ELLIPTIC COHOMOLOGY, p-ADIC MODULAR FORMS AND ATKIN'S OPERATOR U_p

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ABSTRACT *We construct a p-adic version of Elliptic Cohomology whose coefficient ring agrees with Serre's ring of p-adic modular forms. We then construct a stable operation* $\widehat{\mathrm{U}}_p$ *in this theory agreeing with Atkin's operator* U_p *on p-adic modular forms.*

Throughout the paper we assume given a fixed prime $p \geq 5$. We begin as in **[2]** by considering the *universal Weierstrass cubic* (for $\mathbf{Z}_{(p)}$ algebras) $\mathbf{Ell}/R_*$:

$$\mathbf{Ell}\colon Y^2 = 4X^3 - g_2X - g_3$$

where $R_* = \mathbf{Z}_{(p)}[g_2, g_3]$ is the graded ring for which $|g_n| = 4n$. We can also assign gradings 4, 6 to X, Y respectively. Now the *discriminant*

$$\Delta_{\mathbf{Ell}} = g_2^3 - 27g_3^2$$

is non-zero and hence **Ell** is an *elliptic curve* over R_*. Thus we can define an abelian group structure on **Ell** considered as a projective variety- see **[5]**, **[11]**. This has the unique point at infinity $\mathbf{O} = [0, 1, 0]$ as its zero. We can take the local parameter

$$T = -\frac{2X}{Y}$$

and then the group law on **Ell** yields a *formal group law* (commutative and 1 dimensional) $F^{\mathbf{Ell}}$ over R_*. This is explained in detail in for example **[11]**. Associated to this is an *invariant differential*

$$\omega_{\mathbf{Ell}} = \frac{dT}{\frac{\partial}{\partial Y}F^{E\ell\ell}(T, 0)} = \frac{dX}{Y}$$

which can also be written as

$$\omega_{\mathbf{Ell}} = d\log^{F^{E\ell\ell}}(T).$$

The formal group law $F^{E\ell\ell}$ is classified by a unique homomorphism $\varphi\colon L_* \longrightarrow R_*$ where L_* is Lazard's universal ring (given its natural grading). But topologists are aware that L_* is isomorphic to MU_*, the coefficient ring of complex (co)bordism $MU^*(\ )$, and moreover the natural orientation for complex line bundles in this theory has associated to it a universal formal group law F^{MU}. This is all explained in for example **[1]**. Thus we obtain a *genus*

$$\varphi_{E\ell\ell}\colon MU_* \longrightarrow R_*.$$

The ring R_* can be identified with a ring of *modular forms for* $SL_2(\mathbf{Z})$ *which are holomorphic at* $i\infty$ as explained in **[2]**. Under this identification we have

$$R_* \cong S(\mathbf{Z}_{(p)})_*$$

where

$$g_2 \longleftrightarrow \frac{1}{12}E_4 \quad \text{and} \quad g_3 \longleftrightarrow -\frac{1}{216}E_6$$

and E_{2n} denotes the weight $2n$ *Eisenstein function*. We will use this identification without further comment.

Now *Elliptic Cohomology* is usually defined by first localising R_* at the multiplicative set generated by $\Delta_{\mathbf{Ell}}$ which makes $R_*[\Delta_{\mathbf{Ell}}^{-1}]$ universal for elliptic curves. However, this is not necessary if we only worry about the formal group law (in fact such a Weierstrass cubic is always non-singular at $\mathbf{O}$). We define a functor on the category of finite CW complexes by

$$E\ell\ell[1]^*(\ ) = R_*({V_1}^{-1}) \otimes_{MU_*} MU^*(\ )$$

where $V_1 \in R_{2(p-1)}$ is the image of the Eisenstein function E_{p-1} in R_*.

‡ **We would like to thank the SERC for support whilst this research was carried out.**

(1) THEOREM. *The functor $E\ell\ell[1]^*(\ )$ is a multiplicative cohomology theory on $\mathbf{CW}^f$ which is complex oriented in the sense of* **[1]** *by a multiplicative natural transformation*

$$\overline{\varphi_{E\ell\ell}}: MU^*(\ ) \longrightarrow E\ell\ell[1]^*(\ )$$

extending $\varphi_{E\ell\ell}$.

