in Shatz [5]. Our ultimate motivation is the need to have 'change of rings theorems' for such cohomology, allowing reduction to the cohomology of automorphism groups. Devinatz's paper [3] gives an indication of the kind of result required; an application in which the use of topological splittings of profinite groupoids is insufficient because of the topology appears in [1].

1. Profinite groupoids

Let $\mathcal{G}$ be a groupoid (i.e., a small category in which every morphism is invertible). The function $\operatorname{Obj}\mathcal{G} \longrightarrow \operatorname{Mor}\mathcal{G}$ which sends each object to its identity morphism can be viewed as embedding $\operatorname{Obj}\mathcal{G}$ into $\operatorname{Mor}\mathcal{G}$ so we can view $\mathcal{G}$ as consisting of the set of all its morphisms. Later we will use the notation

$$\mathcal{G}(*, x) = \bigcup_{y \in \operatorname{Obj}\mathcal{G}} \mathcal{G}(y, x) \subseteq \mathcal{G}.$$

Recall from Higgins [4] that a subgroupoid $\mathcal{N}$ of a groupoid $\mathcal{G}$ is *normal* if it has the following properties:

A) $\operatorname{Obj}\mathcal{N} = \operatorname{Obj}\mathcal{G}$;

B) for every morphism $x \xrightarrow{f} y$ in $\mathcal{G}$,

$$f\mathcal{N}(x, x)f^{-1} = \mathcal{N}(x, x).$$

We write $\mathcal{N} \lhd \mathcal{G}$ if $\mathcal{N}$ is a normal subgroupoid of $\mathcal{G}$. We can then form the *quotient groupoid* $\mathcal{G}/\mathcal{N}$ whose objects are equivalence classes of objects of $\mathcal{G}$ under the relation

$$x \sim y \iff \exists\, x \xrightarrow{h} y \text{ in } \mathcal{N}.$$

Similarly, the morphisms are equivalence classes of morphisms of $\mathcal{G}$ under the relation

$$x \xrightarrow{f} y \sim x' \xrightarrow{f'} y' \iff \exists\, x \xrightarrow{p} x',\ y \xrightarrow{q} y' \text{ in } \mathcal{N} \text{ s.t. } f' = qfp^{-1}.$$

Composition of equivalence classes is defined

$$[y \xrightarrow{g} z][x \xrightarrow{f} y] = [x \xrightarrow{gf} z]$$

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whenever these classes contain composable elements. This is well defined since given morphisms $x \xrightarrow{p} x'$, $y \xrightarrow{q} y'$, $y \xrightarrow{r} y'$, $z \xrightarrow{s} z'$ in $\mathcal{N}$,

$$(sgr^{-1})(qfp^{-1}) = sg(r^{-1}q)fp^{-1} = (sh)gfp^{-1}$$

where $h = g(r^{-1}q)g^{-1}$ is in $\mathcal{N}(z,z)$. There is an evident quotient functor $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{N}$.

We define a groupoid $\mathcal{G}$ to be *automorphism finite* if for every $x \in \operatorname{Obj}\mathcal{G}$, $\mathcal{G}(x,x)$ is a finite group. We define a groupoid $\mathcal{G}$ to be (*automorphism*) *profinite* if it is the inverse limit of automorphism finite groupoids,

$$\mathcal{G} \cong \varprojlim_{\mathcal{N} \triangleleft \mathcal{G}} \mathcal{G}/\mathcal{N}.$$

Such a groupoid has a natural topology in which basic open sets have the form

$$U(x \xrightarrow{f} y, \mathcal{N}) = \{x \xrightarrow{g} y : gf^{-1} \in \mathcal{N}(y,y)\},$$

where $f \in \operatorname{Mor}\mathcal{G}$ and $\mathcal{G}/\mathcal{N}$ is automorphism finite.

A *topological groupoid* is a groupoid which is a topological space such that the partial composition $\mathcal{G} \underset{\operatorname{Obj}\mathcal{G}}{\times} \mathcal{G} \longrightarrow \mathcal{G}$, inverse function $\mathcal{G} \longrightarrow \mathcal{G}$, domain and codomain functions $\mathcal{G} \longrightarrow \operatorname{Obj}\mathcal{G}$ are continuous, where $\operatorname{Obj}\mathcal{G}$ has the subspace topology. A profinite groupoid in the above sense is a topological groupoid.

