Level 5M Project: Representations of Quivers & Gabriel's Theorem

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Abstract

Gabriel's theorem, first proved by Peter Gabriel in 1972 [1], comes in two parts. Part (i) states that "*a quiver Q is of finite orbit type if and only if each component of its underlying undirected graph $\hat{Q}$ is a simply–laced Dynkin diagram*" and part (ii) states "*let Q be a quiver such that $\hat{Q}$ is a simply–laced Dynkin diagram; then $\underline{n}$ is the dimension of a (unique) indecomposable representation of Q if and only if $\underline{n} \in \Phi^+$*" [2]. This surprising theorem gives a deceptively simply result which links isomorphism classes of representations of a given quiver with an assigned dimension vector, with the root system of the geometric object underlying such a quiver. This project introduces quivers and representation theory to the reader, with the intention of heading towards proving Gabriel's theorem. This is the main result in the project, and so a proof of both parts is provided. On the way, the reader is introduced to quivers, the category of quiver representations and its equivalence with the category of $\mathbb{K}Q$–modules, Dynkin diagrams, root systems and the Weyl group and the Coxeter functors.

Contents

1 Introduction

A quiver Q is a graph with a finite set of vertices Q_0 and a finite set of directed edges Q_1. A representation V of a quiver Q assigns a vector space to each vertex of Q and a linear map to each directed edge of Q. Quivers and their representations are relatively simply objects, but they play a vital role in the representation theory of finite–dimensional algebras [3]. Quivers also have an interesting link with root systems, and this link is explored in Section 4 where we introduce Gabriel's theorem to the reader.

There are two parts Gabriel's theorem. Part (i) states that "*a quiver Q is of finite orbit type if and only if each component of its underlying undirected graph $\hat{Q}$ is a simply–laced Dynkin diagram*" and part (ii) states "*let Q be a quiver such that $\hat{Q}$ is a simply–laced Dynkin diagram; then $\underline{n}$ is the dimension of a (unique) indecomposable representation of Q if and only if $\underline{n} \in \Phi^+$*", where Φ^+ is the set of all positive roots of a root system Φ [2]. A quiver of finite orbit type has finitely many isomorphism classes of indecomposable representations for an assigned dimension vector, and Gabriel's theorem tells us that such quivers are exactly the Dynkin diagrams A_n, D_n, E_6, E_7, E_8, regardless of the orientation of the arrows in the original quiver Q. Gabriel's theorem also gives us that the indecomposable representations of a quiver Q are in one–to–one correspondence with the positive roots in the root system associated with the Dynkin diagram underlying Q. Gabriel's theorem holds over an arbitrary field [3], but in this project we will assume that we are working over algebraically closed fields.

The first section gives a more algebraic approach to quivers and their representations, and introduces category theory to the reader. The categories of finite dimensional quiver representations and finite dimensional $\mathbb{K}Q$–modules are introduced, and their equivalence proven. Section 3 then gives a more geometric interpretation of quiver representations and introduces the reader to the representation space $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ of a quiver Q for an assigned dimension vector $\underline{n}$ and the group action of $\text{GL}(\underline{n}) := \prod_{x \in Q_0} \text{GL}(n_x)$ on $\text{rep}_{\mathbb{K}}(Q, \underline{n})$. Section 4 is dedicated to proving both parts of Gabriel's theorem. In order to prove part (i), it is necessary to understand the link between Dynkin diagrams and the Tits form of a quiver, and so this is covered in detail using [4]. The proof of part (ii) requires the use of the root systems, the Weyl group and the Coxeter functors and so these are also defined in detail.

Since this project is intended for recently graduated Mathematics students, no prior knowledge of representation theory or category theory is assumed, but readers are required to have previous experience of linear algebra, some group theory and algebraic geometry. Standard notation will be used throughout this paper - any notation presumed to be new to the reader will be specified when introduced.

2 Quiver Representations and Categories

2.1 Quivers and Representations

2.1.1 Introduction to Quivers

This first section covers the main definitions that will be used in the project. The following definitions have been adapted from [2] and [3].

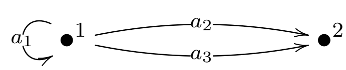
Definition 2.1. A (*finite*) *quiver* $Q = (Q_0, Q_1)$ is a directed graph where $Q_0 = \{1, 2, \cdots, t\}$ ($t \in \mathbb{N}$) is the set of vertices and $Q_1 = \{a_1, a_2, \cdots, a_r\}$ ($r \in \mathbb{N}$) is the set of directed edges where each a_i is an arrow between two vertices. The maps

$$h\colon Q_1 \to Q_0 \quad , \quad t\colon Q_1 \to Q_0$$

send each directed edge to its head and tail, respectively. Quivers allow *loops* (arrows for which $t(a) = h(a)$) and multiple arrows (arrows a_1, a_2 such that $h(a_1) = h(a_2)$ and $t(a_1) = t(a_2)$) between vertices.

A quiver $Q = (Q_0, Q_1)$ is called *connected* if the underlying graph $\hat{Q}$ is connected where $\hat{Q}$ is the graph with vertices Q_0 and edges Q_1, without orientation. In this project, we assume that all quivers are connected.

Example 2.2. Here is an example of a quiver Q where $Q_0 = \{1, 2\}$ and $Q_1 = \{a_1, a_2, a_3\}$:



Definition 2.3. Let Q be a quiver. A *representation* V of Q is a direct sum of finite dimensional $\mathbb{K}$–vector spaces

$$\bigoplus_{x \in Q_0} V_x$$

together with a set of $\mathbb{K}$–linear maps

$$\{v_a\colon V_{t(a)} \to V_{h(a)} \mid a \in Q_1\}.$$

We can assign to V a *dimension vector* $\underline{n} := (n_x)_{x \in Q_0}$, where each $n_x \in \mathbb{N}$ is the dimension of the vector space V_x for each $x \in Q_0$. We write such a representation

$$V := \left(\bigoplus_{x \in Q_0} V_x \ , \ \{v_a\}_{a \in Q_1} \right).$$

Definition 2.4. Let V, W be representations of a quiver Q. A *morphism* $\phi\colon V \to W$ is a collection of $\mathbb{K}$–linear maps

$$\{\phi_x\colon V_x \to W_x \mid x \in Q_0\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{v_a} & V_{h(a)} \\ \downarrow{\scriptstyle \phi_{t(a)}} & & \downarrow{\scriptstyle \phi_{h(a)}} \\ W_{t(a)} & \xrightarrow[w_a]{} & W_{h(a)} \end{array}$$

for all $a \in Q_1$. We say that ϕ is an *isomorphism* if and only if ϕ_x is invertible for all $x \in Q_0$. If $\phi \colon V \to W$ is an isomorphism, then we say that V,W are *isomorphic*, which is denoted $V \cong W$.

2.1.2 The Kroenecker Quiver

Example 2.5. The *r–arrow Kroenecker quiver* K_r is the quiver with set of vertices $K_{r_0} = \{1, 2\}$ and set of arrows $K_{r_1} = \{a_1, \ldots, a_r\}$ where each a_i has $t(a_i) = 1$ and $h(a_i) = 2$ for all $i = 1, \ldots, r$ (see Fig. 1) [3]. The representations of K_r are of the form

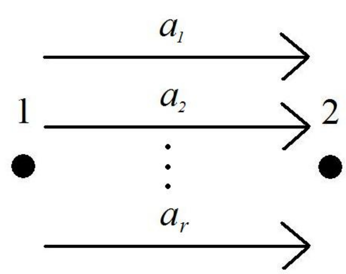


Figure 1: The Kroenecker Quiver K_r

$$V_{K_r} = \big(V_1 \oplus V_2\ ,\ \{v_{a_i}\}_{a \in K_{r_1}}\big)$$

where V_1 and V_2 are the vector spaces corresponding to the vertices 1 and 2, respectively, and $\{v_{a_i}\}$ is the set of linear maps $v_{a_i} : V_1 \to V_2$ for all $i = 1, \ldots, r$. Since the set K_{r_0} consists of just two vertices, then the dimension vectors assigned to the representations of K_r are pairs of non–negative integers $(m, n) := (\dim(V_1), \dim(V_2))$. We may choose bases $(v_1, \ldots, v_m)$ of V_1 and $(w_1, \ldots, w_n)$ of V_2 which correspond to an $m \times m$ matrix A and an $n \times n$ matrix B for each basis, respectively. Since changing basis corresponds to replacing each matrix A, B by a conjugate CAC^{-1}, DBD^{-1}, respectively, where C is an invertible $m \times m$ matrix and D is an invertible $n \times n$ matrix, then it follows from Definition 2.4 that the isomorphism classes of the representations of K_r with dimension vector (m, n) correspond bijectively to the r–tuples of $m \times n$ matrices (up to simultaneous multiplications on the left by invertible $n \times n$ matrices and on the right by invertible $m \times m$ matrices).

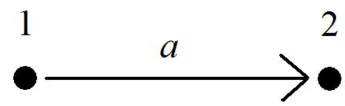


Figure 2: K_1

Example 2.6. We will now consider the simplest case of the r–arrow Kroenecker quiver - namely, when $r = 1$. When this is the case, we have the quiver K_1 illustrated in Fig. 2.

A representation of K_1 is of the form $V_{K_1} = (V_1 \oplus V_2\ ,\ v_a\colon V_1 \to V_2)$. We can represent V_{K_1} on a diagram:

$$V_1 \bullet \xrightarrow{\quad v_a \quad} \bullet V_2.$$

If $\dim(V_1) = n$, $\dim(V_2) = m$ then $V_1 \cong \mathbb{K}^n, V_2 \cong \mathbb{K}^m$ and so we can define the map $v_a\colon \mathbb{K}^n \to \mathbb{K}^m$ by

$$v_a(\underline{v}) = M \cdot \underline{v}$$

where M is an $m \times n$ matrix.

Definition 2.7. Let V be a representation of a quiver Q. A *subrepresentation* V' of V is a representation such that the following two conditions hold:

(i) there exists an inclusion $\iota_x\colon V'_x \to V_x$ for all vertices in V';

(ii) the following diagram commutes for all $a \in Q_1$:

$$\begin{array}{ccc} V'_{t(a)} & \xrightarrow{v'_a} & V'_{h(a)} \\ {\scriptstyle \iota_{t(a)}}\downarrow & & \downarrow{\scriptstyle \iota_{h(a)}} \\ V_{t(a)} & \xrightarrow[v_a]{} & V_{h(a)} \end{array}$$

that is, the maps v'_a are the linear maps of the representation V', and they must be the restriction of v_a to $V'_{t(a)}$.

Example 2.8. Let Q be the quiver illustrated in Fig. 3. Let V be the representation of Q which is defined by the vector space $\mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^2$ together with the set of $\mathbb{K}$–linear maps $v_{a_1}, v_{a_2}, v_{a_3}, v_{a_4}\colon \mathbb{R}^2 \to \mathbb{R}^2$ where each v_{a_i} is the identity map on $\mathbb{R}^2$. Let V' be a representation of the quiver Q' which has vertices $Q'_0 = \{2, 3, 4\}$ and arrows $Q'_1 = \{a_2, a_4\}$ (this is achieved by simply removing the vertex 1 from the quiver in Fig. 3 and the corresponding arrows a_1, a_3 which both have $t(a_i) = 1$). Define V' to be the vector space $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ together with the inclusion map $\iota\colon \mathbb{R} \to \mathbb{R}^2$ for each v'_{a_i}. Then V' is a subrepresentation of V.

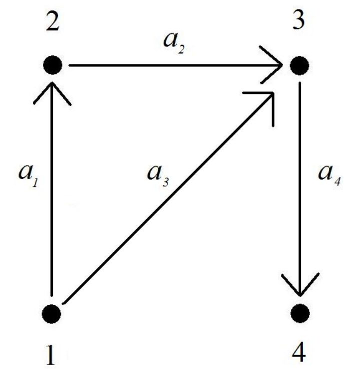


Figure 3: Quiver with $Q_0 = \{1, 2, 3, 4\}$ and $Q_1 = \{a_1, a_2, a_3, a_4\}$.

2.2 Path Algebras

The majority of the following definitions are from [2], but with some input from [3], [5] and [6].

Definition 2.9. A *path* p in a quiver Q is a sequence $a_1 a_2 \ldots a_n$ (composing left to right) of arrows such that $h(a_i) = t(a_{i+1})$ for all $i = 1, 2, \ldots, n-1$. We define the head of the path p to be $h(p) := h(a_n)$ and the tail of the path to be $t(p) := t(a_1)$. So we have

$$t(p) \xrightarrow{a_1} h(a_1) \xrightarrow{a_2} h(a_2) \xrightarrow{a_3} \cdots \xrightarrow{a_n} h(p).$$

We define the *length*, length(p), of a path p to be the number of directed edges in the sequence defining it. Fix $x \in Q_0$. The *trivial path* e_x is defined to be the path where $h(e_x) = t(e_x) = x$. Notice that there is a trivial path associated with every vertex $x \in Q_0$.

Definition 2.10. A path p of length greater than or equal to 1 is called a *cycle* if and only if $t(p) = h(p)$. A path of this kind with length 1 is called a *loop*. A quiver Q is called *acyclic* if it contains no cycles (and therefore no loops).

Example 2.11. Let Q be the quiver illustrated in Fig. 3. Then a simple example of a path in Q is $p = a_1 a_2 a_4$, where $t(p) = 1$ and $h(p) = 4$. This is a path of length 3. This quiver is acyclic since it contains no cycles.

Definition 2.12. Let $\mathbb{K}$ be a field. A $\mathbb{K}$–*algebra* is a ring A together with an identity element, denoted 1_A or just 1, such that A has a $\mathbb{K}$–vector space structure compatible with the multiplication of the ring, that is, such that

$$\lambda(ab) = (a\lambda)b = a(\lambda b) = (ab)\lambda$$

for all $\lambda \in \mathbb{K}$ and $a, b \in A$.

Note that sometimes a $\mathbb{K}$–algebra is described as a ring A together with a mapping $\mathbb{K} \times A \to A$. This mapping is a ring morphism which again is required to be compatible with the multiplication of the ring.

