I_n-LOCAL JOHNSON-WILSON SPECTRA AND THEIR HOPF ALGEBROIDS

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ABSTRACT. We consider a generalization $\mathcal{E}(n)$ of the Johnson-Wilson spectrum $E(n)$ for which $\mathcal{E}(n)_*$ is a local ring with maximal ideal I_n. We prove that the spectra $E(n)$, $\mathcal{E}(n)$ and $\widehat{E(n)}$ are Bousfield equivalent. We also show that the Hopf algebroid $\mathcal{E}(n)_*\mathcal{E}(n)$ is a free $\mathcal{E}(n)_*$-module, generalizing a result of Adams and Clarke for KU_*KU.

INTRODUCTION

For each prime p and $n > 0$, the Johnson-Wilson ring spectrum $E(n)$ provides an important example of a p-local periodic ring spectrum. The associated Hopf algebroid $E(n)_*E(n)$ is well known to be flat over $E(n)_*$, but as far as we are aware there is no proof in the literature that it is a free module for every n. Of course, after passage to the I_n-adic completion $\widehat{E(n)}$, and more drastically the I_n-adic completion of $E(n)_*E(n)$ (see [4, 8]), such problems disappear. On the other hand, for the ring spectrum KU, the associated Hopf algebroid KU_*KU was shown to be free over KU_* by Frank Adams and Francis Clarke [3, 2, 6]. Actually their approach has two parallel interpretations: one purely algebraic involving stably numerical polynomials [5]; the other topological in that it makes use of the cofibre sequence

$$\Sigma^2 kU \xrightarrow{t} kU \longrightarrow H\mathbb{Z}$$

induced by the Bott map $t\colon S^2 \longrightarrow kU$ in connective K-theory.

In this paper we demonstrate an analogous result by constructing an $\mathcal{E}(n)_*$-basis for $\mathcal{E}(n)_*\mathcal{E}(n)$ for a *generalized Johnson-Wilson spectrum* $\mathcal{E}(n)$ whose homotopy ring is the (graded) local ring

$$\mathcal{E}(n)_* = (E(n)_*)_{I_n}.$$

For completeness, in Section 1 we discuss even more general generalized Johnson-Wilson spectra to which appropriate analogues of our results apply, however we only describe the $\mathcal{E}(n)$ case explicitly.

Our main result is the following which has some immediate consequences stated in the Corollary.

Theorem. *$\mathcal{E}(n)_*\mathcal{E}(n)$ is a free $\mathcal{E}(n)_*$-module on a countably infinite basis.*

Corollary.
A) *For every $\mathcal{E}(n)_*$-module M_* and $s > 0$,*

$$\operatorname{Ext}^{s,*}_{\mathcal{E}(n)_*}(\mathcal{E}(n)_*\mathcal{E}(n), M_*) = 0.$$

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In particular,

$$\mathcal{E}(n)^*\mathcal{E}(n) = \mathrm{Hom}^*_{\mathcal{E}(n)_*}(\mathcal{E}(n)_*\mathcal{E}(n), \mathcal{E}(n)_*),$$

and this is a free $\mathcal{E}(n)_*$-module on an uncountably infinite basis.
B) *The $\mathcal{E}(n)$-module spectrum $\mathcal{E}(n) \wedge \mathcal{E}(n)$ is a countable wedge*

$$\mathcal{E}(n) \wedge \mathcal{E}(n) \simeq \bigvee_{\alpha} \Sigma^{2\ell(\alpha)}\mathcal{E}(n),$$

where ℓ is some integer valued function of the index α.

Actually, when $s \geqslant 2$, $\mathrm{Ext}^{s,*}_{\mathcal{E}(n)_*}(\mathcal{E}(n)_*\mathcal{E}(n), M_*) = 0$ for formal reasons. The statement about $\mathcal{E}(n)^*\mathcal{E}(n)$ follows from a version of the Universal Coefficient Spectral Sequence of Adams [1].

Our approach to constructing a basis follows a line of argument suggested by that of Adams [2] which also has a purely algebraic interpretation in Adams and Clarke [3, 6].

Although the technology of brave new ring spectra applies to generalized Johnson-Wilson spectra [7, 15], we have no need of such structure, except perhaps to ensure the existence of the relevant Universal Coefficient Spectral Sequence mentioned above; alternatively, M. Hopkins has shown that such spectral sequences exist for all multiplicative cohomology theories constructed using the Landweber Exact Functor Theorem.

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1. Generalized Johnson-Wilson spectra

Given a prime p and $n \geqslant 1$ we define *generalized Johnson-Wilson spectra* as follows. Begin with a regular sequence $\mathbf{u}\colon u_0 = p, u_1, \ldots, u_k, \ldots$ in BP_* satisfying

$$u_k \in BP_{2(p^k-1)}, \quad (p, u_1, \ldots, u_{k-1}) = I_k \lhd BP_*,$$

where I_k is actually independent of the choice of generators for BP_*. Of course we have

$$I_k = (p, v_1, \ldots, v_{k-1}) = (p, w_1, \ldots, w_{k-1}),$$

where v_j and w_j are the Hazewinkel and Araki generators respectively.

There is a commutative ring spectrum $BP\langle n; \mathbf{u}\rangle$ for which

$$BP\langle n; \mathbf{u}\rangle_* = \pi_* BP\langle n; \mathbf{u}\rangle = BP_*/(u_j : j \geqslant n+1).$$

We will denote by $I_n \lhd BP\langle n; \mathbf{u}\rangle_*$ the image of the ideal $I_n \lhd BP_*$ under the natural ring homomorphism $BP_* \longrightarrow BP\langle n; \mathbf{u}\rangle_*$.

