

Mori Dream Spaces as Fine Moduli of Quiver Representations

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Chapter 1

Introduction

1.1 Introduction

Mori Dream Spaces and their Cox rings have been the subject of a great deal of interest since their introduction by Hu–Keel [19] over a decade ago. From the geometric side, these varieties enjoy the property that all operations of the Mori programme can be carried out by variation of GIT quotient, while from the algebraic side, obtaining an explicit presentation of the Cox ring is an interesting problem in itself. Examples include \mathbb{Q} -factorial projective toric varieties, spherical varieties and log Fano varieties of arbitrary dimension. In this thesis we use the representation theory of quivers to study multigraded linear series on Mori Dream Spaces. Our main results construct Mori Dream Spaces as fine moduli spaces of ϑ -stable representations of bound quivers for a special stability condition ϑ , thereby extending results of Craw–Smith [10] for projective toric varieties.

Let X be a Mori Dream Space and let $\mathcal{L} = (L_0, L_1, \dots, L_r)$ be a collection of effective line bundles on X with $L_0 = \mathcal{O}_X$. In Chapter 3 we show how to construct a quiver of sections for \mathcal{L} . We would like this quiver to encode the sections of $L_j \otimes L_i^{-1}$ for every L_i and L_j in \mathcal{L} , but we are obstructed by the lack of a canonical basis for the space $H^0(X, L_j \otimes L_i^{-1})$. However, every Mori Dream Space admits a natural embedding into a projective toric variety \tilde{X} , whose class group is isomorphic to that of X . We harness a key property of this ambient toric variety, or more precisely of the collection $\tilde{\mathcal{L}} = (E_0, \dots, E_r)$ on \tilde{X} obtained by lifting \mathcal{L} from X . While the spaces $H^0(X, L_j \otimes L_i)$ have no canonical basis, $H^0(\tilde{X}, E_j \otimes E_i^{-1})$ certainly does: the torus invariant sections. We define the quiver of sections for $\tilde{\mathcal{L}}$ on \tilde{X} to be the quiver of sections for $\tilde{\mathcal{L}}$ on \tilde{X} , as given in Craw–Smith [10].

The key difference in the Mori Dream Space case lies in ideal of relations in the path algebra. We define an ideal of relations R in the paths algebra which encodes not only the “toric relations” given in [10], but also all the relations in the Cox ring of X . Indeed, the bound quiver of sections Q for \mathcal{L} is finite, acyclic and the quotient $\mathbb{k}Q/R$ is isomorphic to the endomorphism algebra

$A_{\mathcal{L}} = \text{End}(\bigoplus_{0 \leq i \leq r} L_i)$. Setting aside the ideal of relations for now, we define the *multigraded linear series* of the collection \mathcal{L} to be the toric quiver variety $|\mathcal{L}| = \mathcal{M}_{\vartheta}(Q)$ obtained as the fine moduli space of ϑ -stable representations of Q with dimension vector $(1, \dots, 1)$ for the special weight vector $\vartheta = (-r, 1, \dots, 1)$. This fine moduli space carries a collection of tautological line bundles $(\mathcal{W}_0, \dots, \mathcal{W}_r)$ with $\mathcal{W}_0 = \mathcal{O}_{|\mathcal{L}|}$. Since paths in the quiver arise from sections of line bundles of the form $L_j \otimes L_i^{-1}$ on X , evaluating these sections defines a rational map $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$. Our first main result describes the geometry of this map.

Theorem 1.1.1. *For a collection $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ of effective line bundles on X , the map $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$ is a morphism if and only if each L_i is basepoint-free, in which case the image is presented explicitly as a geometric quotient and the tautological bundles satisfy $\varphi_{|\mathcal{L}|}^*(\mathcal{W}_i) = L_i$.*

If each L_i on X is the restriction of a basepoint-free line bundle on \tilde{X} then this morphism is simply the restriction of the morphism from [10, Theorem 1]. This is typically not the case, however, because the nef cone of X may be the union of the nef cones of a finite collection of ambient toric varieties.

We provide a necessary and sufficient criterion for $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$ to be a closed immersion, and a straightforward application of multigraded regularity due to Hering–Schenck–Smith [17] (see also Maclagan–Smith [22]) provides an efficient way to exhibit many collections that give rise to closed immersions. The resulting geometric quotient constructions of X are new, and while they cannot improve upon the Hu–Keel construction from the birational point of view, it is sometimes possible to encode more refined information of X via \mathcal{L} , such as its bounded derived category of coherent sheaves on X .

In Chapter 4 we give our second main result. This is more algebraic, and provides a fine moduli description of X . The ideal of relations R in the path algebra $\mathbb{k}Q$ defines an ideal I_R in the Cox ring of $|\mathcal{L}|$ that cuts out $\mathcal{M}_{\vartheta}(\text{mod}(A_{\mathcal{L}}))$, the fine moduli space of ϑ -stable $A_{\mathcal{L}}$ -modules with dimension vector $(1, \dots, 1)$. This subscheme contains the image of the morphism $\varphi_{|\mathcal{L}|}$ from Theorem 1.1.1, and in general this inclusion is proper. Nevertheless, by saturating I_R with the irrelevant ideal for the GIT quotient construction of the multigraded linear series, and by comparing the result with the ideal I_Q that cuts out the image of $\varphi_{|\mathcal{L}|}$, we obtain the following algebraic result.

Theorem 1.1.2. *For any Mori Dream Space X , there exist (many) collections \mathcal{L} on X such that the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$ identifies X with the fine moduli space $\mathcal{M}_{\vartheta}(\text{mod}(A_{\mathcal{L}}))$, and the tautological line bundles on $\mathcal{M}_{\vartheta}(\text{mod}(A_{\mathcal{L}}))$ coincide with the line bundles of \mathcal{L} .*

Our proof of this results uses as far as possible the analogous result from [10, Theorem 1.2] for the ambient toric variety, though much remains to be done because I_R can be rather complicated.

More generally, when the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$ is a closed immersion it identifies X with $\mathcal{M}_{\vartheta}(\text{mod}(A_{\mathcal{L}}))$ precisely when the saturation of I_R by the irrelevant ideal coincides with the ideal

I_Q . These ideals can be computed explicitly in any given example (see Chapter 5), so it is possible to check directly whether Theorem 1.1.2 holds (subject to computational limitations).

The final two chapters of this thesis are more computational in nature. In Chapter 5 we give a computational method to find the ideals which cut out $\varphi_{|\mathcal{L}|}(X)$ and $\mathcal{M}_\vartheta(Q, R)$ in both the Mori Dream Space and toric cases. In Chapter 6 we use these computations to verify the results of Chapter 4 in three cases: for two non-toric del Pezzo surfaces and for the Grassmannian $\text{Gr}(2, 4)$.

For a list of line bundles \mathcal{L} , we wish to check whether X is isomorphic to the moduli space of bound quiver representations of the quiver of sections for \mathcal{L} . We will see that this amounts to checking whether $I_Q = I_R : B_Y^\infty$. In Chapter 5 we present a method for computing $\widetilde{I}_R, I_R, \widetilde{I}_Q$ and I_Q for a given quiver Q , and as an application we show that $I_Q = I_R : B_Y^\infty$ for certain collections of line bundles on X_4, X_5 and $\text{Gr}(2, 4)$.

We give code which, given a quiver Q , outputs a list of all paths in Q . To find \widetilde{I}_R as defined in [10], we must simply check through all pairs of paths to find all those with the same head, tail and label. Finding generators for I_R is more complicated. In Lemma 5.1.7, we give a generating set for I_R conducive to calculations, we give pseudocode to compute such a generating set explicitly.

We show in Proposition 5.2.1 that the ideals I_Q and \widetilde{I}_Q can be written as kernels of \mathbb{k} -algebra homomorphisms. They can therefore be computed using Elimination theory. We give Macaulay2 code for computing both \widetilde{I}_Q and I_Q in section 5.2.2.

In chapter 6, we illustrate the method for a pair of del Pezzo surfaces and the Grassmannian $\text{Gr}(2, 4)$. For the del Pezzo surfaces, we choose \mathcal{L} to be a full, strongly exceptional collection of line bundles. Such collections are of particular interest because they freely generate the bounded derived category of coherent sheaves on X , that is, the functor

$$\mathbf{R}\text{Hom}(\mathcal{F}, -) : D^b(\text{coh}(X)) \longrightarrow D^b(\text{mod}(A_{\mathcal{L}}))$$

is an equivalence of bounded derived categories. A result of Bergman–Proudfoot [3] establishes that the del Pezzo surface X is isomorphic to a connected component of $\mathcal{M}_\vartheta(\text{mod}(A_{\mathcal{L}}))$ in each case, and our computations demonstrate that in fact X is isomorphic to the moduli space. For the Grassmannian $X = \text{Gr}(2, 4)$, we show that $X \cong \mathcal{M}_\vartheta(\text{mod}(A_{\mathcal{L}}))$ when $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}(2), \mathcal{O}(4))$.

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mathematical ability. Last but not least, I would like to thank Duncan Somerville for his constant support.

1.3 Declaration

Chapters 1,3 and 4 and section 6.1 are revised versions of material from Craw–Winn [11]. Chapter 2 is expository. To the best of my knowledge, the remainder of this thesis is original work by the author, except where explicitly stated otherwise.

Chapter 2

Background

In this chapter we summarise necessary background material. In section 2.1 we first consider toric varieties. We show how toric varieties are constructed from fans (see Fulton [15], Cox–Little–Schenck [6]) and describe the construction of toric varieties as GIT quotients (see Cox [7], Mukai [24], Dolgachev [13]). Secondly, we introduce a generalisation of projective \mathbb{Q} -factorial toric varieties: Mori Dream Spaces. These will be the primary objects of interest in this thesis. We give background material on Mori Dream Spaces, including their construction as GIT quotients, after Hu–Keel [19], Hassett–Tschinkel [16], Laface–Velasco [21]. In section 2.2, we consider two important families of Mori Dream Spaces. In section 2.2.1 we give background information on Grassmannians (see Mukai [24]) and describe their Cox Rings. In section 2.2.2 we summarise material from Batyrev–Popov [2] and Manin [23] on del Pezzo surfaces. These will be our main source of examples of Mori Dream Spaces. We summarise results due to Batyrev–Popov giving generators and relations for the del Pezzo Surfaces of degree 3,4 and 5. In section 2.2.3 we give an explicit computation of $\text{Cox}(X_5)$, following Dernethal [12].

In section 2.3 we introduce the notion of multigraded regularity for projective toric varieties (see Maclagan–Smith [22], Hering–Schenck–Smith [17]), which will be a crucial component of the proofs in Chapter 4. In section 2.4 we give background information on quivers from Craw–Smith [10].

This thesis continues the programme begun by Craw–Smith in [10], extending from projective toric varieties to the Mori Dream Space case. In section 2.5, we summarise the results of [10]. For a collection of line bundles on a toric variety X , we introduce quivers of sections for toric varieties. We show how this quiver allows us to define a new ambient space for the toric varieties, the multilinear series, and give necessary and sufficient conditions for the existence of a morphism from X to this ambient variety. If the morphism exists, its image is a GIT quotient. It is almost always possible to find a list of line bundles \mathcal{L} such that the morphism is a closed immersion, and the image of X is a moduli space of bound representations of the quiver of sections for \mathcal{L} .

2.1 Mori Dream Spaces

In this section we give background information on our objects of study: Mori Dream Spaces. We first examine a special case, projective toric varieties, paying particular attention to their construction as GIT quotients.

2.1.1 Toric Varieties

We summarise material from Fulton [15] and Cox–Little–Schenck [6].

Let V be a real vector space. A *strongly convex polyhedral cone* in V is the span over \mathbb{R}^+ of a finite collection of vectors which does not contain a line through 0. Let σ be a convex polyhedral cone. We say a hyperplane H is a supporting hyperplane of σ if σ is contained in a halfspace defined by H and $\sigma \cap H \neq 0$. A *face* of σ is the intersection of σ with a supporting hyperplane. Given a cone σ , its *dual cone* σ^\vee is defined to be $\sigma^\vee := \{v \in V^* \mid \langle u, v \rangle \geq 0 \text{ for all } u \in \sigma\}$.

Let $N \cong \mathbb{Z}^r$ be a lattice, let $M := \text{Hom}(N, \mathbb{Z})$ be its dual lattice and let $N_{\mathbb{R}} := N \otimes \mathbb{R}$. We define a *rational strongly convex polyhedral cone* in $N_{\mathbb{R}}$ to be a strongly convex polyhedral cone generated by a finite collection of vectors in N . A fan Σ is a collection of rational convex polyhedral cones in $N_{\mathbb{R}}$ such that the faces of every cone in Σ are also in Σ , and such that every pair of cones in Σ intersects in a common face. We also assume that Σ is non-degenerate in the sense that it is not contained in any vector subspace of $N_{\mathbb{R}}$.

We define a *toric variety* $X = X(\Sigma)$ as follows. For every cone $\sigma \in \Sigma$ we define an open set

$$U_\sigma := \text{Spec}(\mathbb{k}[\sigma^\vee \cap M]).$$

If τ is a face of a cone σ in Σ , then there is a natural embedding

$$U_\tau \hookrightarrow U_\sigma$$

If we consider any two cones $\sigma, \sigma' \in \Sigma$, then their intersection $\tau := \sigma \cap \sigma'$ is a common face of both. Hence U_τ embeds into both U_σ and $U_{\sigma'}$. The toric variety X is defined to be the variety obtained by gluing each pair of open sets U_σ and $U_{\sigma'}$ along the open subset of each isomorphic to U_τ . The subvariety U_0 is an algebraic torus, its natural action on itself extends to an action of the torus on X . We say an n dimensional cone is simplicial if it has precisely n generators. We say a fan is simplicial if every cone in the fan is simplicial.

Proposition 2.1.1. *A toric variety $X(\Sigma)$ is \mathbb{Q} -factorial if and only if Σ is simplicial.*

Let $\Sigma(1)$ denote the set of one dimensional cones (or rays) of the fan Σ , we assume this set has cardinality d and denote the j th element of this set by τ_j . The elements of $\Sigma(1)$ determine irreducible codimension one torus invariant subvarieties of $X(\Sigma)$. These subvarieties generate the

free group of Weil divisors, \mathbb{Z}^d . Let $\text{Cl}(X)$ denote the class group of X : the group of Weil divisors modulo linear equivalence. We obtain a map from $\text{deg} : \mathbb{Z}^d \rightarrow \text{Cl}(X)$ which maps a Weil divisor to its equivalence class. The map deg fits into a short exact sequence:

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^d \longrightarrow \text{Cl}(X) \longrightarrow 0 \quad (2.1.1)$$

The first map sends $u \in M$ to $\sum_{j=1}^d \langle u, v(j) \rangle D_j$, where $v(j)$ is the vector in N of least magnitude which generates τ_j , and D_j is the Weil divisor corresponding to τ_j .

We construct X as a GIT quotient, summarising material from e.g. Cox [7], King [20], Mukai [24], Dolgachev [13], Craw [9].

We define the *Cox ring* of X to be

$$\text{Cox}(X) := \mathbb{k}[x_1, \dots, x_d].$$

This is the semigroup algebra of the effective cone of Weil divisors, $\mathbb{k}[\mathbb{N}^d]$. The map $\mathbb{Z}^d \rightarrow \text{Cl}(X)$ induces a map

$$\text{deg} : \text{Cox}(X) \longrightarrow \text{Cl}(X)$$

which gives a $\text{Cl}(X)$ grading of $\text{Cox}(X)$. We define the D th graded part to be $\text{Cox}(X)_D$. By surjectivity of deg (2.1.1),

$$\text{Cox}(X)_D \cong H^0(X, D) \quad (2.1.2)$$

for any $D \in \text{Cl}(X)$. This grading of X gives a $G := \text{Hom}(\text{Cl}(X), \mathbb{k}^*)$ action on $\text{Spec}(\text{Cox}(X)) \cong \mathbb{A}^d$. Explicitly, in the case that we will be interested in, when $\text{Cl}(X)$ is free of rank ρ (and hence $G = (\mathbb{k}^*)^\rho$), if the lattice map

$$\mathbb{Z}^d \longrightarrow \text{Cl}(X)$$

is given by a matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{\rho 1} & \cdots & a_{\rho d} \end{pmatrix}$$

then the action of $(\lambda_1, \dots, \lambda_\rho) \in (\mathbb{k}^*)^\rho$ on a point $(p_1, \dots, p_d) \in \text{Spec}(\text{Cox}(X))$ is given by

$$(\lambda_1, \dots, \lambda_\rho) \cdot (p_1, \dots, p_d) = (\lambda_1^{a_{11}} \cdots \lambda_\rho^{a_{\rho 1}} p_1, \dots, \lambda_1^{a_{1d}} \cdots \lambda_\rho^{a_{\rho d}} p_d).$$

We can construct X as the GIT quotient of $\text{Spec}(\text{Cox}(X))$ by this action. The finitely generated abelian group $\text{Cl}(X)$ is the *character group* for the action of G on $\text{Spec}(\text{Cox}(X))$. We pick a character $L \in \text{Cl}(X)$ with the additional assumption that L is a very ample line bundle. By (2.1.1.) The \mathbb{k} -algebra of semi-invariant functions $\bigoplus_{m \geq 0} \text{Cox}(X)_{mL}$ satisfies

$$\bigoplus_{m \geq 0} H^0(X, L^m) = \bigoplus_{m \geq 0} \text{Cox}(X)_{mL}$$

and hence

$$\mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(X, L^m)\right) \cong \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}\right),$$

which are also isomorphic to X since L is very ample. Cox [7] gives a description of the L -unstable locus.

Proposition 2.1.2. (*[7]*) *The unstable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ for the action of $(\mathbb{k}^*)^r$ is*

$$\mathbb{V}((H^0(X, L))) = \mathbb{V}(B_X)$$

where

$$B_X = (x^{\hat{\sigma}} \in \mathrm{Cox}(X) \mid \sigma \text{ is a top-dimensional cone in } \Sigma)$$

where $x^{\hat{\sigma}} = \prod_{\substack{1 \leq j \leq d \\ \tau_j \notin \sigma}} x^d$.

Say sections s_0, \dots, s_N generate $\bigoplus_{m \in \mathbb{N}} \mathrm{Cox}(X)_{mL}$. Hence the map

$$\begin{aligned} \pi : \mathrm{Spec}(\mathrm{Cox}(X)) \setminus \mathbb{V}(B_X) &\longrightarrow \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}\right) \\ p &\longmapsto (s_0(p) : \dots : s_N(p)) \end{aligned}$$

is in fact a morphism. It has the property that $\pi(p) = \pi(q)$ if and only if p and q are in the same G orbit since L is very ample, and is thus a *good geometric quotient* - a morphism with the property that the preimage of a point is a G -orbit. We define $\mathrm{Proj}(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}) \cong X$ to be the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$.

2.1.2 Mori Dream Spaces

Let X be a projective \mathbb{Q} -factorial variety. In this thesis we will assume that the divisor class group of X , $\mathrm{Cl}(X)$ is finitely generated and free of rank ρ . Let D_1, \dots, D_ρ be Weil divisors whose classes provide an integral basis of $\mathrm{Cl}(X)$.

Definition 2.1.3. The *Cox ring* of X is defined to be the Λ graded ring

$$\mathrm{Cox}(X, D_1, \dots, D_\rho) := \bigoplus_{(m_1, \dots, m_\rho) \in \mathbb{Z}^\rho} H^0(X, m_1 D_1 + \dots + m_\rho D_\rho).$$

Mori Dream Spaces are defined in Hu–Keel [19] Definition 1.10, however, we state the main theorem of [19] which gives a much simpler necessary and sufficient condition for X to be a Mori Dream Space:

Theorem 2.1.4. (*Prop 2.9, [19]*) *A projective \mathbb{Q} -factorial variety X is a Mori Dream Space if and only if $\mathrm{Cox}(X, D_1, \dots, D_\rho)$ is finitely generated as a \mathbb{k} -algebra.*

Projective \mathbb{Q} -factorial toric varieties are also Mori Dream Spaces:

Theorem 2.1.5. (*Cor 2.10, [19]*) X is a toric variety if and only if $\text{Cox}(X)$ is a polynomial ring.

Remark 2.1.6. By Hu-Keel [19] and Hassett-Tschinkel [16], for any two bases D_1, \dots, D_ρ and E_1, \dots, E_ρ of $\text{Cl}(X)$, the rings $\text{Cox}(X, D_1, \dots, D_\rho)$ and $\text{Cox}(X, E_1, \dots, E_\rho)$ are isomorphic. Therefore X being a Mori Dream Space does not depend on the choice of basis for $\text{Cl}(X)$.

From now on, we will assume X is a Mori Dream Space and pick a presentation

$$\text{Cox}(X) = \mathbb{k}[x_1, \dots, x_d]/I_X \tag{2.1.3}$$

Since this does not depend on the choice of basis for $\text{Cl}(X)$, we will refer to $\text{Cox}(X, D_1, \dots, D_\rho)$ as simply $\text{Cox}(X)$. We will assume that the number of generators in this presentation is as small as possible.

Since we assume that $\text{Cl}(X)$ is finitely generated and free, the ideal I_X is prime by the following theorem due to Elizondo–Kurano–Watanabe:

Theorem 2.1.7. (*[14]*) *Let X be a Mori Dream Space whose class group is finitely generated and free. Then $\text{Cox}(X)$ is a factorial \mathbb{k} -algebra.*

In particular, if we pick a presentation

$$\text{Cox}(X) = \mathbb{k}[x_1, \dots, x_d]/I_X$$

then I_X is a prime ideal.

Remark 2.1.8. We note that Theorem 2.1.7 implies that I_X does not contain any monomials. If it did, then it would also contain a variable, since I_X is prime. This would contradict our assumption that the number of generators d is as small as possible.

We summarise material from Hu–Keel [19], Laface–Velsaco [21] on the construction of Mori Dream Spaces as GIT quotients. The grading map

$$\text{Cox}(X) \longrightarrow \text{Cl}(X)$$

induces a $G := \text{Hom}(\text{Cl}(X), \mathbb{k}^*) = (\mathbb{k}^*)^\rho$ action on $\text{Spec}(\text{Cox}(X)) = \mathbb{V}(I_X) \subseteq \mathbb{A}^d$. We construct X as a GIT quotient of $\text{Spec}(\text{Cox}(X))$ under this action as follows. The abelian group $\text{Cl}(X)$ is the *character group* of X . We pick a character $L \in \text{Cl}(X)$, with the additional assumption that L is a very ample line bundle on X . The \mathbb{k} -algebra $\bigoplus_{m \geq 0} \text{Cox}(X)_{mL}$ of L -semi-invariant functions satisfies

$$\bigoplus_{m \geq 0} H^0(X, L^m) = \bigoplus_{m \geq 0} \text{Cox}(X)_{mL}$$

and hence

$$\mathrm{Proj}\left(\bigoplus_{m \geq 0} H^0(X, L^m)\right) \cong \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}\right),$$

which are also isomorphic to X since L is very ample. The unstable locus of $\mathrm{Spec}(\mathrm{Cox}(X))$ for the action of G is $\mathbb{V}((H^0(X, L)))$.

If sections s_0, \dots, s_N generate $\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}$ then the map

$$\pi : \mathrm{Spec}(\mathrm{Cox}(X)) \setminus \mathbb{V}(B_X) \longrightarrow \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}\right)$$

$$p \mapsto (s_0(p) : \dots : s_N(p))$$

is in fact a morphism. It has the property that $\pi(p) = \pi(q)$ if and only if p and q are in the same G orbit. Hence, after removal of the unstable locus, π is a good geometric quotient. We define $\mathrm{Proj}(\bigoplus_{m \geq 0} \mathrm{Cox}(X)_{mL}) \cong X$ to be the GIT quotient of $\mathrm{Spec}(\mathrm{Cox}(X))$ under the G action induced by the $\mathrm{Cl}(X)$ -grading of $\mathrm{Cox}(X)$.

The map $\pi : \mathbb{k}[x_1, \dots, x_d]/I_X \longrightarrow \mathbb{Z}^\rho$ can be extended to a map

$$\tilde{\pi} : \mathbb{k}[x_1, \dots, x_d] \longrightarrow \mathbb{Z}^\rho$$

by composing π with τ , the canonical surjection $\mathbb{k}[x_1, \dots, x_d] \longrightarrow \mathbb{k}[x_1, \dots, x_d]/I_X$. Hence $\mathbb{k}[x_1, \dots, x_d]$ is \mathbb{Z}^ρ graded and we have a commutative diagram:

$$\begin{array}{ccc} \mathbb{k}[x_1, \dots, x_d] & \xrightarrow{\deg} & \mathbb{Z}^\rho \\ \tau \downarrow & & \downarrow \psi \\ \mathrm{Cox}(X) & \longrightarrow & \mathrm{Cl}(X) \end{array}$$

The action of $(\mathbb{k}^*)^\rho$ on $\mathrm{Spec}(\mathbb{k}[x_1, \dots, x_d]/I_X)$ extends to an action on $\mathrm{Spec}(\mathbb{k}[x_1, \dots, x_d])$.

The ample cone of X has a chamber decomposition. Picking a very ample character L in the interior of a chamber, we obtain a toric variety

$$\tilde{X}_L := \mathrm{Proj}\left(\bigoplus_{m \geq 0} \mathbb{k}[x_1, \dots, x_d]_{m\tilde{L}}\right)$$

where $\tilde{L} := \psi^{-1}(L)$. Picking different characters in the same chamber results in isomorphic toric varieties. The chambers in the decomposition correspond to the ample cones for the toric varieties obtained. The toric variety \tilde{X}_L has $\mathrm{Cox}(\tilde{X}_L) = \mathbb{k}[x_1, \dots, x_d]$, and $\mathrm{Cl}(\tilde{X}_L) \cong \mathrm{Cl}(X)$ via the map $\psi : \tilde{X}_L \rightarrow X$. It is obtained as the GIT quotient of \mathbb{A}^d under the action of G with unstable locus $\mathbb{V}(H^0(\tilde{X}_L, \tilde{L}))$.

Remark 2.1.9. We note that the unstable locus of X is the intersection of the unstable locus of \tilde{X} and $\text{Spec}(\text{Cox}(X))$. Hence the embedding $\text{Spec}(\text{Cox}(X)) \hookrightarrow \text{Spec}(\text{Cox}(\tilde{X}_L))$ descends to an embedding $X \hookrightarrow \tilde{X}_L$.

Remark 2.1.10. We note that \tilde{X}_L does in general depend on the choice of L , or more precisely, on the chamber containing L . However, in what follows it will not matter what choice of L we make, and hence we will refer to \tilde{X}_L as simply \tilde{X} from now on.

We present an example which illustrates the concepts introduced in this section, further details on del Pezzo surfaces can be found in section 2.2.2.

Example 2.1.11. Let X_4 be the del-Pezzo surface obtained as the blow-up of $\mathbb{P}_{\mathbb{k}}^2$ at four points in general position. The Picard group $\text{Pic}(X_4) \cong \mathbb{Z}^5$ has a basis given by l_0 , the pullback to X_4 of the hyperplane class on $\mathbb{P}_{\mathbb{k}}^2$, together with the four exceptional curves l_1, l_2, l_3, l_4 . The semigroup homomorphism $\text{deg}: \mathbb{N}^{10} \rightarrow \text{Pic}(X_4)$ obtained as multiplication by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

induces the grading map $\text{deg}: \mathbb{k}[x_1, \dots, x_{10}] \rightarrow \text{Pic}(X)$, and the $\text{Pic}(X)$ -homogeneous ideal

$$I_{X_4} := \left(\begin{array}{cc} x_2x_5 - x_3x_6 + x_4x_7, & x_1x_5 - x_3x_8 + x_4x_9 \\ x_1x_6 - x_2x_8 + x_4x_{10}, & x_1x_7 - x_2x_9 + x_3x_{10}, & x_5x_{10} - x_6x_9 + x_7x_8 \end{array} \right)$$

determines $\text{Cox}(X_4) = \mathbb{k}[x_1, \dots, x_{10}]/I_{X_4}$ following Batyrev–Popov [2]. To construct an ambient toric variety, note that $\chi = 11l_0 - 5l_1 - 3l_2 - 2l_3 - l_4$ is ample on X_4 and $\mathbb{A}^{10} //_{\chi} T$ is \mathbb{Q} -factorial (see Example 2.11 of Laface–Velasco [21], who note further that $-K_{X_4} = 3l_0 - l_1 - l_2 - l_3 - l_4$ defines a non- \mathbb{Q} -factorial toric quotient, so $-K_{X_4}$ lies in a GIT wall for the action on $\mathbb{A}_{\mathbb{k}}^{10}$). Thus $\chi \in \text{Pic}(X)$ lies in an open GIT chamber for the action of T on $\mathbb{A}_{\mathbb{k}}^{10}$, and we set $\tilde{X}_4 := \mathbb{A}^{10} //_{\chi} T$.

2.2 Cox Rings of Mori Dream Spaces

In this section we introduce two families of del Pezzo surfaces: Grassmannians and del Pezzo surfaces. We study their Cox rings.

2.2.1 Grassmannians

The *Grassmannian* $\text{Gr}(r, n)$ is the scheme which represents the functor mapping a scheme X to the rank d vector subbundles of the trivial rank n vector bundle on X . A rank r element of $\text{Mat}(r, n)$

determines an r dimensional vector subspace of n dimensional space, up to change of basis, i.e. up to multiplication by an element of $\mathrm{GL}(r)$. Hence $\mathrm{Gr}(r, n)$ is the space of $\mathrm{GL}(r)$ orbits of rank r matrices in $\mathrm{Mat}(r, n)$, where the group action is multiplication on the left.

An embedding of $\mathrm{Gr}(r, n)$ into projective space \mathbb{P}^N , where $N = \binom{n}{r} - 1$, is given by the determinantal line bundle on $\mathrm{Gr}(r, n)$. Explicitly, this maps an element of $\mathrm{Mat}(r, n)$ to its $r \times r$ minors (there are $\binom{n}{r}$ such). We note that the action of $\mathrm{GL}(r)$ only changes the $r \times r$ minors by a nonzero scalar multiple, hence this map is well defined. We also note that since each matrix has rank r , at least one of the minors will be nonzero. The image of this map is cut out by the ideal of Plucker relations. We consider the map

$$D : \mathbb{k}[z_1, \dots, z_N] \longrightarrow \mathbb{k}[a_{ij}]$$

mapping z_i to the i th $r \times r$ minor of the generic matrix (a_{ij}) . The ideal of plucker relations is the kernel of D . Hence $\mathbb{k}[z_1, \dots, z_n]/\ker(D)$ is the homogeneous coordinate ring of the Grassmannian $\mathrm{Gr}(r, n)$. By Remark 3.9 of Castravet–Tevelev [4], this ring coincides with $\mathrm{Cox}(\mathrm{Gr}(r, n))$.

We can calculate I_X very easily. It is the kernel of the \mathbb{k} -algebra homomorphism

$$\mathbb{k}[Z_1, \dots, Z_n] \longrightarrow \mathbb{k}[a_{ij} | 1 \leq i \leq r, 1 \leq j \leq n]$$

which maps Z_i to the i th $r \times r$ minor of a generic matrix (a_{ij}) . We can hence calculate I_X using the methods of section 7.2 using Macaulay2. We give code for such a calculation:

```
i1: R = QQ[z_1..z_rn,x_1..x_N, MonomialOrder => Eliminate 8]
i2: M = genericMatrix(R,z_1,r,n)
i3: H = minors(r,M)
i4: K = ideal(x_1-H_0, ..., x_N-H_(N-1) )
i5: G = gens gb K
i6: J = selectInSubring(1,G)
i7: IX= ideal(J)
```

2.2.2 Del Pezzo Surfaces

In this section, we recall some facts from birational geometry before giving essential background information on Mori Dream Spaces. We describe results of Batyrev–Popov [2] giving generators and relations for the Cox rings of certain non-toric del Pezzo surfaces. We conclude by calculating the Cox ring of the del Pezzo surface of degree 4 following the method of Batyrev–Popov [2] and Derenthal [12].

We summarise material on del Pezzo surfaces found in Manin [23] or Batyrev–Popov [2].

We denote the blow-up of a smooth scheme X at a point p by \overline{X} , where \overline{X} is also nonsingular. We denote the blow-up map by π and note that π induces an isomorphism of $X \setminus \{p\}$ with $\overline{X} \setminus \pi^{-1}(p)$.

We call $E := \pi^{-1}(p)$ the *exceptional curve*. The *del Pezzo surface of degree $9 - r$* is the blow up of \mathbb{P}^2 at $0 \leq r \leq 8$ points p_1, \dots, p_r in general position. This is a smooth surface which we denote by X_r . We say r points are in *general position* if no three points lie on a line; no six points lie on a conic; and no cubic with a double point containing seven of the points contains the eighth.

We denote the blow-up map by $\pi_r : X_r \rightarrow \mathbb{P}^2$. The Picard group of X_r satisfies $\text{Pic}(X_r) \cong \mathbb{Z}^{r+1}$, with a basis given by $l_0 := \pi^*(\mathcal{O}(1)), l_1 := \pi^{-1}(p_1), \dots, l_r := \pi^{-1}(p_r)$. The intersection form is given by the following matrix

$$\left(\begin{array}{c|cccc} & l_0 & l_1 & \cdots & l_r \\ \hline l_0 & 1 & 0 & \cdots & 0 \\ l_1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_r & 0 & \cdots & 0 & -1 \end{array} \right)$$

To harmonize with the literature, we denote multiplication in the Picard group using additive (rather than tensor) notation.

The (-1) -curves on X_r are the inverse images of blown up points and the strict transforms of the following curves on \mathbb{P}^2 :

- (i) Lines between pairs of blown up points;
- (ii) Conics containing five blown up points;
- (iii) Cubics with a double point containing seven blown up points;
- (iv) Quartics with three double points containing eight blown up points;
- (v) Quintics with six double points containing eight blown up points;
- (vi) Sextics with seven double points containing eight blown up points.

Every del Pezzo Surface is a Mori Dream Space by Batyrev–Popov [2]. If we blow up $r \leq 3$ points we obtain a smooth projective toric surface. Batyrev–Popov describe generators and relations for the Cox rings of X_4, X_5 and X_6 in [2]. We summarise their results.

The following result can be found in Laface–Velasco [21]

Proposition 2.2.1. (*[21]*) *Let X be a surface. If $E \in \text{Pic}(X)$ is a (-1) -curve then $H^0(X, E)$ is generated by a unique (up to scalar multiplication) section. This section is a generator of $\text{Cox}(X)$.*

By Proposition 2.2.1 the section of any (-1) -curve is a generator of $\text{Cox}(X_r)$. These are the only generators of $\text{Cox}(X_r)$ by the following Theorem due to Batyrev–Popov.

Theorem 2.2.2. (Thm 3.2, [2]) For $3 \leq r \leq 7$, the Cox ring of the del Pezzo surface X_r is generated by global sections of $\mathcal{O}(D)$ where D is a (-1) -curve.

