A CHARACTERISATION OF THE $\mathbb{Z}^n \oplus \mathbb{Z}(\delta)$ LATTICE AND DEFINITE NONUNIMODULAR INTERSECTION FORMS

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Abstract. We prove a generalisation of Elkies' characterisation of the $\mathbb{Z}^n$ lattice to nonunimodular definite forms (and lattices). Combined with inequalities of Frøyshov and of Ozsváth and Szabó, this gives a simple test of whether a rational homology three-sphere may bound a definite four-manifold. As an example we show that small positive surgeries on torus knots do not bound negative-definite four-manifolds.

1. Introduction

The intersection pairing of a smooth compact four-manifold, possibly with boundary, is an integral symmetric bilinear form Q_X on $H_2(X;\mathbb{Z})/\mathit{Tors}$; it is nondegenerate if the boundary of X has first Betti number zero. Characteristic covectors for Q_X are elements of $H^2(X;\mathbb{Z})/\mathit{Tors}$ represented by the first Chern classes of spin^c structures. In the case that Q_X is positive-definite there are inequalities due to Frøyshov (using Seiberg-Witten theory, see [3]) and Ozsváth and Szabó which give lower bounds on the square of a characteristic covector. It is helpful in this context to prove existence of characteristic covectors with small square. The following result was conjectured by the authors in [10].

Theorem 1. *Let Q be an integral positive-definite symmetric bilinear form of rank n and determinant δ. Then there exists a characteristic covector ξ for Q with*

$$\xi^2 \le \begin{cases} n-1+1/\delta & \text{if } \delta \text{ is odd,} \\ n-1 & \text{if } \delta \text{ is even;} \end{cases}$$

this inequality is strict unless $Q = (n-1)\langle 1\rangle \oplus \langle \delta \rangle$. Moreover, the two sides of the inequality are congruent modulo $4/\delta$.

For unimodular forms the first part of the theorem was proved by Elkies [2]. To prove its generalisation we reinterpret the statement in terms of integral lattices. In Section 2 we introduce the necessary notions and then study short characteristic

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covectors of some special types of lattices. The proof of the first part of the theorem is presented in Section 3. The main idea is to use the theory of linking pairings to embed four copies of the lattice into a unimodular lattice and then apply Elkies' theorem along with results of Section 2.

For unimodular forms, a well-known and useful constraint on the square of a characteristic vector is that ξ^2 is congruent modulo 8 to the signature of the form. In Section 4 we give a generalisation of this to nonunimodular forms, which specialises to give the congruence in Theorem 1.

In Section 5 we combine Theorem 1 with an inequality of Ozsváth and Szabó [12, Theorem 9.6] to obtain the following theorem, where $d(Y, \mathfrak{t})$ denotes the correction term invariant defined in [12] for a spin^c structure $\mathfrak{t}$ on a rational homology three-sphere Y:

Theorem 2. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = \delta$. If Y bounds a negative-definite four-manifold X, and if either δ is square-free or there is no torsion in $H_1(X;\mathbb{Z})$, then*

$$\max_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} 4d(Y,\mathfrak{t}) \geq \begin{cases} 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even.} \end{cases}$$

The inequality is strict unless the intersection form of X is $(n-1)\langle -1\rangle \oplus \langle -\delta\rangle$. Moreover, the two sides of the inequality are congruent modulo $4/\delta$.

Some applications of Theorem 2 are discussed in [10]. As a further application we consider manifolds obtained as integer surgeries on knots. Knowing which positive surgeries on a knot bound negative-definite manifolds has implications for existence of fillable contact structures and for the unknotting number of the knot (see e.g. [7, 11]). In particular we consider torus knots; it is well known that $(pq-1)$-surgery on the torus knot $T_{p,q}$ is a lens space. It follows that all large ($\geq pq-1$) surgeries on torus knots bound both positive- and negative-definite four-manifolds (see e.g. [11]). We show in Section 7 that small positive surgeries on torus knots cannot bound negative-definite four-manifolds. In particular, we obtain the following result; for a more precise statement see Corollary 7.2.

Proposition 3. *Let $2 \leq p < q$ and $1 \leq n \leq (p-1)(q-1)+2$. Then $+n$-surgery on $T_{p,q}$ cannot bound a negative-definite four-manifold with no torsion in H_1.*

There has been further progress on the question of which Dehn surgeries on torus knots bound negative-definite manifolds in the time since this paper was originally written. In particular Proposition 3 has been improved in many cases by Greene [4]. A complete answer to the question will appear in [11].

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2. Short characteristic covectors in special cases

An *integral lattice* L of rank n is the free abelian group $\mathbb{Z}^n$ along with a nondegenerate symmetric bilinear form $Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$. Let V denote the tensor product of L with $\mathbb{R}$; thus V is the vector space $\mathbb{R}^n$ to which the form Q extends, and L is a full rank lattice in V. The *signature* $\sigma(L)$ of the lattice L is the signature of Q. We say L is *definite* if $|\sigma(L)| = n$. For convenience denote $Q(x, y)$ by $x \cdot y$ and $Q(y, y)$ by y^2. A set of generators $v_1, \dots, v_n$ for L forms a basis of V satisfying $v_i \cdot v_j \in \mathbb{Z}$. With respect to such a basis the form Q is represented by the matrix $[v_i \cdot v_j]_{i,j=1}^n$. The *determinant* (or *discriminant*) of L is the determinant of the form (or the corresponding matrix).

We will write simply $\mathbb{Z}^n$ for the standard positive-definite unimodular lattice of discriminant 1 (the form on this lattice is the standard inner-product form). In what follows we will use the well known fact that for $n < 8$ this is the only positive-definite unimodular lattice; this also easily follows from Elkies' Theorem (Theorem 1 with $\delta = 1$) and the congruence condition for characteristic vectors.

The *dual lattice* $L' \cong \mathrm{Hom}(L, \mathbb{Z})$ consists of all vectors $x \in V$ satisfying $x \cdot y \in \mathbb{Z}$ for all $y \in L$. A *characteristic covector* for L is an element $\xi \in L'$ with $\xi \cdot y \equiv y^2 \pmod 2$ for all $y \in L$.

We say a lattice is *complex* if it admits an automorphism $\mathbf{i}$ with $\mathbf{i}^2$ given by multiplication by -1. A lattice is *quaternionic* if it admits an action of the quaternionic group $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, with -1 acting by multiplication. Note that the rank of a complex lattice is even, and the rank of a quaternionic lattice is divisible by 4. For any lattice L, let L^m denote the direct sum of m copies of L. There is a standard way to make $L \oplus L$ into a complex lattice and L^4 into a quaternionic lattice; for example, the quaternionic structure is given by

$$\begin{aligned} \mathbf{i} : (x, y, z, w) &\mapsto (-y, x, -w, z) \\ \mathbf{j} : (x, y, z, w) &\mapsto (-z, w, x, -y). \end{aligned}$$

Let $\mathbb{Z}(\delta)$ denote the rank one lattice with determinant δ; in particular $\mathbb{Z} = \mathbb{Z}(1)$.

Lemma 2.1. *Let $\delta \in \mathbb{N}$, and let p be a prime. Let L be an index p sublattice of $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$. Then L has a characteristic covector ξ with $\xi^2 < n - 1$ unless $L \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(p^2\delta)$.*

Proof. We may assume L contains none of the summands of $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$; any such summand of L contributes 1 to the right-hand side of the inequality and at most 1 to the left-hand side.

If $n = 1$ then clearly $L \cong \mathbb{Z}(p^2\delta)$. Now suppose $n > 1$. Let $\{e, e_1, \ldots, e_{n-2}, f\}$ be a basis for $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$, where $e, e_1, \ldots, e_{n-2}$ have square 1 and $f^2 = \delta$. Then multiples of e give coset representatives of L in $\mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$; it follows that a basis for L is given by $\{pe, e_1 + s_1 e, \ldots, e_{n-2} + s_{n-2} e, f + te\}$. Here s_i, t are nonzero residues modulo p in $[1-p, p-1]$, whose parities we may choose if p is odd. With respect to this basis, the bilinear form on L has matrix

$$Q = \begin{pmatrix} p^2 & ps_1 & ps_2 & ps_3 & \ldots & ps_{n-2} & pt \\ ps_1 & 1+s_1^2 & s_1 s_2 & s_1 s_3 & \ldots & s_1 s_{n-2} & s_1 t \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ ps_{n-2} & s_1 s_{n-2} & s_2 s_{n-2} & s_3 s_{n-2} & \ldots & 1+s_{n-2}^2 & s_{n-2} t \\ pt & s_1 t & s_2 t & s_3 t & \ldots & s_{n-2} t & \delta + t^2 \end{pmatrix},$$

with inverse

$$Q^{-1} = \begin{pmatrix} \frac{1+\sum s_i^2 + t^2/\delta}{p^2} & -\frac{s_1}{p} & -\frac{s_2}{p} & -\frac{s_3}{p} & \ldots & -\frac{s_{n-2}}{p} & -\frac{t}{p\delta} \\ -\frac{s_1}{p} & 1 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ -\frac{s_{n-2}}{p} & 0 & 0 & 0 & \ldots & 1 & 0 \\ -\frac{t}{p\delta} & 0 & 0 & 0 & \ldots & 0 & \frac{1}{\delta} \end{pmatrix}.$$

With respect to the dual basis, an element of the dual lattice L' is represented by an n-tuple $\xi \in \mathbb{Z}^n$. An n-tuple corresponds to a characteristic covector if its components have the same parity as the corresponding diagonal elements of Q.