For any multiplicative set $S \subseteq BP\langle n; \mathbf{u}\rangle_*$ containing u_n and having $I_n \cap S = \emptyset$, we can form the localization

$$E(n; \mathbf{u}; S)_* = BP\langle n; \mathbf{u}\rangle_* [S^{-1}].$$

There is a commutative ring spectrum $E(n; \mathbf{u}; S)$ with

$$E(n; \mathbf{u}; S)_* = \pi_* E(n; \mathbf{u}; S) = BP_*/(u_j : j \geqslant n+1)[S^{-1}].$$

Example 1.1. a) When $S = \{u_n^r : r \geqslant 1\}$,

$$E(n; \mathbf{u}; \{u_n^r : r \geqslant 1\})_* = BP\langle n; \mathbf{u}\rangle_* [u_n^{-1}].$$

This ring contains a maximal ideal I_n generated by the image of $I_n \lhd BP\langle n; \mathbf{u}\rangle_*$, whose quotient ring is

$$E(n; \mathbf{u}; \{u_n^r : r \geqslant 1\})_*/I_n = K(n)_*.$$

This is a mild generalization of the original notion of a Johnson-Wilson spectrum. There is also an I_n-adic completion $E(n;\mathbf{u};\{u_n^r : r \geqslant 1\})\widehat{_{I_n}}$ with homotopy ring $(E(n;\mathbf{u};\{u_n^r : r \geqslant 1\})_*)\widehat{_{I_n}}$.
b) When $S = BP\langle n;\mathbf{u}\rangle_* - I_n$,

$$E(n;\mathbf{u};BP\langle n;\mathbf{u}\rangle_* - I_n)_* = (BP\langle n;\mathbf{u}\rangle_*)_{I_n}.$$

This is a (graded) local ring with residue (graded) field

$$E(n;\mathbf{u};BP\langle n;\mathbf{u}\rangle_* - I_n)_*/I_n = K(n)_*.$$

In all cases we have the following which is a consequence of modified versions of standard arguments based on the Landweber Exact Functor Theorem.

Theorem 1.2. *For each spectrum $E(n;\mathbf{u};S)$ the following hold.*
a) *On the category of BP_*BP-comodules, tensoring with the BP_*-module $E(n;\mathbf{u};S)_*$ preserves exactness.*
b) *$E(n;\mathbf{u};S)_*E(n;\mathbf{u};S)$ is a flat $E(n;\mathbf{u};S)_*$-module.*
c) *$(E(n;\mathbf{u};S)_*, E(n;\mathbf{u};S)_*E(n;\mathbf{u};S))$ is a Hopf algebroid over $\mathbb{Z}_{(p)}$.*

Setting $u_k = v_k$, the Hazewinkel generator, for all k, we obtain the standard connective spectrum $BP\langle n\rangle$ and the Johnson-Wilson spectra $E(n)$, $\mathcal{E}(n)$ for which

$$\pi_*E(n) = E(n)_* = BP\langle n\rangle_*[v_n^{-1}],$$
$$\pi_*\mathcal{E}(n) = \mathcal{E}(n)_* = (BP\langle n\rangle_*)_{I_n}.$$

Notice that every unit $u \in \mathcal{E}(n)_*$ has the form

$$u = av_n^r + w, \qquad (1.1)$$

where $a \in \mathbb{Z}_{(p)}^\times$ and $w \in I_n$; in particular, $u \in \mathcal{E}(n)_{2(p^n-1)r}$. Of course, unlike the case of $E(n)$, the multiplicative set inverted to form $\mathcal{E}(n)_*$ from $BP\langle n\rangle_*$ is infinitely generated. However, for every such unit u arising in $BP\langle n\rangle_*$, multiplication by $U = \eta_{\mathrm{R}}(u) \in \mathcal{E}(n)_*BP\langle n\rangle$ preserves $\mathcal{E}(n)_*$-linearly independent sets by courtesy of the following algebraic result (see for example theorem 7.10 of [12]) and Corollary 2.3 which shows that $\mathcal{E}(n)_*BP\langle n\rangle$ is a free $\mathcal{E}(n)_*$-module.

Proposition 1.3. *Let A be a commutative unital local ring with maximal ideal $\mathfrak{m}$. Let M be a flat A-module and $(m_i : i \geqslant 1)$ be a collection of elements in M. Suppose that under the reduction map*

$$q\colon M \longrightarrow \overline{M} = A/\mathfrak{m} \underset{A}{\otimes} M,$$

the resulting collection $(q(m_i) : i \geqslant 1)$ of elements in $\overline{M}$ is $A/\mathfrak{m}$-linearly independent. Then $(m_i : i \geqslant 1)$ is A-linearly independent in M.