The del Pezzo surfaces X_4, X_5 and X_6 are the blow ups of \mathbb{P}^2 at the first four, five and six points respectively from points p_1, \dots, p_6 in general position. We can always pick $p_1 = (1, 0, 0), p_2 = (0, 1, 0), p_3 = (0, 0, 1)$ and $p_4 = (1, 1, 1)$. The (-1) -curves on X_4, X_5 and X_6 are the preimages of blown up points, the strict transforms of lines in \mathbb{P}^2 between pairs of blown up points, and the strict transforms of conics through five blown up points. We describe these explicitly below.

The strict transform of the line containing points p_i and p_j is $l_0 - l_i - l_j$, and the strict transform of the conic containing points p_{i_1}, \dots, p_{i_5} is $2l_0 - l_{i_1} - \dots - l_{i_5}$. All the (-1) -curves are either equal to l_i for some $i \in \{1, \dots, 6\}$, or the strict transforms of lines through pairs of points or conics through five points. Hence we can associate to each (-1) -curve the equation of a line or conic, unless it is the preimage of a blown up point. In that case, it is a useful convention to assign a constant to each l_i . We will always choose that constant to be 1. We denote the homogeneous polynomial (or form) associated to a line bundle l in this way to be $f_l \in \mathbb{k}[z_1, z_2, z_3]$.

We present the generators of X_4 , their degree and the curve in \mathbb{P}^2 of which they are the strict transform in the following table:

Generator	Degree	Point or Curve in \mathbb{P}^2
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	$2l_0 - l_1 - l_2$	line between p_1 and p_2
x_6	$2l_0 - l_1 - l_3$	line between p_1 and p_3
x_7	$2l_0 - l_1 - l_4$	line between p_1 and p_4
x_8	$2l_0 - l_2 - l_3$	line between p_2 and p_3
x_9	$2l_0 - l_2 - l_4$	line between p_2 and p_4
x_{10}	$2l_0 - l_3 - l_4$	line between p_3 and p_4

We present the generators of X_5 , their degree and the curve in \mathbb{P}^2 of which they are the strict transform in the following table:

We present the generators of X_6 , their degree and the curve in \mathbb{P}^2 of which they are the strict transform in the following table:

Generator	Degree	Point or Curve in \mathbb{P}^2
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	l_5	p_5
x_6	$l_0 - l_1 - l_2$	line between p_1 and p_2
x_7	$l_0 - l_1 - l_3$	line between p_1 and p_3
x_8	$l_0 - l_1 - l_4$	line between p_1 and p_4
x_9	$l_0 - l_1 - l_5$	line between p_1 and p_5
x_{10}	$l_0 - l_2 - l_3$	line between p_2 and p_3
x_{11}	$l_0 - l_2 - l_4$	line between p_2 and p_4
x_{12}	$l_0 - l_2 - l_5$	line between p_2 and p_5
x_{13}	$l_0 - l_3 - l_4$	line between p_3 and p_4
x_{14}	$l_0 - l_3 - l_5$	line between p_3 and p_5
x_{15}	$l_0 - l_4 - l_5$	line between p_4 and p_5
x_{16}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	conic containing p_1, p_2, p_3, p_4 and p_5

Figure 2.1: X_5 Case

Generator	Degree	Point or Curve in \mathbb{P}^2
x_1	l_1	p_1
x_2	l_2	p_2
x_3	l_3	p_3
x_4	l_4	p_4
x_5	l_5	p_5
x_6	l_6	p_6
x_7	$l_0 - l_1 - l_2$	line between p_1 and p_2
x_8	$l_0 - l_1 - l_3$	line between p_1 and p_3
x_9	$l_0 - l_1 - l_4$	line between p_1 and p_4
x_{10}	$l_0 - l_1 - l_5$	line between p_1 and p_5
x_{11}	$l_0 - l_1 - l_6$	line between p_1 and p_6
x_{12}	$l_0 - l_2 - l_3$	line between p_2 and p_3
x_{13}	$l_0 - l_2 - l_4$	line between p_2 and p_4
x_{14}	$l_0 - l_2 - l_5$	line between p_2 and p_5
x_{15}	$l_0 - l_2 - l_6$	line between p_2 and p_6
x_{16}	$l_0 - l_3 - l_4$	line between p_3 and p_4
x_{17}	$l_0 - l_3 - l_5$	line between p_3 and p_5
x_{18}	$l_0 - l_3 - l_6$	line between p_3 and p_6
x_{19}	$l_0 - l_4 - l_5$	line between p_4 and p_5
x_{20}	$l_0 - l_4 - l_6$	line between p_4 and p_6
x_{21}	$l_0 - l_5 - l_6$	line between p_5 and p_6
x_{22}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	conic containing p_1, p_2, p_3, p_4 and p_5
x_{23}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_6$	conic containing p_1, p_2, p_3, p_4 and p_6
x_{24}	$2l_0 - l_1 - l_2 - l_3 - l_5 - l_6$	conic containing p_1, p_2, p_3, p_5 and p_6
x_{25}	$2l_0 - l_1 - l_2 - l_4 - l_5 - l_6$	conic containing p_1, p_2, p_4, p_5 and p_6
x_{26}	$2l_0 - l_1 - l_3 - l_4 - l_5 - l_6$	conic containing p_1, p_3, p_4, p_5 and p_6
x_{27}	$2l_0 - l_2 - l_3 - l_4 - l_5 - l_6$	conic containing p_2, p_3, p_4, p_5 and p_6

Having found the generators of $\text{Cox}(X_r)$, we turn our attention to finding the ideal of relations I_{X_r} . Batyrev–Popov first computed I_{X_4} , then used induction on the number of blown up points to obtain I_{X_5} and I_{X_6} .

Proposition 2.2.3. (Prop 4.1, [2]) *The Cox ring of X_4 is isomorphic to the homogeneous coordinate ring of $\text{Gr}(3, 5)$, i.e.*

$$\text{Cox}(X_4) = \mathbb{k}[x_1, \dots, x_{10}]/I_{X_4}$$

where

$$I_{X_4} := \left(\begin{array}{c} x_2x_5 - x_3x_6 + x_4x_7, \quad x_1x_5 - x_3x_8 + x_4x_9 \\ x_1x_6 - x_2x_8 + x_4x_{10}, \quad x_1x_7 - x_2x_9 + x_3x_{10}, \quad x_5x_{10} - x_6x_9 + x_7x_8 \end{array} \right)$$

The terms of each relation in I_{X_4} are sections of a single line bundle, each of which is a *ruling*.

Definition 2.2.4. A *ruling* is a line bundle l such that $l = l_1 + l_2$ where l_1, l_2 are (-1) -curves and $l_1 \cdot l_2 = 1$.

Each ruling L can be written in $r - 1$ ways as a sum of (-1) -curves, i.e.

$$L = L_1 + L'_1 = L_2 + L'_2 = \cdots = L_{r-1} + L'_{r-1}$$

where each L_i and L'_i is a (-1) -curve. A relation in $\text{Cox}(X_r)$ arises from a ruling in the following way. We recall that a form f_l is associated to each (-1) -curve on X_r . The forms

$$f_{L_1}f_{L'_1}, \dots, f_{L_{r-1}}f_{L'_{r-1}}$$

have $r - 3$ relations between them. These lift to give relations between the sections of the ruling L , in the sense that if

$$a_1f_{L_1}f_{L'_1} + \cdots + a_nf_{L_n}f_{L'_n} = 0$$

then

$$a_1x_{L_1}x_{L'_1} + \cdots + a_nx_{L_n}x_{L'_n}$$

is a relation in the Cox ring. In fact, by the following theorem due to Batyrev–Popov [2], these are the only relations.

Proposition 2.2.5. ([2]). For $r = 4, 5$ or 6 , I_{X_R} is the ideal generated by relations between sections of rulings as described above.

Proof. We sketch the proof. Let d_r be the number of generators of $\text{Cox}(X_r)$, let J_{X_r} be the ideal generated by relations between sections of rulings and suppose $\text{Cox}(X_r) = \mathbb{k}[x_1, \dots, x_{d_r}]/I_{X_r}$. We show that $U_r := \text{Spec}(\mathbb{k}[x_1, \dots, x_{d_r}]/I_{X_r})$ is isomorphic to $V_r := \text{Spec}(\mathbb{k}[x_1, \dots, x_{d_r}]/J_{X_r})$. Both contain $0 \in \mathbb{A}^{d_r}$, so it suffices to show that for each (-1) -curve E we have $(U_r \setminus \{0\}) \cap (x_E \neq 0) \cong (V_r \setminus \{0\}) \cap (x_E \neq 0)$, where x_E is the generating section of $H^0(X, E)$.

For $r = 5, 6$, there exist (-1) -curves E' such that $E + E'$ is a ruling and hence a relation between the sections of $E + E'$:

$$x_E x_{E'} = \sum a_i x_{E_i} x_{E'_i},$$

where none of the E_i 's or E'_i 's intersect E , and indeed, none of the other pairs of (-1) -curves whose sum equals the ruling intersect E . The set of (-1) -curves that do not intersect E can be identified with the (-1) -curves on X_{r-1} , and each (-1) -curve D has either $E \cdot D = 1$ or 0 . Hence

$$(V_r \setminus \{0\}) \cap (x_E \neq 0) \cong V_{r-1} \times \mathbb{A}^1 \setminus \{0\}.$$

By Proposition 4.4 of [2], $U_r \cap (x_E \neq 0) \cong U_{r-1} \times \mathbb{A}^1 \setminus \{0\}$, so the proposition is true by induction. \square

2.2.3 Computing $\text{Cox}(X_5)$

In the rest of this section, we use the theory due to Batyrev–Popov [2] summarised above to calculate the ideal I_{X_5} . This calculation can be found in Derenthal [12] using a generic fifth point $(1, \alpha, \beta)$. For our calculation we pick a specific point, and give a little more detail. First we compute the forms associated to the generators of I_{X_5} . We give the rulings, their sections and associated forms for X_5 . Once we have this information we calculate the relations between the associated forms, and hence the relations between sections of rulings. This gives us I_{X_5} .

Let X_5 be the blow up of p_1, \dots, p_5 where

$$p_1 := (1, 0, 0), p_2 := (0, 1, 0), p_3 := (0, 0, 1), p_4 := (1, 1, 1), p_5 := (1, 2, 3)$$

We compute the forms f_l for X_5 as described in Table 2.1. First, it is possible to compute the equations of lines through pairs of points by inspection. To compute the equations of conics through five points we use a simple Maple procedure, `findconics`, described in Appendix B. We present the forms in the following table.

Generator of $\text{Cox}(X_5)$	(-1) -curve l	f_l
x_1	l_1	1
x_2	l_2	1
x_3	l_3	1
x_4	l_4	1
x_5	l_5	1
x_6	$l_0 - l_1 - l_2$	z_3
x_7	$l_0 - l_1 - l_3$	z_2
x_8	$l_0 - l_1 - l_4$	$z_2 - z_3$
x_9	$l_0 - l_1 - l_5$	$2z_3 - 3z_2$
x_{10}	$l_0 - l_2 - l_3$	z_1
x_{11}	$l_0 - l_2 - l_4$	$z_1 - z_3$
x_{12}	$l_0 - l_2 - l_5$	$z_3 - 3z_1$
x_{13}	$l_0 - l_3 - l_4$	$z_1 - z_2$
x_{14}	$l_0 - l_3 - l_5$	$2z_1 - z_2$
x_{15}	$l_0 - l_4 - l_5$	$z_1 - 2z_2 + z_3$
x_{16}	$2l_0 - l_1 - l_2 - l_3 - l_4 - l_5$	$3z_1z_2 - 4z_1z_3 + z_2z_3$

We compute the rulings for X_5 and their sections using the code in Appendix B. For each ruling, we calculate the relations between the forms corresponding to its sections using Maple. There are four forms f_1, f_2, f_3 and f_4 corresponding to sections s_1, s_2, s_3 and s_4 and two relations between

them. To find the relations, we use the Maple command “solve” to find solutions to the pair of equations:

$$af_1 + bf_2 + cf_3 = 0$$

$$df_1 + ef_2 + hf_4 = 0$$

thus we obtain two generators

$$as_1 + bs_2 + cs_3 \text{ and } ds_1 + es_2 + hs_4$$

of I_{X_5} .

We give the rulings, their sections, associated forms in $\mathbb{k}[z_1, z_2, z_3]$ and the relation between them in the following table:

Rulings	Sections	Forms in $\mathbb{k}[z_1, z_2, z_3]$	Relations
$l_0 - l_1$	x_2x_6 x_3x_7 x_4x_8 x_5x_9	z_3 z_2 $z_2 - z_3$ $2z_3 - 3z_2$	$x_2x_6 - x_3x_7 + x_4x_8, 2x_2x_6 - 3x_3x_7 - x_5x_9$
$l_0 - l_2$	x_1x_6 x_3x_{10} x_4x_{11} x_5x_{12}	z_3 z_1 $z_1 - z_3$ $z_3 - 3z_1$	$x_1x_6 - x_3x_{10} + x_4x_{11}, x_1x_6 - 3x_3x_{10} - x_5x_{12}$
$l_0 - l_3$	x_1x_7 x_2x_{10} x_4x_{13} x_5x_{14}	z_2 z_1 $z_1 - z_2$ $2z_1 - z_2$	$x_1x_7 - x_2x_{10} + x_4x_{13}, x_1x_7 - 2x_2x_{10} + x_5x_{14}$
$l_0 - l_4$	x_1x_8 x_2x_{11} x_3x_{13} x_5x_{15}	$z_2 - z_3$ $z_1 - z_3$ $z_1 - z_2$ $z_1 - 2z_2 + z_3$	$x_1x_8 - x_2x_{11} + x_3x_{13}, -2x_1x_8 + x_2x_{11} - x_5x_{15}$
$l_0 - l_5$	x_1x_9 x_2x_{12} x_3x_{14} x_4x_{15}	$2z_3 - 3z_2$ $z_3 - 3z_1$ $2z_1 - z_2$ $z_1 - 2z_2 + z_3$	$-x_1x_9 + 2x_2x_{12} + 3x_3x_{14},$ $2x_1x_9 + x_2x_{12} + 3x_4x_{15}$
$2l_0 - l_1 - l_2 - l_3 - l_4$	x_5x_{16} x_6x_{13} x_7x_{11} x_8x_{10}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_3(z_1 - z_2)$ $z_2(z_1 - z_3)$ $(z_2 - z_3)z_1$	$x_5x_{16} + x_6x_{13} - 3x_8x_{10},$ $x_6x_{13} - x_7x_{11} + x_8x_{10}$
$2l_0 - l_1 - l_2 - l_3 - l_5$	x_4x_{16} x_6x_{14} x_7x_{12} x_9x_{10}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_3(2z_1 - z_2)$ $z_2(z_3 - 3z_1)$ $(2z_3 - 3z_2)z_1$	$x_4x_{16} + 2x_6x_{14} + x_7x_{12},$ $x_4x_{16} + x_6x_{14} + x_9x_{10}$
$2l_0 - l_1 - l_2 - l_4 - l_5$	x_3x_{16} x_6x_{15} x_8x_{12} x_9x_{11}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_3(z_1 - 2z_2 + z_3)$ $-(-z_2 + z_3)(z_3 - 3z_1)$ $(2z_3 - 3z_2)(z_1 - z_3)$	$x_3x_{16} + x_6x_{15} + x_8x_{12},$ $x_3x_{16} + 2x_6x_{15} + x_9x_{11}$

$2l_0 - l_1 - l_3 - l_4 - l_5$	x_2x_{16} x_7x_{15} x_8x_{14} x_9x_{13}	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_2(z_1 - 2z_2 + z_3)$ $(z_2 - z_3)(2z_1 - z_2)$ $(2z_3 - 3z_2)(z_1 - z_2)$	$x_2x_{16} + x_7x_{15} - 2x_8x_{14},$ $x_2x_{16} + 3x_7x_{15} + 2x_9x_{13}$
$2l_0 - l_2 - l_3 - l_4 - l_5$	x_1x_{16} $x_{10}x_{15}$ $x_{11}x_{14}$ $x_{12}x_{13}$	$3z_1z_2 - 4z_1z_3 + z_2z_3$ $z_1(z_1 - 2z_2 + z_3)$ $(z_1 - z_3)(2z_1 - z_2)$ $(z_3 - 3z_1)(z_1 - z_2)$	$x_1x_{16} + 2x_{10}x_{15} - x_{11}x_{14},$ $x_1x_{16} + 3x_{10}x_{15} + x_{12}x_{13}$

Hence,

$$I_{X_5} = \left(\begin{array}{l} x_5x_{16} + x_6x_{13} - 3x_8x_{10}, x_4x_{16} + 2x_6x_{14} + x_7x_{12}, x_4x_{16} + x_6x_{14} + x_9x_{10} \\ x_3x_{16} + x_6x_{15} + x_8x_{12}, x_3x_{16} + 2x_6x_{15} + x_9x_{11}, x_2x_{16} + x_7x_{15} - 2x_8x_{14} \\ x_2x_{16} + 3x_7x_{15} + 2x_9x_{13}, x_1x_{16} + 2x_{10}x_{15} - x_{11}x_{14}, x_1x_{16} + 3x_{10}x_{15} + x_{12}x_{13} \\ x_2x_6 - x_3x_7 + x_4x_8, 2x_2x_6 - 3x_3x_7 - x_5x_9, x_1x_6 - x_3x_{10} + x_4x_{11} \\ x_1x_6 - 3x_3x_{10} - x_5x_{12}, x_1x_7 - x_2x_{10} + x_4x_{13}, x_1x_7 - 2x_2x_{10} + x_5x_{14} \\ x_1x_8 - x_2x_{11} + x_3x_{13}, -2x_1x_8 + x_2x_{11} - x_5x_{15}, -x_1x_9 + 2x_2x_{12} + 3x_3x_{14} \\ -2x_1x_9 + x_2x_{12} + 3x_4x_{15}, x_6x_{13} - x_7x_{11} + x_8x_{10} \end{array} \right).$$

2.3 Multigraded Regularity for Projective Toric Varieties

Maclagan-Smith introduced the notion of multigraded regularity in [22] as a generalisation of Castelnuovo-Mumford regularity. Let X be a projective toric variety, and let $\text{Cox}(X) = \mathbb{k}[x_1, \dots, x_d]$. Multigraded regularity is a useful tool for studying the geometry of X . For example, it gives a bound for the multidegrees of the equations which cut out the subvariety corresponding to an ideal sheaf, and it allows us to test whether an ample line bundle gives a projectively normal embedding of X . In this thesis, we will use multigraded regularity of a line bundle $L = L_1 \otimes \dots \otimes L_k$ with respect to L_1, \dots, L_k to ensure surjectivity of the multiplication map

$$H^0(X, L_1) \otimes_{\mathbb{k}} \dots \otimes_{\mathbb{k}} H^0(X, L_k) \longrightarrow H^0(X, L).$$

We summarise material due to Maclagan–Smith [22], Hering–Schenck–Smith [17].

Let \mathcal{F} be a coherent sheaf, let B and M_1, \dots, M_k be line bundles on X . For a vector $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{N}^k$, we denote $M_1^{u_1} \otimes \dots \otimes M_k^{u_k}$ by $M^{\mathbf{u}}$.

Definition 2.3.1. We say \mathcal{F} is *B-regular* (with respect to M_1, \dots, M_k) if $H^i(X, \mathcal{F} \otimes B \otimes M^{-\mathbf{u}}) = 0$ for all $i > 0$ and all $\mathbf{u} \in \mathbb{N}^k$ satisfying $|\mathbf{u}| := u_1 + \dots + u_k = i$.

The following theorem is due to Maclagan–Smith in the toric case, but was generalised by Hering–Schenck–Smith [17].

Theorem 2.3.2. (*[22], [17]*)

Let \mathcal{F} be a coherent sheaf that is B -regular with respect to M_1, \dots, M_k . For all $\mathbf{u} \in \mathbb{N}^k$ the map

$$H^0(X, \mathcal{F} \otimes B \otimes M^{\mathbf{u}}) \otimes H^0(X, M^{\mathbf{v}}) \longrightarrow H^0(X, \mathcal{F} \otimes B \otimes M^{\mathbf{u}+\mathbf{v}})$$

is surjective for all $\mathbf{v} \in \mathbb{N}^k$.

Now let L_1, \dots, L_k be line bundles and let $L = L_1^{\beta_1} \otimes \dots \otimes L_k^{\beta_k}$ be \mathcal{O}_X regular with respect to L_1, \dots, L_k .

Corollary 2.3.3. *Let L_1, \dots, L_k be line bundles and suppose $L = L_1^{\beta_1} \otimes \dots \otimes L_k^{\beta_k}$ be \mathcal{O}_X regular with respect to L_1, \dots, L_k . The multiplication map*

$$H^0(X, L)^{\otimes d} \longrightarrow H^0(X, L^d)$$

is surjective.

Proof. If L is \mathcal{O}_X regular with respect to L_1, \dots, L_k then it follows immediately from Theorem 2.3.2 (2) that the multiplication map

$$H^0(X, L) \otimes H^0(X, L) \longrightarrow H^0(X, L^2) \tag{2.3.1}$$

is surjective. Indeed, every section of L^d can be written as the product of a section of L^{d-1} and a section of L . Hence by induction, the map

$$H^0(X, L)^{\otimes d} \longrightarrow H^0(X, L^d)$$

is surjective. □

Proposition 2.3.4. *For any line bundles $L_1, \dots, L_k \in \text{Pic}(X)$, if the sublattice of $\text{Pic}(X)$ generated by L_1, \dots, L_k contains an ample bundle then there exist $\beta_1, \dots, \beta_k \in \mathbb{N}$ such that $L := L_1^{\beta_1} \otimes \dots \otimes L_k^{\beta_k}$ is \mathcal{O}_X regular with respect to L_1, \dots, L_k .*

Proof. Since the sublattice of $\text{Pic}(X)$ generated by L_1, \dots, L_k contains an ample line bundle, we can pick $\alpha_1, \dots, \alpha_k \in \mathbb{N}$ such that $L_1^{\alpha_1} \otimes \dots \otimes L_k^{\alpha_k}$ is ample. Suppose X is n dimensional, then $L(\mathbf{u}) := (L_1^{\alpha_1+n} \otimes \dots \otimes L_k^{\alpha_k+n}) \otimes (L_1^{-u_1} \otimes \dots \otimes L_k^{-u_k})$ is also ample for any $\mathbf{u} = (u_1, \dots, u_k)$ with $u_1 + \dots + u_k \leq n$. Therefore by Demazure Vanishing (see e.g. Thm 9.2.3 Cox–Little–Schenck [6]) $H^i(X, L(\mathbf{u})) = 0$ for all $i > 0$ and all \mathbf{u} such that $u_1 + \dots + u_k = i$, since $H^i(X, L(\mathbf{u})) = 0$ for $i > n$. Hence, letting $\beta_i := \alpha_i + n$, we have the statement of the proposition. □

2.4 Quivers and Quiver Representations

A *quiver* can be defined by giving its vertices, its arrows, and the vertices at the head and tail of each arrow. For a quiver Q , define Q_0 to be its set of vertices, Q_1 to be its set of arrows, and define maps

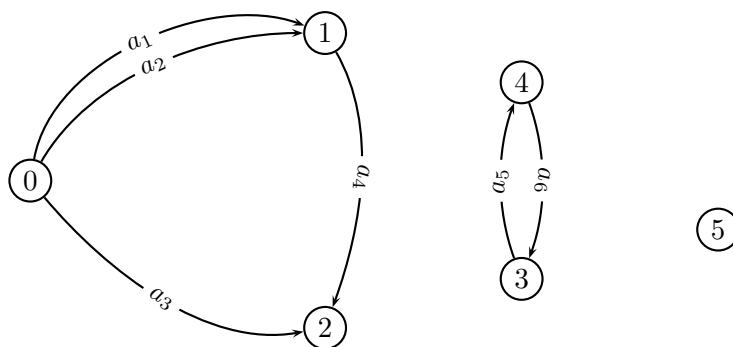
$$h, t : Q_1 \longrightarrow Q_0$$

mapping each arrow to its head and tail respectively. A *path* p is a sequence of arrows

$$p = a_n \dots a_1$$

such that $t(a_i) = h(a_{i-1})$, we define the support of p to be the set $\{a_1, \dots, a_n\}$. The path algebra $\mathbb{k}Q$ is defined to be the \mathbb{k} algebra generated by all paths in Q . The multiplication of two paths is defined to be their concatenation if it exists and zero otherwise. The maps h and t can be extended to $\mathbb{k}Q$ by defining $h(p) = h(a_n)$ and $t(p) = t(a_1)$. A *cycle* is a path where $h(p) = t(p)$, and Q is said to be *acyclic* if none of its nontrivial paths are cycles.

Example 2.4.1. For the quiver below the maps h and t are:



$t(a_1) = 0$	$h(a_1) = 1$
$t(a_2) = 0$	$h(a_2) = 1$
$t(a_3) = 0$	$h(a_3) = 2$
$t(a_4) = 1$	$h(a_4) = 2$
$t(a_5) = 3$	$h(a_5) = 4$
$t(a_6) = 4$	$h(a_6) = 3$

Let Q be a finite, connected quiver. A representation of Q consists of a \mathbb{k} -vector space W_i for $i \in Q_0$ and a \mathbb{k} -linear map $w_a : W_{t(a)} \rightarrow W_{h(a)}$ for $a \in Q_1$. It is convenient to write W as shorthand

for $((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1})$. The dimension vector of W is the vector $\underline{r} \in \mathbb{Z}^{Q_0}$ with components $r_i = \dim_{\mathbb{k}}(W_i)$ for $i \in Q_0$. A map of representations $\psi: W \rightarrow W'$ is a family $\psi_i: W_i \rightarrow W'_i$ of \mathbb{k} -linear maps for $i \in Q_0$ satisfying $w'_a \psi_{t(a)} = \psi_{h(a)} w_a$ for $a \in Q_1$. With composition defined componentwise, we obtain the abelian category of finite dimensional representations of Q . For $\theta \in \mathbb{Z}^{Q_0}$, define $\theta(W) := \sum_{0 \leq i \leq \rho} \theta_i \dim_{\mathbb{k}}(W_i)$. Following King [20], a representation W of Q is θ -semistable if $\theta(W) = 0$ and every subrepresentation $W' \subset W$ satisfies $\theta(W') \geq 0$. Moreover, W is θ -stable if the only subrepresentations W' with $\theta(W') = 0$ are 0 and W .

2.5 Toric Varieties as Fine Moduli of Quiver Representations

In this section we summarise the findings of Craw–Smith in [10]. This paper investigates this link between the existence of an interpretation of a projective toric variety as a fine moduli space of quiver representations and the existence of a strong exceptional collection of line bundles.

We summarise the main results. Let X be a projective toric variety with $\text{Cox}(X) \cong \mathbb{k}[x_1, \dots, x_d]$. Given a list of line bundles $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ on X , Craw-Smith defined the *quiver of sections* for \mathcal{L} . They defined $|\mathcal{L}|$ to be the fine moduli space of representations of this quiver. This is a generalisation of the linear series for a single line bundle, so they refer to $|\mathcal{L}|$ as the multilinear series (or multigraded linear series) for \mathcal{L} . They showed that there exists a natural map $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$, and that this map is a morphism if and only if L_1, \dots, L_r are basepoint free. If this is the case then the image of X is a GIT quotient. Then, whereas strong exceptional collections are comparatively rare, they showed that it is almost always possible to pick line bundles \mathcal{L} such that $\varphi_{|\mathcal{L}|}$ is a closed embedding, and such that its image is the fine moduli space of bound quiver representations of the complete quiver of sections for \mathcal{L} .

2.5.1 Multilinear Series

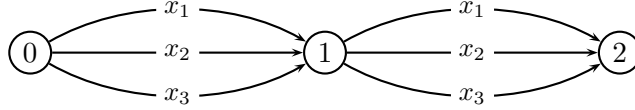
Let $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a list of distinct effective line bundles on X . A torus invariant section $s \in H^0(X, L_j \otimes L_i^{-1})$ is said to be *irreducible* if it does not factor through some L_k with $k \neq i, j$. The *quiver of sections* $Q_{\mathcal{L}}$ for \mathcal{L} is defined to be the quiver whose vertices are in one to one correspondence with bundles in \mathcal{L} , i.e. $Q_0 = \{0, \dots, r\}$, and where the arrows from i to j are in one to one correspondence with irreducible torus invariant sections of $L_j \otimes L_i^{-1}$. We can think of $Q_{\mathcal{L}}$ as being a labelled quiver, where each arrow is labelled by the section it corresponds to. It is possible to assume that the elements in \mathcal{L} are ordered such that if $j < i$ then $L_j \otimes L_i^{-1}$ is not effective.

Since each element in \mathcal{L} is effective, $Q_{\mathcal{L}}$ is rooted at 0 and connected. Since X is projective, it is impossible that both $L_j \otimes L_i^{-1}$ and $L_i \otimes L_j^{-1}$ are both effective, so $Q_{\mathcal{L}}$ is acyclic. Define

$$\text{div} : Q_1 \longrightarrow \mathbb{Z}^d$$

to be the map which sends an arrow a to the *divisor of zeroes* of the torus invariant section labelling a .

Example 2.5.1. Let $X = \mathbb{P}^2$ and let $\mathcal{L} = (\mathcal{O}_X, \mathcal{O}(1), \mathcal{O}(2))$. The quiver of sections for \mathcal{L} is:



The map div can be extended to the path algebra. If a path p has support $\{a_1, \dots, a_n\}$, define

$$\text{div}(p) := \text{div}(a_1) + \dots + \text{div}(a_n).$$

Define the ideal of relations R to be a two sided ideal in the path algebra $\mathbb{k}Q$ generated by all differences $p - p'$ where $\text{t}(p) = \text{t}(p')$, $\text{h}(p) = \text{h}(p')$ and $\text{div}(p) = \text{div}(p')$. The pair (Q, R) is called a *bound quiver of sections*, or a quiver of sections with relations.

Proposition 2.5.2. *If (Q, R) is the complete bound quiver of sections for $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$, then the quotient algebra $\mathbb{k}Q/R$ is isomorphic to $\text{End}(\bigoplus_{i=0}^r L_i)$.*

The incidence map $\text{inc}: \mathbb{Z}^{Q_1} \rightarrow \mathbb{Z}^{Q_0}$ defined by setting $\text{inc}(\chi_a) = \chi_{\text{h}(a)} - \chi_{\text{t}(a)}$ has image equal to the sublattice $\text{Wt}(Q) \subset \mathbb{Z}^{Q_0}$ of functions $\theta: Q_0 \rightarrow \mathbb{Z}$ satisfying $\sum_{i \in Q_0} \theta_i = 0$. The vectors $\{\chi_i - \chi_0 : i \neq 0\}$ form a \mathbb{Z} -basis for $\text{Wt}(Q)$. The $\text{Wt}(Q)$ -grading $\mathbb{k}[y_a : a \in Q_1] \rightarrow \text{Wt}(Q)$ determined by sending y_a to $\text{inc}(\chi_a)$ for $a \in Q_1$ induces a faithful action of the algebraic torus $G := \text{Hom}(\text{Wt}(Q), \mathbb{k}^\times)$ on $\mathbb{A}_{\mathbb{k}}^{Q_1} = \text{Spec } \mathbb{k}[y_a : a \in Q_1]$ in which $g = (g_i)_{i \in Q_0}$ acts on $w = (w_a)_{a \in Q_1}$ as $(g \cdot w)_a = g_{\text{h}(a)} w_a g_{\text{t}(a)}^{-1}$. For $\theta \in \text{Wt}(Q)$, let $\mathbb{k}[y_a : a \in Q_1]_\theta$ denote the θ -graded piece and

$$\mathbb{A}_{\mathbb{k}}^{Q_1} //_\theta G = \text{Proj} \left(\bigoplus_{j \geq 0} \mathbb{k}[y_a : a \in Q_1]_{j\theta} \right)$$

the categorical quotient of the open subset of θ -semistable points in $\mathbb{A}_{\mathbb{k}}^{Q_1}$.

Assume in addition that Q is acyclic with a unique source $0 \in Q_0$. The *toric quiver flag variety* Y_Q is the GIT quotient $\mathbb{A}_{\mathbb{k}}^{Q_1} //_\vartheta G$ linearised by the special weight $\vartheta := \sum_{i \in Q_0} (\chi_i - \chi_0) \in \text{Wt}(Q)$. Such varieties, studied initially by Craw–Smith [10] and in greater generality by Craw [8], can be characterised as follows:

Proposition 2.5.3. *Let Q be a finite, connected, acyclic quiver with a unique source $0 \in Q_0$ and special weight $\vartheta = \sum_{i \in Q_0} (\chi_i - \chi_0)$. The toric quiver flag variety Y_Q coincides with:*

(i) *the GIT quotient $\mathbb{A}_{\mathbb{k}}^{Q_1} //_{\vartheta} G$ linearised by $\vartheta \in \text{Wt}(Q)$;*

(ii) *the geometric quotient of $\mathbb{A}_{\mathbb{k}}^{Q_1} \setminus \mathbb{V}(B_Y)$ by the action of G , where the irrelevant ideal is*

$$B_Y := \left(\prod_{a \in T} y_a : T \text{ is a spanning tree of } Q \text{ rooted at } 0 \right) = \bigcap_{i \in Q_0 \setminus \{0\}} (y_a : \mathbf{h}(a) = i);$$

(iii) *the fine moduli space $\mathcal{M}_{\vartheta}(Q)$ of ϑ -stable representations of the quiver Q of dimension vector $\underline{r} = (1, \dots, 1) \in \mathbb{Z}^{Q_0}$.*

Moreover, Y_Q is a smooth projective toric variety obtained as a tower of projective space bundles over $\text{Spec}(\mathbb{k})$

Proof. See Craw–Smith [10, Proposition 3.8] and Craw [8, Theorem 3.3]. □

Remark 2.5.4. The description of $Y_Q = \mathcal{M}_{\vartheta}(Q)$ as a fine moduli space of representations ensures that it carries a collection of tautological line bundles $\{\mathcal{W}_i : i \in Q_0\}$ with $\mathcal{W}_0 \cong \mathcal{O}_{Y_Q}$ and sheaf homomorphisms $\{\mathcal{W}_{\mathbf{t}(a)} \rightarrow \mathcal{W}_{\mathbf{h}(a)} : a \in Q_1\}$ whose restriction to the fibre over $\mathcal{M}_{\vartheta}(Q)$ encodes the corresponding representation $\{W_{\mathbf{t}(a)} \rightarrow W_{\mathbf{h}(a)} : a \in Q_1\}$. Moreover, the abelian group homomorphism $\text{Wt}(Q) \rightarrow \text{Pic}(Y_Q)$ sending $(\theta_0, \dots, \theta_r)$ to $\mathcal{W}_1^{\theta_1} \otimes \dots \otimes \mathcal{W}_r^{\theta_r}$ is an isomorphism. For more details, see [8, Sections 2-3].