Definition 2.13. Let $\mathbb{K}$ be a field and Q be a quiver. A *path algebra* $\mathbb{K}Q$ is the $\mathbb{K}$–vector space which has as its basis the set of all paths in Q. Multiplication in $\mathbb{K}Q$ is defined to be

$$p \cdot q = \begin{cases} pq & \text{if } h(p) = t(q) \\ 0 & \text{otherwise} \end{cases}$$

where $p, q \in \mathbb{K}Q$.

To show that multiplication in $\mathbb{K}Q$ is associative, let $p, q, r \in \mathbb{K}Q$. We have $p \cdot (q \cdot r) = (p \cdot q) \cdot r$ since

$$p \cdot (q \cdot r) = \begin{cases} p \cdot qr & \text{if } h(q) = t(r) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} pqr & \text{if } h(p) = t(qr) = t(r) \\ 0 & \text{otherwise} \end{cases}$$

and

$$(p \cdot q) \cdot r = \begin{cases} pq \cdot r & \text{if } h(p) = t(q) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} pqr & \text{if } h(p) = t(pq) = t(r) \\ 0 & \text{otherwise} \end{cases}.$$

We can decompose the path algebra $\mathbb{K}Q$ into a direct sum

$$\mathbb{K}Q = \mathbb{K}Q_0 \oplus \mathbb{K}Q_1 \oplus \mathbb{K}Q_2 \oplus \cdots \oplus \mathbb{K}Q_l \oplus \cdots$$

where for each $l \geq 0$, $\mathbb{K}Q_l$ is the subspace of $\mathbb{K}Q$ generated by the set of paths of length l.

Let p, q be paths of length $m, n \geq 0$, respectively. Then either $p \cdot q = 0$ or $p \cdot q = pq$ by the definition of multiplication in $\mathbb{K}Q$. In the former case, the length of the path $p \cdot q$ is 0, and in the latter case the length of $p \cdot q$ is $\text{length}(p) + \text{length}(q) = m + n$. In either case we have that $p \cdot q \in \mathbb{K}Q_{m+n}$ and so

$$(\mathbb{K}Q_m) \cdot (\mathbb{K}Q_n) \subseteq \mathbb{K}Q_{m+n}$$

for all $m, n \geq 0$.

Definition 2.14. The decomposition above defines a *grading* on $\mathbb{K}Q$, and so we say that $\mathbb{K}Q$ is a *$\mathbb{Z}$–graded $\mathbb{K}$–algebra.*

2.3 Introduction to Category Theory

Category theory is a mathematical language for describing a variety of different mathematical structures. It enables us to study these different structures by considering a more general abstract notion. Category theory also makes it possible to translate problems in one area of mathematics to another. We first begin by giving a formal definition of a category. The definitions in this section are from [7].

Definition 2.15. A *category* $\mathcal{C}$ consists of a class $\mathrm{Ob}(\mathcal{C})$ of *objects* and a set $\mathrm{Hom}_{\mathcal{C}}(A,B)$ of *morphisms* $A \to B$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ such that the following conditions are satisfied:

(i) for each $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, there exists a morphism

$$g \circ f \in \mathrm{Hom}_{\mathcal{C}}(A,C)$$

called the *composite morphism* of f and g which has the property that if we also have $h \in \mathrm{Hom}_{\mathcal{C}}(C,D)$ then

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(ii) for each $A \in \mathrm{Ob}(\mathcal{C})$, there exists an element $1_A \in \mathrm{Hom}_{\mathcal{C}}(A,A)$ called the *identity morphism* on A which has the property that if $f \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ then

$$f \circ 1_A = f = 1_B \circ f.$$

When the category $\mathcal{C}$ is known, $\mathrm{Hom}_{\mathcal{C}}(A,B)$ is often denoted $\mathrm{Hom}(A,B)$ for ease of notation.

Note 2.16. It should be noted here that it is important that we have a *class* of objects rather than a set to avoid Russell's Paradox which states that if "R is the set of all sets which are not members of themselves, then R is neither a member of itself nor not a member of itself" [8]. This means that you cannot have the set of all sets. It is this, and similar paradoxes, which forces us to introduce the more formal notion of a class of objects. However, the distinction between a class of objects and a set of objects is beyond the scope of this project.

Example 2.17. There are two important categories that we will be dealing with in Section 2.4. They are as follows:

(i) The category $\mathrm{rep}_{\mathbb{K}}(Q)$ of finite dimensional quiver representations of a quiver Q over a field $\mathbb{K}$. The objects are the quiver representations and the morphisms are the morphisms of quiver representations as defined in Definition 2.4 (see Section 2.1).

(ii) The category $\mathbb{K}Q$–mod of finite dimensional $\mathbb{K}Q$–modules. Here, the objects are modules over the path algebra $\mathbb{K}Q$ and the morphisms are simply $\mathbb{K}Q$–module homomorphisms (see Section 2.2).

We can now define a functor, which is a map between categories. Functors have the property that they preserve objects and morphisms.

Definition 2.18. A *covariant functor* $F\colon \mathcal{A} \to \mathcal{B}$ from a category $\mathcal{A}$ to a category $\mathcal{B}$ is an assignment of objects

$$F\colon \mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$$

and a morphism

$$F(m)\colon F(A_1) \to F(A_2)$$

for each morphism $m\colon A_1 \to A_2$ in $\mathcal{A}$. We require the following conditions to hold:

(i) F preserves identity morphisms, ie. for $A \in \mathcal{A}$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$;

(ii) F preserves composition, ie. $F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$ where m_1, m_2 are morphisms in $\mathcal{A}$.

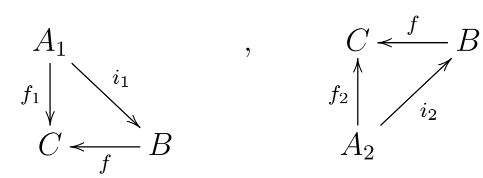
A *contravariant functor* simply reverses the order of the morphisms, so (ii) would become $F(m_1 \circ m_2) = F(m_2) \circ F(m_1)$.

We say that two categories $\mathcal{A}$, $\mathcal{B}$ are *equivalent* if there exists an equivalence between them, that is, if $F\colon \mathcal{A} \to \mathcal{B}$ is a functor between the two categories $\mathcal{A}$, $\mathcal{B}$ then there exists a functor $G\colon \mathcal{B} \to \mathcal{A}$ such that

$$G \circ F \cong \mathrm{id}_{\mathcal{A}} \quad , \quad F \circ G \cong \mathrm{id}_{\mathcal{B}}.$$

For the purposes of this project, we do not need to define this equivalence since the two functors constructed in the categorical equivalence in Section 2.4 are in fact direct inverses to each other, that is, their composition is actually equal to the identity of each category rather than just equivalent to the identity.

Definition 2.19. Let A_1, A_2 be objects in a category $\mathcal{A}$. An object $\mathcal{B} \in \mathrm{Ob}(\mathcal{A})$ together with morphisms $i_1\colon A_1 \to B$, $i_2\colon A_2 \to B$ is called the *coproduct* (or *direct sum*) of A_1 and A_2 if the following two diagrams commute



for all $C \in \mathrm{Ob}(\mathcal{A})$. That is, for every object $C \in \mathrm{Ob}(\mathcal{A})$ and for every two morphisms $f_1\colon A_1 \to C$, $f_2\colon A_2 \to C$, there exists a unique morphism $f\colon B \to C$ such that

$$f_1 = f \circ i_1 \quad , \quad f_2 = f \circ i_2.$$

When this is the case, we write $B = A_1 \oplus A_2$ and this is unique (up to isomorphism). If every two objects in a category $\mathcal{C}$ all have a coproduct, then we say that $\mathcal{C}$ has (*finite*) *coproducts*.

Note 2.20. Note that you cannot "add" in categories, so to say that $A_1 \to B$ where $B = A_1 \oplus A_2$ actually means $a \mapsto (a, 0)$. $A_1 \oplus A_2$ refers to pairs of elements (a_1, a_2) where $a_1 \in A_1$ and $a_2 \in A_2$.

Example 2.21. Let $\mathcal{C} = R$–mod, that is, let $\mathcal{C}$ be the category of modules over a ring R, and let $A_1, A_2 \in \mathrm{Ob}(\mathcal{C})$. Assume that A_1, A_2 are contained in an R–module M and that $A_1 \cap A_2 = \{0\}$. Then define $B := A_1 + A_2$. Since their intersection is trivial, then in fact $B = A_1 \oplus A_2$. Define maps $A_1 \to B$ and $A_2 \to B$ by

$$a_1 \mapsto (a_1, 0)$$

$$a_2 \mapsto (0, a_2)$$

respectively. Now, for any $C \in \mathrm{Ob}(\mathcal{C})$, we have morphisms $f_1\colon A_1 \to C$ where $f_1(a_1)$ makes sense and $f_2\colon A_2 \to C$ where $f_2(a_2)$ makes sense. Define $f := (f_1, f_2)$, where f is a map $B = A_1 \oplus A_2 \to C$, by

$$(a_1, a_2) \mapsto (f_1(a_1), f_2(a_2)).$$

Definition 2.22. A category $\mathcal{C}$ is called *additive* if

(i) it has coproducts (see Definition 2.19);

(ii) for every $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ we have

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in A$, and for all $f, f' \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and all $g, g' \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ we have

$$(f + f') \circ g = (f \circ g) + (f' \circ g)$$

and

$$f \circ (g + g') = (f \circ g) + (f \circ g');$$

(iii) the category has a zero object, $0 \in \mathrm{Ob}(\mathcal{C})$. This is an object with the property that there is exactly one morphism from each object to 0 and exactly one morphism from 0 to any other object.

The category $\mathrm{rep}_{\mathbb{K}}(Q)$ of representations of a quiver Q is an example of an additive category [2].

2.4 Equivalence of Categories

We can now go on to state and prove the first equivalence of categories. The following proof is adapted from [9]. Note that in [9], paths are composed from right to left. With this in mind, we will work with right $\mathbb{K}Q$–modules.

Theorem 2.23. *The category $\mathrm{rep}_{\mathbb{K}}(Q)$ of finite dimensional quiver representations of a quiver Q over a field $\mathbb{K}$ is equivalent to the category $\mathbb{K}Q$–mod of finite dimensional $\mathbb{K}Q$–modules.*

Proof. In this proof, we will only construct the functors $F\colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathbb{K}Q$–mod and $G\colon \mathbb{K}Q\text{–mod} \to \mathrm{rep}_{\mathbb{K}}(Q)$. These functors are inverse to each other, but this is not explicitly shown.

We construct a functor $F\colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathbb{K}Q$–mod by first defining what F does to the objects of $\mathrm{rep}_{\mathbb{K}}(Q)$. Let $V \in \mathrm{Ob}(\mathrm{rep}_{\mathbb{K}}(Q))$. Define $F(V) := M$ as $M = \bigoplus_{x \in Q_0} V_x$. We have that M is a graded $\mathbb{K}$–vector space. Let

$$\iota_x\colon V_x \to M \quad , \quad \pi_x\colon M \to V_x$$

be the canonical inclusion and the canonical projection, respectively, where $x \in Q_0$. Using these maps, we can define the module action of $\mathbb{K}Q$ on M:

(i) for $m \in M$, we define $m \cdot e_x = \iota_x \circ \pi_x(m)$ for all $x \in Q_o$;

(ii) for $m \in M$, we define $m \cdot (a_1 \dots a_n) = \iota_{h(a_n)} \circ (v_{a_n} \cdots v_{a_1}) \circ \pi_{t(a_1)}(m)$

where e_x is the trivial path at the vertex x and $a_1 \dots a_n$ is a path in Q. This defines where the trivial path and where the paths in Q are sent to, and so this can be extended to a module action of all of $\mathbb{K}Q$ on M.

We must now describe what F does to the morphisms of $\text{rep}_{\mathbb{K}}(Q)$. Let $\phi\colon V \to W$ be a morphism of quiver representations V,W. From before, we have that $M := F(V) = \bigoplus_{x \in Q_O} V_x$ and $N := F(W) = \bigoplus_{x \in Q_0} W_x$. Let $(m_1, \dots, m_t) \in M$, where $m_x \in V_x$ for all $x \in Q_0$. Define $\psi\colon M \to N$ by

$$\psi(m_1, \dots, m_t) = (\phi_1(m_1), \dots, \phi_t(m_t)) \in N.$$

We must now show that ψ is a $\mathbb{K}Q$–module homomorphism:

(i) for $(m_1, \dots, m_t), (m'_1, \dots, m'_t) \in M$ we have

$$\begin{aligned} \psi(m_1 + m'_1, \dots, m_t + m'_t) &= (\phi_1(m_1 + m'_1), \dots, \phi_t(m_t + m'_t)) \\ &= (\phi_1(m_1) + \phi_1(m'_1), \dots, \phi_t(m_t) + \phi_t(m'_t)) \\ &= (\phi_1(m_1), \dots, \phi_t(m_t)) + (\phi_1(m'_1), \dots, \phi_t(m'_t)) \\ &= \psi(m_1, \dots, m_t) + \psi(m'_1, \dots, m'_t) \end{aligned}$$

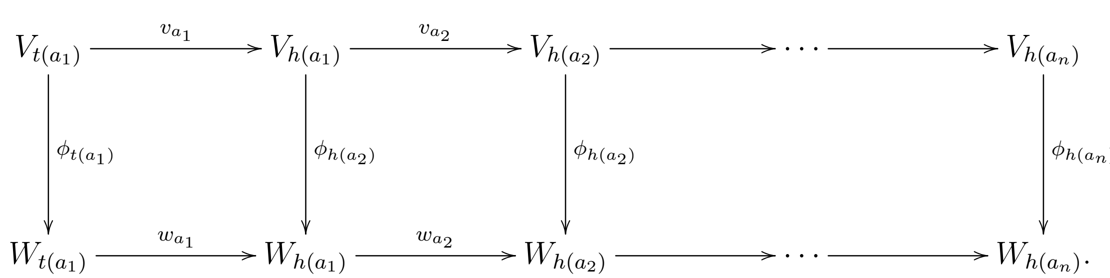
(ii) for $(m_1, \dots, m_t) \in M$ and $a_1 \dots a_n \in \mathbb{K}Q$ we have

$$\begin{aligned} \psi((m_1, \dots, m_t)(a_1 \dots a_n)) &= \psi((0, \dots, 0, v_{a_n} \cdots v_{a_1}(m_{t(a_1)}), 0, \dots, 0)) \\ &= (0, \dots, 0, \phi_{h(a_n)}(v_{a_n} \cdots v_{a_1}(m_{t(a_1)})), 0, \dots, 0) \qquad (1) \end{aligned}$$

where the entry $v_{a_n} \cdots v_{a_1}(m_{t(a_1)})$ from the first row is in the $h(a_n)$–th position, and

$$\begin{aligned} \psi(m_1, \dots, m_t) \cdot (a_1 \dots a_n) &= (\phi_1(m_1), \dots, \phi_t(m_t)) \cdot (a_1 \dots a_n) \\ &= (0, \dots, 0, w_{a_t} \dots w_{a_1}(\phi_{t(a_1)}(m_{t(a_1)})), 0, \dots, 0). \qquad (2) \end{aligned}$$

We have that (1) = (2) since the following diagram commutes (since ϕ is a morphism of quiver representations):



To show that F is indeed a functor, we must show that it preserves identity morphisms and preserves composition. We have the preservation of the identity morphisms trivially since if $\phi_x = \mathrm{id}$ for all $x \in Q_0$ then $\psi(m_1, \ldots, m_t) = (m_1, \ldots, m_t)$. To show that composition is preserved, let $\mu\colon U \to V$ and $\phi\colon V \to W$ be morphisms in $\mathrm{rep}_{\mathbb{K}}(Q)$. We have that

$$\phi \circ \mu := \{\phi_x \circ \mu_x\colon U_x \to W_x \mid x \in Q_0\}.$$

We must show that $F(\phi \circ \mu) = F(\phi) \circ F(\mu)$. Consider

$$F(\phi \circ \mu)(u_1, \ldots, u_t) = (\phi_1 \circ \mu_1(u_1), \ldots, \phi_t \circ \mu_t(u_t))$$

and

$$F(\phi) \circ F(\mu)(u_1, \ldots, u_t) = F(\phi)(\mu_1(u_1), \ldots, \mu_t(u_t)) = (\phi_1 \circ \mu_1(u_1), \ldots, \phi_t \circ \mu_t(u_t)).$$

Since these are equal then $F\colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathbb{K}Q$–mod is a functor.