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Proof: The proof is based on Landweber's Exact Functor Theorem together with the fact that E_{p-1} agrees modulo p with the leading term u in

$$[p]_{F^{E\ell\ell}}(T) \equiv uT^p \pmod{(p, T^{p+1})}.$$

This is well known—see **[4]** for example.

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We can view each group $E\ell\ell[1]_{2a}$ as a subgroup of $\mathbf{Z}_{(p)}[[q]]$, since

$$E_{p-1} = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{m\geq 1} \sigma_{2p-3}(m)q^m \in \mathbf{Z}_{(p)}[[q]].$$

We remark that, as is well known, $B_{p-1}/2(p-1)$ has p-adic valuation exactly -1.

Let A_* be a $\mathbf{Z}$ graded object in some category Γ (e.g. a group, or R module) and let $f_* = \{f_n : A_n \longrightarrow B\}_{n\in\mathbf{Z}}$ be a collection of morphisms in Γ into a fixed object B. Then there is a unique extension of the f_n to a morphism

$$f_{\text{Total}} : A_{\text{Total}} \longrightarrow B$$

where

$$A_{\text{Total}} = \bigoplus_{n\in\mathbf{Z}} A_n$$

(here we assume such direct sums exist in Γ). Now take the case $A_* = E\ell\ell[1]_*$ and for each $n \geq 1$ consider the group homomorphism

$$\rho_m : E\ell\ell[1]_{2m} \longrightarrow \mathbf{Z}/p^n[[q]]$$

obtained by reducing the canonical inclusion $E\ell\ell[1]_{2m} \longrightarrow \mathbf{Z}[[q]]$ modulo p^n. Then as above we have the canonical extension

$$\rho_{\text{Total}} : E\ell\ell[1]_{\text{Total}} \longrightarrow \mathbf{Z}/p^n[[q]].$$

Let $J(n)_{\text{Total}} = \ker \rho_{\text{Total}}$ and consider the quotient $E\ell\ell[1]_{\text{Total}}/J(n)_{\text{Total}}$. We have two Lemmas which give us insight into this quotient.

(2) LEMMA. *For each $n \geq 1$ and $\alpha \in E\ell\ell[1]_{2m}$ with $m \in \mathbf{Z}$ we have*

$$E_{p-1}^{p^{n-1}}\alpha - \alpha \in J(n)_{\text{Total}}.$$

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Proof: We have

$$E_{p-1} = 1 - \frac{2(p-1)}{B_{p-1}} \sum_{k\geq 1} \sigma_{2p-3}(k)q^k$$

and hence $E_{p-1} \equiv 1 \pmod{p}$. From this it is easy to deduce the result.

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(3) LEMMA. *Let $n \geq 1$, $\alpha \in E\ell\ell[1]_{2a}$, $\beta \in E\ell\ell[1]_{2b}$ and $\alpha - \beta \in J(n)_{\mathrm{Total}}$. Then*

$$a \equiv b \pmod{(p-1)p^{n-1}}.$$

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Proof: See **[10]**.

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Thus for each $n \geq 1$ we can recover a $\mathbf{Z}/2(p-1)p^{n-1}$ graded object $\big(E\ell\ell[1]/J(n)\big)_*$ with

$$\big(E\ell\ell[1]/J(n)\big)_{\bar{m}} = \mathrm{image}\ \big[j_n : E\ell\ell[1]_m \longrightarrow E\ell\ell[1]_{\mathrm{Total}}/J(n)_{\mathrm{Total}}\big]$$

for each $m \in \mathbf{Z}$ and where $\bar{m}$ denotes the residue class of $m \pmod{2(p-1)p^{n-1}}$. It is easy to see that there is a homomorphism of graded rings

$$E\ell\ell_* \longrightarrow \big(E\ell\ell[1]/J(n)\big)_*$$

extending the maps j_n and where the gradings are mapped as the natural projection $\mathbf{Z} \longrightarrow \mathbf{Z}/2(p-1)p^{n-1}$.