Suppose now that p is odd. Choose s_i to be odd for all i and $t \equiv \delta \pmod 2$. Then $\xi = (1, 0, \ldots, 0)$ is a characteristic covector whose square satisfies

$$\xi^2 = \frac{1 + \sum s_i^2 + t^2/\delta}{p^2} < n - 1,$$

noting that $|s_i|,\ |t| \leq p - 1$.

Finally if $p = 2$ then $\xi = (0, 0, \ldots, 0, (1+\delta) \bmod 2)$ is a characteristic covector with $\xi^2 < n - 1$. $\square$

Lemma 2.2. *Let $\delta \in \mathbb{N}$ be odd, and let p be a prime with $p = 2$ or $p \equiv 1 \pmod 4$. Let M be an index p complex sublattice of $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$. Then M has a characteristic covector ξ with $\xi^2 < 2n - 2$ unless $M \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(p\delta)^2$.*

Proof. We may assume M contains none of the summands of $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$. Suppose first that $n = 1$. Let $\{e, \mathbf{i}e\}$ be a basis for $\mathbb{Z}(\delta)^2$ with $e^2 = \delta$. Then $\{pe, \mathbf{i}e + se\}$ is a basis for M, for some s. Now

$$\begin{aligned} \mathbf{i}(\mathbf{i}e + se) = -e + s\mathbf{i}e &\in M \\ \implies -e - s^2 e &\in M, \end{aligned}$$

from which it follows that p divides $1 + s^2$. The bilinear form on L has matrix

$$Q = \begin{pmatrix} p^2\delta & ps\delta \\ ps\delta & \delta(1+s^2) \end{pmatrix} = p\delta \begin{pmatrix} p & s \\ s & \frac{1+s^2}{p} \end{pmatrix} \cong p\delta I,$$

since any positive-definite unimodular form of rank 2 is diagonalisable. Thus in this case $M \cong \mathbb{Z}(p\delta)^2$.

Now suppose $n > 1$. Let $\{e, e_1, \ldots, e_{2n-3}, f_1, f_2\}$ be a basis for $\mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta)^2$, where e, e_i have square 1 and $f_i^2 = \delta$. A basis for M is given by $\{pe, e_i + s_i e, f_i + t_i e\}$, where s_i, t_i are nonzero residues modulo p which we may choose to be odd integers in $[1-p, p-1]$. The matrix in this basis is

$$Q = \begin{pmatrix} p^2 & ps_1 & ps_2 & ps_3 & \ldots & ps_{2n-3} & pt_1 & pt_2 \\ ps_1 & 1+s_1^2 & s_1s_2 & s_1s_3 & \ldots & s_1s_{2n-3} & s_1t_1 & s_1t_2 \\ ps_2 & s_1s_2 & 1+s_2^2 & s_2s_3 & \ldots & s_2s_{2n-3} & s_2t_1 & s_2t_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ ps_{2n-3} & s_1s_{2n-3} & s_2s_{2n-3} & s_3s_{2n-3} & \ldots & 1+s_{2n-3}^2 & s_{2n-3}t_1 & s_{2n-3}t_2 \\ pt_1 & s_1t_1 & s_2t_1 & s_3t_1 & \ldots & s_{2n-3}t & \delta+t_1^2 & t_1t_2 \\ pt_2 & s_1t_2 & s_2t_2 & s_3t_2 & \ldots & s_{2n-3}t & t_1t_2 & \delta+t_2^2 \end{pmatrix},$$

with inverse

$$Q^{-1} = \begin{pmatrix} \frac{1+\sum s_i^2+\sum t_i^2/\delta}{p^2} & -\frac{s_1}{p} & -\frac{s_2}{p} & \ldots & -\frac{s_{2n-3}}{p} & -\frac{t_1}{p\delta} & -\frac{t_2}{p\delta} \\ -\frac{s_1}{p} & 1 & 0 & \ldots & 0 & 0 & 0 \\ -\frac{s_2}{p} & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \\ -\frac{s_{2n-3}}{p} & 0 & 0 & \ldots & 1 & 0 & 0 \\ -\frac{t_1}{p\delta} & 0 & 0 & \ldots & 0 & \frac{1}{\delta} & 0 \\ -\frac{t_2}{p\delta} & 0 & 0 & \ldots & 0 & 0 & \frac{1}{\delta} \end{pmatrix}.$$

If $p = 2$ then Q is even, and $\xi = 0$ is characteristic.

Suppose now that p is odd. Then $\xi = (k, l_1, \dots, l_{2n-1})$ is characteristic if k is odd and each l_i is even. By completing squares we have that

$$\xi^2 = \frac{1}{p^2}\left(k^2 + \sum_{i=1}^{2n-3}(ks_i - pl_i)^2 + \frac{1}{\delta}\sum_{i=1}^{2}(kt_i - pl_{2n-3+i})^2\right).$$

Notice that for odd $|k| < p$ and for appropriate choice of even l_i each of the squares in the sum above is less than p^2. Thus it suffices to prove the statement for $\delta = 1$ and $n = 2$; the following sublemma completes the proof.

Sublemma 2.3. *Let* $F(k, l_1, l_2, l_3) = k^2 + \sum_{i=1}^{3}(ks_i - l_ip)^2$, *where* s_i *are odd numbers with* $|s_i| \leq p - 2$. *Then there exist* k *odd,* $|k| < p$, *and* l_i *even for which* $F < 2p^2$.

Proof. Let $K = \{2-p, 4-p, \dots, p-2\}$, and let

$$K_i = \left\{k \in K \;\middle|\; \exists\ l_i \text{ even with } |ks_i - l_ip| < \frac{p-1}{2}\right\}.$$

Note that multiplication by s_i followed by reduction modulo $2p$ induces a permutation of K. Since $p \equiv 1 \pmod 4$ it follows that $|K_i| = \frac{p-1}{2}$.

If there exists $k \in \cap_{i=1}^{3} K_i$ then with this choice of k and the corresponding choices for l_1, l_2, l_3 we find

$$F < p^2 + 3\left(\frac{p-1}{2}\right)^2 < 2p^2.$$

Otherwise let K_{ij} denote $K_i \cap K_j$. Then

$$p - 1 \geq |\cup_{i=1}^{3} K_i| = \sum_{i=1}^{3}|K_i| - \sum_{i<j}|K_{ij}|,$$

from which it follows that $\sum_{i<j}|K_{ij}| \geq \frac{p-1}{2}$. Thus there exists some $k \in K_{ij}$ with $|k| \leq \frac{p+1}{2}$. With this k and appropriate l_i we find

$$F < p^2 + 2\left(\frac{p-1}{2}\right)^2 + \left(\frac{p+1}{2}\right)^2 < 2p^2.$$

□

Lemma 2.4. *Let* $\delta \in \mathbb{N}$ *be odd, and let* q *be a prime with* $q \equiv 3 \pmod 4$. *Let* N *be an index* q^2 *quaternionic sublattice of* $\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4$, *with the quotient group* $(\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4)/N$ *having exponent* q. *Then* N *has a characteristic covector* ξ *with* $\xi^2 < 4n - 4$ *unless* $N \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(q\delta)^4$.