2.5.2 Bound quiver representations

Let Q be a quiver. For any nontrivial path $p = a_k \cdots a_1$ in Q , define the monomial $y_p := y_{a_k} \cdots y_{a_1} \in \mathbb{k}[y_a : a \in Q_1]$, and for any representation W of Q , define $w_p : W_{\mathbf{t}(p)} \rightarrow W_{\mathbf{h}(p)}$ to be the \mathbb{k} -linear map $w_p = w_{a_k} \cdots w_{a_1}$ obtained by composition. Let $J \subset \mathbb{k}Q$ be a two-sided ideal of relations with generators of the form $\sum_{p \in \Gamma} c_p p$, where each Γ is a finite set of paths that share the same head and the same tail. A representation W of Q is a representation of the bound quiver (Q, J) if and only if $\sum_{p \in \Gamma} c_p w_p = 0$ for each Γ arising in the definition of J . A point in representation space $(w_a) \in \mathbb{A}_{\mathbb{k}}^{Q_1}$ defines a representation of (Q, J) if and only if it lies in the subscheme $\mathbb{V}(I_J)$ cut out by the ideal

$$I_J := \left(\sum_{p \in \Gamma} c_p y_p \in \mathbb{k}[y_a : a \in Q_1] \mid \sum_{p \in \Gamma} c_p p \text{ is a generator of } J \right)$$

of relations in $\mathbb{k}[y_a : a \in Q_1]$. The ideal I_J is $\text{Wt}(Q)$ -homogeneous, since we J is generated by sums $\sum_{p \in \Gamma} c_p p$ where the p 's have the same heads and tails. Hence $\mathbb{V}(I_J)$ is G -invariant and the GIT quotient

$$\mathcal{M}_{\vartheta}(Q, J) := \mathbb{V}(I_J) //_{\vartheta} G = \text{Proj} \left(\bigoplus_{j \geq 0} (\mathbb{k}[y_a : a \in Q_1] / I_J)_{j\vartheta} \right) \quad (2.5.1)$$

is the fine moduli space of ϑ -stable representations of (Q, J) with dimension vector $(1, \dots, 1)$. The tautological bundles on $\mathcal{M}_\vartheta(Q, J)$ are obtained from those on $\mathcal{M}_\vartheta(Q)$ by restriction.

Remark 2.5.5. The abelian category of finite-dimensional representations of (Q, J) is equivalent to the category of finitely-generated $\mathbb{k}Q/J$ -modules, so $\mathcal{M}_\vartheta(Q, J)$ is equivalently the fine moduli space of ϑ -stable modules over $\mathbb{k}Q/J$ that are isomorphic as $(\bigoplus_{i \in Q_0} \mathbb{k}e_i)$ -modules to $\bigoplus_{i \in Q_0} \mathbb{k}e_i$.

2.5.3 Morphism to the Multigraded Linear Series

Consider the \mathbb{k} -algebra homomorphism $\tilde{\Phi}: \mathbb{k}[y_a : a \in Q_1] \rightarrow \mathbb{k}[x_1, \dots, x_d]$ sending y_a to $x^{\text{div}(a)}$ for $a \in Q_1$. The actions of the groups $G = \text{Hom}(\text{Wt}(Q), \mathbb{k}^*)$ and $T = \text{Hom}(\Lambda, \mathbb{k}^*)$ on $\mathbb{k}[y_a : a \in Q_1]$ and $\mathbb{k}[x_1, \dots, x_d]$ respectively arise from the horizontal semigroup homomorphisms in the diagram

$$\begin{array}{ccc} \mathbb{N}^{Q_1} & \xrightarrow{\text{inc}} & \text{Wt}(Q) \\ \text{div} \downarrow & & \downarrow \text{pic} \\ \mathbb{N}^d & \xrightarrow{\text{deg}} & \Lambda \end{array} \quad (2.5.2)$$

where the vertical maps satisfy $\text{div}(\chi_a) = \text{div}(a)$ for $a \in Q_1$ and $\text{pic}(\chi_i) = E_i$ for $i \in Q_0$. The map $\tilde{\Phi}$ is equivariant with respect to these actions precisely because (2.5.2) commutes. Hence the rational map $\varphi_{|\mathcal{L}|}$ is a morphism if and only if the preimage of $\mathbb{V}(B_Y)$ is contained in $\mathbb{V}(B_X)$.

Theorem 2.5.6. (*Cor 4.2, [10]*) *We obtain a morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$, if and only if each line bundle in the list \mathcal{L} is basepoint free.*

If each line bundle in \mathcal{L} is basepoint free, then we say the quiver of sections for \mathcal{L} , $Q_{\mathcal{L}}$ is a *basepoint free quiver of sections*.

If this is the case, by Proposition 4.3 of [10], the image of X is given as a GIT quotient:

$$\varphi(X) = \mathbb{V}(I_Q) //_{\vartheta} G$$

where I_Q is the prime ideal

$$I_Q = (f \in \mathbb{k}[y_a | a \in Q_1] | f \text{ is homogeneous and } f \in \ker(\tilde{\Phi})).$$

Craw–Smith gave necessary and sufficient conditions for the morphism φ to be a closed embedding.

Proposition 2.5.7 (Proposition 4.9, [10]). *Let Q be a basepoint free quiver of sections, and let $\vartheta = \sum_{i \in Q_0} (\mathbf{e}_i - \mathbf{e}_0)$. The map $\varphi: X \rightarrow Y$ is a closed embedding if and only if the line bundle $L := L_0^{\vartheta_0} \otimes \dots \otimes L_r^{\vartheta_r}$ is ample and $((\text{Cox}(Y)/I_Q)[y^{-\hat{\sigma}}])_{[0]} \cong ((\text{Cox}(X)[x^{-\hat{\sigma}}])_{[0]})$ for all top dimensional cones σ in the fan defining X .*

We say a quiver of sections Q is *very ample* if it is basepoint free and $\varphi_Q: X \rightarrow Y$ is a closed embedding.

Corollary 2.5.8. (Cor 4.10 [10]) Let \mathcal{L} be a list of basepoint free line bundles and define $L := \bigotimes_{i \in Q_0} L_i$. Assume that the multiplication map $H^0(X, L_1) \otimes \cdots \otimes H^0(X, L_r) \longrightarrow H^0(X, L)$ is surjective. Then $\varphi_Q : X \longrightarrow Y$ is a closed embedding if and only if L is very ample.

2.5.4 Projective Toric Varieties as Fine Moduli

Recall the ideal of relations R in $\mathbb{k}Q$ is generated by all differences of paths $p - p'$ where $\mathfrak{h}(p) = \mathfrak{h}(p')$, $\mathfrak{t}(p) = \mathfrak{t}(p')$ and $\text{div}(p) = \text{div}(p')$. A representation of the bound quiver (Q, R) is a representation $W = (W_i, w_a)$ of Q where $w_p - w_{p'} = 0$ whenever $p - p' \in R$. The fine moduli space of representations of (Q, R) , $\mathcal{M}_\vartheta(Q, R)$, is the GIT quotient of $\mathbb{V}(I_R)$ under the action of G , where

$$I_R = (y_p - y_{p'} \mid \mathfrak{h}(p) = \mathfrak{h}(p'), \mathfrak{t}(p) = \mathfrak{t}(p') \text{ and } \text{div}(p) = \text{div}(p')).$$

The ideal I_R is homogeneous with respect to the $\text{Wt}(Q)$ grading, and hence $\mathbb{V}(I_R)$ is a G -invariant subset of \mathbb{A}^{Q^1} .

If Q is a very ample quiver of sections, then $\mathcal{M}_\vartheta(Q, R) \cong X$ if and only if $\overline{\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)} = \overline{\mathbb{V}(I_R) \setminus \mathbb{V}(B_Y)}$. We can translate this into a statement in commutative algebra:

$$X \cong \mathcal{M}_\vartheta(Q, R) \text{ if and only if } I_Q = I_R : B_Y^\infty.$$

If this is the case, then we say that Q is *fine*. The next theorem shows that it is almost always the case that we can find a list of line bundles \mathcal{L} such that the complete quiver of sections for \mathcal{L} is fine.

Theorem 2.5.9 (Theorem 5.5, [10]). Let L_1, \dots, L_{r-2} be basepoint free line bundles on X . If the subsemigroup of $\text{Pic}(X)$ generated by L_1, \dots, L_{r-2} contains an ample line bundle, then there exist line bundles L_{r-1} and L_r such that the quiver of sections of $\mathcal{L} := \{\mathcal{O}_X, L_1, \dots, L_r\}$ is fine.

Chapter 3

Geometric Results

3.1 Quivers of Sections on Mori Dream Spaces

In this section we introduce the bound quiver of sections for a collection of line bundles on a Mori Dream Space. These bound quivers encode the endomorphism algebra of the direct sum of the sheaves in the collection. For $r \geq 0$, consider a collection of distinct line bundles

$$\mathcal{L} := (L_0, L_1, \dots, L_r) \subset \text{Cl}(X)$$

on the Mori Dream Space X , where $L_0 = \mathcal{O}_X$ and L_1, \dots, L_r are effective. For $0 \leq i \leq r$, define $E_i := \psi^{-1}(L_i)$ using diagram (2.1.2) to obtain a collection

$$\widetilde{\mathcal{L}} := (E_0, E_1, \dots, E_r)$$

of distinct rank one reflexive sheaves on an ambient toric variety \widetilde{X} . For $0 \leq i, j \leq r$, we say that a torus-invariant section $s \in H^0(X, E_j \otimes E_i^{-1}) = \text{Hom}(E_i, E_j)$ is *irreducible* if it does not factor through some E_k with $k \neq i, j$. The following definition extends the notion of a quiver of sections for a collection of line bundles on a projective toric variety due to Craw-Smith [10] introduced in Section 2.5.

Definition 3.1.1. The *quiver of sections* of the collection \mathcal{L} on X is defined to be the quiver of sections of the collection $\widetilde{\mathcal{L}}$ on \widetilde{X} , that is, the quiver Q with vertex set $Q_0 = \{0, \dots, r\}$, and where the arrows from i to j correspond to the irreducible sections in $H^0(X, E_j \otimes E_i^{-1})$.

Remark 3.1.2. 1. Definition 3.1.1 depends a priori on the choice of ambient toric variety \widetilde{X} . However, any two are isomorphic in codimension-one, so Q is independent of the choice.

2. We abuse terminology by calling Q the ‘quiver of sections of \mathcal{L} ’ because paths in Q from i to j are not constructed directly from a basis of $\text{Hom}(L_i, L_j)$ as in the literature, see [8, 10]. We justify this abuse by recovering the Hom spaces in Proposition 3.1.5 below.

Lemma 3.1.3. *The quiver of sections Q is connected, acyclic, and $0 \in Q_0$ is the unique source.*

Proof. Projectivity of \tilde{X} ensures that at most one of $\text{Hom}(E_i, E_j)$ and $\text{Hom}(E_j, E_i)$ is nonzero for $i \neq j$, so Q is acyclic since there cannot be paths from i to j and from j to i . For $i \in Q_0$, the space $\text{Hom}(E_0, E_i)$ has a torus-invariant element since E_1, \dots, E_r are effective and $E_0 \cong \mathcal{O}_{\tilde{X}}$, giving rise to a path in Q from 0 to $i \in Q_0$ so 0 is the unique source. \square

The quiver of sections depends purely on the collection of reflexive sheaves $\tilde{\mathcal{L}}$ on \tilde{X} , but we aim to encode information about the collection of line bundles \mathcal{L} on the Mori Dream Space X . To achieve this, write the Cox ring $\mathbb{k}[x_1, \dots, x_d]$ of the toric variety \tilde{X} as the semigroup algebra of the semigroup \mathbb{N}^d of effective torus-invariant Weil divisors in \tilde{X} . Define the *label* of an arrow $a \in Q_1$, denoted $\text{div}(a) \in \mathbb{N}^d$, to be the divisor of zeroes in \tilde{X} of the defining torus-invariant section $s \in H^0(X, E_j \otimes E_i^{-1})$. More generally, the label of any path p in Q is the torus-invariant divisor $\text{div}(p) := \sum_{a \in \text{supp}(p)} \text{div}(a)$. It is often convenient to consider the corresponding labelling monomial $x^{\text{div}(p)} := \prod_{a \in \text{supp}(p)} x^{\text{div}(a)} \in \mathbb{k}[x_1, \dots, x_d]$ in the Cox ring of \tilde{X} . Recall from (2.1.3) that our chosen presentation of the Cox ring of X determines an ideal $I_X \subset \mathbb{k}[x_1, \dots, x_d]$.

Definition 3.1.4. Consider the two-sided ideal

$$R := \left(\sum_{p \in \Gamma} c_p p \in \mathbb{k}Q \mid \begin{array}{l} \Gamma \text{ is any finite set of paths that share the same} \\ \text{head and tail and satisfy } \sum_{p \in \Gamma} c_p x^{\text{div}(p)} \in I_X \end{array} \right)$$

in the path algebra $\mathbb{k}Q$. The pair (Q, R) is the *bound quiver of sections* of the collection \mathcal{L} .

Proposition 3.1.5. *The quotient algebra $\mathbb{k}Q/R$ is isomorphic to $\text{End}_{\mathcal{O}_X}(\bigoplus_{i \in Q_0} L_i)$, and each vertex $i \in Q_0$ satisfies $e_i(\mathbb{k}Q/R)e_0 \cong H^0(X, L_i)$.*

Proof. The proof of Proposition 3.3 in [10] applies verbatim to the collection of reflexive sheaves $\tilde{\mathcal{L}}$ on \tilde{X} , so the \mathbb{k} -algebra epimorphism $\tilde{\eta}: \mathbb{k}Q \rightarrow \text{End}_{\mathcal{O}_{\tilde{X}}}(\bigoplus_{i \in Q_0} E_i)$ sending $\sum c_p p$ to $\sum c_p x^{\text{div}(p)}$ has kernel the $\text{Cl}(X)$ -homogeneous ideal

$$\tilde{R} := (p - p' \in \mathbb{k}Q \mid \text{h}(p) = \text{h}(p'), \text{t}(p) = \text{t}(p'), \text{div}(p) = \text{div}(p'))$$

in $\mathbb{k}Q$. Embed $\text{End}_{\mathcal{O}_{\tilde{X}}}(\bigoplus_{i \in Q_0} E_i) = \bigoplus_{i, j \in Q_0} H^0(E_j \otimes E_i^{-1})$ in the Cox ring $\mathbb{k}[x_1, \dots, x_d]$ of \tilde{X} . The restriction of the $\text{Cl}(X)$ -graded homomorphism τ from (2.1.2) then defines the right-hand vertical map in the commutative diagram of \mathbb{k} -algebras

$$\begin{array}{ccc} \mathbb{k}Q & \xrightarrow{\tilde{\eta}} & \text{End}_{\mathcal{O}_{\tilde{X}}}(\bigoplus_{i \in Q_0} E_i) \\ \parallel & & \downarrow \tau \\ \mathbb{k}Q & \xrightarrow{\eta} & \text{End}_{\mathcal{O}_X}(\bigoplus_{i \in Q_0} L_i) \end{array} \quad (3.1.1)$$

where the epimorphism η sends $\sum c_p p$ to the class $\sum c_p x^{\text{div}(p)} \bmod I_X$ in the Cox ring of X . The ideal R is the kernel of η , so the first statement holds. The second statement follows from the first since we have $L_0 = \mathcal{O}_X$ and we compose arrows and maps from right to left. \square

3.2 Multilinear Series

In this section we use the quiver of sections of a collection \mathcal{L} of line bundles on a Mori Dream Space X to define the corresponding multilinear series $|\mathcal{L}|$. This variety generalises the classical linear series of a single line bundle in that one obtains a natural map from X to $|\mathcal{L}|$ by evaluating sections of line bundles. We give necessary and sufficient conditions for this map to be a morphism and to be a closed embedding. In the case that the map is a morphism we describe its image as a GIT quotient.

Let $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of effective line bundles on a Mori Dream Space X . Lemma 3.1.3 guarantees that the corresponding quiver of sections Q is finite, connected, acyclic and has a unique source $0 \in Q_0$.

Definition 3.2.1. The *multilinear series* for \mathcal{L} is the toric quiver flag variety $|\mathcal{L}| := Y_Q$ of Q from Proposition 2.5.3. It carries tautological line bundles $\{\mathcal{W}_i : i \in Q_0\}$ with $\mathcal{W}_0 \cong \mathcal{O}_{Y_Q}$.

Remark 3.2.2. Just as Q is not precisely the quiver of sections of \mathcal{L} (see Remark 3.1.2), it is perhaps an abuse of terminology to call Y_Q the multilinear series of \mathcal{L} . Indeed, for the special case $\mathcal{L} = (\mathcal{O}_X, L_1)$ we have that $Y_Q \cong \mathbb{P}(H^0(E_1))$ is a projective space, but it need not coincide with the classical linear series $|L_1|$ because the epimorphism $\tau|_{H^0(\tilde{X}, E_1)} : H^0(\tilde{X}, E_1) \rightarrow H^0(X, L_1)$ from diagram (2.1.2) need not be an isomorphism.

In order to study morphisms from X to the multigraded linear series $|\mathcal{L}|$, define

$$\tilde{\Phi} : \mathbb{k}[y_a : a \in Q_1] \rightarrow \mathbb{k}[x_1, \dots, x_d]$$

to be the \mathbb{k} -algebra homomorphism sending y_a to $x^{\text{div}(a)}$ for $a \in Q_1$. The actions of the groups $G = \text{Hom}(\text{Wt}(Q), \mathbb{k}^*)$ and $T = \text{Hom}(\text{Cl}(X), \mathbb{k}^*)$ on $\mathbb{k}[y_a : a \in Q_1]$ and $\mathbb{k}[x_1, \dots, x_d]$ respectively arise from the horizontal semigroup homomorphisms in the diagram

$$\begin{array}{ccc} \mathbb{N}^{Q_1} & \xrightarrow{\text{inc}} & \text{Wt}(Q) \\ \text{div} \downarrow & & \downarrow \text{pic} \\ \mathbb{N}^d & \xrightarrow{\text{deg}} & \text{Cl}(X) \end{array} \quad (3.2.1)$$

where the vertical maps satisfy $\text{div}(\chi_a) = \text{div}(a)$ for $a \in Q_1$ and $\text{pic}(\chi_i) = E_i$ for $i \in Q_0$. The map $\tilde{\Phi}$ is equivariant with respect to these actions precisely because (3.2.1) commutes. Under the identification of $\text{Wt}(Q)$ with the Picard group of $|\mathcal{L}|$, the subspace of the Cox ring $\mathbb{k}[y_a : a \in Q_1]$ of $|\mathcal{L}|$ spanned by monomials of weight $\theta \in \text{Wt}(Q)$ coincides with $H^0(\mathcal{W}_1^{\theta_1} \otimes \dots \otimes \mathcal{W}_r^{\theta_r})$.

Since the T -action on $\text{Cox}(X)$ is compatible with that on $\mathbb{k}[x_1, \dots, x_d]$ by (2.1.2), the map

$$\Phi := \tau \circ \tilde{\Phi} : \mathbb{k}[y_a : a \in Q_1] \longrightarrow \text{Cox}(X)$$

is equivariant. The induced equivariant morphism $\Phi^*: \mathbb{V}(I_X) \rightarrow \mathbb{A}_{\mathbb{k}}^{Q_1}$ descends to a rational map $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$.

Proposition 3.2.3. *Let $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of effective line bundles on X . The rational map $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$ is a morphism if and only if L_i is basepoint-free for $1 \leq i \leq r$.*

Proof. For $x \in X$ choose any lift $\tilde{x} \in \mathbb{V}(I_X) \setminus \mathbb{V}(B_X)$. The G -orbit of the quiver representation $\Phi^*(\tilde{x}) \in \mathbb{A}_{\mathbb{k}}^{Q_1}$, which is independent of the choice of lift, is obtained by evaluating the labels on arrows at \tilde{x} , that is, by evaluating sections of the bundles $L_{h(a)} \otimes L_{t(a)}^{-1}$ at x . The rational map $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$ is a morphism if and only if every such $\Phi^*(\tilde{x}) \in \mathbb{A}_{\mathbb{k}}^{Q_1}$ is ϑ -stable. Let $W' = ((W'_i)_{i \in Q_0}, (w'_a)_{a \in Q_1})$ be a proper subrepresentation of $\Phi^*(\tilde{x})$. Since $\vartheta_0 = -r$ and $\vartheta_i > 0$ for $i > 0$, the submodule W' of $\Phi^*(\tilde{x})$ is ϑ -destabilising if and only if $\dim_{\mathbb{k}}(W'_0) = 1$ and there exists $i > 0$ such that for every path $p = a_\ell \cdots a_1$ from 0 to i , the composition $w'_{a_\ell} \cdots w'_{a_1}$ is the zero map. In particular, $\Phi^*(\tilde{x}) \in \mathbb{A}_{\mathbb{k}}^{Q_1}$ is ϑ -unstable if and only if there exists $i > 0$ such that the evaluation of every section of L_i at x equals zero. Equivalently, $\Phi^*(\tilde{x}) \in \mathbb{A}_{\mathbb{k}}^{Q_1}$ is ϑ -stable if and only if L_i is basepoint-free for $1 \leq i \leq r$. \square

The Cox ring of X is a unique factorisation domain, so $\ker(\Phi_Q)$ is prime and hence so is the ideal

$$I_Q := \left(f \in \mathbb{k}[y_a : a \in Q_1] : f \in \ker(\Phi_Q) \text{ is } \text{Wt}(Q)\text{-homogeneous} \right) \quad (3.2.2)$$

generated by its $\text{Wt}(Q)$ -homogeneous elements. This ideal can be computed explicitly as the kernel of the \mathbb{k} -algebra homomorphism

$$\Psi: \mathbb{k}[y_a : a \in Q_1] \rightarrow \text{Cox}(X) \otimes_{\mathbb{k}} \mathbb{k}[\text{Wt}(Q)] \quad (3.2.3)$$

satisfying $\Psi(y_a) = t_{h(a)} x^{\text{div}(a)} t_{t(a)}^{-1}$ for $a \in Q_1$; see Chapter 5 for details. This ideal cuts out the image of the morphism constructed in Proposition 3.2.3 as follows.

Proposition 3.2.4. *Let $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of basepoint-free line bundles on X with quiver of sections Q . Then*

- (i) *the image of the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$ is $\mathbb{V}(I_Q) //_{\vartheta} G$; and*
- (ii) *the tautological line bundles on $|\mathcal{L}|$ satisfy $\varphi_{|\mathcal{L}|}^*(\mathcal{W}_i) = L_i$ for $i \in Q_0$.*

Proof. Since X is complete, the image of $\varphi_{|\mathcal{L}|}$ is a closed subscheme of $Y := |\mathcal{L}|$. The geometric quotient construction of Y from Proposition 2.5.3(i) implies that the image is therefore the geometric quotient of a G -invariant closed subscheme of $\mathbb{A}_{\mathbb{k}}^{Q_1} \setminus \mathbb{V}(B_Y)$. The affine variety $\mathbb{V}(\ker(\Phi))$ is the image of the equivariant morphism $\text{Spec}(\text{Cox}(X)) \rightarrow \mathbb{A}_{\mathbb{k}}^{Q_1}$ induced by Φ , and the variety $\mathbb{V}(I_Q)$ cut out by the $\text{Wt}(Q)$ -homogeneous part of $\ker(\Phi)$ is the minimal G -invariant algebraic set in $\mathbb{A}_{\mathbb{k}}^{Q_1}$

containing all G -orbits from $\mathbb{V}(\ker(\Phi))$. The image of $\varphi_{|\mathcal{L}|}$ is therefore the geometric quotient of $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)$ by the action of G . This coincides with the GIT quotient $\mathbb{V}(I_Q) //_{\vartheta} G$ by Proposition 2.5.3, so (i) holds. For part (ii), the tautological bundle \mathcal{W}_i on Y corresponds to the weight $\chi_i - \chi_0 \in \text{Wt}(Q)$ under the isomorphism from Remark 2.5.4. Since the equivariant morphism $\text{Spec}(\text{Cox}(X)) \rightarrow \mathbb{A}_{\mathbb{k}}^{Q_1}$ factors through $\mathbb{A}_{\mathbb{k}}^d$, examining the diagrams (2.1.2) and (3.2.1) shows that $\varphi_{|\mathcal{L}|}^*(\mathcal{W}_i) = (\psi \circ \text{pic})(\chi_i - \chi_0) = \psi(E_i) = L_i$ for $i \in Q_0$. \square

Proof of Theorem 1.1.1. Proposition 3.2.3 establishes that $\varphi_{|\mathcal{L}|}: X \dashrightarrow |\mathcal{L}|$ is a morphism if and only if L_i is basepoint-free for $1 \leq i \leq r$. Proposition 3.2.4 then presents the image explicitly as a geometric quotient, and establishes that the tautological line bundles on $|\mathcal{L}|$ satisfy $\varphi_{|\mathcal{L}|}^*(\mathcal{W}_i) = L_i$ for $i \in Q_0$ as required. \square

Remark 3.2.5. The list of reflexive sheaves $\widetilde{\mathcal{L}}$ on \widetilde{X} determines the ideal

$$\widetilde{I}_Q = \left(f \in \mathbb{k}[y_a : a \in Q_1] : f \in \ker(\widetilde{\Phi}) \text{ is } \text{Wt}(Q)\text{-homogeneous} \right) \quad (3.2.4)$$

obtained as the toric ideal of the semigroup homomorphism $\text{inc} \oplus \text{div}: \mathbb{N}^{Q_1} \rightarrow \text{Wt}(Q) \oplus \mathbb{N}^d$. If each reflexive sheaf in $\widetilde{\mathcal{L}}$ is a basepoint-free line bundle on \widetilde{X} , then Theorem 1 of [10] gives a morphism $\varphi_{|\widetilde{\mathcal{L}}|}: \widetilde{X} \rightarrow \mathbb{V}(I_{\widetilde{Q}}) //_{\vartheta} G$ whose restriction to X is the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow \mathbb{V}(I_Q) //_{\vartheta} G$ from Proposition 3.2.4. However, this is typically not the case as Example 3.2.9.

3.2.1 Criteria for closed immersion

A collection \mathcal{L} is said to be *very ample* if the morphism $\varphi_{|\mathcal{L}|}$ from Proposition 3.2.3 is a closed immersion. We now introduce a necessary and sufficient condition for \mathcal{L} to be very ample. We (enhance and) adapt the proofs of Proposition 5.7 of [8] and Corollary 4.10 of [10] to our situation because Q is not precisely the quiver of sections for \mathcal{L} (see Remarks 3.1.2 and 3.2.2).

Theorem 3.2.6. *Let $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ be a collection of basepoint-free line bundles on X . The following are equivalent:*

- (i) *the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$ is a closed immersion;*
- (ii) *the image of the multiplication map*

$$H^0(L_1) \otimes \cdots \otimes H^0(L_r) \longrightarrow H^0(L_1 \otimes \cdots \otimes L_r). \quad (3.2.5)$$

is a very ample linear series;

- (iii) *the map $\prod_{1 \leq i \leq r} \varphi_{|L_i|}: X \rightarrow |L_1| \times \cdots \times |L_r|$ is a closed immersion.*

Proof. The toric variety $|\mathcal{L}|$ is smooth, so the ample bundle $\vartheta = \mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_r$ determines the closed immersion $\varphi_{|\vartheta|}: |\mathcal{L}| \rightarrow \mathbb{P}^*(H^0(\vartheta))$. The composition $\varphi_{|\vartheta|} \circ \varphi_{|\mathcal{L}|}: X \rightarrow \mathbb{P}^*(H^0(\vartheta))$ is determined by the line bundle $(\varphi_{|\vartheta|} \circ \varphi_{|\mathcal{L}|})^*(\vartheta) = (\psi \circ \text{pic})(\theta) = L_1 \otimes \cdots \otimes L_r$ and the subspace of sections $\Phi(H^0(\vartheta)) \subseteq H^0(L_1 \otimes \cdots \otimes L_r)$. We claim that $\Phi(H^0(\vartheta))$ coincides with the image V of the multiplication map (3.2.5), in which case $\varphi_{|\vartheta|} \circ \varphi_{|\mathcal{L}|}$ coincides with the (a priori rational) map $\varphi_V: X \rightarrow \mathbb{P}^*(V)$ to the classical linear series. Indeed, for $\theta = (\theta_0, \dots, \theta_r) \in \text{Wt}(Q)$, the restriction of Φ to the subspace spanned by monomials of weight θ defines a \mathbb{k} -linear map

$$\Phi_\theta: H^0(\mathcal{W}_1^{\theta_1} \otimes \cdots \otimes \mathcal{W}_r^{\theta_r}) \rightarrow H^0(L_1^{\theta_1} \otimes \cdots \otimes L_r^{\theta_r})$$

because $(\psi \circ \text{pic})(\theta) = L_1^{\theta_1} \otimes \cdots \otimes L_r^{\theta_r}$. In particular, the map Φ_θ for $\vartheta = \sum_{1 \leq i \leq r} (\chi_i - \chi_0)$ and the product $\otimes_{1 \leq i \leq r} \Phi_{(\chi_i - \chi_0)}$ fit in to a commutative diagram of \mathbb{k} -vector spaces

$$\begin{array}{ccc} H^0(\mathcal{W}_1) \otimes \cdots \otimes H^0(\mathcal{W}_r) & \longrightarrow & H^0(\mathcal{W}_1 \otimes \cdots \otimes \mathcal{W}_r) \\ \otimes_{1 \leq i \leq r} \Phi_{(\chi_i - \chi_0)} \downarrow & & \downarrow \Phi_\vartheta \\ H^0(L_1) \otimes \cdots \otimes H^0(L_r) & \longrightarrow & H^0(L_1 \otimes \cdots \otimes L_r) \end{array} \quad (3.2.6)$$

in which the horizontal maps are given by multiplication. For $1 \leq i \leq r$, the map $\Phi_{(\chi_i - \chi_0)}$ can be obtained by composing three surjective maps, namely, the isomorphism $H^0(\mathcal{W}_i) \rightarrow e_i(\mathbb{k}Q)e_0$ from [8, Corollary 3.5], the restriction to $e_i(\mathbb{k}Q)e_0$ of the epimorphism $\tilde{\eta}: \mathbb{k}Q \rightarrow \text{End}(\bigoplus_{i \in Q_0} E_i)$ from the proof of Proposition 3.1.5, and the map $\tau|_{H^0(E_i)}: H^0(E_i) \rightarrow H^0(L_i)$ from diagram (2.1.2). It follows that $\otimes_{1 \leq i \leq r} \Phi_{(\chi_i - \chi_0)}$ is surjective, so commutativity of the diagram implies that the image of Φ_ϑ coincides with the image V of (3.2.5). This proves the claim.

Since V is the image of the multiplication map (3.2.5), the morphism $\varphi_V: X \rightarrow \mathbb{P}^*(V)$ is the composition of the product $\prod_{1 \leq i \leq r} \varphi_{|L_i|}: X \rightarrow |L_1| \times \cdots \times |L_r|$ of morphisms to the classical linear series and the appropriate Segre embedding to $\mathbb{P}^*(V)$. The claim implies that the diagram

$$\begin{array}{ccccc} |L_1| \times \cdots \times |L_r| & \xrightarrow{\text{Segre}} & \mathbb{P}^*(V) & \xrightarrow{\iota} & \mathbb{P}^*(H^0(\vartheta)) \\ \prod_{1 \leq i \leq r} \varphi_{|L_i|} \uparrow & & & & \uparrow \varphi_{|\vartheta|} \\ X & \xrightarrow{\varphi_{|\mathcal{L}|}} & & & |\mathcal{L}| \end{array}$$

commutes, where ι is the closed immersion of projective spaces induced by Φ_ϑ . Three maps in the diagram are closed immersions, so $\varphi_{|\mathcal{L}|}$ is a closed immersion if and only if $\prod_{1 \leq i \leq r} \varphi_{|L_i|}$ is a closed immersion if and only if the linear series V is very ample as required. \square

Remark 3.2.7. Neither of the maps from statements (i) and (iii) of Theorem 3.2.6 factors through the other. Typically $|\mathcal{L}|$ has much lower dimension than $|L_1| \times \cdots \times |L_r|$, so the multigraded linear series is a more efficient multigraded ambient space than the product.

Corollary 3.2.8. *Let L_1, \dots, L_{r-1} be basepoint-free line bundles on X . If the subsemigroup of $\text{Pic}(X)$ generated by L_1, \dots, L_{r-1} contains an ample bundle, then there exists a line bundle L_r such that the quiver of sections for $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$ is very ample.*

Proof. Theorem 3.2.6 implies that $\varphi_{|\mathcal{L}|}$ is a closed immersion if $L_1 \otimes \dots \otimes L_r$ is very ample and the map (3.2.5) is surjective. The proof of [10, Proposition 4.14] now applies verbatim. \square

Example 3.2.9. Continuing Example 2.1.11, let X_4 be the del-Pezzo surface for which the ample linearisation $\chi = 11l_0 - 5l_1 - 3l_2 - 2l_3 - l_4$ defines $\widetilde{X}_4 := \mathbb{A}^{10} //_{\chi} T$. We compute using the intersection pairing that each line bundle in the list

$$\mathcal{L} = (\mathcal{O}_{X_4}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0) \quad (3.2.7)$$

is basepoint-free but not ample. Write $\widetilde{\mathcal{L}} = (E_0, E_1, \dots, E_6)$. Since each E_i is basepoint-free on some ambient toric variety, the code from [21, Example 2.11] computes the irrelevant ideal for the GIT quotient $\mathbb{A}_{\mathbb{k}}^d //_{E_i} T$ determined by the corresponding linearisation $E_i \in \text{Cl}(X)$. By comparing each with the irrelevant ideal of $\chi \in \text{Cl}(X)$ we see that E_2, E_3, E_4 are not basepoint-free line bundles on \widetilde{X}_4 . In particular, we cannot deduce that $\varphi_{|\mathcal{L}|}$ is a morphism simply by restriction from the toric case (compare Remark 3.2.5), though it is a morphism by Proposition 3.2.3.