We can now form the inverse limit

$$\widehat{E\ell\ell[1]}_\bullet = \lim_{\substack{\longleftarrow \\ n}} \big(E\ell\ell[1]/J(n)\big)_*$$

which is a ring graded by

$$\lim_{\substack{\longleftarrow \\ n}}\ \mathbf{Z}/2(p-1)p^n \cong \mathbf{Z}/2(p-1) \times \mathbf{Z}_p$$

and complete with respect to the ideals $J(n)_*$ obtained by intersecting with $J(n)_{\mathrm{Total}}$; moreover the ring $\widehat{E\ell\ell[1]}_\bullet$ agrees with the ring of *p-adic modular forms* of **[10]**. The natural map $E\ell\ell[1]_* \longrightarrow \widehat{E\ell\ell[1]}_\bullet$ induces a genus

$$\widehat{\varphi_{E\ell\ell}} : MU_* \longrightarrow \widehat{E\ell\ell[1]}_\bullet\ .$$

(4) THEOREM. *The functor*

$$\widehat{E\ell\ell[1]}^\bullet(\ ) = \widehat{E\ell\ell[1]}_\bullet \otimes_{MU_*} MU^*(\ )$$

is a multiplicative $\mathbf{Z}/2(p-1) \times \mathbf{Z}_p$ graded cohomology theory on $\mathbf{CW}^f$, complex oriented by a multiplicative natural transformation

$$\widehat{\varphi_{E\ell\ell}} : MU^*(\ ) \longrightarrow \widehat{E\ell\ell[1]}^\bullet(\ )$$

extending $\varphi_{E\ell\ell}$.

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The proof of this result is exactly as for $E\ell\ell[1]^*(\ )$ since E_{p-1} is still a unit in $\widehat{E\ell\ell[1]}_\bullet$.

Now for any $\alpha \in \widehat{E\ell\ell[1]}_{\overline{2a}}$, we can find a sequence

$$\big(\alpha_m \in E\ell\ell[1]_{2a_m}\big)_{m \geq 1}$$

such that the sequence $(a_m)_{m \geq 1}$ is a p-adic Cauchy sequence in the sense that

$$\mathrm{ord}\ (a_{m+1} - a_m) \longrightarrow \infty$$

and we can further require that $a_m \longrightarrow \infty$. To see this, suppose that α is the limit of the sequence

$$\big(\gamma_m \in E\ell\ell[1]_{2c_m}\big)_{m \geq 1}$$

with $\gamma_{m+1} - \gamma_m \in J(m)_{\mathrm{Total}}$. Then

$$E_{p-1}^{p^{m-1}} - 1 \in J(m)_{\mathrm{Total}}$$

and so replacing γ_{m+1} by $\gamma_{m+1} E_{p-1}^{d_m p^{m-1}}$ if necessary, we can assume that $c_{m+1} > c_m$. Observe also tha such an α has a well defined *q-expansion*

$$\alpha(q) = \sum a_n q^n = \lim_{m \longrightarrow \infty} a_{m,n} q^n$$

where

$$\alpha_m(q) = \sum a_{m,n} q^n$$

and $\alpha_m \longrightarrow \alpha$.

We can now define $\widehat{\mathrm{U}}_p : \widehat{E\ell\ell[1]}^\bullet(\ ) \longrightarrow \widehat{E\ell\ell[1]}^\bullet(\ )$. First recall a basic fact from **[10]**.

(5) PROPOSITION. *Let $\alpha \in E\ell\ell[1]_*$ be a modular form and let its q-expansion be*

$$\alpha(q) = \sum a_n q^n.$$

Then

$$(\mathrm{U}_p\alpha)(q) = \sum a_{np} q^n$$

is the q-expansion of a p-adic modular form.

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Proof: Let $\alpha = \lim_{m\longrightarrow\infty} \alpha_m$ with $\alpha_m \in E\ell\ell[1]_{2a_m}$ and $a_m \longrightarrow \infty$. Then eventually we have $a_m > 0$ and so

$$\mathrm{T}_p\alpha_m \in E\ell\ell[1]_{2a_m}$$

exists and has q-expansion

$$\begin{aligned}(\mathrm{T}_p\alpha_m)(q) &= \sum a_{m,np} q^n + p^{a_m-1} \sum a_{m,n} q^{np} \\ &\longrightarrow \sum a_{np} q^n \qquad \text{as } m \longrightarrow \infty.\end{aligned}$$