We will now construct a functor $G\colon \mathbb{K}Q$–mod $\to \mathrm{rep}_{\mathbb{K}}(Q)$. Let us first describe what G does to the objects of $\mathbb{K}Q$–mod. Let $M \in \mathrm{Ob}(\mathbb{K}Q\text{–mod})$ and define $G(M) := V = (\bigoplus_{x \in Q_0} V_x, \{v_a\}_{a \in Q_1})$ where $V_x := Me_x = \{me_x | m \in M\}$ is a $\mathbb{K}$–vector space and $v_a\colon V_{t(a)} \to V_{h(a)}$ is given by

$$v_a(me_{t(a)}) = (me_{t(a)}) \cdot a = (m \cdot a) = (m \cdot a) \cdot e_{h(a)} \in V_{h(a)} = Me_{h(a)}$$

(the right hand side of this expression simply shows that v_a has the correct codomain).

We must now describe what G does to the morphisms of $\mathbb{K}Q$–mod. Let $\psi\colon M \to N$ be a $\mathbb{K}Q$–module homomorphism. Define $G(\psi) := \phi\colon V \to W = \{\phi_x\colon Me_x \to Ne_x \mid x \in Q_0\}$ by

$$\phi_x(me_x) = \phi(m)e_x.$$

We must check that ϕ is a morphism of quiver representations. To do this, we must check that

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{v_a} & V_{h(a)} \\ {\scriptstyle \phi_{t(a)}}\downarrow & & \downarrow{\scriptstyle \phi_{h(a)}} \\ W_{t(a)} & \xrightarrow[w_a]{} & W_{h(a)} \end{array}$$

commutes for all $a \in Q_1$. Let $me_{t(a)} \in V_{t(a)}$. We have

$$\begin{aligned} \phi_{h(a)}(v_a(me_{t(a)})) &= \phi_{h(a)}(m \cdot a) \\ &= \phi_{h(a)}((m \cdot a) \cdot e_{h(a)}) \\ &= \phi(m \cdot a) \cdot e_{h(a)} \\ &= \phi(m) \cdot a \end{aligned}$$

and

$$\begin{aligned} w_a(\phi_{t(a)}(me_{t(a)})) &= w_a(\phi(m) \cdot e_{t(a)}) \\ &= (\phi(m) \cdot e_{t(a)}) \cdot a \\ &= \phi(m) \cdot a \end{aligned}$$

and so the diagram commutes.

We now show that G is a functor. Like for the functor F, the preservation of the identity morphisms is trivial, so we will now consider the preservation of composition. Let $\alpha\colon L \to M$, $\beta\colon M \to N$ be $\mathbb{K}Q$–module homomorphisms. We have $G(\alpha) := \phi^\alpha\colon G(L) \to G(M) = \{\phi^\alpha_x\colon Le_x \to Me_x \mid x \in Q_0\}$ and $G(\beta) := \phi^\beta\colon G(M) \to G(N) = \{\phi^\beta_x\colon Me_x \to Ne_x \mid x \in Q_0\}$. We must show that $G(\beta \circ \alpha) = G(\beta) \circ G(\alpha)$. For each $x \in Q_0$ and for $l \in L$ we have

$$G(\beta_x \circ \alpha_x)(le_x) = \beta \circ \alpha(l)e_x$$

and

$$G(\beta_x) \circ G(\alpha_x)(le_x) = G(\beta_x)(\alpha(l)e_x) = \beta \circ \alpha(l)e_x.$$

Since these are equal then $G\colon \mathbb{K}Q\text{–mod} \to \text{rep}_\mathbb{K}(Q)$ is a functor.

The functors F,G are inverses to each other, that is, $F \circ G = \text{id}_{\mathbb{K}Q\text{–mod}}$ and $G \circ F = \text{id}_{\text{rep}_\mathbb{K}(Q)}$, and so the two categories are equivalent. □

3 Geometric Interpretation of Isomorphism Classes of Quiver Representations

We will now consider a more geometric approach to the equivalence classes of quiver representations, by considering the orbits of a group action.

3.1 The Representation Space

The following section is adapted from [3].

Let Q be a quiver and let V be a representation of Q with an assigned dimension vector $\underline{n}$. By Definition 2.3, we have $\dim(V_x) = n_x$ for all $x \in Q_0$ and so $V_x \cong \mathbb{K}^{n_x}$. Then, like in Example 2.6, we have that each map $v_a \colon V_{t(a)} \to V_{h(a)}$ is in fact a matrix M of size $n_{h(a)} \times n_{t(a)}$. Using this, we can now define the representation space of Q.

Definition 3.1. Let Q be a quiver and let $\underline{n} = (n_1, n_2, \ldots, n_t)$ where $t = |Q_0|$. The *representation space* of Q for the dimension vector $\underline{n}$ is the vector space

$$\operatorname{rep}_{\mathbb{K}}(Q, \underline{n}) := \bigoplus_{a \in Q_1} \operatorname{Hom}(\mathbb{K}^{n_{t(a)}}, \mathbb{K}^{n_{h(a)}}).$$

Using the above, and by letting $\operatorname{mat}_{m \times n}(\mathbb{K})$ denote the vector space of all $m \times n$ matrices with entries from the field $\mathbb{K}$, we can say that

$$\operatorname{rep}_{\mathbb{K}}(Q, \underline{n}) = \bigoplus_{a \in Q_1} \operatorname{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K}).$$

A point in $\operatorname{rep}_{\mathbb{K}}(Q, \underline{n})$ will be denoted $\mathbf{p} := (p_a)_{a \in Q_1}$.

We now calculate the dimension of $\operatorname{rep}_{\mathbb{K}}(Q, \underline{n})$. Fix vertices $1 \leq i, j \leq t$ and $a \in Q_1$. Each basis element E^a_{ij} of the vector space $\operatorname{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K})$ has a 1 in the ij–th entry and zeros elsewhere. There are $n_{t(a)}$ choices of where the 1 can go along the first row, and $n_{h(a)}$ rows and so a basis for $\operatorname{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K})$ consists of $n_{t(a)} n_{h(a)}$ matrices. The representation space is a direct sum of the vector spaces $\operatorname{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K})$ for each $a \in Q_1$, and so the dimension of the representation space is

$$\dim(\operatorname{rep}_{\mathbb{K}}(Q, \underline{n})) = \sum_{a \in Q_1} n_{t(a)} n_{h(a)}.$$

According to [3], a central problem of quiver theory is to "*describe the isomorphism classes of finite–dimensional representations of a prescribed quiver, having a prescribed dimension vector*". We have Definition 2.4 which gives us the definition of isomorphic quiver representations in terms of a commutative diagram, but we can now expand on this by using Definition 3.1, and the fact that $\operatorname{rep}_{\mathbb{K}}(Q, \underline{n}) = \bigoplus_{a \in Q_1} \operatorname{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K})$, to say that two representations V, W of a quiver Q are isomorphic if and only if the matrices corresponding to the linear maps between V and W are related by a change of basis.

3.2 A Group Action on the Representation Space

3.2.1 The General Linear Group

The *general linear group* $\mathrm{GL}(V)$ of a vector space V over a field $\mathbb{K}$ consists of all the invertible linear maps $\phi\colon V \to V$. It is sometimes denoted $\mathrm{Aut}(V)$ since the elements are all the automorphisms of V (bijective linear maps from V to itself). The general linear group of degree n over a field $\mathbb{K}$ is a subset of $\mathrm{mat}_{n\times n}(\mathbb{K})$. It consists of all invertible $n \times n$ matrices, and is denoted $\mathrm{GL}(n)$. The binary operation on $\mathrm{GL}(n)$ is matrix multiplication. Since matrix multiplication is associative, the elements are invertible by choice and there exists a multiplicative identity $\mathbb{I}_n$ (the $n \times n$ matrix with 1's along the diagonal and 0's elsewhere) then $\mathrm{GL}(n)$ is indeed a group. If a vector space V has a fixed dimension n then $\mathrm{GL}(n)$ is isomorphic to $\mathrm{GL}(V)$.

Before we look at another example, we need some definitions.

Definition 3.2. Let G be a group and let X be a set. A *group action* (or just *action*) of G on the set X consists of a function $G \times X \to X$ written

$$(g, x) \mapsto g * x$$

such that the following two conditions hold:

(i) for all $g, h \in G$ and for all $x \in X$,

$$gh * x = g * (h * x);$$

(ii) for all $x \in X$,

$$e * x = x$$

where $e \in G$ is the identity element of G.

In this situation, X is called a *G–set*.

Note 3.3. Note that for ease of notation, the symbol $*$ is often omitted.

Definition 3.4. Let X be a set and let G be a group acting on X. For $x \in X$, the *orbit* of x is defined to be

$$\mathcal{O}_x := \{g * x \mid g \in G\}$$

which is a subset of X.

Let us now consider the general linear group acting on the representation space. We define the group

$$\mathrm{GL}(\underline{n}) := \prod_{x\in Q_0} \mathrm{GL}(n_x)$$

which acts linearly on the representation space $\mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$ by

$$(g_x)_{x\in Q_0} * p_a := g_{h(a)} p_a g^{-1}_{t(a)}$$

where $g_{h(a)} \in \mathrm{GL}(n_{h(a)})$, $p_a \in \mathrm{mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{K})$ and $g_{t(a)}^{-1} \in \mathrm{GL}(n_{t(a)})$. We will abuse the definition slightly and refer to this action as *conjugation*.

We have already calculated the dimension of $\mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$, which was

$$\dim(\mathrm{rep}_{\mathbb{K}}(Q, \underline{n})) = \sum_{a \in Q_1} n_{t(a)} n_{h(a)}.$$

Using a similar method, we can calculate the dimension of $\mathrm{GL}(\underline{n})$, which is

$$\dim(\mathrm{GL}(\underline{n})) = \sum_{x \in Q_0} {n_x}^2.$$

The question now is, "why is it relevant to consider an action of the general linear group on the representation space?". The answer to this, is the result obtained in the following proposition.

Proposition 3.5. *Two representations of a quiver Q with dimension vector $\underline{n}$ are isomorphic if and only if they lie in the same orbit of the general linear group $GL(\underline{n})$ acting on the representation space by conjugation.*

Proof. Let $V, W \in \mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$ be isomorphic representations of the quiver Q with dimension vector $\underline{n}$. Then by Definition 2.4, there exists a collection of invertible linear maps $\{\phi_x : V_x \to W_x \mid x \in Q_0\}$ such that

$$\begin{CD} V_{t(a)} @>{v_a}>> V_{h(a)} \\ @V{\phi_{t(a)}}VV @VV{\phi_{h(a)}}V \\ W_{t(a)} @>>{w_a}> W_{h(a)} \end{CD}$$

commutes for all $a \in Q_1$. By choosing bases, let $V_x \cong \mathbb{K}^{n_x}$, $W_x \cong \mathbb{K}^{n_x}$ for all $x \in Q_0$. By associating matrices with each linear map (see Section 3.1) we get the commutative diagram:

$$\begin{CD} \mathbb{K}^{n_{t(a)}} \cong V_{t(a)} @>{A_a}>> V_{h(a)} \cong \mathbb{K}^{n_{h(a)}} \\ @V{P_{t(a)}}VV @VV{P_{h(a)}}V \\ \mathbb{K}^{n_{t(a)}} \cong W_{t(a)} @>>{B_a}> W_{h(a)} \cong \mathbb{K}^{n_{h(a)}}. \end{CD}$$

We can now say that V, W are isomorphic if and only if A_a, B_a are related by a change of basis, that is,

$$B_a \quad \sim \quad P_{h(a)} A_a P_{t(a)}^{-1} \tag{3}$$

where $P_{t(a)} \in \mathrm{GL}(n_{t(a)})$ and $P_{h(a)} \in \mathrm{GL}(n_{h(a)})$. Now recall that the action of the group $\mathrm{GL}(\underline{n}) := \prod_{x \in Q_0} \mathrm{GL}(n_x)$ on the representation space $\mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$ is precisely the action given in (3). So we can then say that $V, W \in \mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$ are isomorphic if and only if they lie in the same $\mathrm{GL}(\underline{n})$–orbit. $\square$

3.2.2 The Quiver S_r

The following example is from [3].