We end this section with some remarks intended to justify working with $\mathcal{E}(n)$ rather than $E(n)$. For algebraic reasons, our proof of E_*-freeness for E_*E only appears to work for $E = \mathcal{E}(n)$ although we conjecture that the result is true for $E = E(n)$. However, there are sound topological reasons for viewing $\mathcal{E}(n)$ as a substitute for $E(n)$. Notice that

$$E(n)_*/I_n = \mathcal{E}(n)_*/I_n = \widehat{E(n)}_*/I_n = K(n)_*.$$

Theorem 1.4. *The spectra*

$$E(n),\ \mathcal{E}(n),\ \widehat{E(n)}$$

are Bousfield equivalent. More generally, the spectra

$$E(n;\mathbf{u};\{u_n^r : r \geqslant 1\}),\ E(n;\mathbf{u};BP\langle n;\mathbf{u}\rangle_* - I_n),\ E(n;\mathbf{u};\{u_n^r : r \geqslant 1\})\widehat{_{I_n}}$$

are Bousfield equivalent.

Remark 1.5. It is claimed in proposition 5.3 of [10] that $E(n)$ and $\widehat{E(n)}$ are Bousfield equivalent. The proof given there is not correct since the extension $E(n)_* \longrightarrow \widehat{E(n)}_*$ is not faithfully flat because I_n is not contained in the radical of $E(n)_*$. We refer the reader to Matsumura [12], especially theorem 8.14(3), for standard algebraic facts concerning faithful flatness. In the following proof, we provide an alternative argument based on the Landweber Filtration Theorem [11].

Proof. For simplicity we only give the proof for the classical case. Since

$$\widehat{E(n)}_*(X) = \widehat{E(n)}_* \underset{E(n)_*}{\otimes} E(n)_*(X),$$

we need only show that $\widehat{E(n)}_*(X) = 0$ implies $E(n)_*(X) = 0$.

Let M_* a BP_*BP-comodule which is finitely generated as a BP_*-module. Then M_* admits a Landweber filtration by subcomodules

$$0 = M_*^{[0]} \subseteq M_*^{[1]} \subseteq \cdots \subseteq M_*^{[k]} = M_*$$

such that for each $j = 0, \ldots, k$,

$$M_*^{[j]}/M_*^{[j-1]} \cong BP_*/I_{d_j}$$

for some $d_j \geqslant 0$. The $E(n)_*E(n)$-comodule

$$\overline{M}_* = E(n)_* \underset{BP_*}{\otimes} M_*$$

inherits a filtration by subcomodules

$$0 = \overline{M}_*^{[0]} \subseteq \overline{M}_*^{[1]} \subseteq \cdots \subseteq \overline{M}_*^{[k]} = \overline{M}_*$$

satisfying

$$\overline{M}_*^{[j]}/\overline{M}_*^{[j-1]} \cong E(n)_*/I_{d_j},$$

where $E(n)_*/I_{d_j} = 0$ if $d_j > n$. For a BP_*-module N_*,

$$\widehat{E(n)}_* \underset{E(n)_*}{\otimes} E(n)_* \underset{BP_*}{\otimes} N_* \cong \widehat{E(n)}_* \underset{BP_*}{\otimes} N_*.$$

Then writing $\widehat{N}_* = \widehat{E(n)}_* \otimes_{BP_*} N_*$ we have

$$\widehat{M}_*^{[j]}/\widehat{M}_*^{[j-1]} \cong \widehat{E(n)}_*/I_{d_j}.$$

From this it follows that $\overline{M}_* = 0$ if and only if $\widehat{M}_*$. So $\widehat{E(n)}_*$ is faithfully flat in this sense on $E(n)_*$-comodules of the form $\overline{M}_*$ for some finitely generated BP_*BP-comodule.

We can extend this to faithful flatness on all BP_*BP-comodules. Such a comodule N_* is the union of its finitely generated subcomodules, by corollary 2.13 of [13]. For each finitely generated subcomodule $M_* \subseteq N_*$, the short exact sequence

$$0 \to M_* \longrightarrow N_* \longrightarrow N_*/M_* \to 0$$

gives rise to the sequences

$$0 \to \overline{M}_* \longrightarrow \overline{N}_* \longrightarrow \overline{N_*/M_*} \to 0,$$

$$0 \to \widehat{M}_* \longrightarrow \widehat{N}_* \longrightarrow \widehat{N_*/M_*} \to 0.$$

Each of these is short exact since by the Landweber Exact Functor Theorem, tensor product over BP_* with either of $E(n)_*$ or $\widehat{E(n)}_*$ is an exact functor on BP_*-comodules. Suppose that $\widehat{N}_* = 0$; then $\widehat{M}_* = 0$, which implies $\overline{M}_* = 0$. Since

$$\overline{N}_* = \varinjlim_{M_* \subseteq N_*} \overline{M}_*,$$

this gives $\overline{N}_* = 0$. Applying this to the case of $N_* = BP_*(X)$ we obtain the Bousfield equivalence of $E(n)$ with $\widehat{E(n)}$.

In the chain of rings $E(n)_* \subseteq \mathcal{E}(n)_* \subseteq \widehat{E(n)}_*$, the extension $\mathcal{E}(n)_* \longrightarrow \widehat{E(n)}_*$ is faithfully flat, hence $\mathcal{E}(n)$ and $\widehat{E(n)}$ are also Bousfield equivalent. Alternatively, by the Landweber Exact Functor Theorem, tensoring with $\mathcal{E}(n)_*$ is exact on BP_*BP-comodules, so the above proof works as well with $\mathcal{E}(n)$ in place of $E(n)$. □

This result implies that the stable world as seen through the eyes of each of the homology theories $E(n)_*(\ )$, $\mathcal{E}(n)_*(\ )$ and $\widehat{E(n)}_*(\ )$ looks the same; indeed this is true for any generalized Johnson-Wilson spectrum between $BP\langle n\rangle$ and $\mathcal{E}(n)$. The proof of the p-local part of the result of Adams and Clarke [3, 2, 6] also involves working over a (graded) local ring $(KU_*)_{(p)} = \mathbb{Z}_{(p)}[t, t^{-1}]$; of course their result holds over the arithmetically global ring $KU_* = \mathbb{Z}[t, t^{-1}]$.