Proof. We may assume N contains none of the summands of $\mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta)^4$. Suppose first that $n = 1$. Let $\{e, \mathbf{i}e, \mathbf{j}e, \mathbf{k}e\}$ be a basis for $\mathbb{Z}(\delta)^4$ with $e^2 = \delta$. Note that e and $\mathbf{i}e$ represent generators of the quotient $\mathbb{Z}/q \oplus \mathbb{Z}/q$; otherwise we have $e + s\mathbf{i}e \in N$ for some s, and multiplication by $\mathbf{i}$ yields $s^2+1 \equiv 0 \pmod q$, but -1 is not a quadratic residue modulo q. Now a basis for N is given by $\{qe, q\mathbf{i}e, \mathbf{j}e+s_1e+t_1\mathbf{i}e, \mathbf{k}e+s_2e+t_2\mathbf{i}e\}$. The quaternionic symmetry yields

$$s_1 \equiv t_2, \quad s_2 \equiv -t_1, \quad 1+s_1^2+s_2^2 \equiv 0 \pmod q;$$

it follows that the bilinear form Q on N factors as $q\delta$ times a unimodular form. Thus $N \cong \mathbb{Z}(q\delta)^4$.

We now assume $n=2$ and $\delta=1$ (the proof for any other n,δ follows from this case). Let $\{e, \mathbf{i}e, v_1=\mathbf{j}e, v_2=\mathbf{k}e, v_3=f, v_4=\mathbf{i}f, v_5=\mathbf{j}f, v_6=\mathbf{k}f\}$ be a basis of unit vectors for $\mathbb{Z}^8$. Then $\{qe, q\mathbf{i}e, v_1+s_1e+t_1\mathbf{i}e, \dots, v_6+s_6e+t_6\mathbf{i}e\}$ is a basis for N. Invariance of N under multiplication by $\mathbf{i}$ yields

$$s_{i+1} \equiv -t_i, \quad t_{i+1} \equiv s_i \pmod q$$

for $i=1,3,5$. Using further quaternionic symmetry it follows that

$$\begin{aligned} 1+s_1^2+s_2^2 &\equiv 0 \pmod q \\ s_5+s_1s_3+s_2s_4 &\equiv 0 \pmod q \\ s_6+s_2s_3-s_1s_4 &\equiv 0 \pmod q, \end{aligned}$$

hence $s_1, s_2 \not\equiv 0 \pmod q$. From the assumption that no basis vector for $\mathbb{Z}^8$ is contained in N it follows that s_3, s_4 cannot both be divisible by q, and that also s_5, s_6 cannot both be divisible by q. Combining this with the last two congruences above it follows that at most one of $s_3, \dots, s_6$ is divisible by q.

We may choose s_i, t_i in $[1-q, q-1]$ with s_i+t_i odd. Computation of Q, Q^{-1} in the above basis for N now yields that $\xi = (k_1, k_2, l_1, \dots, l_6)$ is characteristic if k_1, k_2 are odd and l_i are even, and

$$\xi^2 = \frac{1}{q^2}\left(k_1^2+k_2^2+\sum_{i=1}^{6}(k_1s_i+k_2t_i-l_iq)^2\right).$$

There are now two cases to consider, depending on whether or not one of the s_i is zero. Existence of a characteristic ξ with $\xi^2<4$ follows in each case from one of the following sublemmas.

Sublemma 2.5. *Let* $F = k_1^2+k_2^2+\sum_{i=1}^{6}(k_1s_i+k_2t_i-l_iq)^2$, *where* s_i *are odd and* t_i *are even integers in* $[1-q, q-1]$. *Then there exist* k_1, k_2 *odd and* l_i *even for which* $F<4q^2$.

Proof. Set $k_2=1$. Let $K=\{2-q, 4-q, \dots, q-2\}$, and let

$$K_i = \left\{ k \in K \;\middle|\; \exists\, l_i \text{ even with } |ks_i+t_i-l_iq| \le \frac{q-1}{2} \right\}.$$

Note that $|K_i| \geq \frac{q-1}{2}$ (in fact K_i either contains $\frac{q-1}{2}$ or $\frac{q+1}{2}$ elements, depending on the value of t_i).

If there is a k which is in the intersection of four of the K_i's, then setting $k_1 = k$ and choosing appropriate l_i we obtain

$$F \leq 1 + 3q^2 + 4\left(\frac{q-1}{2}\right)^2 < 4q^2.$$

Otherwise let K_{ijk} denote the triple intersection of K_i, K_j, K_k, and let K'_{ij} denote $K_i \cap K_j - \cup_k K_{ijk}$. Then

$$q - 1 \geq |\cup_{i=1}^{6} K_i| = \sum_{i=1}^{6} |K_i| - \sum_{i<j} |K'_{ij}| - 2\sum_{i<j<k} |K_{ijk}|,$$

from which it follows that

$$\sum_{i<j} |K'_{ij}| + 2\sum_{i<j<k} |K_{ijk}| \geq 2(q-1),$$

and hence

$$\sum_{i<j<k} |K_{ijk}| \geq \frac{q-1}{2}.$$

Thus there exists some $k \in K_{ijk}$ with $|k| \leq \frac{q-1}{2}$. Setting $k_1 = k$ and choosing corresponding l_i again yields

$$F \leq 1 + 3q^2 + 4\left(\frac{q-1}{2}\right)^2 < 4q^2.$$

□

Sublemma 2.6. *Let* $F = k_1^2 + k_2^2 + \sum_{i=1}^{5}(k_1 s_i + k_2 t_i - l_i q)^2 + (k_2 t_6 - l_6 q)^2$, *where* s_i *and* t_6 *are odd, and* $t_1, \ldots, t_5$ *are even integers in* $[1-q, q-1]$. *Then there exist* k_1, k_2 *odd and* l_i *even for which* $F < 4q^2$.

Proof. First choose k_2 (and l_6) with $|k_2|, |k_2 t_6 - l_6 q| \leq \frac{q-1}{2}$. Again let $K = \{2-q, 4-q, \ldots, q-2\}$, and for each $i \leq 5$ let

$$K_i = \left\{ k \in K \;\middle|\; \exists\, l_i \text{ even with } |k s_i + k_2 t_i - l_i q| \leq \frac{q-1}{2} \right\};$$

then $|K_i| \geq \frac{q-1}{2}$.

If there is a k which is in the intersection of four of the K_i's, then for $k_1 = k$ and appropriate l_i we obtain

$$F \leq 2q^2 + 6\left(\frac{q-1}{2}\right)^2 < 4q^2.$$

Otherwise (with notation as above) we have

$$q-1 \geq | \cup_{i=1}^{5} K_i| = \sum_{i=1}^{5} |K_i| - \sum_{i<j} |K'_{ij}| - 2 \sum_{i<j<k} |K_{ijk}|,$$

from which it follows that

$$\sum_{i<j} |K'_{ij}| + 2 \sum_{i<j<k} |K_{ijk}| \geq \frac{3}{2}(q-1),$$

and hence

$$\sum_{i<j<k} |K_{ijk}| \geq \frac{q+1}{4}.$$

Thus there exists some $k \in K_{ijk}$ with $|k| \leq \frac{3(q+1)}{4}$. Setting $k_1 = k$ and choosing corresponding l_i yields

$$F \leq 2q^2 + 5\left(\frac{q-1}{2}\right)^2 + \left(\frac{3(q+1)}{4}\right)^2 < 4q^2.$$

$\square$

3. Proof of the main theorem

The bilinear form Q on L induces a symmetric bilinear $\mathbb{Q}/\mathbb{Z}$-valued pairing on the finite group L'/L, called the *linking pairing* associated to L. Such pairings λ on a finite group G were studied by Wall [19] (see also [1, 5]). He observed that λ splits into an orthogonal sum of pairings λ_p on the p-subgroups G_p of G. For any prime p let A_{p^k} (resp. B_{p^k}) denote the pairing on $\mathbb{Z}/p^k$ for which p^k times the square of a generator is a quadratic residue (resp. nonresidue) modulo p. If p is odd then λ_p decomposes into an orthogonal direct sum of these two types of pairings on cyclic subgroups. The pairing on G_2 may be decomposed into cyclic summands (there are four equivalence classes of pairings on $\mathbb{Z}/2^k$ for $k \geq 3$), plus two types of pairings on $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$; these are denoted E_{2^k}, F_{2^k}. For an appropriate choice of generators, in E_{2^k} each cyclic generator has square 0, while in F_{2^k} they each have square 2^{1-k}.