Example 3.6. Let S_r denote the quiver with vertices $Q_0 = \{1, 2, \ldots, r, c\}$ and arrows $Q_1 = \{a_1, \ldots, a_r\}$ where $t(a_i) = i$ and $h(a_i) = c$ for all $i = 1, \ldots, r$ (see Fig. 4). The

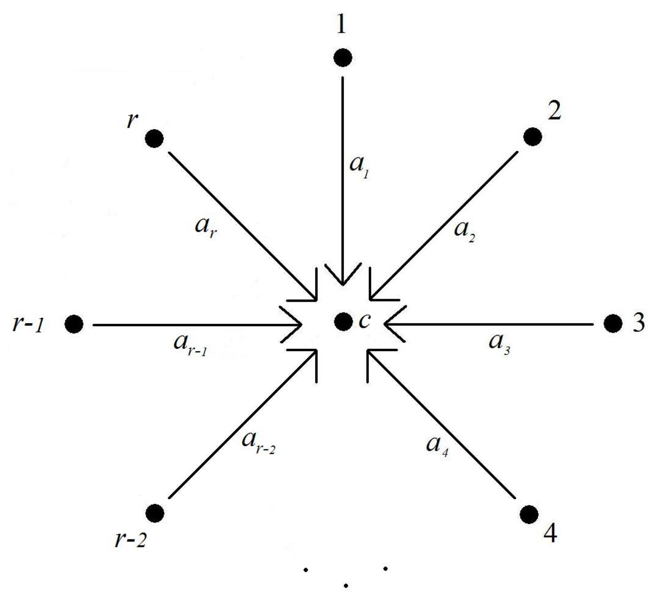


Figure 4: The quiver S_r

representations of S_r are of the form

$$V_{S_r} = (V_1 \oplus V_2 \oplus \cdots \oplus V_r \oplus W \ , \ \{v_{a_i}\}_{a \in Q_1})$$

where $V_1, \ldots, V_r$ are the vector spaces associated with the vertices $1, \ldots, r$ and W is the vector space associated with the central vertex c. The maps $\{v_{a_i}\}_{a \in Q_1}$ are from the vector space V_i to the vector space W for each $i = 1, \ldots, r$. Define $\dim(V_i) := m_i$ and $\dim(W) := n$, so we have $V_i \cong \mathbb{K}^{m_i}$ for all $i = 1, \ldots, r$ and $W \cong \mathbb{K}^n$. We can then assign the dimension vector $(m_1, m_2, \ldots, m_r, n)$ to the representation V_{S_r}.

Let us now consider a map $\Delta \colon \mathbb{S} \to \mathbb{T}$ where $\mathbb{S}$ is the set of all isomorphism classes of representations of S_r with dimension vector $(m_1, \ldots, m_r, n)$, and $\mathbb{T}$ is the set of orbits of the general linear group of the vector space W, $\mathrm{GL}(W)$, acting on the r–tuples $(v_{a_1}(V_1), \ldots, v_{a_r}(V_r))$ of subspaces of $\mathbb{K}^n$ such that $\dim(v_{a_i}(V_i)) \leq m_i$ for all $i = 1, \ldots, r$. We define this map by

$$\begin{aligned} \Delta([V_{S_r}]) &= \mathrm{GL}(W) * (v_{a_1}(V_1), \ldots, v_{a_r}(V_r)) \\ &= \{(g(v_{a_1}(V_1)), \ldots, g(v_{a_r}(V_r))) \mid g \in \mathrm{GL}(W)\} \end{aligned}$$

where $[V_{S_r}]$ denotes the class of representations isomorphic to the representation V_{S_r}.

We need to first check that this map Δ is well–defined, so suppose that $[V_{S_r}] = [V'_{S_r}]$. By Definition 2.4, there exists an invertible map $\phi_i \colon V_i \to V'_i$ such that

$$\begin{array}{ccc} V_{t(a_i)} & \xrightarrow{v_{a_i}} & V_{h(a_i)} \\ {\scriptstyle \phi_{t(a_i)}} \downarrow & & \downarrow {\scriptstyle \phi_{h(a_i)}} \\ V'_{t(a_i)} & \xrightarrow[v'_{a_i}]{} & V'_{h(a_i)} \end{array}$$

commutes for all $a_i \in Q_1$. So $V_i \cong V'_i$ via ϕ_i and $W \cong W'$ via an isomorphism, say ϕ, where W, W' are the vector spaces assigned to the vertex c in the quiver S_r for the representations V_{S_r}, V'_{S_r}, respectively. By considering this commutative diagram at the vertex i, we get the following commutative diagram, which holds for all vertices $i = 1, \dots, r$:

$$\begin{array}{ccc} V_i & \xrightarrow{v_{a_i}} & W \\ {\scriptstyle \phi_i} \downarrow & & \downarrow {\scriptstyle \phi} \\ V'_i & \xrightarrow[v'_{a_i}]{} & W'. \end{array}$$

This gives us that $\phi_i^{-1}(V'_i) \cong V_i$ and $v'_{a_i} = \phi \circ v_{a_i} \circ \phi_i$, since ϕ_i is invertible for all $i = 1, \dots, r$. Now choose an element $(g(v'_{a_1}(V'_1)), \dots, g(v'_{a_r}(V'_r))) \in \mathrm{GL}(W') * (v'_{a_1}(V_1), \dots, v'_{a_r}(V'_r))$. Using the commutativity of the diagram above, we get

$$\begin{aligned} (g(v'_{a_1}(V'_1)), \dots, g(v'_{a_r}(V'_r))) &= (g(\phi v_{a_1} \phi_1^{-1}(V'_1)), \dots, g(\phi v_{a_r} \phi_r^{-1}(V'_r))) \\ &= (g(\phi v_{a_1}(V_1)), \dots, g(\phi v_{a_r}(V_r))) \\ &\in \mathrm{GL}(W) * (v_{a_1}(V_1), \dots, v_{a_r}(V_r)) \end{aligned}$$

since $g\phi \in \mathrm{GL}(W)$. So we have that

$$\mathrm{GL}(W') * (v'_{a_1}(V_1), \dots, v'_{a_r}(V'_r)) \subseteq \mathrm{GL}(W) * (v_{a_1}(V_1), \dots, v_{a_r}(V_r)).$$

To get the reverse inclusion, we simply reverse the argument above. So the map Δ is well–defined.

To show that Δ is injective, suppose that

$$\mathrm{GL}(W) * (v_{a_1}(V_1), \dots, v_{a_r}(V_r)) = \mathrm{GL}(W') * (v'_{a_1}(V_1), \dots, v'_{a_r}(V'_r)) \tag{4}$$

where $V_1, \dots, V_r, W$ are the vector spaces associated to the vertices of S_r in the representation V_{S_r}, and $V'_1, \dots, V'_r, W'$ are the vector spaces associated to the vertices of S_r in the representation V'_{S_r}. The collections of linear maps $\{v_{a_i}\}_{a \in Q_1}$ and $\{v'_{a_i}\}_{a \in Q_1}$ belong to the representations V_{S_r} and V'_{S_r}, respectively. Since the dimensions of W and W' are fixed, namely n, then there exists an invertible linear map $\phi : W \to W'$ and so $\mathrm{GL}(W) \cong \mathrm{GL}(W')$. The same is also true for all other vertices of the quiver, ie. $V_i \cong V_i'$ via an invertible linear map ϕ_i for all $i = 1, \ldots, r$. Recall that $\dim(V_i) = \dim(V_i') = m_i$ and so we have $V_i \cong \mathbb{K}^{m_i}$ and $V_i' \cong \mathbb{K}^{m_i}$ for all $i = 1, \ldots, r$. By Equation (4), there exists $g \in \mathrm{GL}(W)$ such that $g(v_{a_i}(V_i)) = v'_{a_i}(V_i')$, which gives us, together with $V_i \cong \mathbb{K}^{m_i}$ and $V_i' \cong \mathbb{K}^{m_i}$, the following commutative diagram:

$$\begin{array}{ccc} V_i \cong \mathbb{K}^{m_i} & \xrightarrow{v_{a_i}} & W \\ \Big\downarrow \mathrm{id} & & \Big\downarrow g \\ V_i' \cong \mathbb{K}^{m_i} & \xrightarrow[v'_{a_i}]{} & W' \end{array}$$

since $g : W \to W \cong W'$. So the representations V_{S_r}, V'_{S_r} are isomorphic, as required.

Finally we must show that Δ is surjective. Choose an r–tuple $(E_1, \ldots, E_r)$ of subspaces of W. Let $\dim(E_i) = d_i$ for each $i = 1, \ldots, r$. Then we have that $E_i \cong \mathbb{K}^{d_i}$ for each $i = 1, \ldots, r$. We must now show that we can construct a surjective map $\pi\colon \mathbb{K}^{m_i} \to \mathbb{K}^{d_i}$ for each $i = 1, \ldots, r$. We have that $d_i \leq m_i$ since any linear map $f\colon X \to Y$ between vector spaces X, Y has the property that

$$\dim(X) = \dim(\mathrm{im}(f)) + \dim(\ker(f)).$$

We can decompose each $\mathbb{K}^{m_i}$ into a direct sum of vector spaces

$$\mathbb{K}^{m_i} = \mathbb{K}^{d_i} \oplus \mathbb{K}^{m_i - d_i}$$

for each $i = 1, \ldots, r$. We can therefore define a surjective map $\pi\colon \mathbb{K}^{m_i} \to \mathbb{K}^{d_i}$ by

$$\pi|_{\mathbb{K}^{d_i}} = \mathrm{id} \qquad , \qquad \pi|_{\mathbb{K}^{m_i - d_i}} = 0.$$

By changing basis, we then have surjective linear maps from $V_i \to W$ for all $i = 1, \ldots, r$ and so Δ is surjective.

By showing that the map Δ is well–defined and both injective and surjective, we have shown that there exists a bijection between the isomorphism classes of representations with dimension vector $(m_1, \ldots, m_r, n)$ and the orbits of the general linear group $\mathrm{GL}(W)$ acting on the r–tuples $(v_{a_1}(V_1), \ldots, v_{a_r}(V_r))$ of subspaces of $\mathbb{K}^n$ such that $\dim(v_{a_i}(V_i)) \leq m_i$ for all $i = 1, \ldots, r$ via

$$g * (v_{a_1}(V_1), \ldots, v_{a_r}(V_r)) := (g(v_{a_1}(V_1)), \ldots, g(v_{a_r}(V_r))).$$

Example 3.7. From Example 3.6, we can see that classifying the representations of the quiver S_r, is equivalent to classifying the r–tuples of subspaces of a fixed vector space, ie. it is possible to look at the orbits of the general linear group acting on the r–tuples of subspaces of a fixed vector space $\mathbb{K}^n$ to be able to determine the equivalence classes of representations of S_r.

Let us consider the case when $r = 1$. We can see that when $r = 1$, we get the same quiver as K_1 in Example 2.6. The isomorphism classes of the (n, m)–dimensional representations of K_1 correspond bijectively to the maximum number of linearly independent columns (or linearly independent rows, the number of which is equal to that of the columns [10]) in the unique $m \times n$ matrix. This number is called the *rank* of the matrix, and can be at most the lesser of either m or n, and so in particular is finite. So for $r = 1$, there are finitely many isomorphism classes of representations. This same finiteness of equivalence classes occurs for $r = 2$ and $r = 3$, but this finiteness property fails when $r \geq 4$ [3].

We will see later, in Section 4.2, why this finiteness of equivalence classes occurs for $r = 1, 2$ or 3 but not for $r \geq 4$.

4 Gabriel's Theorem

4.1 Dynkin Diagrams and the ADE Classification

Dynkin diagrams, also known as Coxeter-Dynkin diagrams, are ubiquitous throughout mathematics. Dynkin diagrams are named after the Soviet mathematician Eugene Dynkin, who first used them in his papers to classify semisimple Lie algebras in the mid-1940s [11].

Dynkin diagrams are graphs, classified by their structure, which consist of a series of vertices and edges. There can be one, two or three edges between any two vertices, but in this project we will assume that all Dynkin diagrams have only a single edge between any two vertices. Graphs of this nature are called *simply–laced* (or *single–laced*). Simply–laced Dynkin diagrams correspond to what are called *simple roots* in a particular configuration of vectors in Euclidean space, called a *root system* (see Section 4.4.1). The *ADE classification* is a complete list of simply–laced Dynkin diagrams, and they are as follows: A_n, D_n, E_6, E_7 and E_8 (see Fig. 5).

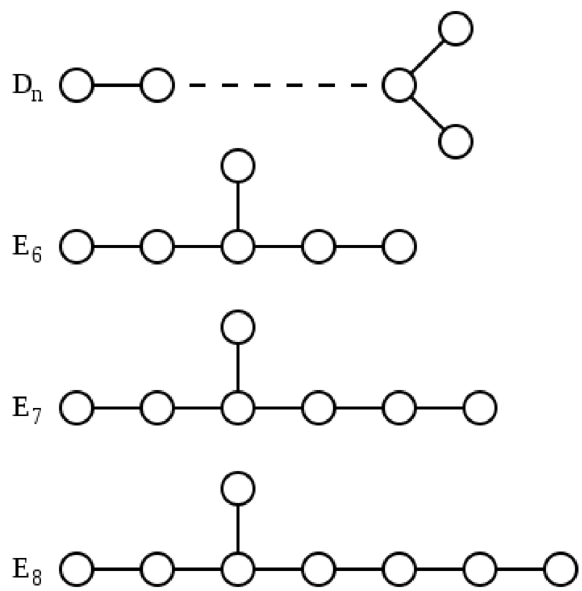


Figure 5: Dynkin diagrams of type ADE [12]. For the A_n diagrams, $n \geq 1$ and for the D_n diagrams, $n \geq 4$ where n is the number of vertices.

Dynkin diagrams can have directed or undirected edges, but in this project we will assume that all edges are undirected. We will also assume that all Dynkin diagrams are finite since we are dealing with quivers which have a finite number of vertices and a finite number of edges.

4.2 Introducing Gabriel's Theorem

Before we can state Gabriel's theorem, we first need some definitions.

Definition 4.1. Let V, W be two representations of the quiver Q. The *direct sum representation* of V and W, denoted $V \oplus W$, is defined to be

$$(V \oplus W)_x := V_x \oplus W_x$$

for all vertices $x \in Q_0$, and

$$(v \oplus w)_a \colon v_{t(a)} \oplus w_{t(a)} \longrightarrow v_{h(a)} \oplus w_{h(a)}$$

for every $a \in Q_1$.