To investigate $\varphi_{|\mathcal{L}|}$ directly in this case, the quiver of sections Q is shown in Figure 6.1, where each arrow is labelled by the torus-invariant section of the relevant reflexive sheaf on \widetilde{X}_4 . Arrows

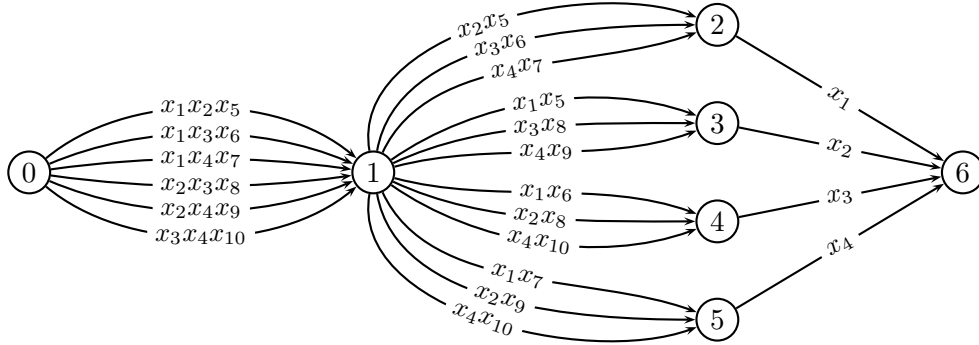


Figure 3.1: A quiver of sections for a collection on X_4

with tail at 0 are listed a_1, \dots, a_6 from the top of Figure 6.1 to the bottom; list those with tail at 1 as a_7, \dots, a_{18} from the top of the figure to the bottom; and list those with head at 6 as a_{19}, \dots, a_{22} from the top to the bottom. Likewise, list the coordinates of $\mathbb{A}_{\mathbb{k}}^{Q_1}$ as y_1, \dots, y_{22} , and compute the

kernel of (3.2.3) to obtain the ideal

$$I_Q = \left(\begin{array}{c} y_{16} - y_{17} + y_{18}, y_{13} - y_{14} + y_{15}, y_{10} - y_{11} + y_{12}, y_7 - y_8 + y_9, y_3 - y_5 + y_6 \\ y_2 - y_4 + y_6, y_1 - y_4 + y_5, y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21} \\ y_9y_{19} - y_{17}y_{22} + y_{18}y_{22}, y_8y_{19} - y_{14}y_{21} + y_{18}y_{22}, y_6y_{17} - y_5y_{18}, y_6y_{14} - y_4y_{15} \\ y_5y_{11} - y_4y_{12}, y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9, y_8y_{15}y_{17} - y_9y_{14}y_{18} - y_8y_{15}y_{18} + y_9y_{15}y_{18} \\ y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, y_9y_{11}y_{17} - y_8y_{12}y_{17} + y_8y_{12}y_{18} - y_9y_{12}y_{18} \\ y_9y_{11}y_{14} - y_8y_{12}y_{14} + y_8y_{11}y_{15} - y_9y_{11}y_{15} \end{array} \right)$$

that cuts out the image of $\varphi_{|\mathcal{L}|}: X_4 \rightarrow |\mathcal{L}|$. We claim that $\varphi_{|\mathcal{L}|}$ is a closed immersion, and hence $X_4 \cong \mathbb{V}(I_Q) //_{\mathfrak{g}} G$. Indeed, for $1 \leq i \leq 4$ we have $L_{i+1} = 2l_0 - l_i$, and the intersection pairing shows that $\varphi_{|L_{i+1}|}: X_4 \rightarrow \mathbb{F}_1$ contracts the (-1) -curves $\{l_j : j \neq i\}$ but not l_i . A simple case-by-case analysis shows that the morphism $\prod_{2 \leq i \leq 5} \varphi_{|L_i|}$ separates all points and tangent vectors of X_4 : a pair of distinct points on X_4 must either both lie on the same exceptional curve, lie on different (non-intersecting) exceptional curves, have one point on an exceptional curve and one off an exceptional curve or have neither lying on an exceptional curve. In each of the above cases, there is an exceptional curve L_j which has neither point on it, and hence the map $\varphi_{|L_j|}$ separates the two points and their tangent vectors. Therefore $\prod_{2 \leq i \leq 5} \varphi_{|L_i|}$ must also separate the points and their tangent vectors, hence so does $\prod_{1 \leq i \leq 6} \varphi_{|L_i|}$. We deduce from Theorem 3.2.6 that $\varphi_{|\mathcal{L}|}: X_4 \rightarrow |\mathcal{L}|$ is a closed immersion.

Chapter 4

Algebraic Results

4.1 Fine Moduli of Bound Quiver Representations

This chapter establishes when the morphism $\varphi_{|\mathcal{L}|}: X \rightarrow |\mathcal{L}|$ induces an isomorphism between the Mori Dream Space X and a fine moduli space $\mathcal{M}_\vartheta(Q, R)$ of ϑ -stable modules over the endomorphism algebra of $\bigoplus_{i \in Q_0} L_i$. Our main algebraic result is an efficient construction for collections of line bundles with this property.

A list \mathcal{L} of line bundles on X defines a pair of two-sided ideals in $\mathbb{k}Q$ and hence a pair of ideals of relations in $\mathbb{k}[y_a : a \in Q_1]$. First, the ideal R from Definition 3.1.4 determines the ideal of relations

$$I_R = \left(\sum_{p \in \Gamma} c_p y_p \in \mathbb{k}[y_a : a \in Q_1] \mid \begin{array}{l} \Gamma \text{ is any set of paths sharing head and} \\ \text{tail for which } \sum_{p \in \Gamma} c_p x^{\text{div}(p)} \in I_X \end{array} \right). \quad (4.1.1)$$

Each generator of I_R is $\text{Wt}(Q)$ -homogeneous and lies in $\ker(\tilde{\Phi})$, so I_R is contained in the prime ideal of equations I_Q from (3.2.2). In Chapter 6 we present code that allows us to compute I_R explicitly.

In addition, the kernel \tilde{R} of the epimorphism $\mathbb{k}Q \rightarrow \text{End}_{\mathcal{O}_{\tilde{X}}}(\bigoplus_{i \in Q_0} E_i)$ obtained by sending p to $x^{\text{div}(p)}$ determines the ideal of relations

$$\tilde{I}_R := I_{\tilde{R}} = \left(\sum_{p \in \tilde{\Gamma}} c_p y_p \in \mathbb{k}[y_a : a \in Q_1] \mid \begin{array}{l} \tilde{\Gamma} \text{ is any set of paths sharing head and} \\ \text{tail for which } \sum_{p \in \tilde{\Gamma}} c_p x^{\text{div}(p)} = 0 \end{array} \right). \quad (4.1.2)$$

We have that \tilde{I}_R is contained both in the ideal of equations \tilde{I}_Q from (3.2.4), and in I_R . We have the following inclusions:

$$\begin{array}{ccc} I_R & \subset & I_Q \\ \cup & & \cup \\ \tilde{I}_R & \subset & \tilde{I}_Q \end{array}$$

Compute the affine varieties in $\mathbb{A}_{\mathbb{k}}^{Q_1}$ cut out by the ideals $\widetilde{I}_R, \widetilde{I}_Q, I_R, I_Q \subset \mathbb{k}[y_a : a \in Q_1]$, remove from each the ϑ -unstable locus $\mathbb{V}(B_Y)$, and compute the geometric quotient by the action of G to obtain the left-hand square in the commutative diagram of GIT quotients

$$\begin{array}{ccccc} \mathbb{V}(\widetilde{I}_Q) //_{\vartheta} G & \longrightarrow & \mathcal{M}_{\vartheta}(Q, \widetilde{R}) & \longrightarrow & \mathbb{A}_{\mathbb{k}}^{Q_1} //_{\vartheta} G \\ \uparrow & & \uparrow & & \parallel \\ \mathbb{V}(I_Q) //_{\vartheta} G & \longrightarrow & \mathcal{M}_{\vartheta}(Q, R) & \longrightarrow & |\mathcal{L}| \end{array} \quad (4.1.3)$$

in which each morphism is a closed immersion.

Theorem 4.1.1. *If \mathcal{L} is a list of basepoint-free line bundles on X , then the induced morphism*

$$\varphi_{|\mathcal{L}|} : X \longrightarrow \mathcal{M}_{\vartheta}(Q, R) \quad (4.1.4)$$

is surjective if and only if I_Q coincides with the saturation $(I_R : B_Y^{\infty})$. In particular, if \mathcal{L} is very ample and $I_Q = (I_R : B_Y^{\infty})$ then (4.1.4) is an isomorphism.

Proof. It suffices by Theorem ?? to show that the closed immersion $\mathbb{V}(I_Q) //_{\vartheta} G \rightarrow \mathbb{V}(I_R) //_{\vartheta} G$ is an isomorphism. Proposition 2.5.3 shows that the ideal B_Y cuts out the ϑ -unstable locus in $\mathbb{A}_{\mathbb{k}}^{Q_1}$, so we need only show that $\mathbb{V}(I_Q) \setminus \mathbb{V}(B_Y)$ is isomorphic to $\mathbb{V}(I_R) \setminus \mathbb{V}(B_Y)$. Since I_Q is prime, this holds if and only if $I_Q = (I_R : B_Y^{\infty})$. The second statement is immediate. \square

Remark 4.1.2. In light of Proposition 3.1.5 and Remark 2.5.5, when the map (4.1.4) is an isomorphism then we describe the Mori Dream Space X as the fine moduli space $\mathcal{M}_{\vartheta}(Q, R)$ of ϑ -stable modules over $\text{End}(\bigoplus_{i \in Q_0} L_i)$ that are isomorphic as $(\bigoplus_{i \in Q_0} \mathbb{k}e_i)$ -modules to $\bigoplus_{i \in Q_0} \mathbb{k}e_i$.

4.2 Main Algebraic Result

We now work towards our main algebraic result which exhibits many collections of line bundles on X for which the morphism from (4.1.4) is an isomorphism, thereby providing a noncommutative algebraic construction of X as in Remark 4.1.2.

We first introduce the collections of interest. Choose generators $g_1, \dots, g_m \in \mathbb{k}[x_1, \dots, x_d]$ of the ideal I_X , set $\delta_0 := \max_{1 \leq j \leq m} \{\text{total degree of } g_j\}$ and define

$$\delta := \begin{cases} \delta_0/2 & \text{if } \delta_0 \text{ is even;} \\ (\delta_0 + 1)/2 & \text{otherwise.} \end{cases} \quad (4.2.1)$$

Consider line bundles L_1, \dots, L_{r-2} on X for which the corresponding rank one reflexive sheaves $E_1 := \psi^{-1}(L_1), \dots, E_{r-2} = \psi^{-1}(L_{r-2})$ on \widetilde{X} are basepoint-free line bundles such that the sub-semigroup of $\text{Pic}(\widetilde{X})$ generated by E_1, \dots, E_{r-2} contains an ample line bundle. Choose sufficiently

positive integers $\beta_1, \dots, \beta_{r-2}$ to ensure that $E := E_1^{\beta_1} \otimes \dots \otimes E_{r-2}^{\beta_{r-2}}$ is \mathcal{O}_X regular with respect to E_1, \dots, E_{r-1} and, moreover, that $E^{2\delta}$ is very ample. We can always find such $\beta_1, \dots, \beta_{r-2}$ by Proposition 2.3.4. Define $E_{r-1} := E^\delta$ and $E_r := E^{2\delta}$. Augment the list L_1, \dots, L_{r-2} on X with $L_0 = \mathcal{O}_X$, $L_{r-1} := \psi(E_{r-1})$ and $L_r := \psi(E_r)$ to obtain a collection

$$\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r) \quad (4.2.2)$$

of basepoint-free line bundles on X . Let Q denote the quiver of sections of \mathcal{L} . The corresponding collection of line bundles $\widetilde{\mathcal{L}} := (\mathcal{O}_{\widetilde{X}}, E_1, \dots, E_r)$ on \widetilde{X} is one of those from Theorem 5.5 of [10], so

$$I_{\widetilde{Q}} = (\widetilde{I}_R : B_Y^\infty). \quad (4.2.3)$$

Thus, the induced morphism $\varphi_{|\widetilde{\mathcal{L}}|} : \widetilde{X} \rightarrow \mathbb{A}^{Q_1} //_{\vartheta} G$ is a closed immersion whose image $\mathbb{V}(\widetilde{I}_Q) //_{\vartheta} G$ is isomorphic to $\mathcal{M}_{\vartheta}(Q, R)$.

Remark 4.2.1. 1. It follows that each collection (4.2.2) determines a commutative diagram

$$\begin{array}{ccccccc} \widetilde{X} & \xrightarrow{\varphi_{|\widetilde{\mathcal{L}}|}} & \mathbb{V}(\widetilde{I}_Q) //_{\vartheta} G & \xrightarrow{\cong} & \mathcal{M}_{\vartheta}(Q, \widetilde{R}) & \longrightarrow & \mathbb{A}_{\mathbb{k}}^{Q_1} //_{\vartheta} G \\ \uparrow & & \uparrow & & \uparrow & & \parallel \\ X & \xrightarrow{\varphi_{|\mathcal{L}|}} & \mathbb{V}(I_Q) //_{\vartheta} G & \longrightarrow & \mathcal{M}_{\vartheta}(Q, R) & \longrightarrow & |\mathcal{L}| \end{array} \quad (4.2.4)$$

in which every morphism is a closed immersion.

2. Since E is $\mathcal{O}_{\widetilde{X}}$ -regular with respect to E_1, \dots, E_{r-2} and each $\beta_i > 0$, Theorem 2.3.2 shows that the multiplication map $H^0(E_{r-1} \otimes E_i^{-1}) \otimes_{\mathbb{k}} H^0(E_{r-1}) \rightarrow H^0(E_r \otimes E_i^{-1})$ is surjective for all $1 \leq i \leq r-1$. This means that for any i , every path from vertex i to r can be decomposed into a path from i to $r-1$ and a path from $r-1$ to r . In particular, every path in Q from 0 to r passes through $r-1$.
3. For clarity in what follows, we work with elements of $\mathbb{k}[y_a : a \in Q_1]$ modulo the relation \sim in which polynomials are equivalent when their difference lies in \widetilde{I}_Q . Since \widetilde{I}_Q is the toric ideal of the semigroup homomorphism $\text{inc} \oplus \text{div} : \mathbb{N}^{Q_1} \rightarrow \text{Wt}(Q) \oplus \mathbb{N}^d$ ([10], Prop 4.3), monomials satisfy $y^{\mathbf{m}} \sim y^{\mathbf{m}'}$ if and only if $\text{inc}(\mathbf{m} - \mathbf{m}') = 0$ and $\text{div}(\mathbf{m} - \mathbf{m}') = 0$, that is, $y^{\mathbf{m}} \sim y^{\mathbf{m}'}$ if and only if they share the same weight in $\text{Wt}(Q)$ and the same image under $\widetilde{\Phi}$.

Before introducing the main result, we present a technical lemma for any list \mathcal{L} as in (4.2.2). Write $\chi = \sum_i \chi_i \mathbf{e}_i \in \mathbb{Z}^{Q_0}$ as $\chi = \chi^+ - \chi^-$ where $\chi^\pm = \sum_i \chi_i^\pm \mathbf{e}_i \in \mathbb{N}^{Q_0}$ have disjoint supports $I_\chi^+ = \{i \in Q_0 : \chi_i > 0\}$ and $I_\chi^- = \{i \in Q_0 : \chi_i < 0\}$. In particular, $\chi \in \text{Wt}(Q)$ gives

$$n_\chi := \sum_{i \in I_\chi^+} \chi_i^+ = \sum_{i \in I_\chi^-} \chi_i^-.$$

For any spanning tree \mathcal{T} in Q , set $y_{\mathcal{T}} := \prod_{a \in \text{supp}(\mathcal{T})} y_a$.

Lemma 4.2.2. *Let \mathcal{T} be a spanning tree in Q and let $\chi \in \text{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$. There exists $\mathbf{m} \in \mathbb{N}^{Q_1}$ such that for any monomial $y^{\mathbf{v}} \in \mathbb{k}[y_a : a \in Q_1]$ of weight χ , we have*

$$(y_{\mathcal{T}})^{2n_{\chi}} y^{\mathbf{v}} \sim y^{\mathbf{m}} \prod_{\alpha=1}^{n_{\chi}} y_{\gamma_{\alpha}} \quad (4.2.5)$$

where $\gamma_1, \dots, \gamma_{n_{\chi}}$ are paths in Q , each with tail at 0 and head at r . Also, $y^{\mathbf{v}}$ divides $\prod_{\alpha=1}^{n_{\chi}} y_{\gamma_{\alpha}}$, and the resulting quotient $\tilde{\Phi}(\prod_{\alpha} y_{\gamma_{\alpha}}) / \tilde{\Phi}(y^{\mathbf{v}})$ depends only on \mathcal{T} and χ .

Proof. We begin by constructing the relevant $\mathbf{m} \in \mathbb{N}^{Q_1}$. The spanning tree \mathcal{T} supports a path q_i from $0 \in Q_0$ to each vertex $i \in Q_0$ and hence to each vertex in I_{χ} . We may therefore write

$$(y_{\mathcal{T}})^{n_{\chi}} = y^{\mathbf{m}_1} \prod_{i \in I_{\chi}^-} (y_{q_i})^{\chi_i^-}. \quad (4.2.6)$$

where $\mathbf{m}_1 \in \mathbb{N}^{Q_1}$ depends only on \mathcal{T} and χ . The tree \mathcal{T} supports a path γ from 0 to r whose label is a torus-invariant section $s \in H^0(E_r)$. Since E_{r-1} is \mathcal{O}_X -regular with respect to E_1, \dots, E_{r-2} and each $\beta_i > 0$, Theorem 2.3.2 implies that the multiplication map

$$H^0(E_{r-1} \otimes E_1^{\beta_1} \otimes \dots \otimes E_{r-2}^{\beta_{r-2}} \otimes E_j^{-1}) \otimes_{\mathbb{k}} H^0(E_j) \rightarrow H^0(E_r) \quad (4.2.7)$$

is surjective. In particular, for each $j \leq r-2$ there exist sections of $E_r \otimes E_j^{-1}$ and E_j whose product is s . Since Q is a complete quiver of sections, there exists a pair of paths labelled by these sections, one from 0 to j denoted q_j'' , and the other from j to r denoted q_j' . Concatenating gives a path $q_j' q_j''$ from 0 to r that passes via j and, by Remark 4.2.1(2), through $r-1$ such that $y_{\gamma} \sim y_{q_j' q_j''} = y_{q_j'} y_{q_j''}$. Multiply by $y_{\mathcal{T}}/y_{\gamma}$ to obtain $y_{\mathcal{T}} \sim y_{q_j'} y^{\mathbf{m}(j)}$ for some $\mathbf{m}(j) \in \mathbb{N}^{Q_1}$ that depends only on \mathcal{T} and j (and on the lift of s via (4.2.7), but we fix one such lift for \mathcal{T} and i). Applying this χ_j^+ -times for each $j \in I_{\chi}^+$ and multiplying gives

$$(y_{\mathcal{T}})^{n_{\chi}} \sim y^{\mathbf{m}_2} \prod_{j \in I_{\chi}^+} (y_{q_j'})^{\chi_j^+}.$$

where $\mathbf{m}_2 \in \mathbb{N}^{Q_1}$ depends only on \mathcal{T} and χ . Multiply by (4.2.6) to see that

$$(y_{\mathcal{T}})^{2n_{\chi}} \sim y^{\mathbf{m}} \prod_{i \in I_{\chi}^-} (y_{q_i})^{\chi_i^-} \prod_{j \in I_{\chi}^+} (y_{q_j'})^{\chi_j^+}. \quad (4.2.8)$$

where $\mathbf{m} := \mathbf{m}_1 + \mathbf{m}_2 \in \mathbb{N}^{Q_1}$ depends only on \mathcal{T} and χ .

To complete the proof, write $\mathbf{v} = \sum_{a \in Q_1} v_a \mathbf{e}_a \in \mathbb{N}^{Q_1}$ where $\text{inc}(\mathbf{v}) = \chi$. Since $\chi \neq 0$ there exists $i \in I_{\chi}^-$, so there exists $a_1 \in Q_1$ with $\mathbf{t}(a_1) = i$ such that $v_{a_1} > 0$. There are two cases. If $\chi_{\mathbf{h}(a_1)} < 0$ then $\mathbf{h}(a_1) \in I_{\chi}^+$, in which case we define $p_1 := a_1$ and repeat the above for $\mathbf{v}' := \mathbf{v} - \mathbf{e}_{a_1}$.

Otherwise, $\chi_{h(a_1)} \leq 0$ in which case there exists $a_2 \in Q_1$ with $t(a_2) = h(a_1)$ such that $v_{a_2} > 0$. Since Q is acyclic we can continue in this way, obtaining a path p_1 that traverses the arrows a_1, a_2, \dots and satisfies $\chi_{h(p_1)} > 0$, that is, $h(p_1) \in I_\chi^+$. As in the first case, we may repeat the above for $\mathbf{v}' := \mathbf{v} - \sum_{a \in \text{supp}(p_1)} \mathbf{e}_a$. In either case, the weight $\chi' := \text{inc}(\mathbf{v}')$ satisfies $n_{\chi'} = n_\chi - 1$, and we obtain by induction a set of paths p_1, \dots, p_{n_χ} satisfying $y^\mathbf{v} = \prod_{\alpha=1}^{n_\chi} y_{p_\alpha}$, where precisely χ_i^- of these paths have tail at $i \in I_\chi^-$ and χ_i^+ have head at $i \in I_\chi^+$. Thus, for $1 \leq \alpha \leq n_\chi$, there exists $i \in I_\chi^-, j \in I_\chi^+$ such that $\gamma_\alpha := q'_j p_\alpha q_i$ is a path in Q from 0 to r and

$$\prod_{\alpha=1}^{n_\chi} y_{\gamma_\alpha} = \prod_{i \in I_\chi^-} (y_{q_i})^{\chi_i^-} \prod_{\alpha=1}^{n_\chi} y_{p_\alpha} \prod_{i \in I_\chi^+} (y_{q'_i})^{\chi_i^+}.$$

Note that $y^\mathbf{v}$ divides $\prod_{\alpha=1}^{n_\chi} y_{\gamma_\alpha}$. Moreover, multiplying (4.2.8) by $y^\mathbf{v}$ gives (4.2.5). The quotient $\tilde{\Phi}(\prod_{\alpha} y_{\gamma_\alpha}) / \tilde{\Phi}(y^\mathbf{v})$ equals $\tilde{\Phi}((y_\mathcal{T})^{2n_\chi}) / \tilde{\Phi}(y^\mathbf{m})$, so depends only on \mathcal{T} and χ as required. \square

Remark 4.2.3. (i) Applying $\tilde{\Phi}(-)$ to (4.2.5) and dividing the resulting equality by $\tilde{\Phi}(y^\mathbf{v})$ shows in addition that the monomial $\tilde{\Phi}(y^\mathbf{m})$ divides $\tilde{\Phi}((y_\mathcal{T})^{2n_\chi})$.

(ii) We draw the reader's attention to the fact that we have also constructed a set of paths p_1, \dots, p_{n_χ} satisfying $y^\mathbf{v} = \prod_{\alpha=1}^{n_\chi} y_{p_\alpha}$, where precisely χ_i^- of these paths have tail at $i \in I_\chi^-$ and χ_i^+ have head at $i \in I_\chi^+$.

We are now in a position to state and prove our main algebraic result.

Theorem 4.2.4. *Let L_1, \dots, L_{r-2} be basepoint-free line bundles on a Mori Dream Space X . If the subsemigroup of $\text{Pic}(X)$ generated by L_1, \dots, L_{r-2} contains an ample line bundle, then there exist line bundles L_{r-1}, L_r such that the induced morphism*

$$\varphi|_{\mathcal{L}}: X \longrightarrow \mathcal{M}_\vartheta(Q, R) \tag{4.2.9}$$

is an isomorphism for $\mathcal{L} = (\mathcal{O}_X, L_1, \dots, L_r)$.

Proof. Define the line bundles L_{r-1} and L_r as described at the start of this section to produce a collection \mathcal{L} of the form (4.2.2). Remark 4.2.1(1) shows that \mathcal{L} is very ample, so by Theorem 4.1.1 it suffices to prove that $I_Q = (I_R : B_Y^\infty)$. To establish one inclusion, let $f \in (I_R : B_Y^\infty)$. Since $I_R \subseteq I_Q$ and hence $(I_R : B_Y^\infty) \subseteq (I_Q : B_Y^\infty)$, we have that $(y_\mathcal{T})^N f \in I_Q$ for some spanning tree \mathcal{T} and $N \in \mathbb{N}$. Since I_Q is prime, we have either $f \in I_Q$ as required, or $B_Y \subseteq I_Q$. B_Y is generated by monomials, and since I_Q is prime, this would imply that I_Q contained a variable. Since $\tilde{\Phi}$ maps variable to monomials, this would imply that I_X contains a monomial, and hence a variable, contradicting our assumption that d is as small as possible. Therefore $(I_R : B_Y^\infty) \subseteq I_Q$. For the opposite inclusion, let $f \in I_Q$ be a homogeneous generator of weight $\chi \in \text{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$ and let \mathcal{T} be a spanning tree in Q . If we can show that $(y_\mathcal{T})^N f \in \widetilde{I_Q} + I_R$ for some $N \in \mathbb{N}$, then by

increasing N if necessary and applying the equality $\widetilde{I}_Q = (\widetilde{I}_R : B_Y^\infty)$ from (4.2.3), we deduce that $(y_T)^N f \in \widetilde{I}_R + I_R = I_R$ and hence $f \in (I_R : B_Y^\infty)$ as required.

In fact we show that $(y_T)^N f \in \widetilde{I}_Q + I_R$ for $N = 2n_\chi$. We proceed in four steps:

STEP 1: *Introduce a set of paths $\{\gamma_{\alpha,\beta}\}$ in Q such that*

$$(y_T)^{2n_\chi} f \sim y^{\mathbf{m}} \left(\sum_{\beta} c_{\beta} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}} \right) \quad (4.2.10)$$

for some $\mathbf{m} \in \mathbb{N}^{Q_1}$ and $c_{\beta} \in \mathbb{k}$, where in addition we have $\widetilde{\Phi}(\sum_{\beta} c_{\beta} \prod_{1 \leq \alpha \leq n_\chi} y_{\gamma_{\alpha,\beta}}) \in I_X$.

Decompose f as a sum of terms $f = \sum_{\beta} c_{\beta} y^{\mathbf{v}_{\beta}}$ for $c_{\beta} \in \mathbb{k}$ and $\mathbf{v}_{\beta} \in \mathbb{N}^{Q_1}$ satisfying $\chi = \text{inc}(\mathbf{v}_{\beta})$. Since $\chi \neq 0$ we apply Lemma 4.2.2 to each monomial $y^{\mathbf{v}_{\beta}}$ to obtain $(y_T)^{2n_\chi} y^{\mathbf{v}_{\beta}} \sim y^{\mathbf{m}} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}$, where \mathbf{m} depends only on \mathcal{T} and χ (not on β) and where each $\gamma_{\alpha,\beta}$ is a path in Q with tail at 0 and head at r . This gives (4.2.10). Also, the quotient $x^{\mathbf{q}} := \widetilde{\Phi}(\prod_{\alpha} y_{\gamma_{\alpha,\beta}}) / \widetilde{\Phi}(y^{\mathbf{v}_{\beta}}) \in \mathbb{k}[x_1, \dots, x_d]$ depends only on \mathcal{T} and χ (not on β). Since $f \in I_Q$, we have $\widetilde{\Phi}(f) \in I_X$ and hence we deduce that $\widetilde{\Phi}(\sum_{\beta} c_{\beta} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}) = x^{\mathbf{q}} (\sum_{\beta} c_{\beta} \widetilde{\Phi}(y^{\mathbf{v}_{\beta}})) = x^{\mathbf{q}} \widetilde{\Phi}(f) \in I_X$ as required.

STEP 2: *Introduce a second set of paths $\{p_{i,j,k,\ell}\}$ in Q such that*

$$\sum_{\beta} c_{\beta} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}} \sim \sum_{i,j,k} c_{i,j,k} \prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}}$$

for some $c_{i,j,k} \in \mathbb{k}$, where for each i,j we have $\widetilde{\Phi}(\sum_k c_{i,j,k} \prod_{1 \leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}}) \in I_X$.

In light of Step 1, expand $\widetilde{\Phi}(\sum_{\beta} c_{\beta} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}}) = \sum_{i,j} h_{i,j} g_i$ in terms of generators of I_X , where each $h_{i,j} \in \mathbb{k}[x_1, \dots, x_d]$ is a nonzero term. Since $\widetilde{\Phi}$ is equivariant and $y_{\gamma_{\alpha,\beta}}$ has weight $\mathbf{e}_r - \mathbf{e}_0 \in \text{Wt}(Q)$, we may assume without loss of generality that each term in this expansion has degree $\widetilde{\text{pic}}(n_\chi(\mathbf{e}_r - \mathbf{e}_0)) = E_r^{n_\chi}$. Thus, expanding each $g_i := g_{i,1} + \dots + g_{i,t_i}$ as a sum of terms for some $t_i \in \mathbb{N}$ gives $h_{i,j} g_{i,k} \in H^0(E_r^{n_\chi})$ for all i,j,k . Since E_{r-1} is \mathcal{O}_X -regular with respect to E_1, \dots, E_{r-2} and $E_r = E_{r-1}^2$, Proposition 2.3.3 implies that the multiplication map $H^0(E_r) \otimes_{\mathbb{k}} \dots \otimes_{\mathbb{k}} H^0(E_r) \rightarrow H^0(E_r^{n_\chi})$ is surjective, so for each i,j,k there exists $c_{i,j,k} \in \mathbb{k}$ and torus-invariant sections $s_{i,j,k,\ell} \in H^0(E_r)$ for $1 \leq \ell \leq n_\chi$ such that $h_{i,j} g_{i,k} = c_{i,j,k} \prod_{\ell=1}^{n_\chi} s_{i,j,k,\ell}$. Since Q is a quiver of sections, there exists a path $p_{i,j,k,\ell}$ in Q from 0 to r whose label is the torus-invariant section $s_{i,j,k,\ell}$, that is, $\widetilde{\Phi}(y_{p_{i,j,k,\ell}}) = s_{i,j,k,\ell}$. For fixed i,j , we therefore obtain

$$h_{i,j} g_{i,k} = c_{i,j,k} \widetilde{\Phi} \left(\prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}} \right). \quad (4.2.11)$$

Summing over $1 \leq k \leq t_i$ gives $h_{i,j} g_i = \widetilde{\Phi}(\sum_k c_{i,j,k} \prod_{1 \leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}})$, and by summing this new expression over all i,j we deduce that

$$\widetilde{\Phi} \left(\sum_{\beta} c_{\beta} \prod_{\alpha=1}^{n_\chi} y_{\gamma_{\alpha,\beta}} \right) = \widetilde{\Phi} \left(\sum_{i,j,k} c_{i,j,k} \prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}} \right) \quad (4.2.12)$$

lies in I_X by Step 1. The main statement of Step 2 now follows from Remark 4.2.1(3) because these polynomials also share the same weight in $\text{Wt}(Q)$, namely $n_\chi(\mathbf{e}_r - \mathbf{e}_0)$.

STEP 3: Introduce a third set of paths $\{q_{i,j,k}\}$ in Q such that

$$\prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}} \sim y^{\mathbf{m}'_{i,j}} y_{q_{i,j,k}} \quad (4.2.13)$$

for some $\mathbf{m}'_{i,j} \in \mathbb{N}^{Q^1}$, where for each i, j we have $\tilde{\Phi}(\sum_k c_{i,j,k} y_{q_{i,j,k}}) \in I_X$.

Fix i, j and simplify notation by suppressing dependence on i, j . To this end, write $c_k := c_{i,j,k}$, $y^{\mathbf{v}^k} := \prod_{1 \leq \ell \leq n_\chi} y_{p_{i,j,k,\ell}}$, $h = h_{i,j}$ and $g_k = g_{i,k}$, in which case (4.2.11) is simply $hg_k = c_k \tilde{\Phi}(y^{\mathbf{v}^k})$. The map $\tilde{\Phi}$ is equivariant and sends monomials to monomials, so $\frac{1}{c_k} hg_k \in H^0(E_r^{n_\chi})$ defines a torus-invariant section. Since $E = E_1^{\beta_1} \otimes \cdots \otimes E_{r-2}^{\beta_{r-2}}$ is \mathcal{O}_X -regular with respect to E_1, \dots, E_{r-2} and $E_r = E_{r-1}^2 = E^{2\delta}$, Proposition 2.3.3 implies that the multiplication map

$$H^0(E) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} H^0(E) \rightarrow H^0(E_r^{n_\chi})$$

is surjective, so $\frac{1}{c_k} hg_k$ is equal to the product of $2\delta n_\chi$ torus-invariant sections of E . Since g_k is a term of a generator of I_X , its total degree is at most $\delta_0 \leq 2\delta$ by (4.2.1), so we may choose 2δ of these sections $s_{k,1}, \dots, s_{k,2\delta} \in H^0(E)$ such that g_k divides $\prod_{1 \leq \mu \leq 2\delta} s_{k,\mu} \in H^0(E_r)$. We now apply the above only for $k = 1$. Since Q is a quiver of sections, there exists a path q_1 in Q from 0 to r satisfying $\tilde{\Phi}(y_{q_1}) = \prod_{1 \leq \mu \leq 2\delta} s_{1,\mu}$, so the section $hg_1/c_1 \tilde{\Phi}(y_{q_1}) \in H^0(E_r^{n_\chi-1})$ is torus-invariant. Surjectivity of the multiplication map $H^0(E_r) \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} H^0(E_r) \rightarrow H^0(E_r^{n_\chi-1})$ determines $n_\chi - 1$ sections of E_r and hence paths $q'_1, \dots, q'_{n_\chi-1}$ in Q from 0 to r labelled by these sections such that $\tilde{\Phi}(y^{\mathbf{m}'}) = hg_1/c_1 \tilde{\Phi}(y_{q_1})$ for $y^{\mathbf{m}'} := \prod_{1 \leq \nu \leq n_\chi-1} y_{q'_\nu}$. In particular,

$$\tilde{\Phi}(y^{\mathbf{v}^1}) = \frac{hg_1}{c_1} = \tilde{\Phi}(y^{\mathbf{m}'} y_{q_1}). \quad (4.2.14)$$

Both monomials $y^{\mathbf{v}^1}$ and $y^{\mathbf{m}'} y_{q_1}$ have weight $n_\chi(\mathbf{e}_r - \mathbf{e}_0) \in \text{Wt}(Q)$, hence $y^{\mathbf{v}^1} \sim y^{\mathbf{m}'} y_{q_1}$. Now reintroduce the indices i, j , setting $\mathbf{m}'_{i,j} := \mathbf{m}'$ and $q_{i,j,1} := q_1$, to obtain (4.2.13) for $k = 1$.