Proposition 3.1. *Let L be a lattice of rank n. Then L^4 may be embedded as a quaternionic sublattice of a unimodular quaternionic lattice U of rank $4n$.*

Proof. Consider an orthogonal decomposition of the linking pairing on L'/L as described above. Let $x \in L'$ represent a generator of a cyclic summand $\mathbb{Z}/p^k$ with $k > 1$. Then $v = p^{k-1}x$ has $v^2 \in \mathbb{Z}$ and $pv \in L$. Adjoining v to L yields a lattice L_1 which contains L as an index p sublattice (so that $\det L = p^2 \det L_1$). Similarly, if $x \in L'$ represents a generator of a cyclic summand of E_{2^k} or F_{2^k} then $v = 2^{k-1}x$ may be adjoined. Finally if x_1, x_2 represent generators of two $\mathbb{Z}/2$ summands then $x_1 + x_2$ may be adjoined. In this way we get a sequence of embeddings

$$L = L_0 \subset L_1 \subset \cdots \subset L_{m_1},$$

where each L_i is an index p sublattice of L_{i+1} for some prime p. Moreover the linking pairing of L_{m_1} decomposes into cyclic summands of prime order, and the determinant of L_{m_1} is either odd or twice an odd number.

Now let $M_0 = L_{m_1} \oplus L_{m_1}$. Note that M_0 is a complex sublattice of $L' \oplus L'$. Let p be a prime which is either 2 or is congruent to 1 modulo 4, so that there exists an integer a with $a^2 \equiv -1 \pmod{p}$. Let $x \in L'$ represent a generator of a $\mathbb{Z}/p$ summand of the linking pairing of L_{m_1}. Then $v = (x, ax) \in L' \oplus L'$ has $v^2 \in \mathbb{Z}$ and $pv \in M_0$. Adjoining v to M_0 yields a lattice M_1; since $\mathbf{i}v + av \in M_0$, M_1 is preserved by $\mathbf{i}$. Continuing in this way we obtain a sequence of embeddings

$$M_0 \subset M_1 \subset \cdots \subset M_{m_2},$$

where each M_i is an index p sublattice of M_{i+1} for some prime p with $p = 2$ or $p \equiv 1 \pmod{4}$. Each M_i is a complex sublattice of $L' \oplus L'$. We may arrange that each M_i with $i > 0$ has odd determinant. The resulting linking pairing of M_{m_2} is the orthogonal sum of pairings on cyclic groups of prime order congruent to 3 modulo 4, and all the summands appear twice.

Finally let $N_0 = M_{m_2} \oplus M_{m_2}$. This is a quaternionic sublattice of $(L')^4$. Let $q \equiv 3 \pmod{4}$ be a prime, and suppose that $x \in L'$ generates a $\mathbb{Z}/q$ summand of the linking pairing of M_{m_2}. There exist integers a, b with $a^2 + b^2 \equiv -1 \pmod{q}$ (take m to be the smallest positive quadratic nonresidue and choose a, b so that $a^2 \equiv m - 1,\ b^2 \equiv -m$). Let $v_1 = (x, 0, ax, bx)$ and let $v_2 = \mathbf{i}v_1 = (0, x, -bx, ax)$. Then $v_i^2, v_1 . v_2 \in \mathbb{Z}$, and $qv_i \in N_0$. Adjoining v_1, v_2 to N_0 yields a quaternionic sublattice N_1 of $(L')^4$; note that $\mathbf{j}v_1 + av_1 - bv_2 \in N_0$. We thus obtain a sequence

$$N_0 \subset N_1 \subset \cdots \subset N_{m_3} = U,$$

where each N_i is an index q^2 sublattice of N_{i+1} for some prime $q \equiv 3 \pmod{4}$, N_{i+1}/N_i has exponent q, each N_i is quaternionic with odd determinant, and U is unimodular. □

Remark 3.2. *Using the methods of Proposition 3.1 one may show that L^2 embeds in a unimodular lattice if and only if the prime factors of* $\det L$ *congruent to 3 mod 4 have even exponents. These embedding results for L^2 and L^4 (without the quaternionic condition in Proposition 3.1) also follow from* [1, Theorem 1.7].

Applying this to rank 1 lattices one recovers the classical fact that any integer is expressible as the sum of 4 squares, and is the sum of 2 squares if and only if its prime factors congruent to 3 mod 4 have even exponents.

Proof of Theorem 1. Let L be the lattice with form Q. Embed L^4 in a unimodular lattice U of rank $4n$ as in Proposition 3.1. By Elkies' theorem [2] either $U \cong \mathbb{Z}^{4n}$ or

U has a characteristic covector ξ with $\xi^2 \leq 4n-8$. In the latter case, restricting to one of the four copies of L yields a covector $\xi|_L$ with $(\xi|_L)^2 \leq n-2$, which easily satisfies the statement of the theorem.

This leaves the case that $U \cong \mathbb{Z}^{4n}$. Let $L_0, \ldots, L_{m_1}, M_0, \ldots, M_{m_2}, N_0, \ldots, N_{m_3}$ be as in the proof of Proposition 3.1. By successive application of Lemma 2.4, we see that either $N_0 \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta_1)^4$ for some $\delta_1 \in \mathbb{N}$, or N_0 has a characteristic covector ξ with $\xi^2 < 4n-4$. In the latter case, restricting ξ to one of the four copies of L yields $(\xi|_L)^2 < n-1$.

If $N_0 \cong \mathbb{Z}^{4n-4} \oplus \mathbb{Z}(\delta_1)^4$ then $M_{m_2} \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_1)^2$. By Lemma 2.2 we find that either $M_0 \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_2)^2$ or M_0 has a characteristic covector ξ with $\xi^2 < 2n-2$. In the latter case, restricting ξ to one of the two copies of L embedded in M_0 yields $(\xi|_L)^2 < n-1$.

Finally if $M_0 \cong \mathbb{Z}^{2n-2} \oplus \mathbb{Z}(\delta_2)^2$ then $L_{m_1} \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta_2)$. Successive application of Lemma 2.1 now yields that either $L \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}(\delta)$ or L has a characteristic covector ξ with $\xi^2 < n-1$.

A proof of the congruence is given in the next section. □

4. A congruence condition on characteristic covectors

Given a positive-definite symmetric bilinear form Q of rank n and determinant δ with $Q \neq (n-1)\langle 1 \rangle \oplus \langle \delta \rangle$, one may ask for an optimal upper bound on the square of a shortest characteristic covector ξ. The square of a characteristic covector of a unimodular form is congruent to the signature modulo 8 (see for example [17]). Thus if $\delta = 1$ we have $\xi^2 \leq n-8$. In this section we give congruence conditions on the square of characteristic covectors of forms of arbitrary determinant. Lattices in this section are not assumed to be definite. (The results in this section may be known to experts but we have not found them in the literature. We also note that the remainder of the paper is independent from this section.)

If ξ_1, ξ_2 are characteristic covectors of a lattice L of determinant δ then their difference is divisible by 2 in L'; it follows that $\xi_1^2 \equiv \xi_2^2$ modulo $\frac{4}{\delta}$. For a lattice with odd determinant the congruence holds modulo $\frac{8}{\delta}$. We will give a formula for this congruence class in terms of the signature and linking pairing of L. For a lattice with determinant δ even or odd we will determine the congruence class of ξ^2 modulo $\frac{4}{\delta}$ in terms of the signature and determinant.

Lemma 4.1. *Suppose* $\delta = \prod_{i=1}^{r} p_i^{k_i} \cdot \prod_{j=1}^{s} q_j^{l_j}$ *where* p_i, q_j *are odd primes, not necessarily distinct. Let* M *be an even lattice of determinant* δ *with linking pairing isomorphic* *to* $\bigoplus_{i=1}^{r} A_{p_i^{k_i}} \oplus \bigoplus_{j=1}^{s} B_{q_j^{l_j}}$. *Then the signature of* M *satisfies the congruence*

$$\sigma(M) \equiv \sum_{k_i \equiv 1\,(2)} (1-p_i) + \sum_{l_j \equiv 1\,(2)} (5-q_j) \qquad (\bmod 8).$$

(For the definition of A_{p^k} *and* B_{p^k} *see the beginning of Section 3.)*

Proof. Let $G(M)$ denote the Gauss sum

$$\frac{1}{\sqrt{\delta}} \sum_{u \in M'/M} e^{i\pi u^2}.$$

(See [18] for more details on Gauss sums and the Milgram Gauss sum formula.) Then $G(M)$ depends only on the linking pairing of M and in fact factors as

$$G(M) = \prod G(A_{p_i^{k_i}}) \cdot \prod G(B_{q_j^{l_j}}).$$

The factors are computed in [18, Theorem 3.9] to be:

$$\begin{aligned} G(A_{p^k}) &= \begin{cases} 1 & \text{if } k \text{ is even,} \\ e^{2\pi i(1-p)/8} & \text{if } k \text{ is odd;} \end{cases} \\ G(B_{q^l}) &= \begin{cases} 1 & \text{if } l \text{ is even,} \\ e^{2\pi i(5-q)/8} & \text{if } l \text{ is odd.} \end{cases} \end{aligned}$$

(Note that in the notation of [18], A_{p^k} corresponds to $(C_p(k); 2)$ and B_{q^l} corresponds to $(C_q(l); 2n_q)$, where n_q is a quadratic nonresidue modulo q.)