A representation U of Q is said to be *decomposable* if and only if

$$U \cong V \oplus W$$

where V, W are non-zero representations of Q. If a non-zero quiver representation is not decomposable, then it is called *indecomposable*.

Definition 4.2. A quiver $Q = (Q_0, Q_1)$ is of *finite orbit type*, or just *finite type*, if and only if it has a finite number of isomorphism classes of indecomposable representations (for an assigned dimension vector $\underline{n}$).

The following statement of Gabriel's Theorem comes from [2].

Theorem 4.3 (Gabriel's Theorem). *(i) A quiver Q is of finite orbit type if and only if each component of its underlying undirected graph $\hat{Q}$ is a simply–laced Dynkin diagram.*

(ii) Let Q be a quiver such that $\hat{Q}$ is a simply–laced Dynkin diagram. Then $\underline{n}$ is the dimension of a (unique) indecomposable representation of Q if and only if $\underline{n} \in \Phi^+$.

For now, let us just focus on part (i). The terms in part (ii) will be explained, and then the statement proved, in Section 4.4.

4.3 Gabriel's Theorem - Part (i)

Gabriel's Theorem now explains why the quiver S_r from Example 3.6 has finitely many isomorphism classes of representations for $r < 4$. When $r = 1$ we get that $\hat{S}_r$ (and therefore $\hat{K}_r$ from Example 2.6) is the Dynkin diagram A_2, when $r = 2$ we obtain the Dynkin diagram A_3 and when $r = 3$ we obtain the Dynkin diagram D_4. When $r \geq 4$, we can see that the underlying undirected graph $\hat{S}_r$ is not one of the simply–laced Dynkin diagram A_n, D_n, E_6, E_7 or E_8. So S_r has infinitely many isomorphism classes of representations for $r \geq 4$, by Gabriel's Theorem, part (i).

4.3.1 Algebraic Groups

Recall from algebraic geometry the following definitions (from [5]).

Let $\mathbb{K}$ be a field. A subset $X \subset \mathbb{K}^n$ is called *algebraic* is there exists polynomials $f_1, \ldots, f_m \in \mathbb{K}[t_1, \ldots, t_n]$ such that

$$X = \{(x_1, \ldots, x_n\} \in \mathbb{K}^n \mid f_i(x_1, \ldots, x_n) = 0 \text{ for all } i = 1, \ldots, m\}.$$

A *morphism* of algebraic sets $X \in \mathbb{K}^n$, $Y \in \mathbb{K}^m$ is a map $X \to Y$ given by $\underline{x} \mapsto (g_1(\underline{x}), \ldots, g_m(\underline{x}))$ where $\underline{x} = (x_1, \ldots, x_n)$ and $g_1, \ldots, g_m \in \mathbb{K}[t_1, \ldots, t_n]$. A morphism $X \to \mathbb{K}$ is called a *regular* function on X. An algebraic set $X \subset \mathbb{K}^n$ is called *irreducible* if and only if its *coordinate ring* $\mathbb{K}[X]$ (the set of all maps $f\colon X \to \mathbb{K}$ such that there exists $f_1 \in \mathbb{K}[t_1, \ldots, t_n]$ for which we have $f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in X$) is an integral domain. An *(affine) algebraic variety* is an irreducible algebraic set.

We can now use these definitions to define an algebraic group (the following definition is from [3]).

Definition 4.4. An *(affine) algebraic group* is an (affine) algebraic variety G, equipped with a group structure such that the multiplication map $\mu\colon G \times G \to G$, where $\mu(g, h) = gh$, and the inverse map $\iota\colon G \to G$, where $\iota(g) = g^{-1}$, are morphisms of varieties.

The general linear group $\mathrm{GL}(\underline{n})$ is an example of an algebraic group. We will now define an action of an algebraic group. The following definition is from [13].

Definition 4.5. Let G be an (affine) algebraic group with identity $e \in G$ and multiplication morphism $m\colon G \times G \to G$. An *action* of G on an algebraic variety X is a regular morphism $a\colon G \times X \to X$ such that $a(e, x) = x$ for all $x \in X$ and the following diagram commutes:

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \mathrm{id}} & G \times X \\ \downarrow{\scriptstyle \mathrm{id} \times a} & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow[a]{} & X. \end{array}$$

Note that this definition of an action of an algebraic group is just a more technical reformulation of Definition 3.2.

Definition 4.6. Let X be a set and let G be a group acting on X. The *stabilizer* of a point $x \in X$ (also known as the *isotropy group*) is defined to be

$$G_x := \{g \in G \mid g * x = x\}.$$

The stabilizer of a point $x \in X$ is a subgroup of G, for all $x \in X$.

The following results are from [3] (Proposition 2.1.7 and Corollary 2.1.8).

The proof of Proposition 4.7 (i) uses more topology than has been introduced in this project, and so we refer the reader to [3] for a proof. The second part of Proposition 4.7 follows from (i), but also uses topology and so once again we refer the reader to [3].

Proposition 4.8 relies on Proposition 2.1.7 from [3]. We will only be using the last statement which refers to the dimensions of the kernel and image of the map, and so we once again refer the reader to [3] for a proof.

Proposition 4.7. *Let X be a vector space equipped with an algebraic action of an algebraic group G and let $x \in X$. Then,*

(i) the orbit $\mathcal{O}_x$ has dimension

$$dim(\mathcal{O}_x) = dim(G) - dim G_x$$

where G_x is the stabilizer of $x \in X$;

(ii) the orbit closure $\bar{\mathcal{O}}_x$ is the union of $\mathcal{O}_x$ and of all orbits of dimension strictly smaller than the dimension of $\mathcal{O}_x$.

Proposition 4.8. *Let $\psi \colon G \to H$ be a homomorphism of algebraic groups. Then the kernel* $\ker \psi$ *is closed (with respect to the Zariski topology) in G and the image* $\mathrm{im}\psi$ *is closed (with respect to the Zariski topology) in H. We have $dim(\ker \psi) + dim(\mathrm{im}\psi) = dim(G)$.*

4.3.2 The Projective General Linear Group

Definition 4.9. The *centre* of a group G is defined to be

$$\mathcal{Z}(G) := \{z \in G \mid gz = zg, \, \forall g \in G\}$$

which is a subgroup of G.

Consider the stabilizer of the group $\mathrm{GL}(\underline{n}) := \prod_{x \in Q_0} \mathrm{GL}(n_x)$ acting on the representation space by conjugation (see Section 3.2.1). This is the subgroup

$$\mathbb{K}^\times \mathbb{I}_{\underline{n}} := \{(\lambda \mathbb{I}_{n_x})_{x \in Q_0} \mid \lambda \in \mathbb{K}^\times\}$$

which is contained in the centre of $\mathrm{GL}(\underline{n})$, where $\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$. This subgroup acts trivially on the representation space $\mathrm{rep}_{\mathbb{K}}(Q, \underline{n})$, and since it is a normal subgroup of $\mathrm{GL}(\underline{n})$, then we can form the quotient group

$$\mathrm{GL}(\underline{n}) / \mathbb{K}^\times \mathbb{I}_{\underline{n}}$$

which is called the *projective general linear group* of degree dimension vector $\underline{n}$, denoted $\mathrm{PGL}(\underline{n})$.

We can use Proposition 4.8 to calculate the dimension of $\mathrm{PGL}(\underline{n})$ since $\mathrm{GL}(\underline{n})$ and $\mathrm{PGL}(\underline{n})$ are both algebraic groups [3]. The morphism $\psi \colon \mathrm{GL}(\underline{n}) \to \mathrm{PGL}(\underline{n})$ is the quotient map since $\mathrm{PGL}(\underline{n}) = \mathrm{GL}(\underline{n}) / \mathbb{K}^\times \mathbb{I}_{\underline{n}}$. Since ψ is surjective, then $\mathrm{im}\psi = \mathrm{PGL}(\underline{n})$. The kernel of ψ consists of all the elements of $\mathrm{GL}(\underline{n})$ which are sent to the identity in $\mathrm{PGL}(\underline{n})$, so $\ker \psi = \mathbb{K}^\times \mathbb{I}_{\underline{n}}$. This group is isomorphic to the multiplicative group GL(1) (via the isomorphism which sends a matrix $\lambda\mathbb{I}_{\underline{n}}$ in the group $\mathbb{K}^\times\mathbb{I}_{\underline{n}}$ of scalar invertible matrices to the 1×1 matrix $[\lambda]$ in GL(1)) which has dimension 1 and so $\dim(\ker\psi) = 1$. This gives us that

$$\dim(\mathrm{PGL}(\underline{n})) = \dim(\mathrm{GL}(\underline{n})) - 1$$

by Proposition 4.8.

4.3.3 The Tits Form of a Quiver

The following definitions are from [14].

Definition 4.10. Let Q be a quiver and let $\underline{m}$, $\underline{n}$ be dimension vectors assigned to two representations of Q. The *Euler form* (or *Ringel form*) of Q is the bilinear form on $\mathbb{Z}^t$ (where t is the cardinality of the set Q_0) given by

$$\langle \underline{m}, \underline{n} \rangle_Q = \sum_{x \in Q_0} m_x n_x - \sum_{a \in Q_1} m_{t(a)} n_{h(a)}.$$

The *symmetric Euler form* (or *Cartan form*) is defined to be

$$(\underline{m}, \underline{n})_Q := \langle \underline{m}, \underline{n} \rangle_Q + \langle \underline{n}, \underline{m} \rangle_Q.$$

Definition 4.11. The *Tits form* of a quiver Q for a dimension vector $\underline{n}$ is the quadratic form of the Euler form and is defined to be $q_Q(\underline{n}) := \langle \underline{n}, \underline{n} \rangle$ and so by Definition 4.10 we have that

$$q_Q(\underline{n}) = \sum_{x \in Q_0} {n_x}^2 - \sum_{a \in Q_1} n_{t(a)} n_{h(a)}.$$

Note that the Tits form (and the symmetric Euler form) of a quiver Q relies only on its underlying undirected graph $\hat{Q}$, and not on the orientation of the arrows.

Definition 4.12. Let Q be a quiver with underlying undirected graph $\hat{Q}$ and let V be a representation of Q with assigned dimension vector $\underline{n}$. The Tits form of Q is called *positive definite* if

$$q_Q(\underline{n}) > 0$$

for all non–zero $\underline{n} = (n_x)_{x \in Q_0}$.

The graph $\hat{Q}$ is called *positive definite* if the Tits form associated with $\hat{Q}$ is positive definite for any dimension vector $\underline{n}$.

Lemma 4.13. *Let Q be a quiver. If $\hat{Q}$ is positive definite then $\hat{Q}$ is simply–laced, that is, has no multiple edges.*

Proof. Let $\hat{Q}$ be positive definite. Then $q_Q(\underline{n})$ is positive definite for all dimension vectors $\underline{n}$, and so in particular is positive definite for the dimension vector $\underline{\alpha} := (1, \dots, 1)$. Let k, l be the number of vertices and the number of edges of $\hat{Q}$, respectively. Then the Tits form of $\hat{Q}$ with dimension vector $\underline{\alpha}$ is

$$q_{\hat{Q}}(\underline{\alpha}) = k - l.$$

So we have $k - l > 0$, that is, $k > l$ and so any two vertices of $\hat{Q}$ can be connected by at most one edge. So $\hat{Q}$ is simply–laced. □

Definition 4.14. A *subgraph* of a given graph $\hat{Q}$ is a graph whose sets of vertices and edges are subsets of those of $\hat{Q}$. A *full subgraph* $\hat{Q}'$ of $\hat{Q}$ is a subgraph such that for any two vertices $x, y \in \hat{Q}'_0$, all arrows in $\hat{Q}$ between x and y also lie in $\hat{Q}'_1$.

The following lemma is from [15] (Lemma 16.3). We remind the reader, that all quivers in this project are connected.

Lemma 4.15. *A subgraph $\hat{Q}'$ of a positive definite graph $\hat{Q}$ is positive definite.*

Proof (sketch). The proof from [15] uses a specific labelling of the graph $\hat{Q}$. Edges of the graph are labelled with integers ≥ 3. If x,y are distinct vertices, then m_{xy} denotes the label on the edge joining x and y. The paper uses the convention that $m_{xx} = 1$ and $m_{xy} = 2$ if x,y are distinct vertices not joined by an edge. [15] then associates to a labelled graph $\hat{Q}$, with vertex set Q_0 of cardinality t, a symmetric $t \times t$ matrix G by setting $g_{xy} := -\cos(\pi/m_{xy})$. The proof then uses this labelling on $\hat{Q}$ and a subgraph $\hat{Q}'$. A subgraph $\hat{Q}'$ of a positive definite graph $\hat{Q}$ is supposed not positive definite and this turns out to be a contradiction. □

The following proposition and its proof is from [4] (Proposition 2.2). The motivation for this result is that it will prove the final implication of the '⇒' direction of part (i) of Gabriel's Theorem. It also has a very instructive proof which provides vital understanding of the link between positive definite graphs and Dynkin diagrams.

Proposition 4.16. *Let Q be a quiver with underlying undirected graph $\hat{Q}$ positive definite. Then $\hat{Q}$ is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8.*

Proof. Let Q be a quiver with $\hat{Q}$ positive definite. We know from Lemma 4.13 that $\hat{Q}$ is simply–laced.

Our aim is to show that if $\hat{Q}$ has certain properties (for example, loops), then we are able to construct a full subgraph $\hat{Q}'$ which is not positive definite, which contradicts Lemma 4.15. This allows us to conclude that Q has none of these properties.

Now suppose that $\hat{Q}$ contains at least one loop, occurring at vertex x. Then $\hat{Q}$ contains a full subgraph $\hat{Q}'$ which consists of the single vertex x and say $k \geq 1$ loops at x. Consider the dimension vector $\underline{n}$ whose dimension at vertex x is 1. Then the Tits form associated with $\hat{Q}$ is

$$q_{\hat{Q}}(\underline{n}) = 1 - k.$$

Since $k \geq 1$, we have $q_{\hat{Q}}(\underline{n}) \leq 0$. So $\hat{Q}$ has no loops.