2. Some bases for $\mathcal{E}(n)_*BP$ and $\mathcal{E}(n)_*BP\langle n\rangle$

We first define a useful basis for $\mathcal{E}(n)_*BP$ which projects to a basis for $\mathcal{E}(n)_*BP\langle n\rangle$ under the natural surjective homomorphism of $\mathcal{E}(n)_*$-algebras

$$q_n \colon \mathcal{E}(n)_*BP \longrightarrow \mathcal{E}(n)_*BP\langle n\rangle.$$

$\mathcal{E}(n)_*BP$ is the polynomial $\mathcal{E}(n)_*$-algebra with the standard generators

$$t_k \in \mathcal{E}(n)_{2(p^k-1)}BP$$

induced from those for BP_*BP described by Adams [1], where

$$\mathcal{E}(n)_*BP = \mathcal{E}(n)_*[t_k : k \geqslant 1].$$

Hence the latter has an $\mathcal{E}(n)_*$-basis consisting of the monomials

$$t_1^{r_1} \cdots t_\ell^{r_\ell} \quad (0 \leqslant r_k).$$

The kernel of q_n is the ideal generated by the elements $V_{n+k} = \eta_{\mathrm{R}}(v_{n+k})$ $(k \geqslant 1)$, where η_{R} is the right unit obtained from the right unit in BP_*BP as the composite

$$BP_* \xrightarrow{\eta_{\mathrm{R}}} BP_*BP \longrightarrow \mathcal{E}(n)_*BP.$$

By well known formulæ for the right unit of BP_*BP, in the ring $\mathcal{E}(n)_*BP$ we have

$$\eta_{\mathrm{R}}(v_{n+k}) = v_n t_k^{p^n} - v_n^{p^k} t_k + \cdots + p t_{n+k} \tag{2.1a}$$

$$\equiv v_n t_k^{p^n} - v_n^{p^k} t_k \bmod I_n. \tag{2.1b}$$

Here the undisplayed terms are polynomials over BP_* in $t_1, \ldots, t_{n+k-1}$.

Remark 2.1. The main source of difficulty in working with $E(n)$ itself in place of $\mathcal{E}(n)$ seems to arise from the fact that the coefficient of $t_j^{p^n}$ in Equation (2.1) is then only a unit modulo I_n, so we can only use monomials involving the $\eta_{\mathrm{R}}(v_{n+k})$ as part of a basis when working over $\mathcal{E}(n)_*$ rather than just $E(n)_*$. This is used crucially in the proof of Proposition 2.2. Perhaps a careful choice of generators in place of the Hazewinkel or Araki generators would overcome this problem.

We will also require an expression for the right unit on v_n:

$$\eta_{\mathrm{R}}(v_n) = v_n + \sum_{1\leqslant j\leqslant n} v_j\theta_j \in \mathcal{E}(n)_*BP, \tag{2.2}$$

where $\theta_j \in \mathcal{E}(n)_{2(p^n-p^j)}BP$ has the form

$$\theta_j = t_{n-j}^{p^j} \bmod I_n.$$

In particular $\theta_0 = t_n \bmod I_n$. Although the θ_j are not unique, the terms $v_j\theta_j \bmod I_n^2$ are well defined. Notice that if $u \in \mathcal{E}(n)_*$ has the form of Equation (1.1), then for the right unit $\eta_{\mathrm{R}}(u)$ on u,

$$\eta_{\mathrm{R}}(u) \equiv av_n^r \bmod I_n.$$

Now we will define some elements that will eventually be seen to form a basis for $\mathcal{E}(n)_*BP$. First we introduce the following elements of $\ker q_n$:

$$\kappa_{r_1,\ldots,r_k;s_1,\ldots,s_\ell} = v_n^{-(s_1+\cdots+s_\ell)} t_1^{r_1}\cdots t_k^{r_k} V_{n+1}^{s_1}\cdots V_{n+\ell}^{s_\ell}, \tag{2.3a}$$

where $0 \leqslant r_j \leqslant p^n-1$ with $r_k \neq 0$ and $\ell > 0$, $s_j \geqslant 0$ and $s_\ell \neq 0$. We also have the elements

$$\kappa_{r_1,\ldots,r_k} = t_1^{r_1}\cdots t_k^{r_k}, \tag{2.3b}$$

where $0 \leqslant r_j \leqslant p^n-1$ with $r_k \neq 0$. The empty sequence corresponds to the element $\kappa_\emptyset = 1$. There are also elements

$$\overline{\kappa}_{r_1,\ldots,r_k} = q_n(\kappa_{r_1,\ldots,r_k}) \in \mathcal{E}(n)_*BP\left\langle n\right\rangle. \tag{2.4}$$

Next we introduce an increasing multiplicative filtration on $\mathcal{E}(n)_*BP$ (apart from a factor of 2 in the indexing, this is the filtration associated with the Atiyah-Hirzebruch spectral sequence for $\mathcal{E}(n)_*BP$),

$$\mathcal{E}(n)_* = \mathcal{E}(n)_*BP^{[0]} \subseteq \cdots \subseteq \mathcal{E}(n)_*BP^{[k]} \subseteq \cdots \subseteq \bigcup_{0\leqslant j} \mathcal{E}(n)_*BP^{[j]} = \mathcal{E}(n)_*BP.$$