For $k > 1$, we have $hg_k = c_k \tilde{\Phi}(y^{\mathbf{v}^k})$. For $1 \leq i \leq m$, the generator g_i of I_X is $\text{Cl}(X)$ -homogeneous, so $g_k := g_{i,k}$ and $g_1 := g_{i,1}$ have the same degree in $\text{Cl}(X)$. Since g_1 divides $\tilde{\Phi}(y_{q_1}) \in H^0(E_r)$, it follows that the term $\tilde{\Phi}(y_{q_1}) g_k / g_1$ also has degree E_r . Divide by its coefficient $c_k / c_1 \in \mathbb{k}$ to obtain a torus-invariant section $\tilde{\Phi}(y_{q_1}) c_1 g_k / c_k g_1 \in H^0(E_r)$ which in turn determines a path q_k in Q with tail at 0 and head at r for which $\tilde{\Phi}(y_{q_k}) = \tilde{\Phi}(y_{q_1}) c_1 g_k / c_k g_1$. Then (4.2.14) gives

$$\tilde{\Phi}(y^{\mathbf{v}^k}) = hg_1 \cdot \frac{g_k}{c_k g_1} = c_1 \tilde{\Phi}(y^{\mathbf{m}'}) \tilde{\Phi}(y_{q_1}) \cdot \frac{g_k}{c_k g_1} = \tilde{\Phi}(y^{\mathbf{m}'} y_{q_k}).$$

It follows that the monomials $y^{\mathbf{v}^k}$ and $y^{\mathbf{m}'} y_{q_k}$ have weight $n_\chi(\mathbf{e}_r - \mathbf{e}_0)$, hence $y^{\mathbf{v}^k} \sim y^{\mathbf{m}'} y_{q_k}$, and we reintroduce the indices i, j , setting $q_{i,j,k} := q_k$, to obtain (4.2.13) for all k . Then

$$\tilde{\Phi}(y^{\mathbf{m}'_{i,j}}) \tilde{\Phi}\left(\sum_k c_{i,j,k} y_{q_{i,j,k}}\right) = \tilde{\Phi}\left(\sum_k c_{i,j,k} \prod_{\ell=1}^{n_\chi} y_{p_{i,j,k,\ell}}\right) \in I_X$$

holds for every i, j by combining (4.2.13) and Step 2. The ideal I_X does not contain the monomial $\tilde{\Phi}(y^{\mathbf{m}'_{i,j}})$ for any i, j , otherwise it would contain a variable of $\mathbb{k}[x_1, \dots, x_d]$ because I_X is prime, this gives a contradiction since we assumed that d is as small as possible. Thus, $\tilde{\Phi}\left(\sum_k c_{i,j,k} y_{q_{i,j,k}}\right) \in I_X$ for every i, j as required.

STEP 4: *Establish that $(y_{\mathcal{T}})^{2n_\chi} f \in \widetilde{I}_Q + I_R$ as required by proving that*

$$(y_{\mathcal{T}})^{2n_\chi} f \sim y^{\mathbf{m}} \left(\sum_{i,j} y^{\mathbf{m}'_{i,j}} \left(\sum_k c_{i,j,k} y_{q_{i,j,k}} \right) \right). \quad (4.2.15)$$

Relation (4.2.15) is immediate from Steps 1-3. For every i, j we also have $\sum_k c_{i,j,k} y_{q_{i,j,k}} \in I_R$ by Step 3, so the right hand side of (4.2.15) also lies in I_R . The definition of \sim given in Remark 4.2.1(3) then implies $(y_{\mathcal{T}})^{2n_\chi} f \in \widetilde{I}_Q + I_R$. This completes the proof of Theorem 4.2.4. \square

Chapter 5

Computing I_R and I_Q

In this chapter we show how to compute $I_R, \widetilde{I}_R, I_Q$ and \widetilde{I}_Q explicitly using Maple and Macaulay2. As an application, in the next chapter we show that $I_Q = I_R : B_Y^\infty$ for certain ample quivers of sections Q on X_4, X_5 and $\text{Gr}(2, 4)$, therefore each is isomorphic to a moduli space of bound quiver representations by Theorem 4.1.1.

We summarise our method for computing \widetilde{I}_R and I_R using Maple below:

1. In section 5.1.1, we show how to input quivers into Maple. We give pseudocode for finding the set of all paths in Q , along with their heads, tails and labels in section 5.1.2.
2. In section 5.1.3, we give pseudocode for finding the generators of \widetilde{I}_R .
3. In section 5.1.4, we prove that there is a choice of generating set for I_R which contains the generating set for \widetilde{I}_R , plus certain additional generators in a form conducive to calculations.
4. In 5.1.5, we give pseudocode for finding the additional generators mentioned above.

We summarise our method for calculating \widetilde{I}_Q and I_Q using Macaulay2:

1. In Appendix A, we give a method for computing the kernels of \mathbb{k} -algebra homomorphisms using Macaulay2.
2. In section 5.2.1, we prove \widetilde{I}_Q and I_Q are kernels of certain \mathbb{k} -algebra homomorphisms. Hence we can apply the results of Appendix A to compute them.
3. In section 5.2.2 we give Macaulay2 code for calculating \widetilde{I}_Q and I_Q .

5.1 Computing \widetilde{I}_R and I_R using Maple

in this section we give a method for calculating \widetilde{I}_R and I_R explicitly.

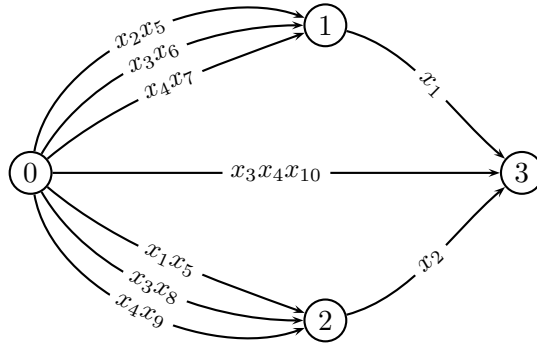
5.1.1 Quivers in Maple

In order to calculate \widetilde{I}_R and I_R , we need to input quivers into Maple. We input quivers as Maple “lists” of all arrows, plus their heads, tails and labels as shown:

$$Q := [[t(a_i), \text{div}(a_i), h(a_i), y_{a_i}] | a_i \in Q_1].$$

We will refer to the i th entry in Q (and more generally in any list) as $Q[i]$. We introduce some notation: for a quiver Q , let $t(Q[i]) := Q[i][1]$, $\text{div}(Q[i]) := Q[i][2]$, $h(Q[i]) := Q[i][3]$ and $y_{Q[i]} := Q[i][4]$. We write $\text{div}(Q[i])$ as an element of \mathbb{N}^d where d is the number of generators of $\text{Cox}(X)$.

Example 5.1.1. On X_4 , $\mathcal{L} = (\mathcal{O}_{X_4}, l_0 - l_1, l_0 - l_2, l_0)$. The quiver of sections for \mathcal{L} , Q , is given below:



Arrows 1-3 are those from 0 to 1. Arrows 4-6 are those from 0 to 2. Arrow 7 goes from 0 to 3. Arrow 8 goes from 1 to 3. Arrow 9 goes from 2 to 3. We input Q into Maple as:

```
[[0, [0, 1, 0, 0, 1, 0, 0, 0, 0, 0], 1, y1], [0, [0, 0, 1, 0, 0, 1, 0, 0, 0, 0], 1, y2], [0, [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], 1, y3]
[0, [1, 0, 0, 0, 1, 0, 0, 0, 0, 0], 2, y4], [0, [0, 0, 1, 0, 0, 0, 0, 1, 0, 0], 2, y5], [0, [0, 0, 0, 1, 0, 0, 0, 0, 1, 0], 2, y6]
[0, [0, 0, 1, 1, 0, 0, 0, 0, 0, 1], 3, y7], [1, [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], 3, y8], [2, [0, 1, 0, 0, 0, 0, 0, 0, 0, 0], 3, y9]]
```

5.1.2 Finding all Paths in Q

The ideals I_R and \widetilde{I}_R are defined in terms of the paths of Q , therefore we need to consider paths as well as arrows of Q . With that in mind, we wrote a Maple procedure “getpaths” which outputs the list of all paths P for a given quiver Q . More specifically, for every path p in Q the output P lists $t(p)$, $\text{div}(p)$, $h(p)$ and y_p . We give pseudocode and a proof of its efficacy.

We define $h(P[i])$, $\text{div}(P[i])$, $t(P[i])$ and $y_{P[i]}$ to be the first, second, third and fourth terms of $P[i]$ respectively. We denote the number of terms in a list L by $|L|$.

Pseudocode 5.1.2. *Input:* Q

Procedure: $P := Q$;, $r := \max\{h(a_i) | a_i \in Q_0\}$, $a := 0$, $b := |Q|$

for i *from* 1 *to* r *do*

for j *from* $|P| - b + 1$ *to* $|P|$ *do*

for k *from* 1 *to* $|Q|$ *do*

if $t(Q[k]) = h(P[j])$ *then* $P := [P, [t(P[j]), \text{div}(P[j]) \text{div}(Q[k]), h(Q[k]), y_{Q[k]}y_{P[j]}]]$ *and* $a := a + 1$.

end if

end do

end do

$b := a$, $a := 0$

end do.

Output: P

Proof. We begin by defining $P := Q$. We work through all the elements of P and Q , and if say the i th element of Q has tail equal to the head of the j th element of P then we add their concatenation (a path of length 2) to P . Once we have worked through all the elements of P and Q in this way we will have added all the paths of length two to P . We record the number of paths we have added (this is the role of a and b).

Next we consider all arrows in Q and all paths of length 2 (i.e. the last b paths in P). If it is possible to concatenate them to form a path of length 3, they are added to P . Again we record the number of additions to P .

We repeat this process r times, where r is the number of vertices in Q . The paths in Q have length at most r , hence after repeating the process r times P lists the details of every path in Q . \square

Example 5.1.3. Let Q be as in example 5.1.1. The output for “getpaths(Q)” is:

$[[0, [0, 1, 0, 0, 1, 0, 0, 0, 0, 0], 1, y_1], [0, [0, 0, 1, 0, 0, 1, 0, 0, 0, 0], 1, y_2], [0, [0, 0, 0, 1, 0, 0, 1, 0, 0, 0], 1, y_3],$
 $[0, [1, 0, 0, 0, 1, 0, 0, 0, 0, 0], 2, y_4], [0, [0, 0, 1, 0, 0, 0, 0, 1, 0, 0], 2, y_5], [0, [0, 0, 0, 1, 0, 0, 0, 0, 1, 0], 2, y_6],$
 $[0, [0, 0, 1, 1, 0, 0, 0, 0, 0, 1], 3, y_7], [1, [1, 0, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_8], [2, [0, 1, 0, 0, 0, 0, 0, 0, 0, 0], 3, y_9],$
 $[0, [1, 1, 0, 0, 1, 0, 0, 0, 0, 0], 3, y_8 y_1], [0, [1, 0, 1, 0, 0, 1, 0, 0, 0, 0], 3, y_8 y_2], [0, [1, 0, 0, 1, 0, 0, 1, 0, 0, 0], 3, y_8 y_3],$
 $[0, [1, 1, 0, 0, 1, 0, 0, 0, 0, 0], 3, y_9 y_4], [0, [0, 1, 1, 0, 0, 0, 0, 1, 0, 0], 3, y_9 y_5], [0, [0, 1, 0, 1, 0, 0, 0, 0, 1, 0], 3, y_9 y_6]]$

5.1.3 Calculating \widetilde{I}_R

We give pseudocode for finding the generators of \widetilde{I}_R and prove its efficacy.

Pseudocode 5.1.4. *Input:* $P := \text{getpaths}(Q)$.

Procedure: $L := []$ (the “empty list”).

for i from 1 to $|P|$ do

for j from 1 to $|P|$ do

If $\mathbf{h}(P[i]) = \mathbf{h}(P[j])$, $\mathbf{t}(P[i]) = \mathbf{t}(P[j])$ and $\text{div}(P[i]) = \text{div}(P[j])$ then $L := [L, y_{P[i]} - y_{P[j]}]$.

end if

end do

end do.

Output: L .

Proof. We check all pairs of paths p_i and p_j . If their heads, tails and labels are equal then $y_{p_i} - y_{p_j}$ is a generator of \widetilde{I}_R so we add $y_{p_i} - y_{p_j}$ to L . All generators are of this form, and since we check all pairs of paths this must give a list of all generators for \widetilde{I}_R . \square

Remark 5.1.5. Note that while L is a generating set for \widetilde{I}_R , it will almost certainly contain many redundancies. In particular, L will probably have many terms equal to zero.

Example 5.1.6. Considering the quiver Q from Example 5.1.1, the output from “zeropart” where input was all paths in Q (i.e. the output from “getpaths”). Output is:

$$[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, y_8 y_1 - y_9 y_4, 0, 0, y_9 y_4 - y_8 y_1, 0, 0, 0].$$

Hence

$$\widetilde{I}_R = (y_8 y_1 - y_9 y_4).$$

5.1.4 A Generating Set for I_R

We give a technical lemma which describes a generating set for I_R .

Lemma 5.1.7. *Fix a presentation $I_X = \langle g_1, \dots, g_m \rangle$. The ideal I_R is generated by $S_1 \cup S_2$ where*

$$S_1 := \{y_p - y_{p'} \mid \mathbf{h}(p) = \mathbf{h}(p'), \mathbf{t}(p) = \mathbf{t}(p'), \text{div}(p) = \text{div}(p')\}$$

and

$$S_2 := \left\{ \sum a_i y_{p_i} \mid \begin{array}{l} \mathbf{h}(p_i) = \mathbf{h}(p_j), \mathbf{t}(p_i) = \mathbf{t}(p_j) \text{ for all } i, j \text{ and } \sum a_i \widetilde{\Phi}(y_{p_i}) = h_{ij} g_i \\ \text{for some } j \text{ where } h_{ij} \text{ is a term in } \mathbb{k}[x_1, \dots, x_d] \end{array} \right\}.$$

Proof. The ideal generated by S_1 and S_2 is contained in I_R . For the converse, let $f = \sum a_i y_{p_i}$ be a generator of I_R (where paths p_i share same head and same tail). It suffices to show that f lies in the ideal generated by S_1 and S_2 .

Since $f \in I_R$, $\tilde{\Phi}(f) = \sum_{i,j} h_{ij} g_i \in I_X$ for some terms $h_{ij} \in \mathbb{k}[x_1, \dots, x_d]$. We proceed in two stages: first we show that f can be written as an element of \widetilde{I}_R plus a linear combination of $\text{Wt}(Q)$ homogeneous y_p 's mapping to terms of the sum $\sum h_{ij} g_i$. All elements of \widetilde{I}_R lie in the ideal generated by S_1 by [10]. Secondly, we show that the remaining part lies in the ideal generated by S_2 .

STEP 1: Identifying vertices of Q with line bundles, let $E = \mathfrak{h}(p_i) \otimes \mathfrak{t}(p_i)^{-1}$ for any i (note that this is independent of i). Since $\tilde{\Phi}$ is a toric homomorphism and maps monomials to monomials, each term $a_i y_{p_i}$ maps to a term $a_i x^{\mathbf{m}_i}$ in $(S_{\widetilde{X}})_E$. So, we can write

$$\tilde{\Phi}\left(\sum_{i=1}^n a_i y_{p_i}\right) = \sum_{i=1}^n a_i x^{\mathbf{m}_i} = a_{i_1} x^{\mathbf{m}_{i_1}} + \dots + a_{i_t} x^{\mathbf{m}_{i_t}}$$
 after cancelling,

where $\{i_1, \dots, i_t\} \subseteq \{1, \dots, n\}$. Hence we can decompose f as

$$f = a_{i_1} y_{p_{i_1}} + \dots + a_{i_t} y_{p_{i_t}} + (f - (a_{i_1} y_{p_{i_1}} + \dots + a_{i_t} y_{p_{i_t}})).$$

We note that $f - (a_{i_1} y_{p_{i_1}} + \dots + a_{i_t} y_{p_{i_t}})$ is homogeneous and in the kernel of $\tilde{\Phi}$. It is therefore an element of \widetilde{I}_R , and lies in the ideal generated by S_1 by [10].

STEP 2 : Redefine $a_{i_\alpha} y_{p_{i_\alpha}} =: a_\alpha y_{p_\alpha}$ and $x^{\mathbf{m}_{i_\alpha}} =: x^{\mathbf{m}_\alpha}$. We show that $\sum_{\alpha=1}^t a_\alpha y_{p_\alpha}$ lies in the ideal generated by S_2 . Since $\sum a_\alpha x^{\mathbf{m}_\alpha} \in I_X$, we can write

$$\sum_{\alpha} a_\alpha x^{\mathbf{m}_\alpha} = \sum_{i,j} h_{ij} g_i$$

where g_i is a generator of I_X and h_{ij} is a term. Since $\tilde{\Phi}$ is equivariant, $\sum_{i,j} h_{ij} g_i$ is homogeneous of degree E . Hence

$$\sum_{i,j} h_{ij} g_i = \sum_{i,j} h_{ij} g_i \cap \mathbb{k}[x_1, \dots, x_d]_E.$$

Since each $h_{ij} g_i$ is a homogeneous polynomial, we may assume without loss of generality that each $h_{ij} g_i$ has degree E . We can decompose $h_{ij} g_i$ into terms, say $h_{ij} g_i = \sum_k h_{ijk} g_{ik}$. For all i, j and k ,

$$h_{ijk} g_{ik} = c_{ijk} x^{\mathbf{v}_{ijk}} \tag{5.1.1}$$

where $c_{ijk} \in \mathbb{k}$ and $x^{\mathbf{v}_{ijk}}$ is a torus invariant section of E . By definition of the quiver of sections there exists a path p_{ijk} from $\mathfrak{t}(p_\alpha)$ to $\mathfrak{h}(p_\alpha)$ labelled by $x^{\mathbf{v}_{ijk}}$. Additionally, we can ensure that $\mathbf{v}_{ijk} = \mathbf{v}_{i'j'k'}$ if and only if $p_{ijk} = p_{i'j'k'}$, and that $p_{ijk} = p_\alpha$ if and only if $x^{\mathbf{v}_{ijk}} = x^{\mathbf{m}_\alpha}$.

Now we will show that

$$\sum_{ijk} c_{ijk} y_{p_{ijk}} = \sum_{\alpha} c_{\alpha} y_{p_{\alpha}}.$$

We prove this is true by examining the coefficient of each term in the left hand side expansion.

$$\sum_{\substack{i'j'k' \\ \text{s.t } p_{i'j'k'}=p_{ijk}}} c_{i'j'k'} y_{p_{i'j'k'}} = c y_{p_{ijk}}$$

if and only if

$$\sum_{\substack{i'j'k' \\ \text{s.t } \mathbf{v}_{i'j'k'}=\mathbf{v}_{ijk}}} c_{i'j'k'} x^{\mathbf{v}_{i'j'k'}} = c x^{\mathbf{v}_{ijk}}.$$

However

$$\sum_{\substack{i'j'k' \\ \text{s.t } \mathbf{v}_{i'j'k'}=\mathbf{v}_{ijk}}} c_{i'j'k'} x^{\mathbf{v}_{i'j'k'}} = \begin{cases} c_{\alpha} x^{\mathbf{m}_{\alpha}} & \text{if } x^{\mathbf{v}_{ijk}} = x^{\mathbf{m}_{\alpha}} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{\substack{i'j'k' \\ \text{s.t } p_{i'j'k'}=p_{ijk}}} c_{i'j'k'} y_{p_{i'j'k'}} = \begin{cases} c_{\alpha} y_{p_{\alpha}} & \text{if } p_{ijk} = p_{\alpha} \\ 0 & \text{otherwise.} \end{cases}$$

So, summing over all i, j and k , we must have $\sum_{ijk} c_{ijk} y_{p_{ijk}} = \sum_{\alpha} c_{\alpha} y_{p_{\alpha}}$. Hence,

$$\sum_{\alpha} c_{\alpha} y_{p_{\alpha}} = \sum_{ij} \left(\sum_k c_{ijk} y_{p_{ijk}} \right). \quad (5.1.2)$$

Crucially, $\sum_k c_{ijk} y_{p_{ijk}}$ is an element of the ideal generated by S_2 . This is because each p_{ijk} has the same head and tail, and by (6.2.1)

$$\widetilde{\Phi}_Q \left(\sum_k c_{ijk} y_{p_{ijk}} \right) = \sum_k c_{ijk} x^{\mathbf{v}_{ijk}} = h_{ij} g_i$$

where $h_{ij} g_i$ is a term times g_i . By (6.2.2), $\sum_{\alpha} c_{\alpha} y_{p_{\alpha}}$ is a sum of elements of S_2 , therefore $\sum_{\alpha} c_{\alpha} y_{p_{\alpha}}$ also lies in the ideal generated by S_2 . \square

5.1.5 Finding Generators of I_R

We show how to calculate I_R for X where I_X is generated by quadratic polynomials with three terms, as in X_4, X_5, X_6 and Grassmannians $\text{Gr}(r, n)$. We give pseudocode and proof of correctness in this case, but it is easily generalisable for cases where I_X does not have this form.

We introduce some notation. Fix a presentation for I_X , we denote the j th generator of I_X as $I_X[j]$, and suppose the number of generators is G . We define $r[i], s[i], t[i], u[i], v[i], w[i], l[i], m[i], n[i]$ such that:

$$I_X[i] = l[i] x_{r[i]} x_{s[i]} + m[i] x_{t[i]} x_{u[i]} + n[i] x_{v[i]} x_{w[i]}.$$

We define $D[i]$, $E[i]$ and $F[i]$ to be vectors of length d with 1's in the $r[i]$ th and $s[i]$ th, $t[i]$ th and $u[i]$ th, and $v[i]$ th and $w[i]$ th positions respectively and zeroes elsewhere.

Pseudocode 5.1.8. *We assume I_X is quadratically generated, and that each generator has three terms.*

Input: To compute I_R for a quiver Q , our input is $P := \text{getpaths}(Q)$.

Procedure:

$S := [] :$

for g from 1 to G do

$L1 := [] :$

$L2 := [] :$

$L3 := [] :$

for i from 1 to $|P|$ do

if $P[i][2][r[g]] > 0$ and $P[i][2][s[g]] > 0$ then $L1 := [L1[], [P[i][1], P[i][2] - D[g], P[i][3], P[i][4]]]$

for j from 1 to $|P|$ do

if $P[j][2][t[g]] > 0$ and $P[j][2][u[g]] > 0$ then $L2 := [L2[], [P[j][1], P[j][2] - E[g], p[j][3], p[j][4]]]$

for k from 1 to $|P|$ do if $P[k][2][v[g]] > 0$ and $P[k][2][w[g]] > 0$ then $L3 := [L3[], [P[k][1], P[k][2] - F[g], P[k][3][4]]]$

for a from 1 to $|L1|$ do

for b from 1 to $|L2|$ do

for c from 1 to $|L3|$ do

if $L1[a][1] = L2[b][1] = L3[c][1]$ and $L1[a][2] = L2[b][2] = L3[c][2]$ and $L1[a][3] = L2[b][3] = L3[c][3]$ then $S := [S[], l[g]L1[a][4] + m[g]L2[b][4] + n[g]L3[c][4]]$

Output: S .

Proof. By Lemma 5.1.7, a generating set for I_R consists of the generators of \widetilde{I}_R , plus elements of $\mathbb{k}[y_a | a \in Q_1]$ of the form $\sum_i a_i y_{p_i}$ where the p_i 's have the same heads and tails and $\widetilde{\Phi}(\sum_i a_i y_{p_i})$ is a monomial times a generator of I_X .

We have already found generators of \widetilde{I}_R , using our Maple procedure “zeropart”. Since assume each generator of I_X has three terms, it remains to find all triples of paths p_1, p_2, p_3 with the same heads and tails, where each p_i is labelled by a monomial times a term of a generator of I_X , modulo constant term a_i say. We then have that $a_1 y_{p_1} + a_2 y_{p_2} + a_3 y_{p_3}$ is a generator of I_R , and by Lemma 5.1.7, once we have found all such generators, we will have a generating set for I_R .

We work through all generators of I_X , for the g th generator, we proceed as follows:

First, we define three empty lists $L1, L2$ and $L3$. We find all paths p whose labels are divisible by

the first, second or third term of $IX[g]$ mod constant, and record their heads, tails, the remainder when we divide their label by a term of $IX[g]$, and y_p in $L1, L2$ or $L3$ respectively. Explicitly:

1. We work through all paths $P[i]$ and see if their labels $P[i][2]$ are divisible by the first term of $IX[g] : x_{r[g]}x_{s[g]}$. If it is divisible, we record $P[i]$'s tail, the remainder when we divide its label by $x_{r[g]}x_{s[g]}$, head and $y_{P[i]}$ in $L1$.
2. We work through all paths $P[j]$ and see if their labels $P[j][2]$ are divisible by the second term of $IX[g] : x_{t[g]}x_{u[g]}$. If it is divisible, we record $P[j]$'s tail, the remainder when we divide its label by $x_{t[g]}x_{u[g]}$, head and $y_{P[j]}$ in $L2$.
3. We work through all paths $P[k]$ and see if their labels $P[k][2]$ are divisible by the third term of $IX[g] : x_{v[g]}x_{w[g]}$. If it is divisible, we record $P[k]$'s tail, the remainder when we divide its label by $x_{v[g]}x_{w[g]}$, head and $y_{P[k]}$ in $L3$.

Secondly, we work through all entries in $L1, L2$ and $L3$. If $L1[a], L2[b]$ and $L3[c]$ have the same first, second and third entries, then they record information about paths p_1, p_2 and p_3 with the same tails, heads and remainder of their label after division by a term of $IX[g]$ mod constant. Hence, replacing constants, $l[g]y_{p_1} + m[g]y_{p_2} + n[g]y_{p_3}$ is a generator of I_R . $L1[a][4] = y_{p_1}, L2[b][4] = y_{p_2}$ and $L3[c][4] = y_{p_3}$. We record $l[g]L1[a][4] + m[g]L2[b][4] + n[g]L3[c][4]$ in S . After working through all such triples for all generators $IX[g]$, S plus the generators from “zeroparts” will give a generating set for I_R . \square

We include verbatim the Maple code “findIR” for computing the extra generators for I_R for X_4 :

```
findIR:=proc(P)
local r, s, t, u, v, w, l, m, n, x, D, E, F, L1, L2, L3, S, i, a, b, c, g:
S:=[]:
l[1] := 1 : l[2] := 1 : l[3] := 1 : l[4] := 1 : l[5] := 1 : m[1] := -1 : m[2] := -1 : m[3] := -1 : m[4] :=
-1 : m[5] := -1 : n[1] := 1 : n[2] := 1 : n[3] := 1 : n[4] := 1 : n[5] := 1 :
r[1] := 2 : r[2] := 1 : r[3] := 1 : r[4] := 1 : r[5] := 5 : s[1] := 5 : s[2] := 5 : s[3] := 6 : s[4] := 7 :
s[5] := 10 : t[1] := 3 : t[2] := 3 : t[3] := 2 : t[4] := 2 : t[5] := 6 :
u[1] := 6 : u[2] := 8 : u[3] := 8 : u[4] := 9 : u[5] := 9 : v[1] := 4 : v[2] := 4 : v[3] := 4 : v[4] := 3 :
v[5] := 7 : w[1] := 7 : w[2] := 9 : w[3] := 10 : w[4] := 10 : w[5] := 8 :
D[1] := [0, 1, 0, 0, 1, 0, 0, 0, 0, 0] : D[2] := [1, 0, 0, 0, 1, 0, 0, 0, 0, 0] : D[3] := [1, 0, 0, 0, 0, 1, 0, 0, 0, 0] :
D[4] := [1, 0, 0, 0, 0, 0, 1, 0, 0, 0] : D[5] := [0, 0, 0, 0, 1, 0, 0, 0, 0, 1] :
E[1] := [0, 0, 1, 0, 0, 1, 0, 0, 0, 0] : E[2] := [0, 0, 1, 0, 0, 0, 0, 1, 0, 0] : E[3] := [0, 1, 0, 0, 0, 0, 0, 1, 0, 0] :
E[4] := [0, 1, 0, 0, 0, 0, 0, 0, 1, 0] : E[5] := [0, 0, 0, 0, 0, 1, 0, 0, 1, 0] :
F[1] := [0, 0, 0, 1, 0, 0, 1, 0, 0, 0] : F[2] := [0, 0, 0, 1, 0, 0, 0, 0, 1, 0] : F[3] := [0, 0, 0, 1, 0, 0, 0, 0, 0, 1] :
```

```

F[4] := [0, 0, 1, 0, 0, 0, 0, 0, 1] : F[5] := [0, 0, 0, 0, 0, 1, 1, 0, 0] :
for g from 1 to 5 do
L1 := [] : L2 := [] : L3 := [] :
for i from 1 to nops(P) do
if op(r[g], op(2,op(i, P))) > 0 and op(s[g], op(2,op(i, P))) > 0 then L1 := [L1[ ], [op(1,op(i, P)),op(2,op(i, P))]-
D[g], op(3,op(i, P)), op(4,op(i, P))]:
else L1 := L1 : end if:
if op(t[g], op(2,op(i, P))) > 0 and op(u[g], op(2,op(i, P))) > 0 then L2 := [L2[ ], [op(1,op(i, P)),op(2,op(i, P))]-
E[g], op(3,op(i, P)), op(4,op(i, P))]:
else L2 := L2 : end if:
if op(v[g], op(2,op(i, P))) > 0 and op(w[g], op(2,op(i, P))) > 0 then L3 := [L3[ ], [op(1,op(i, P)),op(2,op(i, P))]-
F[g], op(3,op(i, P)), op(4,op(i, P))]:
else L3 := L3 : end if:
end do:
for a from 1 to nops(L1) do for b from 1 to nops(L2) do for c from 1 to nops(L3) do
if op(1,op(a, L1))=op(1,op(b, L2)) and op(1,op(a, L1))=op(1,op(c, L3))
and op(2,op(a, L1))=op(2,op(b, L2)) and op(2,op(a, L1))=op(2,op(c, L3))
and op(3,op(a, L1))=op(3,op(b, L2)) and op(3,op(a, L1))=op(3,op(c, L3))
then S := [S[ ], l[g]op(4,op(a, L1))+m[g]op(4,op(b, L2))+n[g]op(4,op(c, L3))]:
else S := S : end if:
end do: end do: end do: end do:
print(S);
end proc;

```

Example 5.1.9. With Q as in Example 5.1.1, the output from $\text{findIR}(Q)$ is:

$$\begin{aligned}
& [y_1 - y_2 + y_3, y_8y_1 - y_8y_2 + y_8y_3, y_9y_4 - y_8y_2 + y_8y_3, y_4 - y_5 + y_6, \\
& y_8y_1 - y_9y_5 + y_9y_6, y_9y_4 - y_9y_5 + y_9y_6, y_8y_2 - y_9y_5 + y_7, y_8y_3 - y_9y_6 + y_7].
\end{aligned}$$

Hence in this case

$$I_R = (y_8y_1 - y_9y_4, y_4 - y_5 + y_6, y_1 - y_2 + y_3, y_3y_8 - y_6y_9 + y_7, y_2y_8 - y_5y_9 + y_7).$$

5.2 Calculating I_Q and \widetilde{I}_Q Using Macaulay2

In this section we give a method for computing \widetilde{I}_Q and I_Q explicitly.

5.2.1 I_Q and \widetilde{I}_Q as Kernels

In this section we use the theory from Appendix A to calculate I_Q using Macaulay2. In order to do this, we show that I_Q is the kernel of a \mathbb{k} -algebra homomorphism ψ :

$$\begin{aligned} \psi : \mathbb{k}[y_a | a \in Q_1] &\longrightarrow \mathbb{k}[x_1, \dots, x_d, t_i, h_i | i \in Q_0] / I_X + A \\ y_a &\mapsto t_{\mathbf{t}(a)} x^{\text{div}(a)} h_{\mathbf{h}(a)} \end{aligned}$$

where

$$A = (t_i h_i - 1 | i \in Q_0).$$

First we need a technical lemma:

Lemma 5.2.1. *Let $f \in \mathbb{k}[y_a | a \in Q_1]$ be homogeneous of weight $\chi \in \text{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$ and let $n := n_\chi$. We consider the map:*

$$\begin{aligned} \overline{\psi} : \mathbb{k}[y_a | a \in Q_1] &\longrightarrow \mathbb{k}[x_1, \dots, x_d, t_i, h_i] / A \\ y_a &\mapsto t_{\mathbf{t}(a)} x^{\text{div}(a)} h_{\mathbf{h}(a)}. \end{aligned}$$

The image of f satisfies

$$\overline{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x)$$

where $i_1, \dots, i_n, j_1, \dots, j_n \in Q_0$.

Proof. Since f is homogeneous, we can decompose f into terms, each of weight χ . By Remark 4.2.3 (ii), for each term we have

$$f = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n y_{p_{\alpha\beta}}$$

where $c_\beta \in \mathbb{k}$, the $p_{\alpha\beta}$'s are paths where χ_i^+ of the $p_{\alpha\beta}$'s have head at $i \in Q_0$ and χ_i^- of the $p_{\alpha\beta}$'s have tail at $i \in Q_0$.

For each $y_{p_{\alpha\beta}}$ we have

$$\overline{\psi}(y_{p_{\alpha\beta}}) = t_{\mathbf{t}(p_{\alpha\beta})} x^{\text{div}(p_{\alpha\beta})} h_{\mathbf{h}(p_{\alpha\beta})}$$

since we are working modulo A . So for any β :

$$\overline{\psi}\left(\prod_{\alpha=1}^k y_{p_{\alpha\beta}}\right) = \prod_{\alpha=1}^n t_{\mathbf{t}(p_{\alpha\beta})} h_{\mathbf{h}(p_{\alpha\beta})} x^{\text{div}(p_{\alpha\beta})}.$$

Now since $\prod_{\alpha=1}^n t_{\mathbf{t}(p_{\alpha\beta})} h_{\mathbf{h}(p_{\alpha\beta})}$ depends only on χ , this is a common factor for $\overline{\psi}\left(\prod_{\alpha=1}^k y_{p_{\alpha\beta}}\right)$ for each β . Hence, summing over β we have:

$$\overline{\psi}(f) = \prod_{\alpha=1}^n (t_{\mathbf{t}(p_{\alpha\beta})} h_{\mathbf{h}(p_{\alpha\beta})}) \times \left(\sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n x^{\text{div}(p_{\alpha\beta})} \right).$$

Letting $g(x) := \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n x^{\text{div}(p_{\alpha\beta})}$ we have the statement of the Lemma. \square

Proposition 5.2.2. *The kernel of ψ is equal to I_Q .*

Proof. We note that the kernel of ψ is precisely the set:

$$\ker \psi = \{f \in \mathbb{k}[y_a | a \in Q_1] | \bar{\psi}(f) \in I_X\}$$

where I_X is considered as an ideal of $\mathbb{k}[x_1, \dots, x_d, t_i, h_i]/A$.

First we show $I_Q \subseteq \ker(\psi)$. Let f be an element of I_Q . We assume f is homogeneous of weight $\chi \in \text{inc}(\mathbb{N}^{Q_1}) \setminus \{0\}$ and let $n := n_\chi$. By Remark 4.2.3 (ii), we have

$$f = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n y_{p_{\alpha\beta}}$$

and by the proof of Lemma 5.2.1 $\bar{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x)$ where $i_1, \dots, i_n, j_1, \dots, j_n \in Q_0$ and where $g(x) = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n x^{\text{div}(p_{\alpha\beta})}$. Now, since $f \in I_Q$, we have that $\tilde{\Phi}(f) \in I_X$. This means

$$\tilde{\Phi}(f) = \tilde{\Phi}\left(\sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n y_{p_{\alpha\beta}}\right) = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n \tilde{\Phi}(y_{p_{\alpha\beta}}) = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n x^{\text{div}(p_{\alpha\beta})} = g(x) \in I_X.$$

Hence $f \in \ker(\psi)$.