Let $\mathcal{G}$ be a groupoid. Recall that $\mathcal{G}$ is *connected* if for every pair of objects x, y in $\mathcal{G}$ there is a morphism $x \xrightarrow{f} y$. More generally, $\mathcal{G}$ is a disjoint union of connected subgroupoids, which we refer to as the *connected components* of $\mathcal{G}$.

2. Proper modules over profinite groupoids

In order to define the cohomology of profinite groupoids we first need to define suitable a notion of module, and we follow the ideas of Galois cohomology, accessibly described in Shatz [5], with Weibel [6] providing a more general cohomological discussion.

For a profinite groupoid $\mathcal{G}$, a *proper $\mathcal{G}$-module* (over a commutative unital ring $\Bbbk$) is a functor $\underline{M}\colon \mathcal{G} \longrightarrow \operatorname{Mod}_{\Bbbk}$ in which for $x, y \in \operatorname{Obj}\mathcal{G}$, $m \in \underline{M}(x)$, $f \in \mathcal{G}(x,y)$, the set

$$\operatorname{Stab}_{\mathcal{G}}(m,f,y) = \{g \in \mathcal{G}(x,y) : \underline{M}(g)m = \underline{M}(f)m\} \subseteq \mathcal{G}(x,y)$$

is open. This generalizes the notion of proper module for a profinite group, for which stabilizers of points are of finite index. We will denote the category of all proper $\mathcal{G}$-modules over $\Bbbk$ by $\operatorname{Mod}_{\Bbbk,\mathcal{G}}$, where the morphisms are natural transformations.

For later use we set

$$\underline{\mathbf{M}} = \bigoplus_{x \in \operatorname{Obj}\mathcal{G}} \underline{M}(x)$$

and view this as a topological space with the discrete topology. We then define a *section* of $\underline{M}$ to be a function $\Phi\colon \operatorname{Obj}\mathcal{G} \longrightarrow \underline{\mathbf{M}}$ with the property that for all $x \in \operatorname{Obj}\mathcal{G}$,

$$\Phi(x) \in \underline{M}(x).$$

We will denote the set of all sections by $\operatorname{Sect}(\mathcal{G}; \underline{M})$; this is a $\Bbbk$-module.

Proposition 2.1. *Let $\mathcal{G}$ be a profinite groupoid and $\Bbbk$ a commutative unital ring.*

a) $\operatorname{Mod}_{\Bbbk,\mathcal{G}}$ *is an abelian category, with structure inherited from that of* $\operatorname{Mod}_{\Bbbk}$.
b) $\operatorname{Mod}_{\Bbbk,\mathcal{G}}$ *has sufficiently many injectives.*

Proof. (a) is straightforward. For (b), suppose that $\underline{M}$ is a proper $\mathcal{G}$-module. Then we need to construct an embedding into an injective object. We follow [5, chapter II §2] in defining the functor $J_{\underline{M}}$ which for $x \in \operatorname{Obj}\mathcal{G}$ is given by

$$J_{\underline{M}}(x) = \{\Phi\colon \mathcal{G}(*,x) \longrightarrow \underline{\mathbf{M}} : \Phi(g) \in \underline{M}(\operatorname{dom} g) \text{ and } \Phi \text{ continuous}\},$$

and for $f \in \operatorname{Mor} \mathcal{G}$,

$$J_{\underline{M}}(f) = \underline{M}(f)_*,$$

the map induced by composition with $\underline{M}(f)$. A modification of the argument of [5, chapter II §2 proposition 4] shows that $J_{\underline{M}}$ is a proper $\mathcal{G}$-module and furthermore there is an evident natural transformation $j\colon \underline{M} \longrightarrow J_{\underline{M}}$ for which

$$j\colon \varphi(x) \longrightarrow J_{\underline{M}}(x);\ j(m)(g) = \varphi(g^{-1})m \quad (\varphi \in \underline{M}, x \in \operatorname{Obj} \mathcal{G}, m \in \varphi(x)).$$

It is now straightforward to verify the following adjunction formula which should be compared with [5, chapter II §2 proposition 4] and [6, example 2.3.13]:

$$\operatorname{Mod}_{\Bbbk,\mathcal{G}}(\underline{N}, J_{\underline{M}}) \cong \prod_{x \in \operatorname{Obj} \mathcal{G}} \operatorname{Mod}_{\Bbbk}(\underline{N}(x), \underline{M}(x)). \tag{2.1}$$