Here the monomial $t_1^{r_1}\cdots t_\ell^{r_\ell}$ has exact filtration $\sum_j r_j(p^j-1)$. Of course each $\mathcal{E}(n)_*BP^{[k]}$ is a finite rank free $\mathcal{E}(n)_*$-module with the basis consisting of all the elements $\kappa_{r_1,\ldots,r_k}$ it contains. There are also compatible filtrations $\ker q_n^{[k]}$, $\mathcal{E}(n)_*BP\left\langle n\right\rangle^{[k]}$ and $K(n)_*BP^{[k]}$ on $\ker q_n$, $\mathcal{E}(n)_*BP\left\langle n\right\rangle$ and $K(n)_*BP$. Notice that for $j \geqslant 0$, V_{n+j} has exact filtration $(p^{n+j}-1)$; more generally, the elements defined in Equations (2.3) satisfy

$$\kappa_{r_1,\ldots,r_k;s_1,\ldots,s_\ell} \in \mathcal{E}(n)_*BP^{[d]} \tag{2.5}$$

whenever

$$d \geqslant \sum_i r_i(p^i-1) + \sum_j s_j(p^{n+j}-1).$$

Proposition 2.2. *The elements*

$$\begin{cases} \kappa_{r_1,\ldots,r_k} & \text{for } 0 \leqslant r_j \leqslant p^n-1,\ r_k \neq 0, \\ \kappa_{r_1,\ldots,r_k;s_1,\ldots,s_\ell} & \text{for } 0 \leqslant r_j \leqslant p^n-1,\ r_k \neq 0,\ 0 \leqslant s_j,\ s_\ell \neq 0,\ \ell > 0, \end{cases} \tag{2.6}$$

form an $\mathcal{E}(n)_*$-basis for $\mathcal{E}(n)_*BP$.

Proof. Since
$$\mathcal{E}(n)_*BP = \bigcup_{j\geqslant 0} \mathcal{E}(n)_*BP^{[m]}$$
it suffices to show that for each $m \geqslant 0$, the κ elements specified in Equation (2.6) and also contained in $\mathcal{E}(n)_*BP^{[m]}$ actually form a basis for $\mathcal{E}(n)_*BP^{[m]}$.

$\mathcal{E}(n)_*BP^{[m]}$ has a natural basis consisting of all the t monomials $t_1^{r_1}\cdots t_k^{r_k}$ $(r_j \geqslant 0)$ it contains. Notice that the number of κ elements in $\mathcal{E}(n)_*BP^{[m]}$ is the same as the number of such monomials, hence is equal to the rank of $\mathcal{E}(n)_*BP^{[m]}$. Let $M(m)$ be the Gram matrix over $\mathcal{E}(n)_*$ expressing the κ elements in terms of the t monomial basis, with suitable orderings on these elements. It suffices to show that $M(m)$ is invertible, and for this we need to show that $\det M(m)$ is a unit in $\mathcal{E}(n)_*$. As $\mathcal{E}(n)_*$ is local, this is true if $\det M(m) \bmod I_n$ is a unit.

We have
$$\begin{aligned}\kappa_{r_1,\ldots,r_k;s_1,\ldots,s_\ell} &\equiv t_1^{r_1}\cdots t_k^{r_k}(t_1^{p^n} - v_n^{p-1}t_1)^{s_1}\cdots(t_\ell^{p^n} - v_n^{p^\ell-1}t_\ell)^{s_\ell} \bmod I_n \\ &\equiv t_1^{r_1+p^n s_1}\cdots t_\ell^{r_\ell+p^n s_\ell} + (\text{terms of lower filtration}) \bmod I_n. \end{aligned} \tag{2.7}$$
Working modulo I_n in terms of the basis of t monomials, the Gram matrix for the κ elements is lower triangular with all diagonal terms being 1, therefore $\det M(m) \equiv 1 \bmod I_n$. So $\det M(m)$ is a unit and $M(m)$ is invertible. Thus the κ elements of $\mathcal{E}(n)_*BP^{[m]}$ form a basis. □

Corollary 2.3. *The short exact sequence of $\mathcal{E}(n)_*$-modules*
$$0 \to \ker q_n \longrightarrow \mathcal{E}(n)_*BP \xrightarrow{q_n} \mathcal{E}(n)_*BP\langle n\rangle \to 0$$
splits so there is an isomorphism of $\mathcal{E}(n)_*$-modules
$$\mathcal{E}(n)_*BP \cong \ker q_n \oplus \mathcal{E}(n)_*BP\langle n\rangle.$$
Also, $\mathcal{E}(n)_*BP\langle n\rangle$ and $\ker q_n$ are free $\mathcal{E}(n)_*$-modules.