Now to show opposite inclusion let $f \in \ker(\psi)$ be homogeneous of weight χ . So

$$f = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n y_{p_{\alpha\beta}}$$

where $c_\beta \in \mathbb{k}$, the $p_{\alpha\beta}$'s are paths where χ_i^+ of the $p_{\alpha\beta}$'s have head at $i \in Q_0$ and χ_i^- of the $p_{\alpha\beta}$'s have tail at $i \in Q_0$. Also, $\bar{\psi}(f) = t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(x) \in I_X$ where $i_1, \dots, i_n, j_1, \dots, j_n \in Q_0$. I_X is generated by $g_1(a_1, \dots, x_d), \dots, g_m(x_1, \dots, x_d)$, so

$$\begin{aligned} & t_{i_1} \cdots t_{i_n} h_{j_1} \cdots h_{j_n} g(\mathbf{x}) = \\ & f_1(\mathbf{x}, \mathbf{t}, \mathbf{h})g_1(\mathbf{x}) + \cdots + f_m(\mathbf{x}, \mathbf{t}, \mathbf{h})g_m(\mathbf{x}) \end{aligned}$$

for some $f_1, \dots, f_m \in \mathbb{k}[x_1, \dots, x_d, t_i, h_i]/A$, where $\mathbf{x} = (x_1, \dots, x_d)$, $\mathbf{t} = (t_0, \dots, t_r)$ and $\mathbf{h} = (h_0, \dots, h_r)$. Substituting $t_i = 1, h_i = 1$ for all $i \in Q_0$ we obtain:

$$g(x) = f_1(\mathbf{x}, 1, \dots, 1)g_1(\mathbf{x}) + \cdots + f_m(\mathbf{x}, 1, \dots, 1)g_m(\mathbf{x})$$

hence $g(\mathbf{x}) \in I_X$.

By the proof of Lemma 5.2.1, we also have

$$g(\mathbf{x}) = \sum_{\beta=1}^k c_\beta \prod_{\alpha=1}^n x^{\text{div}(p_{\alpha\beta})} = \tilde{\Phi}(f) \in I_X.$$

Hence $f \in I_Q$ since it is homogeneous by assumption. □

We note that these results also apply to the toric case by setting $I_X = (0)$.

5.2.2 Macaulay2 Code

We present Macaulay2 code for computing I_Q and \widetilde{I}_Q . Let $q := |Q_1|$ and denote I_Q by IQ, and \widetilde{I}_Q by IQtilde

```
i1: R = QQ[x_1..x_d,t_0..t_r, h_0..h_r, y_1..y_q, MonomialOrder => Eliminate d+2*(r+1)]

i2: K = ideal(y_1-xdiv(a_1),... y_m-xdiv(a_1)).
i3: I = ideal(g_1,...g_m, t_1*h_1-1,..., t_r*h_r-1)
i4: Itilde = ideal(t_0*h_0-1,..., t_r*h_r-1)
i5: H = K+I
i5: G = gens gb H
i6: J = selectInSubring(1,G)
i7: IQ = ideal(J)
i8: Htilde = K+Itilde
i9: Gtilde = gens gb Htilde
i10: Jtilde = selectInSubring(1,Gtilde)
i11: IQtilde = ideal(Jtilde)
```

Chapter 6

Examples of Mori Dream Spaces as Fine Moduli of Quiver Representations

As an application of our main results we illustrate how to reconstruct del Pezzo surfaces directly from the bound quiver of sections of a collection of line bundles whose direct sum is a tilting bundle.

6.1 Tilting bundles on del Pezzo surfaces

Let X be a smooth projective variety over \mathbb{k} and write $\text{coh}(X)$ for the category of coherent sheaves on X . For any vector bundle \mathcal{T} on X , let $A := \text{End}_{\mathcal{O}_X}(\mathcal{T})$ denote its endomorphism algebra and $\text{mod}(A)$ the abelian category of finitely generated right A -modules. We say that \mathcal{T} is a *tilting bundle* on X if the functor

$$\mathbf{R}\text{Hom}(\mathcal{T}, -): D^b(\text{coh}(X)) \longrightarrow D^b(\text{mod}(A))$$

is an exact equivalence of bounded derived categories. If \mathcal{T} decomposes as a direct sum of line bundles $\mathcal{T} = \bigoplus_{0 \leq i \leq r} L_i$ (we need not assume that each L_i has rank one, but we choose to), then after reordering if necessary, the collection (L_0, L_1, \dots, L_r) is a full, strongly exception sequence on X . That is, the line bundles in the collection generate $D^b(\text{coh}(X))$ and they satisfy appropriate Ext-vanishing conditions, namely, that $\text{Hom}(L_j, L_i) = 0$ for $j > i$ and that $\text{Ext}^k(L_i, L_j) = 0$ for $k > 0$ and all $0 \leq i, j \leq r$.

For $0 \leq k \leq 8$, let X_k denote the del Pezzo surface obtained as the blow-up of $\mathbb{P}_{\mathbb{k}}^2$ at k points in general position. The Picard group $\text{Cl}(X_k) \cong \mathbb{Z}^{k+1}$ has a basis given by l_0 , the pullback to X_d of the hyperplane class on $\mathbb{P}_{\mathbb{k}}^2$, together with the k exceptional curves l_1, \dots, l_k . Consider the sequence

of basepoint-free line bundles

$$\mathcal{L}_k := (\mathcal{O}_{X_k}, H, 2l_0 - l_1, \dots, 2l_0 - l_k, 2l_0) \quad (6.1.1)$$

on X_k , and write $L_0 = \mathcal{O}_{X_k}$, $L_1 = l_0$, $L_{i+1} = 2l_0 - l_i$ for $1 \leq i \leq k$, and $L_{k+2} = 2l_0$. The following result is well known, but for convenience we guide the reader towards a proof.

Lemma 6.1.1. *The sequence of line bundles (6.1.1) on X_k is full and strongly exceptional, so the vector bundle $\mathcal{T}_k := \bigoplus_{0 \leq i \leq k+2} L_i$ is tilting.*

Proof. We use the technology of toric systems developed by Hille–Perling [18]. Beginning with the unique toric system l_0, l_0, l_0 on $\mathbb{P}_{\mathbb{k}}^2$, construct a toric system on each X_k as follows: choose $l_0, l_0 - l_1, l_1, l_0 - l_1$ on X_1 , then repeat for $k \geq 2$, introducing l_k in the second-last position while subtracting l_k from each neighbouring divisor to obtain the toric system

$$l_0, l_0 - l_1, l_1 - l_2, l_2 - l_3, \dots, l_{k-1} - l_k, l_k, l_0 - \sum_{1 \leq i \leq k} l_i$$

on X_k . List these divisors from left to right as D_1, \dots, D_{k+3} . Observe that for $1 \leq i \leq k+2$ we have $L_i = \mathcal{O}(D_1 + \dots + D_i)$, and $-K_{X_k} = \mathcal{O}(D_1 + \dots + D_{k+3})$, and Theorem 5.7 of Hille–Perline [18] establishes that the sequence $(L_0, L_1, \dots, L_{k+2})$ is full and strongly exceptional as required. \square

Let (Q_k, J_k) denote the bound quiver of sections of the collection \mathcal{L}_k on X_k . For $k \leq 3$, the variety X_k is toric, in which case $\mathcal{L} = \widetilde{\mathcal{L}}_k$ and the method of Craw–Smith [10] shows that the morphism $\varphi|_{\mathcal{L}_k}: X_k \rightarrow \mathcal{M}_\vartheta(Q_k, J_k)$ is an isomorphism. We now consider the cases where $k = 4$ and 5. We were unable to compute the case $k = 6$ due to computational complexity.

We also consider a collection of line bundles \mathcal{L} on $\text{Gr}(2, 4)$ which gives an isomorphism with the moduli space of bound quiver representations for the quiver of sections of \mathcal{L} .

6.2 X_4 Tilting Example

On X_4 , a strong exceptional collection of line bundles is $\mathcal{L} := (\mathcal{O}_{X_4}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0)$ where notation is as in Section 2.2.2. The quiver of sections for \mathcal{L} is: Arrows with tail at 0 are listed a_1, \dots, a_6 from the top of Figure 6.1 to the bottom; list those with tail at 1 as a_7, \dots, a_{18} from the top of the figure to the bottom; and list those with head at 6 as a_{19}, \dots, a_{22} from the top to the bottom. Likewise, list the coordinates of $\mathbb{A}_{\mathbb{k}}^{Q_1}$ as y_1, \dots, y_{22} .

Using the methods described in sections 5.1 and 5.2, we calculated $\widetilde{I}_R, I_R, \widetilde{I}_Q$, and I_Q . We compute B_Y by computing the intersection

$$B_Y = \bigcap_{i \in Q_0} (y_{a_j} \in \mathbb{k}[y_a | a \in Q_1] | \mathbf{h}(a_j) = i).$$

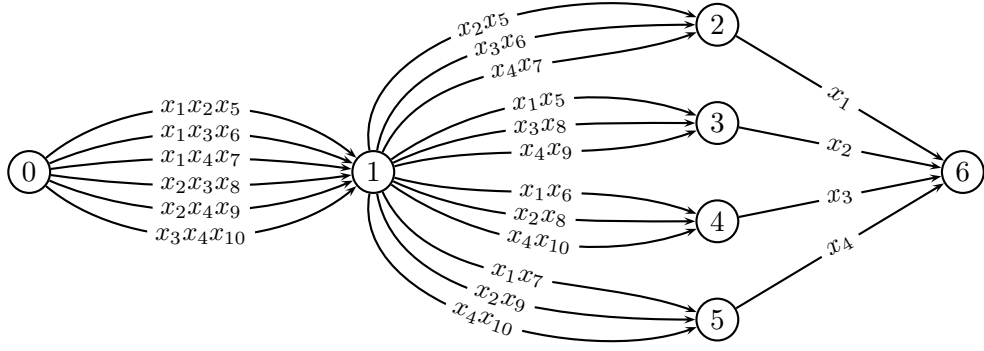


Figure 6.1: A quiver of sections for a collection on X_4

We then compared $\widetilde{I}_R : B_Y^\infty$ to \widetilde{I}_Q , and $I_R : B_Y^\infty$ to I_Q . The results were as follows:

$$\widetilde{I}_R = \left(\begin{array}{l} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, \\ y_5y_{16} - y_3y_{17}, y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, \\ y_5y_{10} - y_1y_{12}, y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, \\ y_3y_{14}y_{21} - y_4y_{16}y_{22}, y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22} \end{array} \right)$$

$$I_R = \left(\begin{array}{l} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, \\ y_5y_{16} - y_3y_{17}, y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, \\ y_5y_{10} - y_1y_{12}, y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, \\ y_3y_{14}y_{21} - y_4y_{16}y_{22}, y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, \\ y_{16} - y_{17} + y_{18}, y_{13} - y_{14} + y_{15}, y_{10} - y_{11} + y_{12}, y_7 - y_8 + y_9, y_3 - y_5 + y_6, \\ y_2 - y_4 + y_6, y_1 - y_4 + y_5, y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, \\ y_9y_{19} - y_{17}y_{22} + y_{18}y_{22}, y_8y_{19} - y_{14}y_{21} + y_{18}y_{22}, y_6y_{17} - y_5y_{18}, \\ y_6y_{14} - y_4y_{15}, y_5y_{11} - y_4y_{12}, y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9 \end{array} \right)$$

$$\widetilde{I}_Q = \left(\begin{array}{l} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, y_5y_{16} - y_3y_{17}, \\ y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, y_5y_{10} - y_1y_{12}, \\ y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, y_3y_{14}y_{21} - y_4y_{16}y_{22}, \\ y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, \\ y_8y_{15}y_{16} - y_9y_{13}y_{18}, y_7y_{12}y_{16} - y_9y_{10}y_{17}, y_7y_{11}y_{13} - y_8y_{10}y_{14}, \\ y_8y_{10}y_{15}y_{17} - y_7y_{12}y_{13}y_{18}, y_7y_{11}y_{15}y_{16} - y_9y_{10}y_{14}y_{18}, \\ y_8y_{12}y_{14}y_{16} - y_9y_{11}y_{13}y_{17} \end{array} \right)$$

$$I_Q = \left(\begin{array}{l} y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{16}y_{22}, \\ y_8y_{19} - y_{13}y_{21}, y_7y_{19} - y_{10}y_{20}, y_6y_{17} - y_5y_{18}, y_6y_{16} - y_3y_{18}, y_5y_{16} - y_3y_{17}, \\ y_6y_{14} - y_4y_{15}, y_6y_{13} - y_2y_{15}, y_4y_{13} - y_2y_{14}, y_5y_{11} - y_4y_{12}, y_5y_{10} - y_1y_{12}, \\ y_4y_{10} - y_1y_{11}, y_3y_8 - y_2y_9, y_3y_7 - y_1y_9, y_2y_7 - y_1y_8, y_3y_{14}y_{21} - y_4y_{16}y_{22}, \\ y_5y_{13}y_{21} - y_2y_{17}y_{22}, y_6y_{10}y_{20} - y_1y_{18}y_{22}, y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, \\ y_8y_{15}y_{16} - y_9y_{13}y_{18}, y_7y_{12}y_{16} - y_9y_{10}y_{17}, y_7y_{11}y_{13} - y_8y_{10}y_{14}, \\ y_8y_{10}y_{15}y_{17} - y_7y_{12}y_{13}y_{18}, y_7y_{11}y_{15}y_{16} - y_9y_{10}y_{14}y_{18}, \\ y_8y_{12}y_{14}y_{16} - y_9y_{11}y_{13}y_{17}, y_{16} - y_{17} + y_{18}, y_{13} - y_{14} + y_{15}, y_{10} - y_{11} + y_{12}, \\ y_7 - y_8 + y_9, y_3 - y_5 + y_6, y_2 - y_4 + y_6, y_1 - y_4 + y_5, y_{15}y_{21} - y_{18}y_{22}, y_{12}y_{20} - y_{17}y_{22}, \\ y_{11}y_{20} - y_{14}y_{21}, y_9y_{19} - y_{17}y_{22} + y_{18}y_{22}, y_8y_{19} - y_{14}y_{21} + y_{18}y_{22}, \\ y_6y_{17} - y_5y_{18}, y_6y_{14} - y_4y_{15}, y_5y_{11} - y_4y_{12}, y_5y_8 - y_6y_8 - y_4y_9 + y_6y_9, \\ y_{11}y_{15}y_{17} - y_{12}y_{14}y_{18}, y_8y_{15}y_{17} - y_9y_{14}y_{18} - y_8y_{15}y_{18} + y_9y_{15}y_{18}, \\ y_9y_{11}y_{17} - y_8y_{12}y_{17} + y_8y_{12}y_{18} - y_9y_{12}y_{18}, \\ y_9y_{11}y_{14} - y_8y_{12}y_{14} + y_8y_{11}y_{15} - y_9y_{11}y_{15} \end{array} \right)$$

B_Y is the intersection of the ideals:

$$(y_1, \dots, y_6), (y_7, y_8, y_9), (y_{10}, y_{11}, y_{12}), (y_{13}, y_{14}, y_{15}), (y_{16}, y_{17}, y_{18}) \text{ and } (y_{19}, y_{20}, y_{21}, y_{22}).$$

We present Macaulay2 code for computing I_Q and \widetilde{I}_Q .

```
i1: R = QQ[x_1..x_10,t_0..t_6, h_0..h_6, y_1..y_22, MonomialOrder => Eliminate 24
]
i2: H = K+I
i4: G = gens gb H
i5: J = selectInSubring(1,G)
i6: IQ = ideal(J)
i6: Htilde= K +Itilde
i7: Gtilde = gens gb K
i8: Jtilde = selectInSubring(1,Gtilde)
i9: IQtilde = ideal(Jtilde)
```

where

$$K = \left(\begin{array}{l} y_1 - t_0h_1x_1x_2x_5, y_2 - t_0h_1x_1x_3x_6, y_3 - t_0h_1x_1x_4x_7, y_4 - t_0h_1x_2x_3x_8, \\ y_5 - t_0h_1x_2x_4x_9, y_6 - t_0h_1x_3x_4x_{10}, y_7 - t_1h_2x_2x_5, y_8 - t_1h_2x_3x_6, \\ y_9 - t_1h_2x_4x_7, y_{10} - t_1h_3x_1x_5, y_{11} - t_1h_3x_3x_8, y_{12} - t_1h_3x_4x_9, \\ y_{13} - t_1h_4x_1x_6, y_{14} - t_1h_4x_2x_8, y_{15} - t_1h_4x_4x_{10}, y_{16} - t_1h_5x_1x_7, \\ y_{17} - t_1h_5x_2x_9, y_{18} - t_1h_5x_3x_{10}, y_{19} - t_2h_6x_1, y_{20} - t_3h_6x_2, y_{21} - t_4h_6x_3, y_{22} - t_5h_6x_4 \end{array} \right),$$

$$I = \left(\begin{array}{l} x_2x_5 - x_3x_6 + x_4x_7, x_1x_5 - x_3x_8 + x_4x_9, x_1x_6 - x_2x_8 + x_4x_{10}, \\ x_1x_7 - x_2x_9 + x_3x_{10}, x_5x_{10} - x_6x_9 + x_7x_8, t_0h_0 - 1, \dots, t_6h_6 - 1 \end{array} \right)$$

and

$$\tilde{I} = \left(t_0h_0 - 1, \dots, t_6h_6 - 1 \right).$$

In Macaulay2, we calculate the saturation of I_R and I_Q with B_Y using the command “saturate”,

```
i1: IQQ = saturate(IQ,BY)
i2: IRR = saturate(IR,BY)
i3: IRR == IQQ
o3: true
```

In the same way we obtain $\tilde{I}_Q = \tilde{I}_R : B_Y^\infty$. Example 3.2.9 showed that \mathcal{L}_4 is very ample, so Theorem 4.1.1 implies that $\varphi_{|\mathcal{L}_4|}: X_4 \rightarrow \mathcal{M}_\vartheta(\text{mod}(A_{\mathcal{L}_4}))$ is an isomorphism.

6.3 X_5 Tilting Example

On X_5 , a strong exceptional collection of line bundles is $\mathcal{L} := (\mathcal{O}_{X_5}, l_0, 2l_0 - l_1, 2l_0 - l_2, 2l_0 - l_3, 2l_0 - l_4, 2l_0 - l_5, 2l_0)$ where notation as in section 2.2.2. The quiver of sections Q is shown in Figure 6.2 (in fact we omit one arrow labelled $x_1x_2x_4x_5x_{16}$ with tail at 0 and head at 4 to prevent the figure from becoming illegible). Arrows with tail at 0 and head at 1 are listed a_1, \dots, a_{10} from the top of Figure 6.2 to the bottom; list those with tail at 1 as a_{11}, \dots, a_{30} from top to bottom; list those with head at 7 as a_{31}, \dots, a_{35} from top to bottom; and list those with tail at 0 and head at

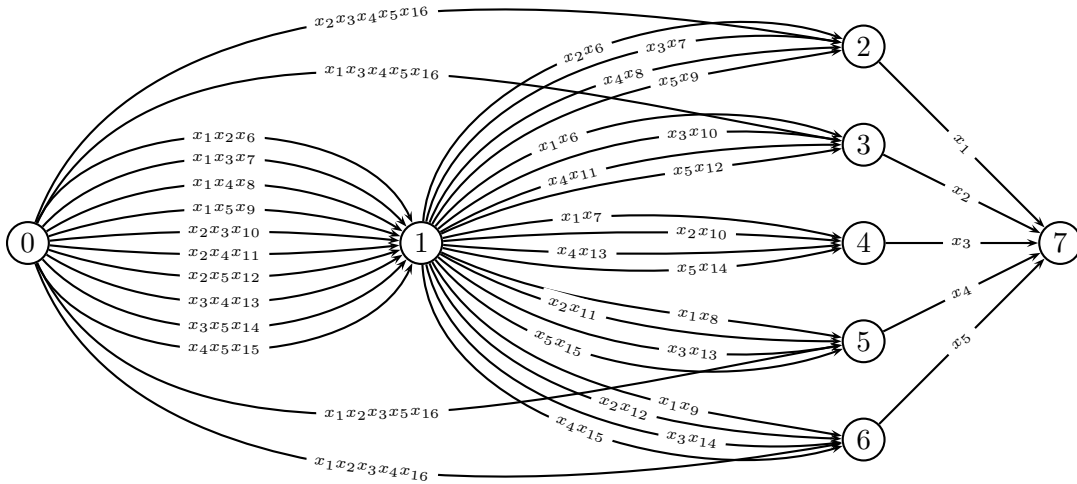


Figure 6.2: A quiver of sections for a full strongly exceptional collection on X_5

those with tail at 0 and head at

$i \geq 2$ as a_{36}, \dots, a_{40} from top to bottom, where the arrow omitted from the figure is a_{38} . List the coordinates of $\mathbb{A}_k^{Q_1}$ as y_1, \dots, y_{40}

Using the methods described in sections 5.1 and 5.2, we calculated $\widetilde{I}_R, I_R, \widetilde{I}_Q, I_Q, B_Y$, and compared $\widetilde{I}_R : B_Y^\infty$ to \widetilde{I}_Q , and $I_R : B_Y^\infty$ to I_Q . The results were as follows:

$$\widetilde{I}_R = \left(\begin{array}{l} y_{34}y_{39} - y_{35}y_{40}, y_{33}y_{38} - y_{35}y_{40}, y_{32}y_{37} - y_{35}y_{40}, \\ y_{31}y_{36} - y_{35}y_{40}, y_{26}y_{34} - y_{30}y_{35}, y_{22}y_{33} - y_{29}y_{35}, \\ y_{21}y_{33} - y_{25}y_{34}, y_{18}y_{32} - y_{28}y_{35}, y_{17}y_{32} - y_{24}y_{34}, \\ y_{16}y_{32} - y_{20}y_{33}, y_{14}y_{31} - y_{27}y_{35}, y_{13}y_{31} - y_{23}y_{34}, \\ y_{12}y_{31} - y_{19}y_{33}, y_{11}y_{31} - y_{15}y_{32}, y_{10}y_{29} - y_9y_{30}, \\ y_{10}y_{28} - y_7y_{30}, y_9y_{28} - y_7y_{29}, y_{10}y_{27} - y_4y_{30}, \\ y_9y_{27} - y_4y_{29}, y_7y_{27} - y_4y_{28}, y_{10}y_{25} - y_8y_{26}, \\ y_{10}y_{24} - y_6y_{26}, y_8y_{24} - y_6y_{25}, y_{10}y_{23} - y_3y_{26}, \\ y_8y_{23} - y_3y_{25}, y_6y_{23} - y_3y_{24}, y_9y_{21} - y_8y_{22}, y_9y_{20} - y_5y_{22}, \\ y_8y_{20} - y_5y_{21}, y_9y_{19} - y_2y_{22}, y_8y_{19} - y_2y_{21}, y_5y_{19} - y_2y_{20}, \\ y_7y_{17} - y_6y_{18}, y_7y_{16} - y_5y_{18}, y_6y_{16} - y_5y_{17}, y_7y_{15} - y_1y_{18}, \\ y_6y_{15} - y_1y_{17}, y_5y_{15} - y_1y_{16}, y_4y_{13} - y_3y_{14}, y_4y_{12} - y_2y_{14}, \\ y_3y_{12} - y_2y_{13}, y_4y_{11} - y_1y_{14}, y_3y_{11} - y_1y_{13}, y_2y_{11} - y_1y_{12}, \\ y_7y_{25}y_{34} - y_8y_{28}y_{35}, y_4y_{25}y_{34} - y_8y_{27}y_{35}, \\ y_9y_{24}y_{34} - y_6y_{29}y_{35}, y_4y_{24}y_{34} - y_6y_{27}y_{35}, \\ y_9y_{23}y_{34} - y_3y_{29}y_{35}, y_7y_{23}y_{34} - y_3y_{28}y_{35}, \\ y_{10}y_{20}y_{33} - y_5y_{30}y_{35}, y_4y_{20}y_{33} - y_5y_{27}y_{35}, \\ y_3y_{20}y_{33} - y_5y_{23}y_{34}, y_{10}y_{19}y_{33} - y_2y_{30}y_{35}, \\ y_7y_{19}y_{33} - y_2y_{28}y_{35}, y_6y_{19}y_{33} - y_2y_{24}y_{34}, \\ y_{10}y_{15}y_{32} - y_1y_{30}y_{35}, y_9y_{15}y_{32} - y_1y_{29}y_{35}, y_8y_{15}y_{32} - y_1y_{25}y_{34} \end{array} \right)$$

$$I_R = \left(\begin{array}{l} y_7 + 2y_9 - y_{10}, y_6 - 2y_8 + y_{10}, y_5 + y_8 - y_9, y_4 + y_9 - 2y_{10}, y_3 - y_8 + y_{10} \\ y_2 + 2y_8 - y_9, y_1 + 3y_8 - y_9 - y_{10}, y_{28} + 2y_{29} - y_{30}, y_{27} + y_{29} - 2y_{30} \\ y_{24} - 2y_{25} + y_{26}, y_{23} - y_{25} + y_{26}, y_{20} + y_{21} - y_{22}, y_{19} + 2y_{21} - y_{22} \\ 2y_{16} + y_{17} + y_{18}, 2y_{15} + 3y_{17} + y_{18}, y_{12} + 2y_{13} + y_{14}, y_{11} + 3y_{13} + y_{14} \\ y_{10}y_{29} - y_9y_{30}, 2y_8y_{29} + 2y_8y_{30} - 3y_9y_{30} - y_{40}, y_{10}y_{25} - y_8y_{26} \\ 2y_9y_{25} + 2y_8y_{26} - 3y_9y_{26} - y_{39}, 2y_{10}y_{21} + 2y_8y_{22} - 3y_{10}y_{22} - y_{38} \\ y_9y_{21} - y_8y_{22}, 3y_{10}y_{17} - 2y_8y_{18} + 3y_{10}y_{18} - 2y_{37}, 3y_9y_{17} + 2y_8y_{18} - y_{37} \\ 6y_{10}y_{13} - 2y_8y_{14} + 3y_{10}y_{14} - y_{36}, 3y_9y_{13} + y_8y_{14} - y_{36}, y_{26}y_{34} - y_{30}y_{35} \\ y_{22}y_{33} - y_{29}y_{35}, y_{21}y_{33} - y_{25}y_{34}, y_{18}y_{32} + 2y_{29}y_{35} - y_{30}y_{35} \\ y_{17}y_{32} - 2y_{25}y_{34} + y_{30}y_{35}, y_{14}y_{31} + y_{29}y_{35} - 2y_{30}y_{35} \\ y_{13}y_{31} - y_{25}y_{34} + y_{30}y_{35} \end{array} \right)$$

$$\widetilde{I}_Q = \left(\begin{array}{c} 2y_{17}y_{26}y_{29} + 2y_{18}y_{25}y_{30} - y_{17}y_{26}y_{30} - y_{18}y_{26}y_{30}, \\ y_{13}y_{26}y_{29} + y_{14}y_{25}y_{30} - 2y_{13}y_{26}y_{30} - y_{14}y_{26}y_{30}, \\ 2y_{18}y_{21}y_{29} + 2y_{17}y_{22}y_{29} - y_{17}y_{22}y_{30} - y_{18}y_{22}y_{30}, \\ y_{14}y_{21}y_{29} + y_{13}y_{22}y_{29} - 2y_{13}y_{22}y_{30} - y_{14}y_{22}y_{30}, \\ 2y_{14}y_{17}y_{29} - 2y_{13}y_{18}y_{29} - y_{14}y_{17}y_{30} + 4y_{13}y_{18}y_{30} + y_{14}y_{18}y_{30}, \\ 2y_{18}y_{21}y_{25} + 2y_{17}y_{22}y_{25} - y_{17}y_{21}y_{26} - y_{18}y_{21}y_{26}, \\ y_{14}y_{21}y_{25} + y_{13}y_{22}y_{25} - 2y_{13}y_{21}y_{26} - y_{14}y_{21}y_{26}, \\ 2y_{14}y_{17}y_{25} - 2y_{13}y_{18}y_{25} - 3y_{13}y_{17}y_{26} - 2y_{14}y_{17}y_{26} + y_{13}y_{18}y_{26} \\ y_{14}y_{17}y_{21} - 4y_{13}y_{18}y_{21} - y_{14}y_{18}y_{21} - 3y_{13}y_{17}y_{22} - 2y_{14}y_{17}y_{22} + y_{13}y_{18}y_{22} \end{array} \right)$$

$$I_Q = \left(\begin{array}{c} y_7 + 2y_9 - y_{10}, y_6 - 2y_8 + y_{10}, y_5 + y_8 - y_9, y_4 + y_9 - 2y_{10}, y_3 - y_8 + y_1, \\ y_2 + 2y_8 - y_9, y_1 + 3y_8 - y_9 - y_{10}, y_{28} + 2y_{29} - y_{30}, y_{27} + y_{29} - 2y_{30}, \\ y_{24} - 2y_{25} + y_{26}, y_{23} - y_{25} + y_{26}, y_{20} + y_{21} - y_{22}, \\ y_{19} + 2y_{21} - y_{22}, 2y_{16} + y_{17} + y_{18}, 2y_{15} + 3y_{17} + y_{18}, \\ y_{12} + 2y_{13} + y_{14}, y_{11} + 3y_{13} + y_{14}, 10y_{29} - y_9y_{30}, \\ 2y_8y_{29} + 2y_8y_{30} - 3y_9y_{30} - y_{40}, y_{10}y_{25} - y_8y_{26}, \\ 2y_9y_{25} + 2y_8y_{26} - 3y_9y_{26} - y_{39}, 2y_{10}y_{21} + 2y_8y_{22} - 3y_{10}y_{22} - y_{38}, \\ y_9y_{21} - y_8y_{22}, 3y_{10}y_{17} - 2y_8y_{18} + 3y_{10}y_{18} - 2y_{37}, \\ 3y_9y_{17} + 2y_8y_{18} - y_{37}, 6y_{10}y_{13} - 2y_8y_{14} + 3y_{10}y_{14} - y_{36}, \\ 3y_9y_{13} + y_8y_{14} - y_{36}, y_{30}y_{35} - y_{10}, y_{29}y_{35} - y_9, y_{26}y_{34} - y_{10}, \\ y_{25}y_{34} - y_8, y_{22}y_{33} - y_9, y_{21}y_{33} - y_8, y_{18}y_{32} + 2y_9 - y_{10}, y_{17}y_{32} - 2y_8 + y_{10}, \\ y_{14}y_{31} + y_9 - 2y_{10}, y_{13}y_{31} - y_8 + y_{10}, y_{21}y_{26}y_{29} - y_{22}y_{25}y_{30} \end{array} \right)$$

We present Macaulay2 code for computing I_Q and \widetilde{I}_Q .

```
i1: R = QQ[x_1..x_16,t_0..t_7,h_0..h_7,y_1..y_40,MonomialOrder => Eliminate 32]
i2: H = I+K
i3: G = gens gb K+I
i4: J = selectInSubring(1,G)
i5: IQ = ideal(J)
i6: Htilde = K+Itilde
i7: Gtilde = gens gb Htilde
i8: Jtilde = selectInSubring(1,Gtilde)
i9: IQtilde = ideal(Jtilde)
```

where

$$K = \left(\begin{array}{l} y_1 - t_0 h_1 x_1 x_2 x_6, y_2 - t_0 h_1 x_1 x_3 x_7, y_3 - t_0 h_1 x_1 x_4 x_8, y_4 - t_0 h_1 x_1 x_5 x_9, y_5 - t_0 h_1 x_2 x_3 x_{10}, \\ y_6 - t_0 h_1 x_2 x_4 x_{11}, y_7 - t_0 h_1 x_2 x_5 x_{12}, y_8 - t_0 h_1 x_3 x_4 x_{13}, y_9 - t_0 h_1 x_3 x_5 x_{14}, y_{10} - t_0 h_1 x_4 x_5 x_{15}, \\ y_{11} - t_1 h_2 x_2 x_6, y_{12} - t_1 h_2 x_3 x_7, y_{13} - t_1 h_2 x_4 x_8, y_{14} - t_1 h_2 x_5 x_9, y_{15} - t_1 h_3 x_1 x_6, \\ y_{16} - t_1 h_3 x_3 x_{10}, y_{17} - t_1 h_3 x_4 x_{11}, y_{18} - t_1 h_3 x_5 x_{12}, y_{19} - t_1 h_4 x_1 x_7, y_{20} - t_1 h_4 x_2 x_{10}, \\ y_{21} - t_1 h_4 x_4 x_{13}, y_{22} - t_1 h_4 x_5 x_{14}, y_{23} - t_1 h_5 x_1 x_8, y_{24} - t_1 h_5 x_2 x_{11}, y_{25} - t_1 h_5 x_3 x_{13}, \\ y_{26} - t_1 h_5 x_5 x_{15}, y_{27} - t_1 h_6 x_1 x_9, y_{28} - t_1 h_6 x_2 x_{12}, y_{29} - t_1 h_6 x_3 x_{14}, y_{30} - t_1 h_6 x_4 x_{15}, \\ y_{31} - t_2 h_7 x_1, y_{32} - t_3 h_7 x_2, y_{33} - t_4 h_7 x_3, y_{34} - t_5 h_7 x_4, y_{35} - t_6 h_7 x_5, y_{36} - t_0 h_2 x_2 x_3 x_4 x_5 x_{16}, \\ y_{37} - t_0 h_3 x_1 x_3 x_4 x_5 x_{16}, y_{38} - t_0 h_4 x_1 x_2 x_4 x_5 x_{16}, y_{39} - t_0 h_5 x_1 x_2 x_3 x_5 x_{16}, y_{40} - t_0 h_6 x_1 x_2 x_3 x_4 x_{16} \end{array} \right)$$

$$I = \left(\begin{array}{l} x_5 x_{16} + x_6 x_{13} - 3x_8 x_{10}, x_4 x_{16} + 2x_6 x_{14} + x_7 x_{12}, \\ x_4 x_{16} + x_6 x_{14} + x_9 x_{10}, x_3 x_{16} + x_6 x_{15} + x_8 x_{12}, x_3 x_{16} + 2x_6 x_{15} + x_9 x_{11}, \\ x_2 x_{16} + x_7 x_{15} - 2x_8 x_{14}, x_2 x_{16} + 3x_7 x_{15} + 2x_9 x_{13}, \\ x_1 x_{16} + 2x_{10} x_{15} - x_{11} x_{14}, x_1 x_{16} + 3x_{10} x_{15} + x_{12} x_{13}, x_2 x_6 - x_3 x_7 + x_4 x_8, \\ 2x_2 x_6 - 3x_3 x_7 - x_5 x_9, x_1 x_6 - x_3 x_{10} + x_4 x_{11}, x_1 x_6 - 3x_3 x_{10} - x_5 x_{12}, \\ x_1 x_7 - x_2 x_{10} + x_4 x_{13}, x_1 x_7 - 2x_2 x_{10} + x_5 x_{14}, x_1 x_8 - x_2 x_{11} + x_3 x_{13}, \\ -2x_1 x_8 + x_2 x_{11} - x_5 x_{15}, -x_1 x_9 + 2x_2 x_{12} + 3x_3 x_{14}, \\ -2x_1 x_9 + x_2 x_{12} + 3x_4 x_{15}, x_6 x_{13} - x_7 x_{11} + x_8 x_{10} \\ t_0 h_0 - 1, \dots, t_7 h_7 - 1 \end{array} \right)$$

and

$$\tilde{I} = \left(t_0 h_0 - 1, \dots, t_7 h_7 - 1 \right)$$

In Macaulay2, we calculate the saturation of I_R and I_Q with B_Y using the command “saturate”,

```
i1: IQQ = saturate(IQ,BY)
i2: IRR = saturate(IR,BY)
i3: IRR == IQQ
o3: true
```

where B_Y is the intersection of:

$$\begin{aligned} & (y_1, \dots, y_{10}), (y_{11}, \dots, y_{14}, y_{36}), (y_{15}, \dots, y_{18}, y_{37}), (y_{19}, \dots, y_{22}, y_{38}), \\ & (y_{23}, \dots, y_{26}, y_{39}), (y_{27}, \dots, y_{30}, y_{40}) \text{ and } (y_{31}, \dots, y_{35}) \end{aligned}$$

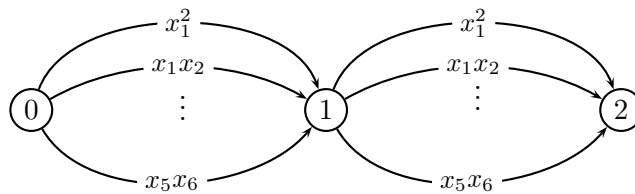
In the same way we obtain $\tilde{I}_Q = \tilde{I}_R : B_Y^\infty$. The collection \mathcal{L}_5 is very ample, so Theorem 4.1.1 implies that $\varphi|_{\mathcal{L}_5}: X_5 \rightarrow \mathcal{M}_\theta(A_{\mathcal{L}_5})$ is an isomorphism.