Now suppose that a vertex x in $\hat{Q}$ is connected to more than 3 vertices, say vertices y_1, y_2, y_3, y_4. Consider the full subgraph $\hat{Q}'$ which consists of these vertices, together with the vertex x. This full subgraph has 5 vertices and so can have at most 4 edges by the proof of Lemma 4.13. Since x is connected to the vertices y_i for all $i = 1, \ldots, 4$ by assumption, then the 4 edges of $\hat{Q}'$ must look like

$$x\bullet \text{------} \bullet y_i$$

for $i = 1, \ldots, 4$. Now consider the dimension vector $\underline{n}$ whose dimension at vertex x is 2 and at vertices y_i is 1 for all $i = 1, \ldots, 4$. We then get

$$q_{\hat{Q}}(\underline{n}) = (2^2 + 1 + 1 + 1 + 1) - (2 + 2 + 2 + 2) = 0.$$

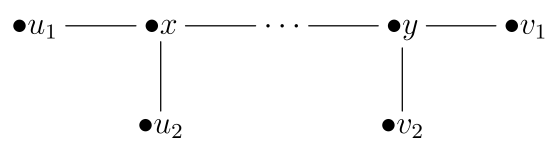
So every vertex of $\hat{Q}$ must be connected to at most 3 other vertices.

Now suppose that $\hat{Q}$ contains no vertices that are connected to 3 other vertices. Take a vertex x in $\hat{Q}$. We have from before that x can be connected to at most two other vertices, say x_{-1} and x_1. The vertices x_{-1}, x_1 can both only be connected to one other additional vertex each, say x_{-2}, x_2, since they are both already connected to x. We continue this process for all the remaining vertices. We obtain that x_{-i} and x_i must be pairwise different for all i (since otherwise we would not have the number of edges in $\hat{Q}$ strictly less than the number of vertices in $\hat{Q}$ which contradicts the proof of Lemma 4.13). We only have a finite number of vertices, so we obtain the (undirected) graph

$$\bullet \text{------} \cdots \text{------} \bullet x_{-2} \text{------} \bullet x_{-1} \text{------} \bullet x \text{------} \bullet x_1 \text{------} \bullet x_2 \text{------} \cdots \text{------} \bullet$$

which is exactly the Dynkin diagram A_n (see Fig. 5).

Now suppose that $\hat{Q}$ has 2 vertices x, y which are connected to 3 others. Then $\hat{Q}$ contains a full subgraph $\hat{Q}'$

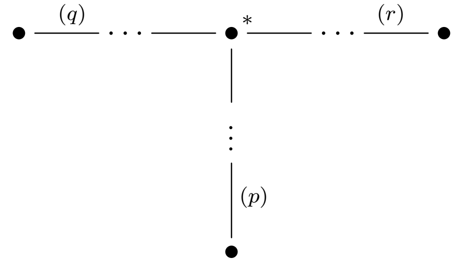


with say, k vertices. Now consider the dimension vector $\underline{n}$ whose dimension at the vertices u_1, u_2, v_1, v_2 is 1 and at the remaining vertices $x, \ldots, y$ is 2. We then get

$$q_{\hat{Q}}(\underline{n}) = (1 + 1 + 1 + 1 + 2^2(k-4)) - (2 + 2 + 2 + 2 + (k - 4 - 1)(2 \cdot 2) = 0.$$

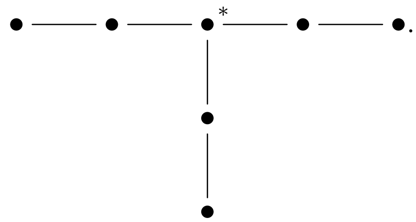
So $\hat{Q}$ can have at most one vertex connected to 3 other vertices.

We can now say that since $\hat{Q}$ can only have one vertex which is connected to 3 other vertices, then $\hat{Q}$ must be of the form



where the number of vertices in each branch of $\hat{Q}$ (not including the central vertex $*$) is p, q, r. We can assume without loss of generality that $p \leq q \leq r$. If one of either p, q, r is equal to 0 then we obtain the Dynkin diagram of type A_n, so now assume that $p, q, r > 0$.

We will now suppose that $p > 1$. Since $p \leq q \leq r$, $\hat{Q}$ contains a full subgraph $\hat{Q}'$

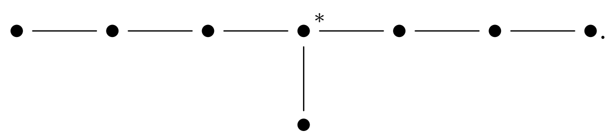


Consider the dimension vector $\underline{n}$ whose dimension at the vertex $*$ is 3, at the vertices adjacent to $*$ is 2 and at the outer vertices is 1. We then get

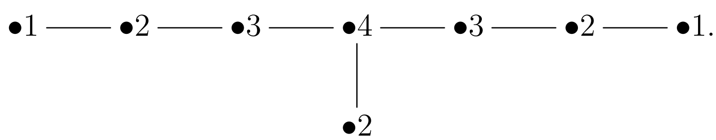
$$q_{\hat{Q}}(\underline{n}) = (3 \cdot 1 + 3 \cdot 2^2 + 3^2) - (2 + 2 + 2 + 6 + 6 + 6) = 0.$$

So $\hat{Q}$ must have $p = 1$.

Let us now consider the branch of $\hat{Q}$ with r vertices. Suppose that $r > 2$. Then $\hat{Q}$ contains the full subgraph $\hat{Q}'$



Consider the dimension vector $\underline{n}$ whose dimensions at the vertices of $\hat{Q}'$ are shown below:

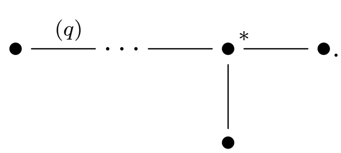


We then get

$$q_{\hat{Q}}(\underline{n}) = (2 \cdot 1 + 3 \cdot 2^2 + 2 \cdot 3^2 + 4^2) - (2 \cdot 2 + 2 \cdot 6 + 2 \cdot 12 + 8) = 0.$$

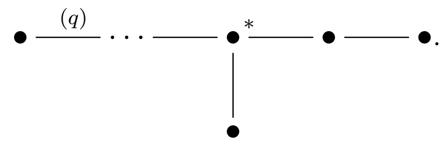
So $\hat{Q}$ must have $r \leq 2$.

We now just have two cases left to consider - when $r = 1$ and when $r = 2$. When $r = 1$, we get the graph

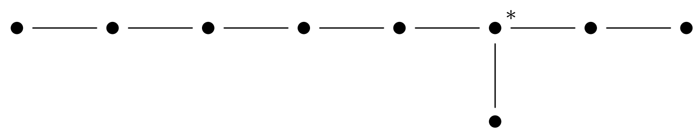


Here, q can be arbitrary, which gives the Dynkin diagram of type D_n.

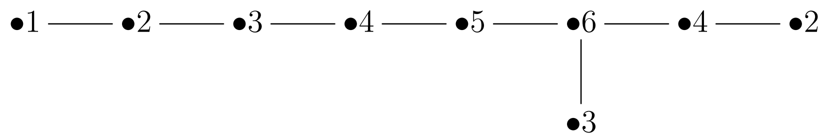
When $r = 2$, we get the graph



Suppose that $q > 4$. Then $\hat{Q}$ contains the full subgraph $\hat{Q}'$



Consider the dimension vector $\underline{n}$ whose dimensions at the vertices of $\hat{Q}'$ are shown below:



We then get

$$q_{\hat{Q}}(\underline{n}) = (1 + 2 \cdot 2^2 + 2 \cdot 3^2 + 2 \cdot 4^2 + 5^2 + 6^2) - (2 + 6 + 12 + 20 + 30 + 24 + 8 + 18) = 0.$$

So $\hat{Q}$ must have $q \leq 4$ when $r = 2$. When $q = 1$ we get D_5, when $q = 2$ we get E_6, when $q = 3$ we get E_7 and finally when $q = 4$ we get E_8.

So when Q has underlying undirected graph $\hat{Q}$ positive definite, then $\hat{Q}$ is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8, as required. □

The above proposition is in fact an if and only if statement, but the proof of the '⇒' direction of part (i) of Gabriel's Theorem only uses the implication proven above.

The following definition is from [16].

Definition 4.17. An open subset D of a topological space X is called *dense* if its closure $\bar{D}$ is equal to the whole space X. Equivalently, we have that a set D is *dense* if and only if it intersects every non–empty open set U of X.

We now need a technical lemma, the result of which will be used in one direction of the proof of part (i) of Gabriel's Theorem.

Lemma 4.18. *Let U, V be non–empty open dense sets in a set X. Then their intersection $U \cap V$ is dense. In particular, $U \cap V$ is non–empty.*

Proof. Let U, V be non–empty open dense sets and let W be a non–empty open set. To show that $U \cap V$ is dense, we need to show that $W \cap (U \cap V)$ is non–empty, by Definition 4.17. Since both U, W are open, then their intersection $W \cap U$ is open. Since U is dense and W is non–empty and open then $W \cap U$ is non–empty. We have that $(W \cap U) \cap V$ is non–empty since V is dense, but this is the same as $W \cap (U \cap V)$, and so $U \cap V$ is dense, as required.

The closure of the empty set is the empty set itself which is not equal to the whole set X and so $U \cap V \neq \emptyset$ by definition. □

4.3.4 Gabriel's Theorem - proof of part (i)

We only prove the '$\Rightarrow$' direction of part (i) for now, since the '$\Leftarrow$' direction follows immediately from part (ii) of Gabriel's theorem (see Section 4.4.3). The following proof is based on the proof from [3].

Proof. (Gabriel's Theorem - part (i))

Let Q be a quiver of finite orbit type (recall that we are assuming in this project that all quivers are connected, and so Q consists of just a single component). By Definition 4.2, we obtain that Q has finitely many isomorphism classes of indecomposable representations for an assigned dimension vector $\underline{n}$. By reformulating Proposition 3.5, $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ has finitely many orbits of $\text{GL}(\underline{n})$. Since there are only finitely many orbits, then let us denote them by $\mathcal{O}_1, \ldots, \mathcal{O}_l$. Now consider the closures of each $\mathcal{O}_i$, denoted $\bar{\mathcal{O}}_1, \ldots, \bar{\mathcal{O}}_l$. These closures are Zariski–closed subsets of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$. Since the closure $\bar{\mathcal{O}}$ of an orbit $\mathcal{O}$ is the union of the orbit itself together with the orbits of dimension strictly less than the dimension of $\mathcal{O}$ by Proposition 4.7, then we have that $\mathcal{O}_i \subseteq \bar{\mathcal{O}}_i$, for all $i = 1, \ldots, l$. Every point of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ has an orbit, and since $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ consists of an infinite union of points, then

$$\text{rep}_{\mathbb{K}}(Q, \underline{n}) = \bigcup_{i=1}^{l} \mathcal{O}_i \subseteq \bigcup_{i=1}^{l} \bar{\mathcal{O}}_i.$$

This union is finite because we have a finite number of orbits (*not* a finite number of points). However, since these orbit closures are themselves contained in $\text{rep}_{\mathbb{K}}(Q, \underline{n})$, we actually have equality, ie.

$$\text{rep}_{\mathbb{K}}(Q, \underline{n}) = \bigcup_{i=1}^{l} \bar{\mathcal{O}}_i,$$

and so we have a finite union of Zariski–closed sets which equals $\text{rep}_{\mathbb{K}}(Q, \underline{n})$.

Now, either all of these Zariski–closed sets are proper in $\text{rep}_{\mathbb{K}}(Q, \underline{n})$, that is they are all strictly contained in $\text{rep}_{\mathbb{K}}(Q, \underline{n})$, or at least one of these orbit closures is *not* proper and so is equal to the whole of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$. Let us assume then that the all of the Zariski–closed sets are proper in $\text{rep}_{\mathbb{K}}(Q, \underline{n})$.

In the Zariski topology, all proper closed sets are of the form

$$V(\mathcal{I}) := \{(x_1, \ldots, x_n) \in \mathbb{K}^n \mid f(x_1, \ldots, x_n) = 0 \, \forall f \in \mathcal{I}\}$$

where $\mathcal{I}$ is an ideal contained in the polynomial ring in n variables $\mathbb{K}[t_1, \ldots, t_n]$. These proper closed subsets have negligible area, and so the complements of all proper closed sets in the Zariski topology are dense, by Definition 4.17. Since the finite union of these proper closed subsets is equal to the whole of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ by assumption, then the finite intersection of the complements of these proper closed subsets gives us the empty set as an intersection of a finite number of dense sets, which is a contradiction by Lemma 4.18 and so our assumption was incorrect.

So we must be in the case where at least one of these closed sets $\bar{\mathcal{O}}_i$ is equal to the whole of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$, which implies that the orbit $\mathcal{O}_i$ is dense, by Definition 4.17.

Let $\mathcal{O}_{\mathbf{p}}$ be the orbit whose closure $\bar{\mathcal{O}}_{\mathbf{p}}$ is equal to the whole of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$, for a point $\mathbf{p} \in \text{rep}_{\mathbb{K}}(Q, \underline{n})$. Let $\text{G} := \text{GL}(\underline{n})$. We have that

$$\dim(\text{rep}_{\mathbb{K}}(Q, \underline{n})) = \dim(\bar{\mathcal{O}}_{\mathbf{p}}) = \dim(\mathcal{O}_{\mathbf{p}})$$

and

$$\dim(\mathcal{O}_{\mathbf{p}}) = \dim(\text{G}) - \dim(\text{G}_{\mathbf{p}})$$

by Proposition 4.7, where $\text{G}_{\mathbf{p}}$ is the stabilizer of $\mathbf{p}$. Since $\text{id} \in \text{G}_{\mathbf{p}}$, then we have that $\dim(\text{G}_{\mathbf{p}}) \geq 1$ and so

$$\dim(\text{rep}_{\mathbb{K}}(Q, \underline{n})) < \dim(\text{G})$$

or equivalently,

$$\dim(\text{rep}_{\mathbb{K}}(Q, \underline{n})) \leq \dim(\text{PGL}(\underline{n}))$$

since $\dim(\text{PGL}(\underline{n})) = \dim(\text{G}) - 1$ by the discussion at the end Section 4.3.2. Using this, the dimension of G given in Section 3.2.1, the dimension of $\text{rep}_{\mathbb{K}}(Q, \underline{n})$ given at the end of Section 3.1 and Definition 4.11 this gives us that

$$\begin{aligned}
\dim(\text{PGL}(\underline{n})) - \dim(\text{rep}_{\mathbb{K}}(Q, \underline{n})) &\geq 0 \\
\dim(\text{G}) - 1 - \dim(\text{rep}_{\mathbb{K}}(Q, \underline{n})) &\geq 0 \\
\sum_{x \in Q_0} {n_x}^2 - 1 - \sum_{a \in Q_1} n_{t(a)} n_{h(a)} &\geq 0 \\
q_Q(\underline{n}) - 1 &\geq 0 \\
q_Q(\underline{n}) &\geq 1
\end{aligned}$$

and so by Proposition 4.16, we have that $\hat{Q}$ is a simply–laced Dynkin diagram, as required. □

4.4 Gabriel's Theorem - Part (ii)

4.4.1 Root Systems and the Weyl Group

Since we are now working towards proving the second part of Gabriel's Theorem, we will assume from now on that all quivers are acyclic (that is, contain no cycles, whether orientated or not) since a quiver whose underlying undirected graph contains a cycle is not of ADE type. It is in fact possible to see that a quiver which contains a cycle cannot be of finite orbit type. Let a quiver Q contain a cycle in its underlying undirected graph $\hat{Q}$. Consider the dimension vector $\underline{n}$ whose dimension at all the vertices of the cycle is 1. Let all the arrows (except one) in the cycle be the identity map in a representation of the quiver, and at this remaining arrow put an arbitrary scalar $\lambda \in \mathbb{R}$. There are infinitely many values for λ and so there are infinitely many non–isomorphic representations [17].