3. $\mathcal{E}(n)_*\mathcal{E}(n)$ as a limit

In this section we will give a description of $\mathcal{E}(n)_*\mathcal{E}(n)$ as a colimit. Although we proceed algebraically, we note that this limit has topological origins since for each $u \in BP\langle n\rangle_{2(p^n-1)r}$ with $r > 0$ and which is a unit in $\mathcal{E}(n)_*$, there is a cofibre sequence
$$\Sigma^{2(p^n-1)r}BP\langle n\rangle \xrightarrow{u} BP\langle n\rangle \longrightarrow BP\langle n-1;u\rangle$$
and $\mathcal{E}(n)$ is the telescope
$$\mathcal{E}(n) = \operatorname*{Tel}_u BP\langle n\rangle.$$
On applying the functor $\mathcal{E}(n)_*(\ )$, there is a short exact sequence
$$0 \to \mathcal{E}(n)_*BP\langle n\rangle \xrightarrow{U} \mathcal{E}(n)_*BP\langle n\rangle \longrightarrow \mathcal{E}(n)_*BP\langle n-1;u\rangle \to 0,$$
and limit
$$\mathcal{E}(n)_*\mathcal{E}(n) \cong \varinjlim_U \mathcal{E}(n)_*BP\langle n\rangle,$$
in which U denotes multiplication by the right unit on u. Since $u \equiv av_n^r \bmod I_n$ in the notation of Equation (1.1), application of the functor $K(n)_*(\ )$ induces another exact sequence and limit
$$0 \to K(n)_*BP\langle n\rangle \xrightarrow{U} K(n)_*BP\langle n\rangle \longrightarrow K(n)_*BP\langle n-1;u\rangle = 0,$$
$$K(n)_*\mathcal{E}(n) \cong \varinjlim_U K(n)_*BP\langle n\rangle.$$

There are also algebraic identities

$$\begin{aligned}
\mathcal{E}(n)_*\mathcal{E}(n) &\cong \mathcal{E}(n)_* \underset{BP_*}{\otimes} BP_*BP \underset{BP_*}{\otimes} \mathcal{E}(n)_*,\\
\mathcal{E}(n)_*BP\left<n\right> &\cong \mathcal{E}(n)_*BP/\ker q_n,\\
K(n)_*BP\left<n\right> &\cong K(n)_* \underset{\mathcal{E}(n)_*}{\otimes} \mathcal{E}(n)_*BP\left<n\right> \cong K(n)_* \underset{BP_*}{\otimes} BP_*BP\left<n\right>,
\end{aligned}$$

which allow us to work without direct reference to the underlying topology.

First we describe a directed system $(\Lambda, \preccurlyeq)$. Recall that $BP\left<n\right>_*$ is a graded unique factorization domain, with group of units $BP\left<n\right>_*^\times = \mathbb{Z}_{(p)}^\times$. Define the sets

$$\Lambda_r = \{(u) \lhd BP\left<n\right>_* : u \in BP\left<n\right>_{2(p^n-1)r},\ u \in \mathcal{E}(n)_* \text{ is a unit}\} \quad (r \geqslant 0), \quad \Lambda = \bigcup_{r\geqslant 0} \Lambda_r.$$

We will often abuse notation and identify (u) with a generator u; this can be made precise by specifying a choice function to select a generator of each such principal ideal. Of course, $(u) = (v)$ if and only if there is a unit $a \in \mathbb{Z}_{(p)}^\times$ for which $u = av$, i.e., if $u \mid v$ and $v \mid u$ in $BP\left<n\right>_*$. We will write $u \preccurlyeq v$ if $(v) \subseteq (u)$, i.e., if $u \mid v$. We will also write $u \prec v$ if $u \preccurlyeq v$ and $(u) \neq (v)$. The directed system $(\Lambda, \preccurlyeq)$ is filtered since for $u, v \in \Lambda$, $u \preccurlyeq uv$ and $v \preccurlyeq uv$.

Remark 3.1. For later use we will need a cofinal subset of Λ and we now describe some obvious examples. Since $BP\left<n\right>_*$ is a countable unique factorization domain, we may list the distinct *prime* ideals lying in Λ as $(w_1), (w_2), (w_3), \ldots$ say. Now inductively define

$$u_0 = 1, \quad u_k = u_{k-1}^k w_k.$$

Then $u_{k-1} \mid u_k$ and indeed $u_{k-1} \prec u_k$. Also, for every element $(u) \in \Lambda$ there is a k such that $u \mid u_k$, hence $u \preccurlyeq u_k$. So the u_k form a cofinal sequence in Λ.

Now form the directed system consisting of pairs of the form $(BP\left<n\right>_*, u)$ with $u \in \Lambda$. If $u, v \in \Lambda$, the morphism $(BP\left<n\right>_*, u) \longrightarrow (BP\left<n\right>_*, uv)$ is multiplication by v,

$$BP\left<n\right>_* \xrightarrow{v} BP\left<n\right>_*.$$

On setting $V = \eta_{\mathrm{R}}(v)$, there is also an homomorphism

$$\mathcal{E}(n)_*BP\left<n\right> \xrightarrow{V} \mathcal{E}(n)_*BP\left<n\right>.$$

These give rise to limits

$$\mathcal{E}(n)_* = \varinjlim_{u\in\Lambda} BP\left<n\right>_* = (BP\left<n\right>_*)_{I_n}, \tag{3.1}$$

$$\mathcal{E}(n)_*\mathcal{E}(n) = \varinjlim_{u\in\Lambda} \mathcal{E}(n)_*BP\left<n\right> = (\mathcal{E}(n)_*BP\left<n\right>)_{\eta_{\mathrm{R}} I_n}. \tag{3.2}$$

Remark 3.2. In describing $\mathcal{E}(n)_*\mathcal{E}(n)$ as a limit, it suffices to replace each map V by

$$\mathcal{E}(n)_*BP\left<n\right> \xrightarrow{v^{-1}V} \mathcal{E}(n)_*BP\left<n\right>,$$

which is of degree 0 and satisfies

$$v^{-1}V \equiv 1 \bmod I_n. \tag{3.3}$$

This will simplify the description of our basis. Notice that if $(v) = (w) \lhd BP\left<n\right>_*$, then

$$v^{-1}V = w^{-1}W,$$

providing another reason for using $v^{-1}V$ in place of V. From now on we will consider $\mathcal{E}(n)_*\mathcal{E}(n)$ as the limit over such maps $v^{-1}V$ rather than the limit of Equation (3.2).