6.4 Gr(2, 4) Example

Let $X = \text{Gr}(2, 4)$. We can compute the Cox ring of X using the method of Section 2.2.1 to obtain

$$\text{Cox}(X) = \mathbb{k}[x_1, \dots, x_6] / (x_1x_6 - x_2x_5 + x_3x_4).$$

We recall that $\text{Pic}(X) \cong \mathbb{Z}$ is generated by the determinantal line bundle on X . Let $\mathcal{L} := (\mathcal{O}_X, \mathcal{O}(2), \mathcal{O}(4))$, the quiver of sections for \mathcal{L} is:



1. Arrows 1-21 are those from 0 to 1, they are labeled by all monomials in $\mathbb{k}[x_1, \dots, x_6]$ of degree 2.
2. Arrows 22-42 are those from 1 to 2, they are also labelled by all monomials of degree 2.

$$\begin{aligned}
I_R = & \left(\begin{aligned}
& y_{32} - y_{35} + y_{37}, y_{11} - y_{14} + y_{16}, y_{21}y_{41} - y_{20}y_{42}, y_{19}y_{41} - y_{4}y_{42}, \\
& y_{17}y_{41} - y_{16}y_{42}, y_{14}y_{41} - y_{13}y_{42}, y_{10}y_{41} - y_{9}y_{42}, y_{5}y_{41} - y_{19}y_{42}, \\
& y_{21}y_{40} - y_{19}y_{42}, y_{20}y_{40} - y_{4}y_{42}, y_{18}y_{40} - y_{16}y_{42}, y_{15}y_{40} - y_{13}y_{42}, \\
& y_{14}y_{40} - y_{16}y_{40} - y_{9}y_{42}, y_{6}y_{40} - y_{20}y_{42}, y_{21}y_{39} - y_{18}y_{42}, y_{20}y_{39} - y_{18}y_{41}, \\
& y_{19}y_{39} - y_{16}y_{42}, y_{17}y_{39} - y_{3}y_{42}, y_{16}y_{39} - y_{3}y_{41}, y_{14}y_{39} - y_{12}y_{42}, \\
& y_{13}y_{39} - y_{12}y_{41}, y_{10}y_{39} - y_{8}y_{42}, y_{9}y_{39} - y_{8}y_{41}, y_{5}y_{39} - y_{17}y_{42}, \\
& y_{4}y_{39} - y_{16}y_{41}, y_{21}y_{38} - y_{17}y_{42}, y_{20}y_{38} - y_{16}y_{42}, y_{19}y_{38} - y_{17}y_{40}, \\
& y_{18}y_{38} - y_{3}y_{42}, y_{16}y_{38} - y_{3}y_{40}, y_{15}y_{38} - y_{12}y_{42}, y_{14}y_{38} - y_{3}y_{40} - y_{8}y_{42}, \\
& y_{13}y_{38} - y_{12}y_{40}, y_{9}y_{38} - y_{8}y_{40}, y_{6}y_{38} - y_{18}y_{42}, y_{4}y_{38} - y_{16}y_{40}, \\
& y_{21}y_{37} - y_{16}y_{42}, y_{20}y_{37} - y_{16}y_{41}, y_{19}y_{37} - y_{16}y_{40}, y_{18}y_{37} - y_{3}y_{41}, \\
& y_{17}y_{37} - y_{3}y_{40}, y_{16}y_{37} - y_{12}y_{40} + y_{8}y_{41}, y_{15}y_{37} - y_{12}y_{41}, y_{14}y_{37} - y_{12}y_{40}, \\
& y_{13}y_{37} - y_{2}y_{40} + y_{7}y_{41}, y_{12}y_{37} - y_{2}y_{38} + y_{7}y_{39}, y_{10}y_{37} - y_{8}y_{40}, \\
& y_{9}y_{37} - y_{7}y_{40} + y_{1}y_{41}, y_{8}y_{37} - y_{7}y_{38} + y_{1}y_{39}, y_{6}y_{37} - y_{18}y_{41}, \\
& y_{5}y_{37} - y_{17}y_{40}, y_{4}y_{37} - y_{13}y_{40} + y_{9}y_{41}, y_{3}y_{37} - y_{12}y_{38} + y_{8}y_{39}, \\
& y_{21}y_{36} - y_{15}y_{42}, y_{20}y_{36} - y_{15}y_{41}, y_{19}y_{36} - y_{13}y_{42}, y_{18}y_{36} - y_{15}y_{39}, \\
& y_{17}y_{36} - y_{12}y_{42}, y_{16}y_{36} - y_{12}y_{41}, y_{14}y_{36} - y_{2}y_{42}, y_{13}y_{36} - y_{2}y_{41}, \\
& y_{12}y_{36} - y_{2}y_{39}, y_{10}y_{36} - y_{7}y_{42}, y_{9}y_{36} - y_{7}y_{41}, y_{8}y_{36} - y_{7}y_{39}, \\
& y_{5}y_{36} - y_{14}y_{42}, y_{4}y_{36} - y_{13}y_{41}, y_{3}y_{36} - y_{12}y_{39}, y_{21}y_{35} - y_{14}y_{42}, \\
& y_{20}y_{35} - y_{13}y_{42}, y_{19}y_{35} - y_{16}y_{40} - y_{9}y_{42}, y_{18}y_{35} - y_{12}y_{42}, \\
& y_{17}y_{35} - y_{3}y_{40} - y_{8}y_{42}, y_{16}y_{35} - y_{12}y_{40}, y_{15}y_{35} - y_{2}y_{42}, \\
& y_{14}y_{35} - y_{12}y_{40} - y_{7}y_{42}, y_{13}y_{35} - y_{2}y_{40}, y_{12}y_{35} - y_{2}y_{38}, y_{10}y_{35} - y_{8}y_{40} - y_{1}y_{42}, \\
& y_{9}y_{35} - y_{7}y_{40}, y_{8}y_{35} - y_{7}y_{38}, y_{7}y_{35} - y_{1}y_{36} - y_{7}y_{37}, y_{6}y_{35} - y_{15}y_{42}, \\
& y_{5}y_{35} - y_{17}y_{40} - y_{10}y_{42}, y_{4}y_{35} - y_{13}y_{40}, y_{3}y_{35} - y_{12}y_{38}, y_{2}y_{35} - y_{7}y_{36} - y_{2}y_{37}, \\
& y_{21}y_{34} - y_{13}y_{42}, y_{20}y_{34} - y_{13}y_{41}, y_{19}y_{34} - y_{13}y_{40}, y_{18}y_{34} - y_{12}y_{41}, \\
& y_{17}y_{34} - y_{12}y_{40}, y_{16}y_{34} - y_{2}y_{40} + y_{7}y_{41}, y_{15}y_{34} - y_{2}y_{41}, y_{14}y_{34} - y_{2}y_{40}, \\
& y_{12}y_{34} - y_{2}y_{37}, y_{10}y_{34} - y_{7}y_{40}, y_{8}y_{34} - y_{7}y_{37}, y_{6}y_{34} - y_{15}y_{41}, \\
& y_{5}y_{34} - y_{16}y_{40} - y_{9}y_{42}, y_{3}y_{34} - y_{2}y_{38} + y_{7}y_{39}, y_{21}y_{33} - y_{12}y_{42}, \\
& y_{20}y_{33} - y_{12}y_{41}, y_{19}y_{33} - y_{12}y_{40}, y_{18}y_{33} - y_{12}y_{39}, y_{17}y_{33} - y_{12}y_{38}, \\
& y_{16}y_{33} - y_{2}y_{38} + y_{7}y_{39}, y_{15}y_{33} - y_{2}y_{39}, y_{14}y_{33} - y_{2}y_{38}, y_{13}y_{33} - y_{2}y_{37}, \\
& y_{10}y_{33} - y_{7}y_{38}, y_{9}y_{33} - y_{7}y_{37}, y_{6}y_{33} - y_{15}y_{39}, y_{5}y_{33} - y_{3}y_{40} - y_{8}y_{42}, \\
& y_{4}y_{33} - y_{2}y_{40} + y_{7}y_{41}, y_{21}y_{31} - y_{10}y_{42}, y_{20}y_{31} - y_{9}y_{42}, y_{19}y_{31} - y_{10}y_{40}, \\
& y_{18}y_{31} - y_{8}y_{42}, y_{17}y_{31} - y_{10}y_{38}, y_{16}y_{31} - y_{8}y_{40}, y_{15}y_{31} - y_{7}y_{42}, \\
& y_{14}y_{31} - y_{8}y_{40} - y_{1}y_{42}, y_{13}y_{31} - y_{7}y_{40}, y_{12}y_{31} - y_{7}y_{38}, y_{9}y_{31} - y_{1}y_{40}, \\
& y_{8}y_{31} - y_{1}y_{38}, y_{7}y_{31} - y_{1}y_{35}, y_{6}y_{31} - y_{14}y_{42} + y_{16}y_{42}, y_{4}y_{31} - y_{9}y_{40}, \\
& y_{3}y_{31} - y_{8}y_{38}, y_{2}y_{31} - y_{1}y_{36} - y_{7}y_{37}, y_{21}y_{30} - y_{9}y_{42}, y_{20}y_{30} - y_{9}y_{41}, \\
& y_{19}y_{30} - y_{9}y_{40}, y_{18}y_{30} - y_{8}y_{41}, y_{17}y_{30} - y_{8}y_{40}, y_{16}y_{30} - y_{7}y_{40} + y_{1}y_{41}, \\
& y_{15}y_{30} - y_{7}y_{41}, y_{14}y_{30} - y_{7}y_{40}, y_{13}y_{30} - y_{9}y_{34}, y_{12}y_{30} - y_{7}y_{37},
\end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
& y_{10}y_{30} - y_{1}y_{40}, y_8y_{30} - y_1y_{37}, y_7y_{30} - y_1y_{34}, y_6y_{30} + y_{16}y_{41} - y_{13}y_{42}, y_5y_{30} - y_{10}y_{40}, \\
& y_3y_{30} - y_7y_{38} + y_1y_{39}, y_2y_{30} - y_7y_{34}, y_{21}y_{29} - y_8y_{42}, y_{20}y_{29} - y_8y_{41}, \\
& y_{19}y_{29} - y_8y_{40}, y_{18}y_{29} - y_8y_{39}, y_{17}y_{29} - y_8y_{38}, y_{16}y_{29} - y_7y_{38} + y_1y_{39}, \\
& y_{15}y_{29} - y_7y_{39}, y_{14}y_{29} - y_7y_{38}, y_{13}y_{29} - y_7y_{37}, y_{12}y_{29} - y_8y_{33}, y_{10}y_{29} - y_1y_{38}, \\
& y_9y_{29} - y_1y_{37}, y_7y_{29} - y_1y_{33}, y_6y_{29} + y_3y_{41} - y_{12}y_{42}, y_5y_{29} - y_{10}y_{38}, \\
& y_4y_{29} - y_7y_{40} + y_1y_{41}, y_2y_{29} - y_7y_{33}, y_{21}y_{28} - y_7y_{42}, y_{20}y_{28} - y_7y_{41}, \\
& y_{19}y_{28} - y_7y_{40}, y_{18}y_{28} - y_7y_{39}, y_{17}y_{28} - y_7y_{38}, y_{16}y_{28} - y_7y_{37}, \\
& y_{15}y_{28} - y_7y_{36}, y_{14}y_{28} - y_1y_{36} - y_7y_{37}, y_{13}y_{28} - y_7y_{34}, y_{12}y_{28} - y_7y_{33}, \\
& y_{10}y_{28} - y_1y_{35}, y_9y_{28} - y_1y_{34}, y_8y_{28} - y_1y_{33}, y_6y_{28} + y_{12}y_{41} - y_2y_{42}, \\
& y_5y_{28} - y_8y_{40} - y_1y_{42}, y_4y_{28} - y_9y_{34}, y_3y_{28} - y_8y_{33}, y_{21}y_{27} - y_6y_{42}, \\
& y_{20}y_{27} - y_6y_{41}, y_{19}y_{27} - y_{20}y_{42}, y_{18}y_{27} - y_6y_{39}, y_{17}y_{27} - y_{18}y_{42}, \\
& y_{16}y_{27} - y_{18}y_{41}, y_{15}y_{27} - y_6y_{36}, y_{14}y_{27} - y_{15}y_{42}, y_{13}y_{27} - y_{15}y_{41}, \\
& y_{12}y_{27} - y_{15}y_{39}, y_{10}y_{27} - y_{14}y_{42} + y_{16}y_{42}, y_9y_{27} + y_{16}y_{41} - y_{13}y_{42}, \\
& y_8y_{27} + y_3y_{41} - y_{12}y_{42}, y_7y_{27} + y_{12}y_{41} - y_2y_{42}, y_5y_{27} - y_{21}y_{42}, y_4y_{27} - y_{20}y_{41}, \\
& y_3y_{27} - y_{18}y_{39}, y_2y_{27} - y_{15}y_{36}, y_1y_{27} + y_8y_{41} - y_7y_{42}, y_{21}y_{26} - y_5y_{42}, \\
& y_{20}y_{26} - y_{19}y_{42}, y_{19}y_{26} - y_5y_{40}, y_{18}y_{26} - y_{17}y_{42}, y_{17}y_{26} - y_5y_{38}, \\
& y_{16}y_{26} - y_{17}y_{40}, y_{15}y_{26} - y_{14}y_{42}, y_{14}y_{26} - y_{17}y_{40} - y_{10}y_{42}, \\
& y_{13}y_{26} - y_{16}y_{40} - y_9y_{42}, y_{12}y_{26} - y_3y_{40} - y_8y_{42}, y_{10}y_{26} - y_5y_{31}, y_9y_{26} - y_{10}y_{40}, \\
& y_8y_{26} - y_{10}y_{38}, y_7y_{26} - y_8y_{40} - y_1y_{42}, y_6y_{26} - y_{21}y_{42}, y_4y_{26} - y_{19}y_{40}, \\
& y_3y_{26} - y_{17}y_{38}, y_2y_{26} - y_{12}y_{40} - y_7y_{42}, y_1y_{26} - y_{10}y_{31}, y_{21}y_{25} - y_4y_{42}, \\
& y_{20}y_{25} - y_4y_{41}, y_{19}y_{25} - y_4y_{40}, y_{18}y_{25} - y_{16}y_{41}, y_{17}y_{25} - y_{16}y_{40}, \\
& y_{16}y_{25} - y_{13}y_{40} + y_9y_{41}, y_{15}y_{25} - y_{13}y_{41}, y_{14}y_{25} - y_{13}y_{40}, y_{13}y_{25} - y_4y_{34}, \\
& y_{12}y_{25} - y_2y_{40} + y_7y_{41}, y_{10}y_{25} - y_9y_{40}, y_9y_{25} - y_4y_{30}, y_8y_{25} - y_7y_{40} + y_1y_{41}, \\
& y_7y_{25} - y_9y_{34}, y_6y_{25} - y_{20}y_{41}, y_5y_{25} - y_{19}y_{40}, y_3y_{25} - y_{12}y_{40} + y_8y_{41}, \\
& y_2y_{25} - y_{13}y_{34}, y_1y_{25} - y_9y_{30}, y_{21}y_{24} - y_3y_{42}, y_{20}y_{24} - y_3y_{41}, y_{19}y_{24} - y_3y_{40}, \\
& y_{18}y_{24} - y_3y_{39}, y_{17}y_{24} - y_3y_{38}, y_{16}y_{24} - y_{12}y_{38} + y_8y_{39}, y_{15}y_{24} - y_{12}y_{39}, \\
& y_{14}y_{24} - y_{12}y_{38}, y_{13}y_{24} - y_2y_{38} + y_7y_{39}, y_{12}y_{24} - y_3y_{33}, y_{10}y_{24} - y_8y_{38}, \\
& y_9y_{24} - y_7y_{38} + y_1y_{39}, y_8y_{24} - y_3y_{29}, y_7y_{24} - y_8y_{33}, y_6y_{24} - y_{18}y_{39}, y_5y_{24} - y_{17}y_{38}, \\
& y_4y_{24} - y_{12}y_{40} + y_8y_{41}, y_2y_{24} - y_{12}y_{33}, y_1y_{24} - y_8y_{29}, y_{21}y_{23} - y_2y_{42}, y_{20}y_{23} - y_2y_{41}, \\
& y_{19}y_{23} - y_2y_{40}, y_{18}y_{23} - y_2y_{39}, y_{17}y_{23} - y_2y_{38}, y_{16}y_{23} - y_2y_{37}, y_{15}y_{23} - y_2y_{36}, \\
& y_{14}y_{23} - y_7y_{36} - y_2y_{37}, y_{13}y_{23} - y_2y_{34}, y_{12}y_{23} - y_2y_{33}, y_{10}y_{23} - y_1y_{36} - y_7y_{37}, \\
& y_9y_{23} - y_7y_{34}, y_8y_{23} - y_7y_{33}, y_7y_{23} - y_2y_{28}, y_6y_{23} - y_{15}y_{36}, \\
& y_5y_{23} - y_{12}y_{40} - y_7y_{42}, y_4y_{23} - y_{13}y_{34}, y_3y_{23} - y_{12}y_{33}, y_1y_{23} - y_7y_{28}, \\
& y_{21}y_{22} - y_1y_{42}, y_{20}y_{22} - y_1y_{41}, y_{19}y_{22} - y_1y_{40}, y_{18}y_{22} - y_1y_{39}, y_{17}y_{22} - y_1y_{38}, \\
& y_{16}y_{22} - y_1y_{37}, y_{15}y_{22} - y_1y_{36}, y_{14}y_{22} - y_1y_{35}, y_{13}y_{22} - y_1y_{34}, y_{12}y_{22} - y_1y_{33}, \\
& y_{10}y_{22} - y_1y_{31}, y_9y_{22} - y_1y_{30}, y_8y_{22} - y_1y_{29}, y_7y_{22} - y_1y_{28}, y_6y_{22} + y_8y_{41} - y_7y_{42}, \\
& y_5y_{22} - y_{10}y_{31}, y_4y_{22} - y_9y_{30}, y_3y_{22} - y_8y_{29}, y_2y_{22} - y_7y_{28}
\end{aligned}$$

$$\begin{aligned}
I_Q = & \left(\begin{aligned}
& y_{32} - y_{35} + y_{37}, y_{11} - y_{14} + y_{16}, y_{40}y_{41} - y_{25}y_{42}, y_{38}y_{41} - y_{37}y_{42}, \\
& y_{35}y_{41} - y_{34}y_{42}, y_{31}y_{41} - y_{30}y_{42}, y_{26}y_{41} - y_{40}y_{42}, \\
& y_{21}y_{41} - y_{20}y_{42}, y_{19}y_{41} - y_{4}y_{42}, y_{17}y_{41} - y_{16}y_{42}, \\
& y_{14}y_{41} - y_{13}y_{42}, y_{10}y_{41} - y_{9}y_{42}, y_{5}y_{41} - y_{19}y_{42}, \\
& y_{39}y_{40} - y_{37}y_{42}, y_{36}y_{40} - y_{34}y_{42}, y_{35}y_{40} - y_{37}y_{40} - y_{30}y_{42}, \\
& y_{27}y_{40} - y_{41}y_{42}, y_{21}y_{40} - y_{19}y_{42}, y_{20}y_{40} - y_{4}y_{42}, \\
& y_{18}y_{40} - y_{16}y_{42}, y_{15}y_{40} - y_{13}y_{42}, y_{14}y_{40} - y_{16}y_{40} - y_{9}y_{42}, \\
& y_6y_{40} - y_{20}y_{42}, y_{38}y_{39} - y_{24}y_{42}, y_{37}y_{39} - y_{24}y_{41}, y_{35}y_{39} - y_{33}y_{42}, \\
& y_{34}y_{39} - y_{33}y_{41}, y_{31}y_{39} - y_{29}y_{42}, y_{30}y_{39} - y_{29}y_{41}, y_{26}y_{39} - y_{38}y_{42}, \\
& y_{25}y_{39} - y_{37}y_{41}, y_{21}y_{39} - y_{18}y_{42}, y_{20}y_{39} - y_{18}y_{41}, \\
& y_{19}y_{39} - y_{16}y_{42}, y_{17}y_{39} - y_{3}y_{42}, y_{16}y_{39} - y_{3}y_{41}, \\
& y_{14}y_{39} - y_{12}y_{42}, y_{13}y_{39} - y_{12}y_{41}, y_{10}y_{39} - y_8y_{42}, y_9y_{39} - y_8y_{41}, \\
& y_5y_{39} - y_{17}y_{42}, y_4y_{39} - y_{16}y_{41}, y_{37}y_{38} - y_{24}y_{40}, y_{36}y_{38} - y_{33}y_{42}, \\
& y_{35}y_{38} - y_{24}y_{40} - y_{29}y_{42}, y_{34}y_{38} - y_{33}y_{40}, y_{30}y_{38} - y_{29}y_{40}, \\
& y_{27}y_{38} - y_{39}y_{42}, y_{25}y_{38} - y_{37}y_{40}, y_{21}y_{38} - y_{17}y_{42}, \\
& y_{20}y_{38} - y_{16}y_{42}, y_{19}y_{38} - y_{17}y_{40}, y_{18}y_{38} - y_3y_{42}, y_{16}y_{38} - y_3y_{40}, \\
& y_{15}y_{38} - y_{12}y_{42}, y_{14}y_{38} - y_3y_{40} - y_8y_{42}, y_{13}y_{38} - y_{12}y_{40}, y_9y_{38} - y_8y_{40}, \\
& y_6y_{38} - y_{18}y_{42}, y_4y_{38} - y_{16}y_{40}, y_{37}^2 - y_{33}y_{40} + y_{29}y_{41}, \\
& y_{36}y_{37} - y_{33}y_{41}, y_{35}y_{37} - y_{33}y_{40}, y_{34}y_{37} - y_{23}y_{40} + y_{28}y_{41}, \\
& y_{33}y_{37} - y_{23}y_{38} + y_{28}y_{39}, y_{31}y_{37} - y_{29}y_{40}, y_{30}y_{37} - y_{28}y_{40} + y_{22}y_{41}, \\
& y_{29}y_{37} - y_{28}y_{38} + y_{22}y_{39}, y_{27}y_{37} - y_{39}y_{41}, y_{26}y_{37} - y_{38}y_{40}, \\
& y_{25}y_{37} - y_{34}y_{40} + y_{30}y_{41}, y_{24}y_{37} - y_{33}y_{38} + y_{29}y_{39}, y_{21}y_{37} - y_{16}y_{42}, \\
& y_{20}y_{37} - y_{16}y_{41}, y_{19}y_{37} - y_{16}y_{40}, y_{18}y_{37} - y_3y_{41}, y_{17}y_{37} - y_3y_{40}, \\
& y_{16}y_{37} - y_{12}y_{40} + y_8y_{41}, y_{15}y_{37} - y_{12}y_{41}, y_{14}y_{37} - y_{12}y_{40}, \\
& y_{13}y_{37} - y_2y_{40} + y_7y_{41}, y_{12}y_{37} - y_2y_{38} + y_7y_{39}, y_{10}y_{37} - y_8y_{40}, \\
& y_9y_{37} - y_7y_{40} + y_1y_{41}, y_8y_{37} - y_7y_{38} + y_1y_{39}, y_6y_{37} - y_{18}y_{41}, \\
& y_5y_{37} - y_{17}y_{40}, y_4y_{37} - y_{13}y_{40} + y_9y_{41}, y_3y_{37} - y_{12}y_{38} + y_8y_{39}, \\
& y_{35}y_{36} - y_{23}y_{42}, y_{34}y_{36} - y_{23}y_{41}, y_{33}y_{36} - y_{23}y_{39}, y_{31}y_{36} - y_{28}y_{42}, \\
& y_{30}y_{36} - y_{28}y_{41}, y_{29}y_{36} - y_{28}y_{39}, y_{26}y_{36} - y_{35}y_{42}, \\
& y_{25}y_{36} - y_{34}y_{41}, y_{24}y_{36} - y_{33}y_{39}, y_{21}y_{36} - y_{15}y_{42}, \\
& y_{20}y_{36} - y_{15}y_{41}, y_{19}y_{36} - y_{13}y_{42}, y_{18}y_{36} - y_{15}y_{39}, y_{17}y_{36} - y_{12}y_{42}, \\
& y_{16}y_{36} - y_{12}y_{41}, y_{14}y_{36} - y_2y_{42}, y_{13}y_{36} - y_2y_{41}, y_{12}y_{36} - y_2y_{39}, \\
& y_{10}y_{36} - y_7y_{42}, y_9y_{36} - y_7y_{41}, y_8y_{36} - y_7y_{39}, y_5y_{36} - y_{14}y_{42}, \\
& y_4y_{36} - y_{13}y_{41}, y_3y_{36} - y_{12}y_{39}, y_{35}^2 - y_{33}y_{40} - y_{28}y_{42}, \\
& y_{34}y_{35} - y_{23}y_{40}, y_{33}y_{35} - y_{23}y_{38}, y_{31}y_{35} - y_{29}y_{40} - y_{22}y_{42}, \\
& y_{30}y_{35} - y_{28}y_{40}, y_{29}y_{35} - y_{28}y_{38}, y_{28}y_{35} - y_{22}y_{36} - y_{28}y_{37}, \\
& y_{27}y_{35} - y_{36}y_{42}, y_{26}y_{35} - y_{38}y_{40} - y_{31}y_{42}, y_{25}y_{35} - y_{34}y_{40},
\end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
& y_{24}y_{35} - y_{33}y_{38}, y_{23}y_{35} - y_{28}y_{36} - y_{23}y_{37}, y_{21}y_{35} - y_{14}y_{42}, \\
& y_{20}y_{35} - y_{13}y_{42}, y_{19}y_{35} - y_{16}y_{40} - y_{9}y_{42}, y_{18}y_{35} - y_{12}y_{42}, \\
y_{17}y_{35} - y_{3}y_{40} - y_{8}y_{42}, y_{16}y_{35} - y_{12}y_{40}, y_{15}y_{35} - y_{2}y_{42}, y_{14}y_{35} - y_{12}y_{40} - y_{7}y_{42}, \\
& y_{13}y_{35} - y_{2}y_{40}, y_{12}y_{35} - y_{2}y_{38}, y_{10}y_{35} - y_{8}y_{40} - y_{1}y_{42}, y_{9}y_{35} - y_{7}y_{40}, \\
& y_{8}y_{35} - y_{7}y_{38}, y_{7}y_{35} - y_{1}y_{36} - y_{7}y_{37}, y_{6}y_{35} - y_{15}y_{42}, \\
& y_{5}y_{35} - y_{17}y_{40} - y_{10}y_{42}, y_{4}y_{35} - y_{13}y_{40}, y_{3}y_{35} - y_{12}y_{38}, \\
& y_{2}y_{35} - y_{7}y_{36} - y_{2}y_{37}, y_{33}y_{34} - y_{23}y_{37}, y_{31}y_{34} - y_{28}y_{40}, \\
& y_{29}y_{34} - y_{28}y_{37}, y_{27}y_{34} - y_{36}y_{41}, y_{26}y_{34} - y_{37}y_{40} - y_{30}y_{42}, \\
y_{24}y_{34} - y_{23}y_{38} + y_{28}y_{39}, y_{21}y_{34} - y_{13}y_{42}, y_{20}y_{34} - y_{13}y_{41}, y_{19}y_{34} - y_{13}y_{40}, \\
& y_{18}y_{34} - y_{12}y_{41}, y_{17}y_{34} - y_{12}y_{40}, y_{16}y_{34} - y_{2}y_{40} + y_{7}y_{41}, y_{15}y_{34} - y_{2}y_{41}, \\
& y_{14}y_{34} - y_{2}y_{40}, y_{12}y_{34} - y_{2}y_{37}, y_{10}y_{34} - y_{7}y_{40}, y_{8}y_{34} - y_{7}y_{37}, \\
y_{6}y_{34} - y_{15}y_{41}, y_{5}y_{34} - y_{16}y_{40} - y_{9}y_{42}, y_{3}y_{34} - y_{2}y_{38} + y_{7}y_{39}, y_{31}y_{33} - y_{28}y_{38}, \\
& y_{30}y_{33} - y_{28}y_{37}, y_{27}y_{33} - y_{36}y_{39}, y_{26}y_{33} - y_{24}y_{40} - y_{29}y_{42}, \\
& y_{25}y_{33} - y_{23}y_{40} + y_{28}y_{41}, y_{21}y_{33} - y_{12}y_{42}, y_{20}y_{33} - y_{12}y_{41}, \\
& y_{19}y_{33} - y_{12}y_{40}, y_{18}y_{33} - y_{12}y_{39}, y_{17}y_{33} - y_{12}y_{38}, \\
y_{16}y_{33} - y_{2}y_{38} + y_{7}y_{39}, y_{15}y_{33} - y_{2}y_{39}, y_{14}y_{33} - y_{2}y_{38}, y_{13}y_{33} - y_{2}y_{37}, \\
& y_{10}y_{33} - y_{7}y_{38}, y_{9}y_{33} - y_{7}y_{37}, y_{6}y_{33} - y_{15}y_{39}, y_{5}y_{33} - y_{3}y_{40} - y_{8}y_{42}, \\
y_{4}y_{33} - y_{2}y_{40} + y_{7}y_{41}, y_{30}y_{31} - y_{22}y_{40}, y_{29}y_{31} - y_{22}y_{38}, y_{28}y_{31} - y_{22}y_{35}, \\
& y_{27}y_{31} - y_{35}y_{42} + y_{37}y_{42}, y_{25}y_{31} - y_{30}y_{40}, y_{24}y_{31} - y_{29}y_{38}, \\
& y_{23}y_{31} - y_{22}y_{36} - y_{28}y_{37}, y_{21}y_{31} - y_{10}y_{42}, y_{20}y_{31} - y_{9}y_{42}, \\
& y_{19}y_{31} - y_{10}y_{40}, y_{18}y_{31} - y_{8}y_{42}, y_{17}y_{31} - y_{10}y_{38}, \\
& y_{16}y_{31} - y_{8}y_{40}, y_{15}y_{31} - y_{7}y_{42}, y_{14}y_{31} - y_{8}y_{40} - y_{1}y_{42}, \\
& y_{13}y_{31} - y_{7}y_{40}, y_{12}y_{31} - y_{7}y_{38}, y_{9}y_{31} - y_{1}y_{40}, y_{8}y_{31} - y_{1}y_{38}, \\
y_{7}y_{31} - y_{1}y_{35}, y_{6}y_{31} - y_{14}y_{42} + y_{16}y_{42}, y_{4}y_{31} - y_{9}y_{40}, y_{3}y_{31} - y_{8}y_{38}, \\
& y_{2}y_{31} - y_{1}y_{36} - y_{7}y_{37}, y_{29}y_{30} - y_{22}y_{37}, y_{28}y_{30} - y_{22}y_{34}, \\
& y_{27}y_{30} + y_{37}y_{41} - y_{34}y_{42}, y_{26}y_{30} - y_{31}y_{40}, y_{24}y_{30} - y_{28}y_{38} + y_{22}y_{39}, \\
& y_{23}y_{30} - y_{28}y_{34}, y_{21}y_{30} - y_{9}y_{42}, y_{20}y_{30} - y_{9}y_{41}, y_{19}y_{30} - y_{9}y_{40}, \\
y_{18}y_{30} - y_{8}y_{41}, y_{17}y_{30} - y_{8}y_{40}, y_{16}y_{30} - y_{7}y_{40} + y_{1}y_{41}, y_{15}y_{30} - y_{7}y_{41}, \\
& y_{14}y_{30} - y_{7}y_{40}, y_{13}y_{30} - y_{9}y_{34}, y_{12}y_{30} - y_{7}y_{37}, y_{10}y_{30} - y_{1}y_{40}, \\
& y_{8}y_{30} - y_{1}y_{37}, y_{7}y_{30} - y_{1}y_{34}, y_{6}y_{30} + y_{16}y_{41} - y_{13}y_{42}, y_{5}y_{30} - y_{10}y_{40}, \\
y_{3}y_{30} - y_{7}y_{38} + y_{1}y_{39}, y_{2}y_{30} - y_{7}y_{34}, y_{28}y_{29} - y_{22}y_{33}, y_{27}y_{29} + y_{24}y_{41} - y_{33}y_{42}, \\
& y_{26}y_{29} - y_{31}y_{38}, y_{25}y_{29} - y_{28}y_{40} + y_{22}y_{41}, y_{23}y_{29} - y_{28}y_{33}, \\
& y_{21}y_{29} - y_{8}y_{42}, y_{20}y_{29} - y_{8}y_{41}, y_{19}y_{29} - y_{8}y_{40}, y_{18}y_{29} - y_{8}y_{39}, \\
& y_{17}y_{29} - y_{8}y_{38}, y_{16}y_{29} - y_{7}y_{38} + y_{1}y_{39}, y_{15}y_{29} - y_{7}y_{39}, \\
& y_{14}y_{29} - y_{7}y_{38}, y_{13}y_{29} - y_{7}y_{37}, y_{12}y_{29} - y_{8}y_{33}, y_{10}y_{29} - y_{1}y_{38}, \\
y_{9}y_{29} - y_{1}y_{37}, y_{7}y_{29} - y_{1}y_{33}, y_{6}y_{29} + y_{3}y_{41} - y_{12}y_{42}, y_{5}y_{29} - y_{10}y_{38}, \\
& y_{4}y_{29} - y_{7}y_{40} + y_{1}y_{41}, y_{2}y_{29} - y_{7}y_{33}, y_{27}y_{28} + y_{33}y_{41} - y_{23}y_{42},
\end{aligned}$$