The congruence on the signature of M now follows from the Milgram formula:

$$G(M) = e^{2\pi i \sigma(M)/8}.$$

□

Proposition 4.2. *Let* L *be a lattice of odd determinant* δ *with linking pairing isomorphic to* $\bigoplus_{i=1}^{r} A_{p_i^{k_i}} \oplus \bigoplus_{j=1}^{s} B_{q_j^{l_j}}$. *Let* ξ *be a characteristic covector for* L. *Then*

$$\xi^2 \equiv \sigma(L) - \sum_{k_i \equiv 1\,(2)} (1-p_i) - \sum_{l_j \equiv 1\,(2)} (5-q_j) \qquad (\bmod 8/\delta).$$

Proof. Let M be an even lattice with the same linking pairing as L. (Existence of M is proved by Wall in [19].) Then by [6, Satz 3] L and M are stably equivalent; that is to say, there exist unimodular lattices U_1, U_2 such that $L \oplus U_1 \cong M \oplus U_2$. Let ξ be a characteristic covector for $L \oplus U_1$. Then we have decompositions

$$\xi = \xi_L + \xi_{U_1} = \xi_M + \xi_{U_2}.$$

Taking squares we find

$$\begin{aligned}
{\xi_L}^2 &\equiv {\xi_M}^2 + {\xi_{U_2}}^2 - {\xi_{U_1}}^2 \\
&\equiv \sigma(U_2) - \sigma(U_1) \\
&\equiv \sigma(L) - \sigma(M) \\
&\equiv \sigma(L) - \sum_{k_i \equiv 1\,(2)} (1 - p_i) - \sum_{l_j \equiv 1\,(2)} (5 - q_j) \qquad \pmod{8/\delta},
\end{aligned}$$

where the last line follows from Lemma 4.1. □

Corollary 4.3. *Let L be a lattice of determinant $\delta \in \mathbb{N}$, and let ξ be a characteristic covector for L. Then*

$$\xi^2 \equiv \begin{cases} \sigma(L) - 1 + 1/\delta & \text{if } \delta \text{ is odd} \\ \sigma(L) - 1 & \text{if } \delta \text{ is even} \end{cases} \qquad \pmod{4/\delta}.$$

Proof. If δ is odd then Proposition 4.2 shows that the congruence class of ξ^2 modulo $\frac{4}{\delta}$ depends only on the signature and determinant; the formula then follows by taking L to be the lattice with the form $r\langle 1\rangle \oplus s\langle -1\rangle \oplus \langle \delta\rangle$ where $r + s = n - 1$ and $r - s = \sigma(L) - 1$.

If δ is even then as in the proof of Proposition 3.1 we find that either L or $L \oplus \mathbb{Z}(2)$ embeds as an index 2^k sublattice of a lattice with odd determinant. It again follows that the congruence class of ξ^2 modulo $\frac{4}{\delta}$ depends only on the signature and determinant. □

5. Proof of Theorem 2

We begin by noting the following restatement of Theorem 1 for negative-definite forms:

Theorem 5.1. *Let Q be an integral negative-definite symmetric bilinear form of rank n and determinant of absolute value δ. Then there exists a characteristic covector ξ for Q with*

$$\xi^2 \geq \begin{cases} -n + 1 - 1/\delta & \text{if } \delta \text{ is odd,} \\ -n + 1 & \text{if } \delta \text{ is even;} \end{cases}$$

this inequality is strict unless $Q = (n-1)\langle -1\rangle \oplus \langle -\delta\rangle$. Moreover, the two sides of the inequality are congruent modulo $4/\delta$.

Let Y be a rational homology three-sphere and X a smooth negative-definite four-manifold bounded by Y, with $b_2(X) = n$. For any Spin^c structure $\mathfrak{t}$ on Y let $d(Y, \mathfrak{t})$ denote the correction term invariant of Ozsváth and Szabó [12]. It is shown in [12, Theorem 9.6] that for each Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(X)$,

$$c_1(\mathfrak{s})^2 + n \leq 4d(Y, \mathfrak{s}|_Y). \tag{1}$$

Also from [12, Theorem 1.2], the two sides of (1) are congruent modulo 8.

The image of $c_1(\mathfrak{s})$ in $H^2(X;\mathbb{Z})/\mathit{Tors}$ is a characteristic covector for the intersection pairing Q_X on $H_2(X;\mathbb{Z})/\mathit{Tors}$. Let δ denote the absolute value of the determinant of Q_X and $\mathrm{Im}(\mathrm{Spin}^c(X))$ the image of the restriction map from $\mathrm{Spin}^c(X)$ to $\mathrm{Spin}^c(Y)$. Then combining (1) with Theorem 5.1 yields

$$\max_{\mathfrak{t}\in\mathrm{Im}(\mathrm{Spin}^c(X))} 4d(Y,\mathfrak{t}) \geq \begin{cases} 1-1/\delta & \text{if } \delta \text{ is odd,} \\ 1 & \text{if } \delta \text{ is even,} \end{cases}$$

with strict inequality unless the intersection form of X is $(n-1)\langle -1\rangle \oplus \langle -\delta\rangle$.

Theorem 2 follows immediately since if either δ is square-free or if there is no torsion in $H_1(X;\mathbb{Z})$, then the restriction map from $\mathrm{Spin}^c(X)$ to $\mathrm{Spin}^c(Y)$ is surjective, and $|H_1(Y;\mathbb{Z})| = \delta$.

Similar reasoning yields the following variant of Theorem 2:

Proposition 5.2. *Let Y be a rational homology sphere with $|H_1(Y;\mathbb{Z})| = rs^2$, with r square-free. If Y bounds a negative-definite four-manifold X, then*

$$\max_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} 4d(Y,\mathfrak{t}) \geq \begin{cases} 1-1/r & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases}$$

Remark 5.3. *Suppose Y and X are as in the statement of Theorem 2 with δ even. If in fact*

$$\max_{\mathfrak{t}\in\mathrm{Im}(\mathrm{Spin}^c(X))} 4d(Y,\mathfrak{t}) = 1,$$

then it is not difficult to see that this maximum must be attained at a spin structure.

6. Surgeries on L-space knots

Let K be a knot in the three-sphere and

$$\Delta_K(T) = a_0 + \sum_{j>0} a_j(T^j + T^{-j})$$

its Alexander polynomial. Torsion coefficients of K are defined by

$$t_i(K) = \sum_{j>0} j a_{|i|+j}\,.$$

Note that $t_i(K) = 0$ for $i \geq N_K$, where N_K denotes the degree of the Alexander polynomial of K. For any $n \in \mathbb{Z}$ denote the n-surgery on K by K_n. K is called an *L-space knot* if for some $n > 0$, K_n is an L-space. Recall from [14] that a rational homology sphere is called an *L-space* if $\widehat{HF}(Y,\mathfrak{s}) \cong \mathbb{Z}$ for each spin^c structure $\mathfrak{s}$ (so its Heegaard Floer homology groups resemble those of a lens space).

There are a number of conditions coming from Heegaard Floer homology that an L-space knot K has to satisfy. In particular (see [14, Theorem 1.2]), its Alexander polynomial has the form

$$\Delta_K(T) = (-1)^k + \sum_{j=1}^{k} (-1)^{k-j}(T^{n_j} + T^{-n_j}). \tag{2}$$

From this it follows easily that the torsion coefficients $t_i(K)$ are given by

$$\begin{array}{ll} t_i(K) = n_k - n_{k-1} + \cdots + n_{k-2j} - i & n_{k-2j-1} \leq i \leq n_{k-2j} \\ t_i(K) = n_k - n_{k-1} + \cdots + n_{k-2j} - n_{k-2j-1} & n_{k-2j-2} \leq i \leq n_{k-2j-1}, \end{array} \tag{3}$$

where $j = 0, \ldots, k-1$ and $n_{-j} = -n_j$, $n_0 = 0$. In particular, $t_i(K)$ is nonincreasing in i for $i \geq 0$.

The following formula for d-invariants of surgeries on such a knot is based on results in [15] and [14]. The proof was outlined to us by Peter Ozsváth. Note that for sufficiently large values of n the formula for $d(K_n, i)$ is contained in [16, Theorem 1.2].