4. Some bases for $\mathcal{E}(n)_*BP\langle n\rangle$ and $\mathcal{E}(n)_*\mathcal{E}(n)$

For each pair (u,s) with $u\in\Lambda_r$ and s a non-negative integer, set

$$M(u;s)_* = \mathcal{E}(n)_*BP\langle n\rangle^{[s+r(p^n-1)]}.$$

By Corollary 2.3, $M(u;s)_*$ is free on the images under q_n of the $\kappa_{r_1,\ldots,r_k}$ defined in Proposition 2.2 and we refer to this as the $q_n\kappa$-basis. There are inclusion maps

$$\mathrm{inc}\colon M(u;s)_* \longrightarrow M(u;s+1)_*.$$

For $v\in\Lambda_t$ and $V=\eta_{\mathrm{R}}(v)$, there is a multiplication by $v^{-1}V$ map

$$v^{-1}V\colon M(u;s)_* \longrightarrow M(uv;s)_*.$$

By Equation (2.2), $v^{-1}V$ raises filtration by $t(p^n-1)$. Equation (3.3) and Proposition 1.3 imply that $v^{-1}V$ is also injective; indeed we have the following result.

Proposition 4.1. *Let $s\geqslant 0$ and $u,v\in\Lambda$. The $\mathcal{E}(n)_*$-submodule*

$$v^{-1}VM(u;s)_* \subseteq M(uv;s)_*$$

is a summand. Furthermore, if $\mathcal{B}$ is a basis for $M(u;s)_*$ then $M(uv;s)_*$ has a basis consisting of the elements

$$v^{-1}Vb \quad (b\in\mathcal{B}), \quad \overline{\kappa}_{r_1,\ldots,r_k}\in M(uv;s)_* - v^{-1}VM(u;s)_*.$$

Proof. $M(u;s)_*$ and $M(uv;s)_*$ each have the $q_n\kappa$-bases. After reduction modulo I_n, the stated elements in $K(n)_*BP\langle n\rangle$ satisfy

$$v^{-1}Vb = b\in K(n)_*BP\langle n\rangle^{[d+s]}, \quad \overline{\kappa}_{r_1,\ldots,r_k}\in K(n)_*BP\langle n\rangle^{[d+h+s]} - K(n)_*BP\langle n\rangle^{[d+s]},$$

where u and v have exact filtrations d and h. These elements are clearly $K(n)_*$-linearly independent, so by Equation (3.3) and Proposition 1.3 they are $\mathcal{E}(n)_*$-linearly independent. Thus they form a basis, so the exact sequence

$$0\to M(u;s)_* \xrightarrow{v^{-1}V} M(uv;s)_* \longrightarrow M(uv;s)_*/v^{-1}VM(u;s)_* \to 0$$

splits and there is a direct sum decomposition

$$M(uv;s)_* = v^{-1}VM(u;s)_* \oplus M(uv;s)_*/v^{-1}VM(u;s)_*. \qquad \square$$

The $\mathcal{E}(n)_*$-linear maps $v^{-1}V$ and inc commute and together form a doubly directed system. Then we have

$$\begin{aligned}\mathcal{E}(n)_*\mathcal{E}(n) &= \varinjlim_{(u,s)} M(u;s)_* \\ &= \varinjlim_{u}\varinjlim_{s} M(u;s)_* \\ &= \varinjlim_{s}\varinjlim_{u} M(u;s)_*.\end{aligned}$$

Each $M(u;s)_*$ is a finitely generated free $\mathcal{E}(n)_*$-module, with a basis consisting of the $\overline{\kappa}$ elements it contains; we will refer to this as its $\overline{\kappa}$-basis. $M(u;s)_*$ also has another useful basis which we will now define.

Choose a cofinal sequence u_k in Λ, for example by the process described in Remark 3.1. For convenience we will assume that $u_0 = 1$. Of course

$$\begin{aligned}\mathcal{E}(n)_*\mathcal{E}(n) &= \varinjlim_{(r,s)} M(u_r;s)_* \\ &= \varinjlim_{r}\varinjlim_{s} M(u_r;s)_* \\ &= \varinjlim_{s}\varinjlim_{r} M(u_r;s)_*.\end{aligned}$$

When $r = 0$, we take the $\overline{\kappa}$-basis for $M(1;s)_*$, denoting its elements by $\overline{\kappa}^{1;s}_{r_1,\ldots,r_k}$. Now for $r \geqslant 1$, suppose that we have defined a basis $\overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_k}$ for $M(u_{r-1};s)_*$. For $M(u_r;s)_*$, replace each $\overline{\kappa}^{r-1;s}_{r_1,\ldots,r_k}$ of this basis by

$$\begin{aligned}\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k} &= w_r^{-1}W_r\overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_k} \\ &\equiv \overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_k} \bmod I_n\end{aligned} \tag{4.1}$$

whenever this element is also in $M(u_r;s)_*$. For $w_r^{-1}W_r\overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_k} \notin M(u_r;s)_*$, set

$$\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k} = \overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_k}. \tag{4.2}$$