$$\begin{aligned}
& y_{26}y_{28} - y_{29}y_{40} - y_{22}y_{42}, y_{25}y_{28} - y_{30}y_{34}, y_{24}y_{28} - y_{29}y_{33}, \\
& y_{21}y_{28} - y_{7}y_{42}, y_{20}y_{28} - y_{7}y_{41}, y_{19}y_{28} - y_{7}y_{40}, y_{18}y_{28} - y_{7}y_{39}, \\
& y_{17}y_{28} - y_{7}y_{38}, y_{16}y_{28} - y_{7}y_{37}, y_{15}y_{28} - y_{7}y_{36}, \\
& y_{14}y_{28} - y_{1}y_{36} - y_{7}y_{37}, y_{13}y_{28} - y_{7}y_{34}, y_{12}y_{28} - y_{7}y_{33}, \\
& y_{10}y_{28} - y_{1}y_{35}, y_{9}y_{28} - y_{1}y_{34}, y_{8}y_{28} - y_{1}y_{33}, y_{6}y_{28} + y_{12}y_{41} - y_{2}y_{42}, \\
& y_{5}y_{28} - y_{8}y_{40} - y_{1}y_{42}, y_{4}y_{28} - y_{9}y_{34}, y_{3}y_{28} - y_{8}y_{33}, y_{26}y_{27} - y_{42}^2, \\
& y_{25}y_{27} - y_{41}^2, y_{24}y_{27} - y_{39}^2, y_{23}y_{27} - y_{36}^2, y_{22}y_{27} + y_{29}y_{41} - y_{28}y_{42}, \\
& y_{21}y_{27} - y_{6}y_{42}, y_{20}y_{27} - y_{6}y_{41}, y_{19}y_{27} - y_{20}y_{42}, y_{18}y_{27} - y_{6}y_{39}, \\
& y_{17}y_{27} - y_{18}y_{42}, y_{16}y_{27} - y_{18}y_{41}, y_{15}y_{27} - y_{6}y_{36}, y_{14}y_{27} - y_{15}y_{42}, \\
& y_{13}y_{27} - y_{15}y_{41}, y_{12}y_{27} - y_{15}y_{39}, y_{10}y_{27} - y_{14}y_{42} + y_{16}y_{42}, \\
& y_{9}y_{27} + y_{16}y_{41} - y_{13}y_{42}, y_{8}y_{27} + y_{3}y_{41} - y_{12}y_{42}, y_{7}y_{27} + y_{12}y_{41} - y_{2}y_{42}, \\
& y_{5}y_{27} - y_{21}y_{42}, y_{4}y_{27} - y_{20}y_{41}, y_{3}y_{27} - y_{18}y_{39}, y_{2}y_{27} - y_{15}y_{36}, \\
& y_{1}y_{27} + y_{8}y_{41} - y_{7}y_{42}, y_{25}y_{26} - y_{40}^2, y_{24}y_{26} - y_{38}^2, y_{23}y_{26} - y_{33}y_{40} - y_{28}y_{42}, \\
& y_{22}y_{26} - y_{31}^2, y_{21}y_{26} - y_{5}y_{42}, y_{20}y_{26} - y_{19}y_{42}, y_{19}y_{26} - y_{5}y_{40}, \\
& y_{18}y_{26} - y_{17}y_{42}, y_{17}y_{26} - y_{5}y_{38}, y_{16}y_{26} - y_{17}y_{40}, y_{15}y_{26} - y_{14}y_{42}, \\
& y_{14}y_{26} - y_{17}y_{40} - y_{10}y_{42}, y_{13}y_{26} - y_{16}y_{40} - y_{9}y_{42}, \\
& y_{12}y_{26} - y_{3}y_{40} - y_{8}y_{42}, y_{10}y_{26} - y_{5}y_{31}, y_{9}y_{26} - y_{10}y_{40}, y_{8}y_{26} - y_{10}y_{38}, \\
& y_{7}y_{26} - y_{8}y_{40} - y_{1}y_{42}, y_{6}y_{26} - y_{21}y_{42}, y_{4}y_{26} - y_{19}y_{40}, y_{3}y_{26} - y_{17}y_{38}, \\
& y_{2}y_{26} - y_{12}y_{40} - y_{7}y_{42}, y_{1}y_{26} - y_{10}y_{31}, y_{24}y_{25} - y_{33}y_{40} + y_{29}y_{41}, \\
& y_{23}y_{25} - y_{34}^2, y_{22}y_{25} - y_{30}^2, y_{21}y_{25} - y_{4}y_{42}, y_{20}y_{25} - y_{4}y_{41}, \\
& y_{19}y_{25} - y_{4}y_{40}, y_{18}y_{25} - y_{16}y_{41}, y_{17}y_{25} - y_{16}y_{40}, \\
& y_{16}y_{25} - y_{13}y_{40} + y_{9}y_{41}, y_{15}y_{25} - y_{13}y_{41}, y_{14}y_{25} - y_{13}y_{40}, \\
& y_{13}y_{25} - y_{4}y_{34}, y_{12}y_{25} - y_{2}y_{40} + y_{7}y_{41}, y_{10}y_{25} - y_{9}y_{40}, \\
& y_{9}y_{25} - y_{4}y_{30}, y_{8}y_{25} - y_{7}y_{40} + y_{1}y_{41}, y_{7}y_{25} - y_{9}y_{34}, y_{6}y_{25} - y_{20}y_{41}, \\
& y_{5}y_{25} - y_{19}y_{40}, y_{3}y_{25} - y_{12}y_{40} + y_{8}y_{41}, y_{2}y_{25} - y_{13}y_{34}, y_{1}y_{25} - y_{9}y_{30}, \\
& y_{23}y_{24} - y_{33}^2, y_{22}y_{24} - y_{29}^2, y_{21}y_{24} - y_{3}y_{42}, y_{20}y_{24} - y_{3}y_{41}, \\
& y_{19}y_{24} - y_{3}y_{40}, y_{18}y_{24} - y_{3}y_{39}, y_{17}y_{24} - y_{3}y_{38}, y_{16}y_{24} - y_{12}y_{38} + y_{8}y_{39}, \\
& y_{15}y_{24} - y_{12}y_{39}, y_{14}y_{24} - y_{12}y_{38}, y_{13}y_{24} - y_{2}y_{38} + y_{7}y_{39}, \\
& y_{12}y_{24} - y_{3}y_{33}, y_{10}y_{24} - y_{8}y_{38}, y_{9}y_{24} - y_{7}y_{38} + y_{1}y_{39}, y_{8}y_{24} - y_{3}y_{29}, \\
& y_{7}y_{24} - y_{8}y_{33}, y_{6}y_{24} - y_{18}y_{39}, y_{5}y_{24} - y_{17}y_{38}, y_{4}y_{24} - y_{12}y_{40} + y_{8}y_{41}, \\
& y_{2}y_{24} - y_{12}y_{33}, y_{1}y_{24} - y_{8}y_{29}, y_{22}y_{23} - y_{28}^2, y_{21}y_{23} - y_{2}y_{42}, \\
& y_{20}y_{23} - y_{2}y_{41}, y_{19}y_{23} - y_{2}y_{40}, y_{18}y_{23} - y_{2}y_{39}, y_{17}y_{23} - y_{2}y_{38}, \\
& y_{16}y_{23} - y_{2}y_{37}, y_{15}y_{23} - y_{2}y_{36}, y_{14}y_{23} - y_{7}y_{36} - y_{2}y_{37}, y_{13}y_{23} - y_{2}y_{34}, \\
& y_{12}y_{23} - y_{2}y_{33}, y_{10}y_{23} - y_{1}y_{36} - y_{7}y_{37}, y_{9}y_{23} - y_{7}y_{34}, y_{8}y_{23} - y_{7}y_{33}, \\
& y_{7}y_{23} - y_{2}y_{28}, y_{6}y_{23} - y_{15}y_{36}, y_{5}y_{23} - y_{12}y_{40} - y_{7}y_{42}, y_{4}y_{23} - y_{13}y_{34}, \\
& y_{3}y_{23} - y_{12}y_{33}, y_{1}y_{23} - y_{7}y_{28}, y_{21}y_{22} - y_{1}y_{42}, y_{20}y_{22} - y_{1}y_{41}, \\
& y_{19}y_{22} - y_{1}y_{40}, y_{18}y_{22} - y_{1}y_{39}, y_{17}y_{22} - y_{1}y_{38}, y_{16}y_{22} - y_{1}y_{37},
\end{aligned}$$

$$\begin{aligned}
& y_{15}y_{22} - y_1y_{36}, y_{14}y_{22} - y_1y_{35}, y_{13}y_{22} - y_1y_{34}, y_{12}y_{22} - y_1y_{33}, \\
& y_{10}y_{22} - y_1y_{31}, y_9y_{22} - y_1y_{30}, y_8y_{22} - y_1y_{29}, y_7y_{22} - y_1y_{28}, \\
& y_6y_{22} + y_8y_{41} - y_7y_{42}, y_5y_{22} - y_{10}y_{31}, y_4y_{22} - y_9y_{30}, y_3y_{22} - y_8y_{29}, \\
& y_2y_{22} - y_7y_{28}, y_{19}y_{20} - y_4y_{21}, y_{17}y_{20} - y_{16}y_{21}, y_{14}y_{20} - y_{13}y_{21}, \\
& y_{10}y_{20} - y_9y_{21}, y_5y_{20} - y_{19}y_{21}, y_{18}y_{19} - y_{16}y_{21}, y_{15}y_{19} - y_{13}y_{21}, \\
& y_{14}y_{19} - y_{16}y_{19} - y_9y_{21}, y_6y_{19} - y_{20}y_{21}, y_{17}y_{18} - y_3y_{21}, \\
& y_{16}y_{18} - y_3y_{20}, y_{14}y_{18} - y_{12}y_{21}, y_{13}y_{18} - y_{12}y_{20}, y_{10}y_{18} - y_8y_{21}, \\
& y_9y_{18} - y_8y_{20}, y_5y_{18} - y_{17}y_{21}, y_4y_{18} - y_{16}y_{20}, y_{16}y_{17} - y_3y_{19}, \\
& y_{15}y_{17} - y_{12}y_{21}, y_{14}y_{17} - y_3y_{19} - y_8y_{21}, y_{13}y_{17} - y_{12}y_{19}, y_9y_{17} - y_8y_{19}, \\
& y_6y_{17} - y_{18}y_{21}, y_4y_{17} - y_{16}y_{19}, y_{16}^2 - y_{12}y_{19} + y_8y_{20}, y_{15}y_{16} - y_{12}y_{20}, \\
& y_{14}y_{16} - y_{12}y_{19}, y_{13}y_{16} - y_2y_{19} + y_7y_{20}, y_{12}y_{16} - y_2y_{17} + y_7y_{18}, \\
& y_{10}y_{16} - y_8y_{19}, y_9y_{16} - y_7y_{19} + y_1y_{20}, y_8y_{16} - y_7y_{17} + y_1y_{18}, \\
& y_6y_{16} - y_{18}y_{20}, y_5y_{16} - y_{17}y_{19}, y_4y_{16} - y_{13}y_{19} + y_9y_{20}, \\
& y_3y_{16} - y_{12}y_{17} + y_8y_{18}, y_{14}y_{15} - y_2y_{21}, y_{13}y_{15} - y_2y_{20}, y_{12}y_{15} - y_2y_{18}, \\
& y_{10}y_{15} - y_7y_{21}, y_9y_{15} - y_7y_{20}, y_8y_{15} - y_7y_{18}, y_5y_{15} - y_{14}y_{21}, y_4y_{15} - y_{13}y_{20}, \\
& y_3y_{15} - y_{12}y_{18}, y_{14}^2 - y_{12}y_{19} - y_7y_{21}, y_{13}y_{14} - y_2y_{19}, y_{12}y_{14} - y_2y_{17}, \\
& y_{10}y_{14} - y_8y_{19} - y_1y_{21}, y_9y_{14} - y_7y_{19}, y_8y_{14} - y_7y_{17}, y_7y_{14} - y_1y_{15} - y_7y_{16}, \\
& y_6y_{14} - y_{15}y_{21}, y_5y_{14} - y_{17}y_{19} - y_{10}y_{21}, y_4y_{14} - y_{13}y_{19}, y_3y_{14} - y_{12}y_{17}, \\
& y_2y_{14} - y_7y_{15} - y_2y_{16}, y_{12}y_{13} - y_2y_{16}, y_{10}y_{13} - y_7y_{19}, y_8y_{13} - y_7y_{16}, \\
& y_6y_{13} - y_{15}y_{20}, y_5y_{13} - y_{16}y_{19} - y_9y_{21}, y_3y_{13} - y_2y_{17} + y_7y_{18}, \\
& y_{10}y_{12} - y_7y_{17}, y_9y_{12} - y_7y_{16}, y_6y_{12} - y_{15}y_{18}, y_5y_{12} - y_3y_{19} - y_8y_{21}, \\
& y_4y_{12} - y_2y_{19} + y_7y_{20}, y_9y_{10} - y_1y_{19}, y_8y_{10} - y_1y_{17}, y_7y_{10} - y_1y_{14}, \\
& y_6y_{10} - y_{14}y_{21} + y_{16}y_{21}, y_4y_{10} - y_9y_{19}, y_3y_{10} - y_8y_{17}, y_2y_{10} - y_1y_{15} - y_7y_{16}, \\
& y_8y_9 - y_1y_{16}, y_7y_9 - y_1y_{13}, y_6y_9 + y_{16}y_{20} - y_{13}y_{21}, y_5y_9 - y_{10}y_{19}, \\
& y_3y_9 - y_7y_{17} + y_1y_{18}, y_2y_9 - y_7y_{13}, y_7y_8 - y_1y_{12}, y_6y_8 + y_3y_{20} - y_{12}y_{21}, \\
& y_5y_8 - y_{10}y_{17}, y_4y_8 - y_7y_{19} + y_1y_{20}, y_2y_8 - y_7y_{12}, y_6y_7 + y_{12}y_{20} - y_2y_{21}, \\
& y_5y_7 - y_8y_{19} - y_1y_{21}, y_4y_7 - y_9y_{13}, y_3y_7 - y_8y_{12}, y_5y_6 - y_{21}^2, y_4y_6 - y_{20}^2, \\
& y_3y_6 - y_{18}^2, y_2y_6 - y_{15}^2, y_1y_6 + y_8y_{20} - y_7y_{21}, y_4y_5 - y_{19}^2, y_3y_5 - y_{17}^2, \\
& y_2y_5 - y_{12}y_{19} - y_7y_{21}, y_1y_5 - y_{10}^2, y_3y_4 - y_{12}y_{19} + y_8y_{20}, \\
& y_2y_4 - y_{13}^2, y_1y_4 - y_9^2, y_2y_3 - y_{12}^2, y_1y_3 - y_8^2, y_1y_2 - y_7^2
\end{aligned}$$

We present Macaulay2 code for computing I_Q .

```

i1: R = QQ[x_1..x_6,t_0..t_2, h_0..h_2, y_1..y_42, MonomialOrder => Eliminate 12 ]
i2: G = gens gb K+I
i3: J = selectInSubring(1,G)
i4: IQ = ideal(J)

```

where

$$K = \left(\begin{array}{l} y_1 - t_1x_1^2h_1, y_2 - t_1x_2^2h_1, y_3 - t_1x_3^2h_1, y_4 - t_1x_4^2h_1, y_5 - t_1x_5^2h_1, \\ y_6 - t_1x_6^2h_1, y_7 - t_1x_1x_2h_1, y_8 - t_1x_1x_3h_1, y_9 - t_1x_1x_4h_1, y_{10} - t_1x_1x_5h_1, \\ y_{11} - t_1x_1x_6h_1, y_{12} - t_1x_2x_3h_1, y_{13} - t_1x_2x_4h_1, y_{14} - t_1x_2x_5h_1, \\ y_{15} - t_1x_2x_6h_1, y_{16} - t_1x_3x_4h_1, y_{17} - t_1x_3x_5h_1, y_{18} - t_1x_3x_6h_1, \\ y_{19} - t_1x_4x_5h_1, y_{20} - t_1x_4x_6h_1, y_{21} - t_1x_5x_6h_1, y_{22} - t_2x_1^2h_2, \\ y_{23} - t_2x_2^2h_2, y_{24} - t_2x_3^2h_2, y_{25} - t_2x_4^2h_2, y_{26} - t_2x_5^2h_2, \\ y_{27} - t_2x_6^2h_2, y_{28} - t_2x_1x_2h_2, y_{29} - t_2x_1x_3h_2, y_{30} - t_2x_1x_4h_2, \\ y_{31} - t_2x_1x_5h_2, y_{32} - t_2x_1x_6h_2, y_{33} - t_2x_2x_3h_2, y_{34} - t_2x_2x_4h_2, \\ y_{35} - t_2x_2x_5h_2, y_{36} - t_2x_2x_6h_2, y_{37} - t_2x_3x_4h_2, y_{38} - t_2x_3x_5h_2, \\ y_{39} - t_2x_3x_6h_2, y_{40} - t_2x_4x_5h_2, y_{41} - t_2x_4x_6h_2, y_{42} - t_2x_5x_6h_2 \end{array} \right)$$

and

$$I = \left(x_3x_4 - x_2x_5 + x_1x_6, t_0h_0 - 1, t_1h_1 - 1, t_2h_2 - 1 \right).$$

In Macaulay2, we calculate the saturation of I_R and I_Q with B_Y using the command "saturate",

```
i1: IQQ = saturate(IQ,BY)
i2: IRR = saturate(IR,BY)
i3: IRR == IQQ
o3: true
```

where B_Y is the intersection of

$$(y_1, \dots, y_{21}) \text{ and } (y_{22}, \dots, y_{42}).$$

It is also possible to calculate \widetilde{I}_R and \widetilde{I}_Q and that $\widetilde{I}_Q = \widetilde{I}_R : B_Y^\infty$ but we omit the calculations here. By our Macaulay2 calculation, the collection \mathcal{L} is very ample, so Theorem 4.1.1 implies that $\varphi_{|\mathcal{L}|} : \text{Gr}(2, 4) \longrightarrow \mathcal{M}_\vartheta(A_{\mathcal{L}})$ is an isomorphism.

Chapter 7

Appendix A: Computing Kernels of \mathbb{k} -Algebra Homomorphisms

7.0.1 Kernels of \mathbb{k} -Algebra Homomorphisms.

In order to calculate the Mori Dream Space analogue of I_Q from [10], we will need to be able to compute kernels of \mathbb{k} -algebra homomorphisms efficiently. Theorem 7.0.2 gives us a way to write kernels, then using Elimination Theory we can compute kernels using Macaulay2.

7.0.2 Kernels

Material from this section can be found in Adams–Loustaunau [1]. Let $\varphi : \mathbb{k}[y_1, \dots, y_m] \rightarrow \mathbb{k}[x_1, \dots, x_n]$ be the \mathbb{k} algebra homomorphism mapping y_i to some $f_i(x_1, \dots, x_n) \in \mathbb{k}[x_1, \dots, x_n]$ for each i . We want to compute $\ker(\varphi)$. First we need a technical lemma.

Lemma 7.0.1. *Let R be a commutative ring. If $a_1, \dots, a_n, b_1, \dots, b_n \in R$, then $a_1 \cdots a_n - b_1 \cdots b_n$ is contained in the ideal*

$$(a_1 - b_1, \dots, a_n - b_n).$$

Proof. $a_1 \cdots a_n - b_1 \cdots b_n = a_1(a_2 \cdots a_n - b_2 \cdots b_n) + b_2 \cdots b_n(a_1 - b_1)$, hence by induction $a_1 \cdots a_n - b_1 \cdots b_n$ can be written as $\sum g_i(a_i - b_i)$, for $g_i \in R$. \square

Now we are able to prove Theorem 7.0.2.

Theorem 7.0.2. *Let $K = (y_1 - f_1, \dots, y_m - f_m) \subseteq \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m]$. The kernel of φ satisfies,*

$$\ker(\varphi) = K \cap \mathbb{k}[y_1, \dots, y_m].$$

Proof. First let $g \in K \cap \mathbb{k}[y_1, \dots, y_m]$, we will show $g \in \ker(\varphi)$. Since $g \in K$ and $g \in \mathbb{k}[y_1, \dots, y_m]$, we must have

$$g(y_1, \dots, y_m) = \sum_{i=1}^n (y_i - f_i(x_1, \dots, x_n)) h_i(y, x)$$

for some $h_i \in \mathbb{k}[x_1, \dots, x_m, y_1, \dots, y_m]$. Hence the image of g under φ is

$$g(f_1, \dots, f_m) = \sum_{i=1}^n (f_i(x_1, \dots, x_n) - f_i(x_1, \dots, x_n)) h_i = 0,$$

and therefore $g \in \ker(\varphi)$.

Conversely, let $g \in \ker(\varphi)$, we can write

$$g = \sum c_{\mathbf{v}} y^{\mathbf{v}}$$

for some $\mathbf{v} \in \mathbb{N}^m$, $c_{\mathbf{v}} \in \mathbb{k}$.

$$g(f_1, \dots, f_m) = 0 \Rightarrow g = g - g(f_1, \dots, f_m) = \sum c_{\mathbf{v}} (y^{\mathbf{v}} - f^{\mathbf{v}})$$

By the lemma, this shows g is in the ideal K . □

Corollary 7.0.3. *Let $\varphi : \mathbb{k}[y_1, \dots, y_m] \longrightarrow \mathbb{k}[x_1, \dots, x_n]/I$ be the \mathbb{k} -algebra homomorphism mapping y_i to $f_i \in \mathbb{k}[x_1, \dots, x_n]/I$ for each i . The kernel of φ is*

$$K + I \cap \mathbb{k}[y_1, \dots, y_m]$$

where we consider

$$K = (y_1 - f_1, \dots, y_m - f_m)$$

and I to be ideals of $\mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m]$.

Proof. Let $g \in (K + I) \cap \mathbb{k}[y_1, \dots, y_m]$, then $g = h + j$, where $h \in K$ and $j \in I$. We can write $j = \sum c_{\mathbf{u}\mathbf{v}j} x^{\mathbf{u}} y^{\mathbf{v}} g_j$, where $g_j \in \mathbb{k}[x_1, \dots, x_n]$ is a generator of I . Hence

$$g(f_1, \dots, f_m) = h(f_1, \dots, f_m) + \sum c_{\mathbf{u}\mathbf{v}j} x^{\mathbf{u}} f^{\mathbf{v}} g_j$$

By Theorem 7.0.2,

Suppose $g \in \ker(\varphi) \subseteq \mathbb{k}[y_1, \dots, y_m]$, and hence $g(f_1, \dots, f_m) \in I \subseteq \mathbb{k}[x_1, \dots, x_n]$. We can write

$$g(y_1, \dots, y_m) = g(y_1 - f_1, \dots, y_m - f_m) + g(f_1, \dots, f_m) \in K + I.$$

□

7.0.3 Elimination Theory and Macaulay2 Calculations

Theorem 7.0.2 and Corollary 7.0.3 give us a way of writing kernels in a \mathbb{k} -algebra $\mathbb{k}[y_1, \dots, y_m]$ as the intersection of an ideal in a larger ring with $\mathbb{k}[y_1, \dots, y_m]$ considered as a subring. In order to compute these intersections, we need the Elimination Theorem. Following Cox–Little–O’Shea [5], we define the *kth elimination ideal* I_k of $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ to be

$$I \cap \mathbb{k}[x_{k+1}, \dots, x_n].$$

Theorem 7.0.4 (The Elimination Theorem). *Let $I \subseteq \mathbb{k}[x_1, \dots, x_n]$ be an ideal, and let G be a groebner basis of I with respect to lex order where $x_1 < x_2 < \dots < x_n$. Then, for every $k \leq n$, the set*

$$G_k = G \cap \mathbb{k}[x_1, \dots, x_n]$$

is a groebner basis of the kth elimination ideal I_k .

In Macaulay2, once we have computed the groebner basis of $K + I$ as in Corollary 7.0.3, we can compute the intersection with $\mathbb{k}[y_1, \dots, y_m]$ using the command “selectInSubring”. Explicitly:

```
i1: R = QQ[x_1..x_n,y_1..y_m, MonomialOrder => Eliminate n]
i2: K = ideal(y_1-f_1,...y_m-f_m)
i3: I = ideal(g_1,...g_k)
i4: G = gens gb K+I
i5: J = selectInSubring(1,G)
i6: kernel = ideal(J)
```

Chapter 8

Appendix B: Computing $\text{Cox}(X_5)$ after Batyrev–Popov and Derenthal

We give code used to compute I_{X_5} in section 2.2.3 following the method of Batyrev–Popov [2] and Derenthal [12].

We present Maple code for finding the equation of a conic containing 5 points in \mathbb{P}^2 .

Pseudocode 8.0.5. *Input: coordinates of five points*

$p = (p_1, p_2, p_3), q = (q_1, q_2, q_3), r = (r_1, r_2, r_3), s = (s_1, s_2, s_3), t = (t_1, t_2, t_3)$.

Procedure:

$L := [ap_1^2 + bp_2^2 + cp_3^2 + dp_1p_2 + ep_1p_3 + fp_2p_3,$
 $aq_1^2 + bq_2^2 + cq_3^2 + dq_1q_2 + eq_1q_3 + fq_2q_3,$
 $ar_1^2 + br_2^2 + cr_3^2 + dr_1r_2 + er_1r_3 + fr_2r_3,$
 $as_1^2 + bs_2^2 + cs_3^2 + ds_1s_2 + es_1s_3 + fs_2s_3,$
 $at_1^2 + bt_2^2 + ct_3^2 + dt_1t_2 + et_1t_3 + ft_2t_3]$

Let L_i denote the i th term of the list L .

$S := [L_1 = 0, L_2 = 0, L_3 = 0, L_4 = 0, L_5 = 0];$

$\text{solve}(S, [a, b, c, d, e, f]);$

Output: solution of S for a, \dots, f .

Proof. A general conic in three variables z_1, z_2, z_3 has the form

$$az_1^2 + bz_2^2 + cz_3^2 + dz_1z_2 + ez_1z_3 + fz_2z_3$$

for $a, b, c, d, e, f \in \mathbb{k}$. This procedure finds coefficients a, \dots, f for such a conic which contains p, \dots, t . This is because S contains the equations of the general conic above evaluated at p, \dots, t set equal to zero, and the Maple command “solve” solves the list of equations S for a, \dots, f . \square

We present lattice maps which induce the $\text{Pic}(X_r)$ grading of $\text{Cox}(X_r)$. We use these maps to calculate all monomials in $H^0(X_r, D)$ for a line bundle $D \in \text{Pic}(X_r)$. We give pseudocode for calculating these monomials when $r = 4$, the cases where $r = 5$ or 6 are similar.

For a Mori Dream Space X where $\text{Cox}(X) = \mathbb{k}[x_1, \dots, x_d]/I_X$ and $\text{Cl}(X) \cong \mathbb{Z}^\rho$, there exists a lattice map

$$\pi : \mathbb{Z}^d \longrightarrow \mathbb{Z}^\rho$$

which induces the grading map

$$\text{deg} : \text{Cox}(X) \longrightarrow \text{Cl}(X).$$

Sections of a line bundle D are therefore elements of $\text{deg}^{-1}(D)$. We give the lattice maps π_4, π_5 and π_6 which induce deg for X_4 and X_5 respectively:

$$\pi_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{pmatrix}$$

$$\pi_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 \end{pmatrix}$$

For X_6 , the matrix π_6 takes up too much space. So we write a list of the degrees in \mathbb{Z}^7 of the 27 variables of $\text{Cox}(X_6)$:

$$\begin{aligned} & \{0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 1\} \\ & \{1, -1, -1, 0, 0, 0, 0\}, \{1, -1, 0, -1, 0, 0, 0\}, \{1, -1, 0, 0, -1, 0, 0\}, \{1, -1, 0, 0, 0, -1, 0\}, \{1, -1, 0, 0, 0, 0, -1\} \\ & \{1, 0, -1, -1, 0, 0, 0\}, \{1, 0, -1, 0, -1, 0, 0\}, \{1, 0, -1, 0, 0, -1, 0\}, \{1, 0, -1, 0, 0, 0, -1\}, \{1, 0, 0, -1, -1, 0, 0\} \\ & \{1, 0, 0, -1, 0, -1, 0\}, \{1, 0, 0, -1, 0, 0, -1\}, \{1, 0, 0, 0, -1, -1, 0\}, \{1, 0, 0, 0, -1, 0, -1\}, \{1, 0, 0, 0, 0, -1, -1\} \\ & \{2, -1, -1, -1, -1, -1, 0\}, \{2, -1, -1, -1, -1, 0, -1\}, \{2, -1, -1, -1, 0, -1, -1\} \\ & \{2, -1, -1, 0, -1, -1, -1\}, \{2, -1, 0, -1, -1, -1, -1\}, \{2, 0, -1, -1, -1, -1, -1\} \end{aligned}$$

We present pseudocode for computing torus invariant sections of a line bundle L on X_4 (i.e. for finding elements of $\text{deg}^{-1}(L)$) and a proof of efficacy. The code for X_5 and X_6 is similar.

Pseudocode 8.0.6. We show how to compute the sections of a line bundle $a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4 \in \text{Pic}(X_4)$. Our method is to find all the elements of $\pi_4^{-1}(a_0, \dots, a_4)$. We let $L[i]$ denote the i th term in a list L , and $L[i][j]$ denote the j th term in $L[i]$, and let $|L|$ denote the number of elements of L .

Input: $L := [a_0, a_1, a_2, a_3, a_4]$

Procedure:

$L1 := []$: (i.e. the “empty list”)

for t_5 from 0 to a_0 do

for t_6 from 0 to a_0 do

for t_7 from 0 to a_0 do

for t_8 from 0 to a_0 do

for t_9 from 0 to a_0 do

for t_{10} from 0 to a_0 do

if $t_5 + t_6 + t_7 + t_8 + t_9 + t_{10} = l_0$ then $L1 := [L1, [t_5, t_6, t_7, t_8, t_9, t_{10}]]$:

$L2 := []$:

for i from 1 to $|L1|$ do

$c_5 := L1[i][1]$

$c_6 := L1[i][2]$

$c_7 := L1[i][3]$

$c_8 := L1[i][4]$

$c_9 := L1[i][5]$

$c_{10} := L1[i][6]$

$c_1 := a_1 + c_5 + c_6 + c_7$

$c_2 := a_2 + c_5 + c_8 + c_9$

$c_3 := a_3 + c_6 + c_8 + c_{10}$

$c_4 := a_4 + c_7 + c_9 + c_{10}$:

if $c_1 \geq 0$ and $c_2 \geq 0$ and $c_3 \geq 0$ and $c_4 \geq 0$ then $L2 := [L2, x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} x_5^{c_5} x_6^{c_6} x_7^{c_7} x_8^{c_8} x_9^{c_9} x_{10}^{c_{10}}]$:

Output: $L2$.

Proof. By considering the matrix π_4 , we see that every torus invariant section in $H^0(X, a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4)$ is of the form $x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} x_5^{c_5} x_6^{c_6} x_7^{c_7} x_8^{c_8} x_9^{c_9} x_{10}^{c_{10}}$ where

$$c_5 + c_6 + c_7 + c_8 + c_9 + c_{10} = a_0 \quad (8.0.1)$$

$$c_1 - c_5 - c_6 - c_7 = a_1 \quad (8.0.2)$$

$$c_2 - c_5 - c_8 - c_9 = a_2 \quad (8.0.3)$$

$$c_3 - c_6 - c_8 - c_{10} = a_3 \quad (8.0.4)$$

$$c_4 - c_7 - c_9 - c_{10} = a_4 \quad (8.0.5)$$

$$(8.0.6)$$

We construct $L1$ to contain all the solutions to (5.3.1) (with t standing in for c) for non-negative integers t_5, \dots, t_{10} . Then we work through all the possible solutions to (5.3.1) (indexed by i) by defining c_5, \dots, c_{10} to be the first up to sixth terms respectively in the i th possible solution to 5.3.1. Given c_5, \dots, c_{10} , we define c_1, \dots, c_4 according to (5.3.2), (5.3.3), (5.3.4) and (5.3.5) respectively. We check if this gives $c_1, \dots, c_4 \geq 0$ and hence a section $x_1^{c_1} x_2^{c_2} x_3^{c_3} x_4^{c_4} x_5^{c_5} x_6^{c_6} x_7^{c_7} x_8^{c_8} x_9^{c_9} x_{10}^{c_{10}}$. We gather all such monomials in $L2$, and hence our output $L2$ contains every point in $\pi_4^{-1}(a_0, \dots, a_4)$ as required. \square

Recall that a ruling is the sum of two (-1) -curves whose intersection number is 1. We give pseudocode for finding rulings on X_4 and a proof of efficacy. The code for X_5 and X_6 is similar.

Pseudocode 8.0.7. *First we write a list of all (-1) -curves $a_0l_0 + a_1l_1 + a_2l_2 + a_3l_3 + a_4l_4$ on X_4 by listing the corresponding elements of $\mathbb{Z}^5 : [a_0, a_1, a_2, a_3, a_4]$. In this format, the list of (-1) -curves L is:*

$$L := [[0, 1, 0, 0, 0], [0, 0, 1, 0, 0], [0, 0, 0, 1, 0], [0, 0, 0, 0, 1], [1, -1, -1, 0, 0], [1, -1, 0, -1, 0], \\ [1, -1, 0, 0, -1], [1, 0, -1, -1, 0], [1, 0, -1, 0, -1], [1, 0, 0, -1, -1]].$$

Again, we denote the i th element of L by $L[i]$, and the number of elements in L by $|L|$. We find all rulings as follows:

Input: the list L .

Procedure:

$$S := [] :$$

$$f[1] := l_1$$

$$f[2] := l_2$$

$$f[3] := l_3$$

$$f[4] := l_4$$

$$f[5] := l_0 - l_1 - l_2$$

$$f[6] := l_0 - l_1 - l_3$$

```

f[7] := l0 - l1 - l4
f[8] := l0 - l2 - l3
f[9] := l0 - l2 - l4
f[10] := l0 - l3 - l4
for i from 1 to |L| - 1 do
for j from i + 1 to |L| do
if L[i][1]L[j][1] - L[i][2]L[j][2] - L[i][3]L[j][3] - L[i][4]L[j][4] - L[i][5]L[j][5] = 1 then S := [S, f[i] +
f[j]]
S := convert(convert(S, set), list)
Output: S.

```

Proof. We define the $f[i]$'s to be the (-1) -curves on X_4 . We work through all the (-1) -curves, indexed by i and j and compute their intersection number:

$$L[i][1]L[j][1] - L[i][2]L[j][2] - L[i][3]L[j][3] - L[i][4]L[j][4] - L[i][5]L[j][5].$$

By definition, if the intersection number of a pair of (-1) -curves is 1, then their sum is a ruling. We collect the sum of all pairs of generators with intersection number 1 in the list S . To avoid repetitions, we convert S to a set and then back to a list. \square

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