Theorem 6.1. *Let $K \subset S^3$ be an L-space knot and let $t_i(K)$ denote its torsion coefficients. Then for any $n > 0$ the d-invariants of the $\pm n$ surgery on K are given by*

$$d(K_n, i) = d(U_n, i) - 2t_i(K), \quad d(K_{-n}, i) = -d(U_n, i)$$

for $|i| \leq n/2$, where U_n denotes the n surgery on the unknot U.

Before sketching a proof of the theorem we need to explain the notation. The d-invariants are usually associated to spin^c structures on the manifold. The set of spin^c structures on a three-manifold Y is parameterized by $H^2(Y;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$. In the case of interest we have $H_1(K_{\pm n};\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$, hence spin^c structures can be labelled by the elements of $\mathbb{Z}/n\mathbb{Z}$. Such a labelling is assumed in the above theorem and described explicitly as follows. Attaching a 2-handle with framing n to $S^3 \times [0,1]$ along $K \subset S^3 \times \{1\}$ gives a cobordism W_n from S^3 to K_n. Note that $H_2(W_n;\mathbb{Z})$ is generated by the homology class of the core of the 2-handle attached to a Seifert surface for K; denote this generator by f_n. A spin^c structure $\mathfrak{t}$ on K_n is labelled by i if it admits an extension $\mathfrak{s}$ to W_n satisfying

$$\langle c_1(\mathfrak{s}), f_n \rangle - n \equiv 2i \pmod{2n}.$$

If $K = U$ is the unknot, the surgery is the lens space which bounds the disk bundle over S^2 with Euler number n. Using either [12, Proposition 4.8] or [13, Corollary 1.5], the d-invariants of U_n for $n > 0$ and $|i| < n$ are given by

$$d(U_n, i) = \frac{(n - 2|i|)^2}{4n} - \frac{1}{4}.$$

Proof of Theorem 6.1. According to [15, Theorem 4.1] the Heegaard Floer homology groups $HF^+(K_n, i)$ for $n \neq 0$ can be computed from the knot Floer homology of K. The knot complex $C = CFK^\infty(S^3, K)$ is a $\mathbb{Z}^2$-filtered chain complex, which is a finitely generated free module over $\mathcal{T} := \mathbb{Z}[U, U^{-1}]$. We write (i, j) for the components of bidegree, and let $C(i,j)$ denote the subgroup generated by elements with filtration level (i,j). Here U denotes a formal variable in degree -2 that decreases the bidegree by $(1,1)$. The homology of the complex C is $HF^\infty(S^3)=\mathcal{T}$, the homology of the quotient complex $B^+ := C\{i\geq 0\}$ is $HF^+(S^3)=\mathbb{Z}[U,U^{-1}]/U\cdot\mathbb{Z}[U] =: \mathcal{T}^+$ and the homology of $C\{i=0\}$ is $\widehat{HF}(S^3)=\mathbb{Z}$. The complexes $C\{j\geq 0\}$ and B^+ are quasi-isomorphic and we fix a chain homotopy equivalence from $C\{j\geq 0\}$ to B^+.

We now recall the description of $HF^+(K_{\pm n})$ $(n>0)$ in terms of the knot Floer homology of K. For $s\in\mathbb{Z}$, let A_s^+ denote the quotient complex $C\{i\geq 0 \text{ or } j\geq s\}$. Let $v_s^+ : A_s^+\to B^+$ denote the projection and $h_s^+ : A_s^+ \to B^+$ the chain map defined by first projecting to $C\{j\geq s\}$, then applying U^s to identify with $C\{j\geq 0\}$ and finally applying the chain homotopy equivalence to B^+. For any $\sigma\in\{0,1,\ldots,n-1\}$ let $\mathbb{A}_\sigma^+ = \oplus_{s\in\sigma+n\mathbb{Z}} A_s^+$ and $\mathbb{B}_\sigma^+=\oplus_{s\in\sigma+n\mathbb{Z}} B_s^+$, where $B_s^+=B^+$ for all $s\in\mathbb{Z}$. Let $D_{\pm n}^+ : \mathbb{A}_\sigma^+\to\mathbb{B}_\sigma^+$ be a homomorphism that on A_s^+ acts by $D_{\pm n}^+(a_s) = (v_s^+(a_s), h_s^+(a_s))\in B_s^+\oplus B_{s\pm n}^+$. Then $HF^+(K_{\pm n},\sigma)$ is isomorphic to the homology of the mapping cone of $D_{\pm n}^+$, and hence there is a short exact sequence

$$0 \longrightarrow \operatorname{coker}\left(D_{\pm n}^+\right)_* \longrightarrow HF^+(K_{\pm n},\sigma) \longrightarrow \ker\left(D_{\pm n}^+\right)_* \longrightarrow 0, \tag{4}$$

where

$$\left(D_{\pm n}^+\right)_* : H_*(\mathbb{A}_\sigma^+) \longrightarrow H_*(\mathbb{B}_\sigma^+)$$

is the induced map on homology. Moreover, the isomorphism from the homology of the mapping cone of $D_{\pm n}^+$ to $HF^+(K_{\pm n},\sigma)$ is a homogeneous map of degree $\pm d(U_n,\sigma)$. When computing for K_n, the grading on B_s^+, where $s=\sigma+nk$, is such that $U^0\in H_*(B_s^+)$ has grading $2k\sigma+nk(k-1)-1$. In case of $-n$-surgery, $U^0\in H_*(B_{\sigma+kn}^+)$ has grading $-2(k+1)\sigma-nk(k+1)$. In either case $\mathbb{A}_\sigma^+$ is graded so that $D_{\pm n}^+$ has degree -1. (See [15] for more details.)

Suppose now that K is an L-space knot with Alexander polynomial as in (2). Define $\delta_k := 0$ and

$$\delta_l := \begin{cases} \delta_{l+1} - 2(n_{l+1}-n_l)+1 & \text{if } k-l \text{ is odd} \\ \delta_{l+1}-1 & \text{if } k-l \text{ is even} \end{cases}$$

for $l=k-1,k-2,\ldots,-k$, where as above $n_{-l}=-n_l$. Then as in the proof of [14, Theorem 1.2] (compare also [15, §5.1]), $C\{i=0\}$ is (up to quasi-isomorphism) equal to the free abelian group with one generator x_l in bidegree $(0,n_l)$ for $l=-k,\ldots,k$ and the grading of x_l is δ_l. To determine the differentials note that the homology of $C\{i=0\}$ is $\mathbb{Z}$ in grading 0, so generated by (the homology class of) x_k. It follows that the differential on $C\{i=0\}$ is a collection of isomorphisms $C_{0,n_{k-2l+1}}\to C_{0,n_{k-2l}}$ for $l=1,\ldots,k$. Similarly we see that the differential on $C\{j=0\}$ is given by a collection of isomorphisms $C_{n_{k-2l+1},0}\to C_{n_{k-2l},0}$ for $l=1,\ldots,k$. This, together with U-equivariance, determines the filtered chain homotopy type of the complex C. (For an example, see Figure 1.)

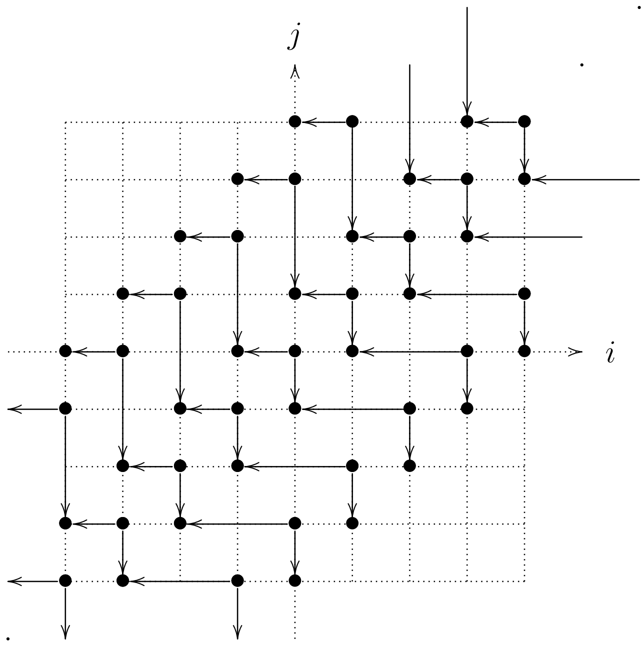

FIGURE 1. **The knot Floer complex** $CFK^\infty(T_{3,5})$**.** Each bullet represents a $\mathbb{Z}$, and each arrow is an isomorphism. The groups and differentials on the axes are determined by the Alexander polynomial

$$\Delta_{T_{3,5}}(T) = -1 + (T + T^{-1}) - (T^3 + T^{-3}) + (T^4 + T^{-4}),$$

and these in turn determine the entire complex.