Hence, we see that in fact

$$\mathrm{U}_p\alpha = \lim_{m\longrightarrow\infty} \mathrm{T}_p\alpha_m.$$

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To define $\widehat{\mathrm{U}}_p$ we mimic the construction of **[2]**. Let

$$\tau \in \mathcal{H} = \{\tau \in \mathbf{C} : \mathrm{im}\ \tau > 0\}$$

and let $L_\tau = < 1, \tau > \subset \mathbf{C}$ be the lattice generated by τ. Consider a lattice L' containing L_τ with index $[L', L] = p$ and also *not* containing $1/p$. Putting $L' = < 1, \tau' >$, we can assume that

$$\tau' = \frac{(j + \tau)}{p}$$

for some j in the range $1 \leq j \leq p$. Now for each such j we have a homomorphism

$$h_j : MU_* \longrightarrow \mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]]$$

defined as the composition obtained from the genus

$$\varphi_{\mathbf{Ell}(q)} : MU_* \xrightarrow{\varphi_{E\ell\ell}} E\ell\ell[1]_* \hookrightarrow \mathbf{Z}_{(p)}[[q]]$$

followed by the homomorphism

$$\theta_j : \mathbf{Z}_{(p)}[[q]] \longrightarrow \mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]];\ q \longmapsto {\zeta_p}^j q^{1/p}$$

in which $\zeta_p = e^{2\pi \mathrm{i}/p}$ and $q^{1/p} = e^{2\pi\mathrm{i}\tau/p}$. As explained in **[2]**, the theory of *Tate curves* shows that there is a strict isomorphism of formal group laws over the ring $\mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]]$,

$$F^{\mathbf{Ell}(q)} \xrightarrow{\cong} F^{({\zeta_p}^j q^{1/p})}$$

where $F^{\mathbf{Ell}(q)}$ is the formal group law induced by $\varphi_{\mathbf{Ell}(q)}$.

Now it is well known that the topologically defined ring MU_*MU is a *Hopf algebroid* which classifies strict isomorphisms of formal group laws—see **[7]** for example. There is thus an extension of each homomorphism h_j to a ring homomorphism

$$H_j : MU_*MU \longrightarrow \mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]]$$

which in turn extends to

$$\theta_j \otimes H_j : E\ell\ell[1]_* \otimes_{MU_*} MU_*MU \longrightarrow \mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]].$$

We remark that this is a right MU_* module map using the genus $\varphi_{\mathbf{Ell}(q)}$ to define the module structure.

It is now easily seen that this construction passes to

$$\widehat{H}_j : \widehat{E\ell\ell[1]}_\bullet \otimes_{MU_*} MU_*MU \longrightarrow \mathbf{Z}_{(p)}(\zeta_p)[[q^{1/p}]].$$

We now define the function

$$\widehat{H} = \frac{1}{p} \sum_{1 \le j \le p} \widehat{H}_j.$$

The image of an element under $\widehat{H}$ is invariant under $\zeta_p \mapsto {\zeta_p}^j$ for $p \not| j$ and is in $\mathbf{Z}_p[[q]]$. Indeed if $\alpha \in \widehat{E\ell\ell[1]}_\bullet$ has expansion $\alpha(q) = \sum a_n q^n$ then

$$\widehat{H}(\alpha)(q) = \sum a_{np} q^n.$$

Hence we can define $\widehat{\mathrm{U}}_p$ to be the natural transformation

$$\widehat{H} \otimes \mathrm{Id} : \widehat{E\ell\ell[1]}_\bullet \otimes_{MU_*} MU^*(\ ) \longrightarrow \widehat{E\ell\ell[1]}_\bullet \otimes_{MU_*} MU^*(\ )$$

which agrees with U_p on the coefficient ring $\widehat{E\ell\ell[1]}_\bullet$.

(6) THEOREM. *There is a degree 0 stable operation*

$$\widehat{\mathrm{U}}_p : \widehat{E\ell\ell[1]}^\bullet(\ ) \longrightarrow \widehat{E\ell\ell[1]}^\bullet(\ )$$

agreeing with U_p *on the coefficient ring* $\widehat{E\ell\ell[1]}_\bullet$.