The following section is adapted from [2] and [17], with some definitions from [18], [19] and [20].

Definition 4.19. Every quiver Q has a representation $Z = (\oplus_{x\in Q_0} Z_x, \{z_a\}_{a\in Q_1})$ with $Z_x = 0$ for all $x \in Q_0$ and $z_a = 0$ for all $a \in Q_1$. This is called the *zero representation.*

A non–zero representation V of Q is called *irreducible* (or *simple*) if the only subrepresentations of V are the zero representation and V itself.

Let Q be a quiver and fix $x \in Q_0$. We can now define the irreducible representation $E^x = (\bigoplus_{y\in Q_0} E^x_y\ , \{e^x_a\}_{a\in Q_1})$ as follows. For every $y \in Q_0$, define the vector spaces E^x_y to be

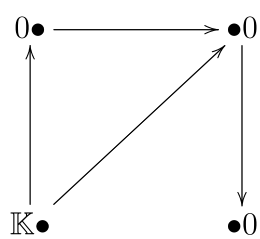
$$E^x_y = \begin{cases} \mathbb{K} & \text{if } y = x \\ 0 & \text{otherwise,} \end{cases}$$

and let e^x_a be the zero map for all $a \in Q_1$. We define the dimension vector of the vector space E^x_y to be

$$\underline{\epsilon}^x_y = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

We write $\underline{\epsilon}^x := \sum_{y\in Q_0} \underline{\epsilon}^x_y$ for the dimension vector of the irreducible representation E^x.

Example 4.20. Let Q be the quiver illustrated in Fig. 3. Then, for example, the irreducible representation E^1 of Q is



where all arrows are the zero map. The dimension vectors at each vertex are $\underline{\epsilon}^1_1 = (1,0,0,0)$, $\underline{\epsilon}^1_2 = (0,0,0,0)$, $\underline{\epsilon}^1_3 = (0,0,0,0)$ and $\underline{\epsilon}^1_4 = (0,0,0,0)$. So we have the dimension vector $\underline{\epsilon}^1 = (1,0,0,0)$ for the representation E^1.

Let Q be a quiver and let $E := \mathbb{R}^t$ (where t is the cardinality of the set Q_0) be an $\mathbb{R}$–vector space equipped with a positive definite symmetric bilinear form $(-,-)$ (like positive definite defined earlier for the Tits form of a quiver, we mean that $(x,x) > 0$ for all non–zero $x \in E$).

Definition 4.21. Let Q be a quiver and fix $x \in Q_0$. If $\underline{\epsilon}^x \in E \setminus \{0\}$ then the reflection $s^x := s^{\underline{\epsilon}^x} : E \to E$, called the *simple reflection*, in the hyperplane orthogonal to $\underline{\epsilon}^x$ is defined to be

$$s^x(\underline{\epsilon}^y) = \underline{\epsilon}^y - \frac{2(\underline{\epsilon}^y, \underline{\epsilon}^x)}{(\underline{\epsilon}^x, \underline{\epsilon}^x)}\underline{\epsilon}^x = \begin{cases} -\underline{\epsilon}^x & \text{if } y = x,\ y \in Q_0 \\ \underline{\epsilon}^x + \underline{\epsilon}^y & \text{if } x, y \text{ are connected by an edge} \\ \underline{\epsilon}^y & \text{otherwise.} \end{cases}$$

Let us consider the simple reflections for the dimension vectors in Example 4.20. We can compute $\underline{\epsilon}^2$, $\underline{\epsilon}^3$ and $\underline{\epsilon}^4$ in the same way as we computed $\underline{\epsilon}^1$, and we can then say that

$$\begin{aligned} s^1(\underline{\epsilon}^1) = -\underline{\epsilon}^1 &= (-1,0,0,0), \\ s^1(\underline{\epsilon}^2) = \underline{\epsilon}^1 + \underline{\epsilon}^2 &= (1,1,0,0), \\ s^1(\underline{\epsilon}^3) = \underline{\epsilon}^1 + \underline{\epsilon}^3 &= (1,0,1,0), \\ s^1(\underline{\epsilon}^4) = \underline{\epsilon}^4 &= (0,0,0,1). \end{aligned}$$

Definition 4.22. A *root system* is a subset $\Phi \subset E$ such that

(i) Φ is finite, Φ spans the vector space E and $0 \notin \Phi$;

(ii) if $\underline{\epsilon}^x \in \Phi$ then $\mathbb{R}\underline{\epsilon}^x \cap \Phi = \{\underline{\epsilon}^x, -\underline{\epsilon}^x\}$;

(iii) if $\underline{\epsilon}^x \in \Phi$ then $s^x\Phi = \Phi$;

(iv) if $\underline{\epsilon}^x, \underline{\epsilon}^y \in \Phi$ then $2(\underline{\epsilon}^y, \underline{\epsilon}^x)/(\underline{\epsilon}^x, \underline{\epsilon}^x) \in \mathbb{Z}$.

The elements of Φ are called *roots*.

Definition 4.23. A subset Π of a root system Φ is called a *basis* (or *simple system*) if

(i) Π is a vector space basis for the span of Φ;

(ii) every element of Φ is a linear combination of elements of Π where all the coefficients have the same sign, ie. for all roots $\underline{\epsilon}^y \in \Phi$ we can write

$$\underline{\epsilon}^y = \sum_{\underline{\epsilon}^x \in \Pi} k_{\underline{\epsilon}^x} \underline{\epsilon}^x$$

where either all $k_{\underline{\epsilon}^x}$ are non–negative integers (in which case we call $\underline{\epsilon}^y$ a *positive root*), or all $k_{\underline{\epsilon}^x}$ are non–positive integers (in which case we call $\underline{\epsilon}^y$ a *negative root*), and $\underline{\epsilon}^x \in \Pi$.

The elements of Π are called *simple roots*.

We denote the set of positive roots by Φ^+, and the set of negative roots by Φ^-. We obtain the root system Φ as a disjoint union $\Phi = \Phi^+ \sqcup \Phi^-$. Every root system Φ has a basis Π [2].

So the dimension vectors $\underline{\epsilon}^x$ for each $x \in Q_0$ in Example 4.20 are simple roots.

Definition 4.24. The *Weyl group* $\mathcal{W}$ is the group generated by the simple reflections $s^x \colon E \to E$, where $\underline{\epsilon}^x \in \Phi$ and $x \in Q_0$. So $\mathcal{W} := \langle s^1, \cdots, s^t \rangle$.

Since the Weyl group permutes the elements of Φ, and Φ is finite and spans E by the definition of a root system, then we have that $\mathcal{W}$ is finite [2].

4.4.2 Coxeter Functors

The following section has been adapted from [2] and [17] with some definitions coming from [18], [21], [22] and [23]. Before we will be able to define the Coxeter functors themselves, we first need some general definitions.

Definition 4.25. Let Q be a quiver and fix $x \in Q_0$. We define the action of a reflection s^x on Q as follows;

(i) if $y \in Q_0$ then $s^x(y) = y$;

(ii) if $a \in Q_1$, then

$$s^x(a) = \begin{cases} a & \text{if } t(a), h(a) \neq x \\ a^{\text{op}} & \text{if } t(a) = x \text{ or } h(a) = x \end{cases}$$

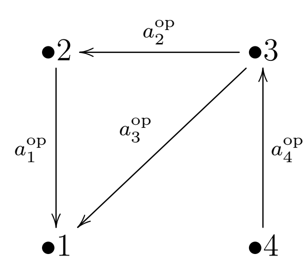
where a^{op} is the arrow a with opposite orientation, that is, $t(a^{\text{op}}) = h(a)$ and $h(a^{\text{op}}) = t(a)$. By applying the reflection s^x to a quiver Q, we obtain the *reflected quiver*

$$s^x Q := (s^x Q_0, s^x Q_1) = (Q_0, s^x Q_1)$$

which differs from Q only by orientation.

Note that we use the same notation s^x for the reflection of a quiver as for the simple reflections in the $\mathbb{R}$–vector space E.

Example 4.26. Let Q be the quiver illustrated in Fig. 3. Then $s^x Q$ is the quiver



Definition 4.27. Let Q be a quiver. A vertex $x \in Q_0$ is called

(i) a *sink* if $t(a) \neq x$ for all $a \in Q_1$;

(ii) a *source* if $h(a) \neq x$ for all $a \in Q_1$.

Our aim now is to extend reflections of dimension vectors at a vertex (see Section 4.4.1) to reflections of representations at vertices. Representations cannot be reflected over arbitrary vertices, but only at vertices which are either a sink or a source [17].

We are now in a position to define the Coxeter functors.

Definition 4.28. Let Q be a quiver, let $* \in Q_0$ be a sink and let $Q' := s^*Q$. We define the *left partial Coxeter functor*

$$C_*^+ \colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathrm{rep}_{\mathbb{K}}(Q')$$

between the categories of finite dimensional $\mathbb{K}$–linear representations of the quivers Q, Q' as follows.

Objects: Let V be an object in $\mathrm{rep}_{\mathbb{K}}(Q)$. We define the object

$$C_*^+ V = (\bigoplus_{x \in Q_0} V'_x \, , \, \{v'_a\}_{a \in Q_1})$$

in $\mathrm{rep}_{\mathbb{K}}(Q')$ as follows:

(i) For all $x \in Q_0$ with $x \neq *$, we let $V'_x = V_x$. For $x \in Q_0$ with $x = *$, we define V'_* to be the kernel of the $\mathbb{K}$–linear map

$$\bigoplus_{a \colon t(a) \to *} V_{t(a)} \to V_*$$

(here, the direct sum is being taken over all the $a \in Q_1$ with $h(a) = *$).

(ii) For all $a \in Q_1$ with $h(a) \neq *$, we let $v'_a = v_a$. For all $a \in Q_1$ with $h(a) = *$, we define the map $v'_a \colon V'_* \to V'_{t(a)} = V_{t(a)}$ to be the composition $v'_a := \pi_a \circ \iota$, where ι is the natural embedding of V'_* into $\bigoplus_{a \colon t(a) \to *} V_{t(a)}$ and π_a is the projection of this direct sum onto the term $V_{t(a)}$ for each $a \colon t(a) \to *$. So $*$ becomes a source in Q'.

Morphisms: Let $\phi \colon V \to W$ be a morphism in $\mathrm{rep}_{\mathbb{K}}(Q)$. We define the morphism

$$C_*^+ \phi = \phi' = (\phi'_x)_{x \in Q_0} \colon C_*^+ V \to C_*^+ W$$

in $\mathrm{rep}_{\mathbb{K}}(Q')$ as follows. For all $x \in Q_0$ with $x \neq *$, we let $\phi'_x = \phi_x$. For $x \in Q_0$ with $x = *$, we define ϕ'_* to be the unique $\mathbb{K}$–linear map which makes the following diagram commutative:

$$\begin{array}{ccccccc}
0 & \longrightarrow & (C_*^+ V)_* & \xrightarrow{\iota} & \bigoplus_{a \colon t(a) \to *} V_{t(a)} & \xrightarrow{(v_a)_a} & V_* \\
 & & \Big\downarrow{\scriptstyle \phi'_*} & & \Big\downarrow{\scriptstyle \bigoplus_{a \in Q_1} \phi_{t(a)}} & & \Big\downarrow{\scriptstyle \phi_*} \\
0 & \longrightarrow & (C_*^+ W)_* & \xrightarrow{\iota} & \bigoplus_{a \colon t(a) \to *} W_{t(a)} & \xrightarrow{(w_a)_a} & W_*.
\end{array}$$

We can define the analogous functor for a source vertex in Q, but first we need to recall a definition from linear algebra. The *cokernel* of a linear map $f \colon U \to U'$ of vector spaces U, U' is the quotient space of the codomain of f by the image of f, that is,

$$\mathrm{coker}(f) := U'/\mathrm{im}(f).$$

Definition 4.29. Let Q' be a quiver, let $* \in Q'_0$ be a source and let $Q := s^*Q'$. We define the *right partial Coxeter functor*

$$C_*^- \colon \mathrm{rep}_{\mathbb{K}}(Q') \to \mathrm{rep}_{\mathbb{K}}(Q)$$

between the categories of finite dimensional $\mathbb{K}$–linear representations of the quivers Q', Q as follows.

Objects: Let V' be an object in $\mathrm{rep}_{\mathbb{K}}(Q')$. We define the object

$$C_*^- V' = \Big(\bigoplus_{x \in Q_0} V_x \;,\; \{v_a\}_{a \in Q_1}\Big)$$

in $\mathrm{rep}_{\mathbb{K}}(Q)$ as follows:

(i) For all $x \in Q'_0$ with $x \neq *$, we let $V_x = V'_x$. For $x \in Q'_0$ with $x = *$, we define V_* to be the cokernel of the $\mathbb{K}$–linear map

$$V'_x \to \bigoplus_{a \colon * \to h(a)} V'_{h(a)}$$

(here, the direct sum is being taken over all the $a \in Q'_1$ with $t(a) = *$).