In order to compute the terms in the short exact sequence (4) we need to understand the maps induced by v_s^+ and h_s^+ on homology. For notational convenience we will use the same symbols v_s^+ and h_s^+ for these induced maps in what follows.

Suppose that the homology H of a quotient complex of C is isomorphic to $\mathcal{T}^+ = \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U]$. We say H starts at (i, j) if the element U^0 has a representative of bidegree (i, j). With this terminology $H_*(B_s^+)$ starts at $(0, n_k)$ and $H_*(C\{j \geq s\})$ starts at $(s, s + n_k)$. It remains to consider $H_*(A_s^+)$. Note that these groups are also isomorphic to $\mathcal{T}^+$. For $s \geq n_k$ the homology of A_s^+ starts at $(0, n_k)$, so $v_s^+ = id$ and $h_s^+ = U^s$. For $n_{k-1} \leq s < n_k$, $H_*(A_s^+)$ starts at $(s - n_k, s)$, so $v_s^+ = U^{n_k - s}$ and

$h^+_s = U^{n_k}$. For $n_{k-2} \leq s < n_{k-1}$ it starts at $(n_{k-1} - n_k, n_{k-1})$, hence $v^+_s = U^{n_k - n_{k-1}}$ and $h^+_s = U^{n_k - n_{k-1} + s}$. It is now easy to observe that for $s \geq 0$ the homology $H_*(A^+_s)$ starts at $(t_s, t_s + n_k)$ and thus $v^+_s = U^{t_s}$ and $h^+_s = U^{t_s + s}$, where t_s is a torsion coefficient of K (compare with equation (3)). Similarly for $s < 0$ we obtain $v^+_s = U^{t_s - s}$ and $h^+_s = U^{t_s}$.

Note that since the spin^c structures corresponding to i and $-i$ are conjugate (so their d invariants are equal) we may restrict to those in the range $\{0, 1, \ldots, \lfloor n/2 \rfloor\}$. Consider now K_n ($n > 0$) and choose some $\sigma \in \{0, 1, \ldots, \lfloor n/2 \rfloor\}$. Writing $(D^+_n)_*$ relative to the decompositions

$$H_*(\mathbb{A}^+_\sigma) = \bigoplus_{l \in \mathbb{Z}} H_*(A^+_{\sigma + ln}) \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{T}^+, \quad H_*(\mathbb{B}^+_\sigma) = \bigoplus_{l \in \mathbb{Z}} H_*(B^+_{\sigma + ln}) \cong \bigoplus_{l \in \mathbb{Z}} \mathcal{T}^+,$$

we obtain for $l > 0$:

$$\begin{aligned}
b_{\sigma+ln} &= U^{t_{\sigma+ln}} a_{\sigma+ln} + U^{t_{\sigma+(l-1)n} + \sigma + (l-1)n} a_{\sigma+(l-1)n} \\
&\qquad \Longrightarrow\ U^{t_{\sigma+ln}} a_{\sigma+ln} = b_{\sigma+ln} - U^{t_{\sigma+(l-1)n} + \sigma + (l-1)n} a_{\sigma+(l-1)n} \\
b_\sigma &= U^{t_\sigma} a_\sigma + U^{t_{n-\sigma}} a_{\sigma-n} \\
&\qquad \Longrightarrow\ U^{t_{n-\sigma}} a_{\sigma-n} = b_\sigma - U^{t_\sigma} a_\sigma \\
b_{\sigma-ln} &= U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln} + U^{t_{(l+1)n-\sigma}} a_{\sigma-(l+1)n} \\
&\qquad \Longrightarrow\ U^{t_{(l+1)n-\sigma}} a_{\sigma-(l+1)n} = b_{\sigma-ln} - U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln}.
\end{aligned}$$

It is straightforward to use the equations above to find an element of $H_*(\mathbb{A}^+_\sigma)$ with any chosen value for a_σ mapping to a given element of $H_*(\mathbb{B}^+_\sigma)$ under $(D^+_n)_*$. It follows that $(D^+_n)_*$ is surjective and its kernel contains one $\mathcal{T}^+$ summand, isomorphic to $H_*(A^+_\sigma)$. Since D^+_n shifts grading by -1, $U^0 \in H_*(B^+_\sigma)$ has grading -1, and the component of $(D^+_n)_*$ from $H_*(A^+_\sigma)$ to $H_*(B^+_\sigma)$ is equal to U^{t_σ}, it follows that $U^0 \in \mathcal{T}^+ \subset \ker D^+_n$ has grading $-2t_\sigma$. The formula for $d(K_n, \sigma)$ now follows using the degree shift between $HF^+(K_n, \sigma)$ and $\ker(D^+_n)_*$.

Finally consider K_{-n} ($n > 0$) and choose $\sigma \in \{0, 1, \ldots, \lfloor n/2 \rfloor\}$. Writing $(D^+_{-n})_*$ with respect to direct sum decompositions as above yields:

$$\begin{aligned}
b_{\sigma+ln} &= U^{t_{\sigma+ln}} a_{\sigma+ln} + U^{t_{\sigma+(l+1)n} + \sigma + (l+1)n} a_{\sigma+(l+1)n} && (l \geq 0) \\
&\qquad \Longrightarrow\ U^{t_{\sigma+ln}} a_{\sigma+ln} = b_{\sigma+ln} - U^{t_{\sigma+(l+1)n} + \sigma + (l+1)n} a_{\sigma+(l+1)n} \\
b_{\sigma-n} &= U^{t_\sigma + \sigma} a_\sigma + U^{t_{n-\sigma} + n - \sigma} a_{\sigma-n} \\
&\qquad \Longrightarrow\ U^{t_\sigma + \sigma} a_\sigma = b_{\sigma-n} - U^{t_{n-\sigma} + n - \sigma} a_{\sigma-n} \\
b_{\sigma-ln} &= U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln} + U^{t_{(l-1)n-\sigma}} a_{\sigma-(l-1)n} && (l \geq 2) \\
&\qquad \Longrightarrow\ U^{t_{(l-1)n-\sigma}} a_{\sigma-(l-1)n} = b_{\sigma-ln} - U^{t_{ln-\sigma} + ln - \sigma} a_{\sigma-ln}\,.
\end{aligned}$$

The top and bottom equations determine a_s (or more precisely $U^N a_s$ for $N >> 0$) for all $s \in \sigma + n\mathbb{Z}$, so the middle equation cannot be fulfilled in general. It follows that $(D^+_{-n})_*$ has cokernel isomorphic to $H_*(B^+_{\sigma-n}) = \mathcal{T}^+$; the grading of U^0 in this group is 0. The formula for $d(K_{-n}, \sigma)$ again follows from the degree shift. $\square$

Combining Theorems 2 and 6.1 yields

Theorem 6.2. *Let $n > 0$ and let K be an L-space knot whose torsion coefficients satisfy*

$$t_i(K) > \begin{cases} \frac{(n-2i)^2+1}{8n} - \frac{1}{4} & \text{if } n \text{ is odd} \\ \frac{(n-2i)^2}{8n} - \frac{1}{4} & \text{if } n \text{ is even} \end{cases}$$

for $0 \leq i \leq n/2$. Then for $0 < m \leq n$, K_m cannot bound a negative-definite four-manifold with no torsion in H_1.

Proof. The formulas follow from the above-mentioned theorems. To see that obstruction for n-surgery to bound a negative-definite manifold implies obstruction for all m-surgeries with $0 < m \leq n$ observe that for fixed i the right-hand side of the inequality is an increasing function of n for $n \geq 2i$. □

Note that the surgery coefficient m in Theorem 6.2 is an integer. In [11] we will consider in more detail the question of which surgeries (including Dehn surgeries) on a knot K can be given as the boundary of a negative-definite four-manifold. In particular it will follow from results in that paper that Theorem 6.2 holds as stated with $m \in \mathbb{Q}$.

7. Example: Surgeries on torus knots

In this section we consider torus knots $T_{p,q}$; we assume $2 \leq p < q$. Right-handed torus knots are L-space knots since for example $pq - 1$ surgery on $T_{p,q}$ yields a lens space [8]. Let $N = (p-1)(q-1)/2$ denote the degree of the Alexander polynomial of $T_{p,q}$. The following proposition gives a simple function which approximates the torsion coefficients of a torus knot.