Notice that by repeated applications of Equation (3.3), we have for all basis elements,

$$\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k} \equiv \overline{\kappa}_{r_1,\ldots,r_k} \bmod I_n. \tag{4.3}$$

Next we consider the effect of raising s by considering the extension

$$M(u_r;s)_* \subseteq M(u_r;s+1)_*.$$

Clearly $M(u_r;s+1)_*$ contains all the elements $\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k}$ together with its $\overline{\kappa}$-basis elements of exact filtration $d_r + s + 1$, where d_r is the exact filtration of u_r. Reducing modulo I_n these elements are $K(n)_*$-linearly independent, so by Equation (4.3) and Proposition 1.3 these are $\mathcal{E}(n)_*$-linearly independent and hence form a basis, showing that this extension splits. We have demonstrated the following.

Proposition 4.2. *For $r, s \geqslant 0$, the $\mathcal{E}(n)_*$-module $M(u_r;s)_*$ is free with the following two bases:*
- *$\mathcal{B}^{u_r;s}_1$ consisting of the elements $\overline{\kappa}_{r_1,\ldots,r_k}$ contained in $M(u_r;s)_*$;*
- *$\mathcal{B}^{u_r;s}_2$ consisting of the elements $\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k}$.*

Now we can state our main result.

Theorem 4.3. *$\mathcal{E}(n)_*\mathcal{E}(n)$ is $\mathcal{E}(n)_*$-free with a basis consisting of the images of the non-zero elements of the form*

$$\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k} \in M(u_r;s)_* - w_r^{-1}W_rM(u_{r-1};s)_* \quad (r,s \geqslant 0)$$

under the natural map $M(u_r;s)_* \longrightarrow \mathcal{E}(n)_*\mathcal{E}(n)$.

Proof. We begin by showing that these elements span $\mathcal{E}(n)_*\mathcal{E}(n)$. Let $z \in \mathcal{E}(n)_*\mathcal{E}(n)$ and suppose that t is the image of $z_r \in M(u_r;s)_*$ under the natural map

$$M(u_r;s)_* \longrightarrow \mathcal{E}(n)_*\mathcal{E}(n).$$

Then z_r can be uniquely expressed as an $\mathcal{E}(n)_*$-linear combination

$$z_r = \sum_{r_1,\ldots,r_k} \lambda_{r_1,\ldots,r_k}\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k}.$$

We can split up this sum as

$$z_r = \left(\sum_{r_1,\ldots,r_\ell} \lambda_{r_1,\ldots,r_\ell} \overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_\ell}\right) + w_r^{-1} W_r \left(\sum_{s_1,\ldots,s_k} \lambda_{s_1,\ldots,s_k} \overline{\kappa}^{u_{r-1};s}_{s_1,\ldots,s_k}\right).$$

Since

$$\sum_{r_1,\ldots,r_\ell} \lambda_{r_1,\ldots,r_\ell} \overline{\kappa}^{u_{r-1};s}_{r_1,\ldots,r_\ell} \in M(u_{r-1};s)_*, \qquad \sum_{s_1,\ldots,s_k} \lambda_{s_1,\ldots,s_k} \overline{\kappa}^{u_{r-1};s}_{s_1,\ldots,s_k} \in M(u_r;s)_*$$

map to linear combinations of the asserted basis elements in the images of $M(u_{r-1};s)_*$ and $M(u_{r-1};s)_*$ in $\mathcal{E}(n)_*\mathcal{E}(n)$, z is also a linear combination of those basis elements.

Now we show that these elements are linearly independent over $\mathcal{E}(n)_*\mathcal{E}(n)$. We know that $\mathcal{E}(n)_*\mathcal{E}(n)$ is $\mathcal{E}(n)_*$-flat, and also that

$$K(n)_* \underset{\mathcal{E}(n)_*}{\otimes} \mathcal{E}(n)_*\mathcal{E}(n) = K(n)_*\mathcal{E}(n)$$
$$(= K(n)_*K(n) \text{ in the standard but misleading notation})$$

which has a $K(n)_*$-basis consisting of the reductions of the elements

$$t_1^{r_1} \cdots t_k^{r_k} \quad (0 \leqslant r_j \leqslant p^n - 1).$$

Now $t_1^{r_1} \cdots t_k^{r_k}$ is the image of $\overline{\kappa}^{u_r;s}_{r_1,\ldots,r_k} \in M(u_r;s)$ under the natural map. Careful book keeping shows that the asserted basis elements do indeed account for all the t_j-monomials in this basis of $K(n)_*\mathcal{E}(n)$. These are linearly independent in $\mathcal{E}(n)_*\mathcal{E}(n)$ by Proposition 1.3. □

The following useful consequence of our construction is immediate on taking

$$\mathcal{E}(n)_*BP\langle n\rangle = \varinjlim_s M(1;s)_*.$$

Corollary 4.4. *The natural map*

$$\mathcal{E}(n)_*BP\langle n\rangle \longrightarrow \mathcal{E}(n)_*\mathcal{E}(n)$$

is a split monomorphism of $\mathcal{E}(n)_*$-modules.

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