In particular, if each of the $\underline{M}(x)$ is an injective $\Bbbk$-module, then $J_{\underline{M}}$ is an injective $\mathcal{G}$-module. □

3. Continuous cohomology of profinite groupoids

We will consider two types of functor $\operatorname{Mod}_{\Bbbk,\mathcal{G}} \longrightarrow \operatorname{Mod}_{\Bbbk}$. The first is a sort of fixed point construction

$$(\ )^{\mathcal{G}}\colon \underline{M} \rightsquigarrow \underline{M}^{\mathcal{G}} = \{\Phi \in \operatorname{Sect}(\mathcal{G}; \underline{M}) : \forall f \in \mathcal{G},\ \underline{M}(f)\Phi(\operatorname{dom} f) = \Phi(\operatorname{codom} f)\}.$$

This functor is left exact, so has right derived functors $\mathrm{R}^n(\ )^{\mathcal{G}}$ for $n \geqslant 0$, which can be viewed as the continuous cohomology of $\mathcal{G}$, $\mathrm{H}^n_c(\mathcal{G};\ )$.

For the second type of functor, we fix $x_0 \in \operatorname{Obj} \mathcal{G}$ and define a kind of 'localization at x_0',

$$(\ )_{x_0}\colon \underline{M} \rightsquigarrow \underline{M}(x_0)^{\mathcal{G}(x_0,x_0)}$$

obtained by restricting to $\underline{M}(x_0)$ and taking the fixed points under the action of the automorphism group of x_0. Again this is left exact and has right derived functors $\mathrm{R}^n(\ )_{x_0}$ which we denote by $\mathrm{H}^n_{x_0}(\mathcal{G};\ )$.

Proposition 3.1. *If the profinite groupoid $\mathcal{G}$ is connected, then for any $x_0 \in \operatorname{Obj} \mathcal{G}$, there is a natural equivalence of functors*

$$(\ )_{x_0} \cong (\ )^{\mathcal{G}}.$$

Hence there are natural equivalences of functors

$$\mathrm{H}^n_c(\mathcal{G};\ ) \cong \mathrm{H}^n_{x_0}(\mathcal{G};\ ) \quad (n \geqslant 0).$$

Proof. For the first part, we produce a $\Bbbk$-isomorphism $F\colon \underline{M}_{x_0} \xrightarrow{\cong} \underline{M}^{\mathcal{G}}$. For $m \in \underline{M}_{x_0}$, define $F(m) = \Phi_m$ by

$$\Phi_m(x) = \underline{M}(f)m \quad (x \in \operatorname{Obj} \mathcal{G})$$

where we choose *any* $f \in \mathcal{G}(x_0, x)$; this choice does not affect the outcome since for a second choice $g \in \mathcal{G}(x_0, x)$, $f^{-1}g \in \mathcal{G}(x_0, x_0)$ and therefore

$$\underline{M}(g)m = \underline{M}(f)\underline{M}(f^{-1}g)m = \underline{M}(f)m.$$

This is easily verified to be an isomorphism.

The second part is an immediate consequence of the first by a standard result on δ-functors, see for example [6, chapter 2]. □

This reduces the calculation of the continuous cohomology $\mathrm{H}^*_c(\mathcal{G}; \underline{M})$ of a proper module $\underline{M}$ to continuous group cohomology, allowing application of the standard techniques described in [5].

Theorem 3.2. *If $\mathcal{G}$ is a connected profinite groupoid and $\underline{M}$ a proper $\mathcal{G}$-module over $\Bbbk$, then for any $x_0 \in \operatorname{Obj} \mathcal{G}$,*

$$\mathrm{H}^*_c(\mathcal{G}; \underline{M}) \cong \mathrm{H}^*_c(\mathcal{G}(x_0, x_0); \underline{M}(x_0)).$$

In the general case we have the following.

Theorem 3.3. *If $\mathcal{G}$ is a profinite groupoid and $\underline{M}$ a proper $\mathcal{G}$-module over $\Bbbk$, then for each $n \geqslant 0$,*

$$\mathrm{H}_c^n(\mathcal{G};\underline{M}) \cong \prod_C \mathrm{H}_c^n(\mathcal{G}(x_C, x_C); \underline{M}(x_C)),$$

where C ranges over the connected components of $\mathcal{G}$ and x_C is a chosen object of C.

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