(ii) For all $a \in Q'_1$ with $t(a) \neq *$, we let $v_a = v'_a$. For all $a \in Q'_1$ with $t(a) = *$, we define the map $v_a \colon V_{h(a)} = V'_{h(a)} \to V_*$ to be the composition $v_a := \pi \circ \iota_a$, where ι_a is the embedding of $V'_{h(a)}$ into $\bigoplus_{a \colon * \to h(a)} V'_{h(a)}$ and π is the natural cokernel projection of this direct sum onto the term V_*. So $*$ becomes a sink in Q.

Morphisms: Let $\phi' \colon V' \to W'$ be a morphism in $\mathrm{rep}_{\mathbb{K}}(Q')$. We define the morphism

$$C_*^- \phi' = \phi = (\phi_x)_{x \in Q_0} \colon C_*^- V' \to C_*^- W'$$

in $\mathrm{rep}_{\mathbb{K}}(Q)$ as follows. For all $x \in Q'_0$ with $x \neq *$, we let $\phi_x = \phi'_x$. For $x \in Q'_0$ with $x = *$, we define ϕ_* to be the unique $\mathbb{K}$–linear map which makes the following diagram commutative:

$$\begin{array}{ccccccc}
V'_* & \xrightarrow{\ \iota_a\ } & \bigoplus_{a \colon * \to h(a)} V'_{h(a)} & \xrightarrow{\ (v_a)_a\ } & (C_*^- V')_* & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle \phi'_*} & & \Big\downarrow{\scriptstyle \bigoplus_{a \in Q_1} \phi'_{h(a)}} & & \Big\downarrow{\scriptstyle \phi_*} & & \\
W'_* & \xrightarrow{\ \iota_a\ } & \bigoplus_{a \colon * \to h(a)} W'_{h(a)} & \xrightarrow{\ (w_a)_a\ } & (C_*^- W')_* & \longrightarrow & 0.
\end{array}$$

The following theorem of Bernstein and Gel'fand can be found in [24] (Theorem 1.1) or as the version stated below in [2] (Lecture 5, Theorem 4).

Theorem 4.30 (Berstein and Gel'fand). *(i) Let Q be a quiver, let V be an indecomposable representation of Q and let $x \in Q_0$ be a sink. There are two cases:*

(a) $V = E^x$ *if and only if* $C_x^+ V = 0$*;*

(b) $V \neq E^x$*, in which case*

i. $C_x^+ V$ *is indecomposable;*

ii. $C_x^- C_x^+ V \cong V$*;*

iii. $\dim(C_x^+ V) = s^x \dim(V)$.

(ii) Now let $x \in Q_0$ be a source. There are two cases:

(a) $V = E^x$ *if and only if* $C_x^- V = 0$*;*

(b) $V \neq E^x$*, in which case*

i. $C_x^- V$ *is indecomposable;*

ii. $C_x^+ C_x^- V \cong V$*;*

iii. $\dim(C_x^- V) = s^x \dim(V)$.

The proof from [2] defines the natural transformations $i^x \colon C_x^- C_x^+ \to \mathrm{id}$, $p^x \colon \mathrm{id} \to C_x^+ C_x^- \to \mathrm{id}$ of functors $\mathrm{rep}_{\mathbb{K}}(Q) \to \mathrm{rep}_{\mathbb{K}}(Q)$. These maps are then defined for a representation V of $\mathrm{rep}_{\mathbb{K}}(Q)$ and some properties are listed and subsequently proven. The proof is technical, but not instructive, and so we refer the reader to [2], [18] or [24].

The main result given by this theorem is that the partial Coxeter functors are almost inverse to each other ('almost' in the sense that we have this restriction that the vertex must be either a sink or a source).

We are still aiming to describe the reflections of representations of a fixed quiver Q, but the Coxeter functors change the original quiver. However, we can do the following to ultimately allow us to define two endofunctors (a functor from a category to itself) on $\mathrm{rep}_{\mathbb{K}}(Q)$.

Let Q be a quiver and number the vertices $1, \ldots, t$ such that for all $a \in Q_1$ we have $t(a) > h(a)$. Such a numbering exists because we are assuming that $\hat{Q}$ contains no cycles [22]. We can then define sequences of quiver reflections (composing right to left)

$$c = c^+ := s^t s^{t-1} \cdots s^2 s^1,$$

$$c^- := c^{-1} = s^1 s^2 \ldots s^t$$

where s^x, $x \in Q_0$, is the reflection on Q. The sequence c is called the *Coxeter transformation* corresponding to the chosen numbering of the vertices in Q.

Definition 4.31. Let Q be a quiver and fix $x \in Q_0$. We say that x is (+)–*admissible* if x is a sink, that is, all arrows $a \in Q_1$ which contain x have $h(a) = x$. We say that a sequence of vertices $1, 2, \ldots, t$, where $1, \ldots, t \in Q_0$, is (+)–*admissible* with respect to the orientation of Q if:

- 1 is a sink with respect to the orientation of Q,
- 2 is a sink with respect to the orientation of s^1Q,
- $\vdots$
- t is a sink with respect to the orientation of $s^{t-1}\cdots s^2s^1Q$.

We say that x is $(-)$–*admissible* if x is a source, that is, all arrows $a \in Q_1$ which contain x have $t(a) = x$. We say that a sequence of vertices $1, 2, \ldots, t$, where $1, \ldots, t \in Q_0$, is $(-)$–*admissible* with respect to the orientation of Q if:

- t is a source with respect to the orientation of Q,
- $t-1$ is a source with respect to the orientation of s^tQ,
- $\vdots$
- 1 is a source with respect to the orientation of $s^ts^{t-1}\cdots s^2Q$.

We can now see that the sequence c is $(+)$–admissible and the sequence c^{-1} is $(-)$–admissible. Also notice that $cQ = Q$ since each arrow has its orientation changed exactly twice. The analogously defined element $c \in \mathcal{W}$ is called the *Coxeter element* of $\mathcal{W}$ [17].

We now define two endofunctors C^+, C^- on $\mathrm{rep}_{\mathbb{K}}(Q)$ as follows:

$$C^+ := C_t^+ \circ C_{t-1}^+ \circ \cdots \circ C_1^+ \colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathrm{rep}_{\mathbb{K}}(Q),$$

$$C^- := C_1^- \circ C_2^- \circ \cdots \circ C_t^- \colon \mathrm{rep}_{\mathbb{K}}(Q) \to \mathrm{rep}_{\mathbb{K}}(Q).$$

These endofunctors are called the *Coxeter functors*, and are well–defined because of the choice of numbering [2]. These functors are independent of the choice of numbering. The following lemma proves as much, and can be found in [22] (Lemma 4.2) or [24] (Lemma 2.1, (3)).

Lemma 4.32. *The Coxeter functors C^+ and C^- are independent of the choice of admissible numbering.*

Proof. We only give a proof for the functor C^+ since the proof for C^- is similar.

Let $x_1, \ldots, x_t$ and $y_1, \ldots, y_t$ be $(+)$–admissible sequences of vertices for the same quiver Q. Note that if i, j are both $(+)$–admissible vertices in Q_0 with $i \neq j$, then we have $C_i^+ \circ C_j^+ = C_j^+ \circ C_i^+$. Let $y_k = x_1$. Then x_1 is not equal to any of the vertices $y_1, \ldots, y_{k-1}$ and hence

$$C_{x_1}^+ \circ C_{y_{k-1}}^+ \circ \cdots \circ C_{y_1}^+ = C_{y_{k-1}}^+ \circ \cdots \circ C_{y_1}^+ \circ C_{x_1}^+.$$

Now let $y_k = x_2$. Then x_2 is not equal to x_1 or any of the vertices $y_1, \ldots, y_{k-1}$ and hence

$$C_{x_2}^+ \circ C_{x_1}^+ \circ C_{y_{k-1}}^+ \circ \cdots \circ C_{y_1}^+ = C_{y_{k-1}}^+ \circ \cdots \circ C_{y_1}^+ \circ C_{x_2}^+ \circ C_{x_1}^+.$$

We can continue this process for the vertices $x_3, \ldots, x_t$ to prove that

$$C^+_{y_t} \circ \cdots \circ C^+_{y_1} = C^+_{x_t} \circ \cdots \circ C^+_{x_t}.$$

□

We now give a corollary of Theorem 4.30 (the statement of which can be found in [18], [23] (Corollary I.4.2) or [24] (Corollary 3.1)).

Corollary 4.33. *Let Q be a quiver and let $V \in rep_{\mathbb{K}}(Q)$ be an indecomposable representation of Q. Suppose that the sequence $x_1, \ldots, x_t$ of vertices is $(+)$–admissible with respect to the orientation of Q. Set $V_k := C^+_{x_k} \circ \cdots \circ C^+_{x_1}(V)$ and $\underline{m}_k := s^{x_k} \cdots s^{x_1}\underline{n}$ for $0 \leq k \leq t$. If i is the last index such that $\underline{m}_k > 0$ for $k \leq i$, then*

(i) each V_k is an indecomposable representation of Q for $k \leq i$ and $V = C^-_{x_1} \circ \cdots \circ C^-_{x_k}(V_k)$;

(ii) if $i < t$, then

- *$V_{i+1} = V_{i+2} = \cdots = V_t = 0$;*
- *$V_i = E^{i+1}$;*
- *$V = C^-_{x_1} \circ \cdots \circ C^-_{x_i}(E^{i+1})$.*

A similar statement is true for $(-)$–admissible sequences.

We now just need one more result before we can go on to prove part (ii) of Gabriel's Theorem. The following lemma is from [2] (Lecture 5, Lemma 1).

Lemma 4.34. *Let Q be a quiver with underlying undirected graph $\hat{Q}$ a Dynkin diagram of type ADE. Suppose that $\underline{n}$ is a non–zero dimension vector.*

(i) $c\underline{n} \neq \underline{n}$;

(ii) there exists a $k \in \mathbb{N}$ such that $c^k\underline{n}$ is not positive.

[2] uses the formula $\langle \underline{n}, c\underline{m} \rangle = -\langle \underline{m}, \underline{n} \rangle$ to prove (i). The proof then assumes that $c\underline{n} = \underline{n}$ and then uses the formula to derive a contradiction. The proof of (ii) from [2] uses the finiteness of the Weyl group to say that c must have finite order. This proof is short and straightforward to understand and so we refer the reader to there (or [18] (Lemma 5.33)) for a proof.

4.4.3 Gabriel's Theorem - proof of part (ii)

We are now in a position to prove part (ii) of Gabriel's Theorem. We will use the proof from [24] (which is similar to the proofs found in [2], [17], [18] and [23]). Proving this part of the theorem will prove the '$\Leftarrow$' direction of part (i) (see Section 4.3) since it will give us that if Q is a quiver such that $\hat{Q}$ is of ADE type, then $\underline{n}$ is the dimension of a (unique) indecomposable representation only if $\underline{n} \in \Phi^+$, and since there is only a finite number of $\underline{n}$'s (because $\Phi = \Phi^+ \sqcup \Phi^-$ is finite) then there is only a finite number of isomorphism classes of indecomposable representations of Q, and so Q is of finite orbit type.

Proof. (Gabriel's Theorem - part (ii)) We first need to show that if $V \in \text{rep}_{\mathbb{K}}(Q)$ is an indecomposable representation of a quiver Q with $\hat{Q}$ of ADE type with assigned dimension vector $\underline{n}$, then $\underline{n}$ is in fact a root.

Let V be such a representation. Choose a numbering $x_1, \ldots, x_t$ of the vertices of Q such that for any directed edge $a \in Q_1$ we have $t(a) > h(a)$. Let

$$c = s^{x_t} s^{x_{t-1}} \cdots s^{x_2} s^{x_1}$$

be the corresponding Coxeter transformation. By Lemma 4.34, there exists a $k \in \mathbb{N}$ such that $c^k \underline{n}$ is not positive. If we consider the (+)–admissible sequence

$$y_1, y_2, \ldots, y_{tk} = (x_1, \ldots, x_t, x_1, \ldots, x_t, \ldots, x_1, \ldots, x_t) \ (k \text{ times})$$

then we have

$$s^{y_{tk}} \cdots s^{y_1} \underline{n} = c^k \underline{n} \not> 0.$$

By Corollary 4.33, it follows that there exists an index $i < tk$ (which depends only on the dimension vector $\underline{n}$) such that

$$V = C^-_{y_1} \circ \cdots \circ C^-_{y_i}(E^{i+1})$$

and

$$\underline{n} = s^{y_1} \cdots s^{y_i}(\underline{\epsilon}^{i+1}).$$

So it follows that the dimension vector $\underline{n}$ is a positive root, and V is determined by $\underline{n}$.

For the converse, let $\underline{r}$ be a positive root. Consider the sequence

$$s^1 \underline{r}, s^2 s^1 \underline{r}, s^3 s^2 s^1 \underline{r}, \ldots .$$

Let i be the last index for which

$$\underline{r}' := s^i s^{i_1} \cdots s^1 \underline{r} > 0$$

(this sequence has a negative root by Lemma 4.34, and so $\underline{r}'$ is the element preceding the first negative root).

So $\underline{r}'$ is a positive root and $s^{i+1}\underline{r}'$ is a negative root. However, since the simple reflections only change one coordinate, we get

$$\underline{r}' = \underline{\epsilon}^{i+1}$$

and

$$s^t s^{t-1} \cdots s^i \underline{r} = \underline{\epsilon}^{i+1}.$$

Recall that E^{i+1} is the representation with dimension vector $\underline{\epsilon}^{i+1}$. We define

$$V := C^-_1 \circ \cdots \circ C^-_i(E^{i+1})$$

which is an indecomposable representation with dimension vector $\underline{r}$, by Corollary 4.33. $\square$

5 Conclusion

In this project, we set out to introduce the reader to quivers and their representations and then to go on and prove a remarkable theorem about their isomorphism classes. Gabriel's theorem was generalised to arbitrary quivers in Kac's Theorem by Victor Kac, which explained that, for example, the dimension vectors of indecomposable representations of a quiver are positive roots of the associated Kac-Moody Lie algebra [25].

Quivers are not restricted to just representation theory, but can be found in algebraic geometry, non–commutative geometry, Lie algebra and physics [25]. This project is by no means exhaustive, but aims to give an introduction to those studying quivers for the first time.

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