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We can also construct a operation $\widehat{\mathrm{V}}_p$ agreeing with the operator V_p of **[9]** on the coefficients $\widehat{E\ell\ell[1]}_\bullet$. Here the effect of V_p on a q-expansion $\sum a_n q^n$ is given by

$$\mathrm{V}_p\big(\sum a_n q^n\big) = \sum a_n q^{np}$$

and V_p is *multiplicative* on $\widehat{E\ell\ell[1]}_\bullet$. To construct $\widehat{\mathrm{V}}_p$ we use the lattice $< 1/p, \tau >$ containing $< 1, \tau >$ with index p and its scaling $< 1, p\tau >$. Notice that on the rational ring $\widehat{E\ell\ell[1]}_\bullet \otimes \mathbf{Q}$ we have the identity

$$\mathrm{T}_p\big(\sum a_n q^n\big) = \mathrm{U}_p\big(\sum a_n q^n\big) + p^{k-1}\mathrm{V}_p\big(\sum a_n q^n\big)$$

if $\sum a_n q^n$ is a modular form of weight k.

(7) THEOREM. *There is a degree 0 multiplicative stable operation*

$$\widehat{\mathrm{V}}_p : \widehat{E\ell\ell[1]}^\bullet(\ ) \longrightarrow \widehat{E\ell\ell[1]}^\bullet(\ )$$

agreeing with V_p *on the coefficient ring* $\widehat{E\ell\ell[1]}_\bullet$.

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Although U_p is not a multiplicative operation its *image* is a subring. This follows from the calculation

$$\begin{aligned} \mathrm{U}_p(\sum a_m q^m)\mathrm{U}_p(\sum b_n q^n) &= \mathrm{U}_p\mathrm{V}_p\big(\mathrm{U}_p(\sum a_m q^m)\mathrm{U}_p(\sum b_n q^n)\big) \\ &= \mathrm{U}_p\big(\mathrm{V}_p\mathrm{U}_p(\sum a_m q^m)\mathrm{V}_p\mathrm{U}_p(\sum b_n q^n)\big) \end{aligned}.$$

This remains true on replacing U_p by U_p^N for $N \geq 1$ and the same is true for $\widehat{\mathrm{U}}_p$. Hence the limit (in an appropriate p-adic sense)

$$\mathrm{U}_p^\infty = \lim_{n \longrightarrow \infty} \mathrm{U}_p^N$$

is a subring. On p-adic Elliptic Cohomology the limit

$$\widehat{\mathrm{U}}_p^\infty = \lim_{n \longrightarrow \infty} \widehat{\mathrm{U}}_p^N$$

appears to give an interesting summand theory. It is known for example that U_p is a *contraction operator* on whose image U_p acts bijectively. This summand may be worth further study, especially if it can be shown to exist without first inverting E_{p-1}.

We end with some remarks on the significance of our results for the algebraic topology of Elliptic Cohomology.

a) The p-adic theory we have defined is closely related to K- theory. Indeed it can be constructed by taking the Moore spectra $M(p^n) = S^0 \cup_{p^n} e^1$, considering them as forming an inverse system under the reduction maps $M(p^{n+1}) \longrightarrow M(p^n)$ and then forming the theory

$$\lim_{n \longrightarrow \infty} \left(E\ell\ell[1] \wedge M(p^n)\right)^*(\ )$$

where $E\ell\ell$ denotes a spectrum representing $E\ell\ell^*(\ )$. We could also replace the Moore spectra by their K-localisations $L_K M(p^n)$ and use the non-periodic version of Elliptic Cohomology with coefficient ring isomorphic to the ring of holomorphic modular forms. Either way we would get our theory $\widehat{E\ell\ell[1]}^\bullet(\ )$. We could also use the theory $E\ell\ell^*(\ )$ of **[2]** to get a doubly periodic theory.

b) The theory of modular forms with both p and E_{p-1} killed is considered for example in **[8]**. This seems a worthwhile area of study since it is much more likely that genuinely v_2-periodic phenomena will be found than in the situation with $E_{p-1} \approx v_1$ inverted. We consider this *supersingular* case in **[3]**.

c) It would be interesting to construct other operations in $E\ell\ell^*(\ )$, for example the ∂ operator of **[10]** (which is a derivation on $E\ell\ell_*$ and *increases* weight by 2) may very well be the restriction of an operation. In particular $\partial(\Delta^N) = 0$ and hence respects the periodicity.

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