Proposition 7.1. *The torsion coefficients of $T_{p,q}$ are given by*

$$t_i = \#\{(a,b) \in \mathbb{Z}^2_{\geq 0} \mid ap + bq < N - i\}$$

and they satisfy $t_i \geq g(N-i)$ for $0 \leq i \leq N$, where $x \mapsto g(x)$ is a piecewise linear continuous function, which equals 0 for $x \leq 0$ and whose slope on the interval $[(k-1)q, kq]$ is k/p.

Proof. The (unsymmetrised) Alexander polynomial of $K = T_{p,q}$ is

$$\tilde{\Delta}_K(T) = \frac{(1-T^{pq})(1-T)}{(1-T^p)(1-T^q)},$$

which is a polynomial of degree $2N$. Writing $\tilde{\Delta}_K$ as a formal power series in T we obtain

$$\begin{aligned} \tilde{\Delta}_K(T) &= (1-T^{pq})(1-T)\sum_{a\geq 0} T^{ap} \sum_{b \geq 0} T^{bq} \\ &= (1-T)\sum_{a\geq 0} T^{ap} \sum_{b\geq 0} T^{bq} - T^{pq}(1-T)\sum_{a\geq 0} T^{ap}\sum_{b\geq 0} T^{bq}. \end{aligned}$$

Clearly only the terms in the first product of the last line contribute to the nonzero coefficients of Δ_K. A power T^k appears with coefficient $+1$ in this term whenever $k = ap + bq$ for some $a, b \in \mathbb{Z}_{\geq 0}$, and with coefficient -1 whenever $k - 1 = ap + bq$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Since the coefficient a_j in $\Delta_K(T)$ is the coefficient of T^{N-j} in $\tilde{\Delta}_K(T)$, we obtain $a_j = m_j - m_{j+1}$, where

$$m_j := \#\{(a,b) \in \mathbb{Z}^2_{\geq 0} \,|\, ap + bq = N - j\}\,.$$

Then for $i \geq 0$

$$t_i = \sum_{j>0} j a_{i+j} = \sum_{j>0} m_{i+j} = \#\{(a,b) \in \mathbb{Z}^2_{\geq 0} \,|\, ap + bq < N - i\}\,.$$

In what follows it is convenient to replace t_i by

$$s(i) := t_{N-i} = \#\{(a,b) \in \mathbb{Z}^2_{\geq 0} \,|\, ap + bq < i\}.$$

Define

$$\bar{g}(i) := \begin{cases} i/p & \text{if } i \geq 0 \\ 0 & \text{if } i \leq 0 \end{cases}$$

and $\bar{s}(i) := \lceil g(i) \rceil$; clearly $\bar{s}(i) \geq \bar{g}(i)$ for every i. Separating the set appearing in the definition of $s(i)$ into subsets with fixed value of b we obtain

$$s(i) = \sum_{b \geq 0} \#\{a \in \mathbb{Z}_{\geq 0} \,|\, ap < i - bq\} = \sum_{b \geq 0} \bar{s}(i - bq) \geq \sum_{b \geq 0} \bar{g}(i - bq) =: g(i)\,.$$

□

The following corollary describes the range of surgeries on $T_{p,q}$ that cannot bound negative-definite manifolds according to Theorem 6.2 (and using Proposition 7.1). To obtain the result of Proposition 3 note that all the lower bounds in the corollary allow for $m = 2$. Note that Lisca and Stipsicz have shown that Dehn surgery on $T_{2,2n+1}$ with positive framing r bounds a negative-definite manifold (possibly with torsion in H_1) if and only if $r \geq 4n$ [7].

Corollary 7.2. *Let $2 \leq p < q$ and $N = (p-1)(q-1)/2$.*

- *If p is even and m is less than the minimum of*

$$\begin{array}{l} 1 + \sqrt{4N} \\ 2 + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta) - 4} - \frac{1}{2}\alpha \\ q - p + 3 \end{array}$$

  where $\alpha = q(p-2) + 2$ and $\beta = q - p + 1$

- *or if p is odd and m is less than the minimum of*

$$\begin{array}{l} 1 + \sqrt{4N} \\ 2 + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta) - 4} - \frac{1}{2}\alpha - \frac{q(p-3)}{p} \\ q - p + 5 - \frac{q+2}{p} \end{array}$$

where $\alpha = q(p-4) + 2 + 3q/p$ *and* $\beta = 2q - p + 1$

- *and* $1 \le n \le 2N + m$,

then $+n$*-surgery on* $T_{p,q}$ *cannot bound a negative definite four-manifold with no torsion in* H_1.

Proof. Write $n = 2N + m$; we may assume $n < pq - 1$. It suffices to show that

$$h(x) := \frac{(m+2x)^2+1}{8n} - \frac{1}{4} < g(x) \tag{5}$$

for $-m/2 \le x \le N$, where g is the function appearing in Proposition 7.1.

Since h is convex and g is piecewise linear, it is enough to check the inequality for $x = kq$ with $k = 0, 1, \ldots, \lfloor N/q \rfloor$ and for $x = N$. Consider first $x = kq$. From the definition of g we obtain $g(kq) = qk(k+1)/2p$. Substituting this into (5) we obtain

$$4k^2(q^2 - n\frac{q}{p}) + 4k(qm - n\frac{q}{p}) + m^2 + 1 - 2n < 0\,.$$

Since $q - n/p > 1/p$, it suffices to consider only $k = 0$ and $k = \lfloor N/q \rfloor$. For $k = 0$ the last inequality yields $m < 1 + \sqrt{4N}$.

Assume first p is even; then $\lfloor N/q \rfloor = p/2 - 1$ and substituting this into (5) gives

$$(m + q(p-2))^2 + 1 < n(2 + q(p-2)).$$

Writing $\mu = m + q(p-2)$, $\alpha = q(p-2) + 2$ and $\beta = q - p + 1$ the last inequality becomes

$$\mu^2 - \alpha\mu + 1 - \alpha\beta < 0\,,$$

which implies

$$\mu < \frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha(\alpha + 4\beta) - 4}\,;$$

this is equivalent to the second condition on m in the statement of the corollary.

Since the slope of g on the interval from $(p/2-1)q$ to N is $1/2$, we get $g(N) = g((p/2-1)q) + (N - (p/2-1)q)/2 = (pq - 2p + 2)/8$. Subsituting this into (5) gives

$$n^2 - n(pq - 2p + 4) + 1 < 0\,,$$

which holds if $n < pq - 2p + 4$ or $m < q - p + 3$.

Assume now p is odd; then $\lfloor N/q \rfloor = (p-3)/2$ and substituting this into (5) gives

$$(m + q(p-3))^2 + 1 < n(2 + \frac{q}{p}(p-1)(p-3)).$$

Writing $\mu = m + q(p-3)$, $\alpha = q(p-4) + 2 + 3q/p$ and $\beta = 2q - p + 1$ the last inequality becomes

$$\mu^2 - \alpha\mu + 1 - \alpha\beta < 0\,,$$

from which the second condition on m in the statement of the corollary follows.

Now the slope of g on the interval from $q(p-3)/2$ to N is $(p-1)/2p$ and $g(N) = (pq - 2p + 4 - (q+2)/p)/8$. Subsituting this into (5) gives

$$n^2 - n(pq - 2p + 6 - \frac{q+2}{p}) + 1 < 0\,,$$

which holds if $n < pq - 2p + 6 - \frac{q+2}{p}$ or $m < q - p + 5 - \frac{q+2}{p}$. $\square$

References

[1] J. P. Alexander, G. C. Hamrick & J. W. Vick. *Linking forms and maps of odd prime order*, Trans. Amer. Math. Soc. **221** (1976), 169–185.

[2] N. D. Elkies. *A characterization of the Z^n lattice*, Math. Res. Lett. **2** (1995), 321–326.

[3] K. A. Frøyshov. *The Seiberg-Witten equations and four-manifolds with boundary*, Math. Res. Lett. **3** (1996), 373–390.

[4] J. E. Greene. *L-space surgeries, genus bounds, and the cabling conjecture*, arXiv:1009.1130, 2010.

[5] A. Kawauchi & S. Kojima. *Algebraic classification of linking pairings on 3-manifolds*, Math. Ann. **253** (1980), 29–42.

[6] M. Kneser & D. Puppe. *Quadratische Formen und Verschlingungsinvarienten von Knoten*, Math. Zeitschr. **58** (1953), 376–384.

[7] P. Lisca & A. I. Stipsicz. *Ozsváth-Szabó invariants and tight contact structures, I*, Geometry and Topology **8** (2004), 925–945.

[8] L. Moser. *Elementary surgery along a torus knot*, Pacific Journal of Math. **38** (1971), 737–745.

[9] B. Owens & S. Strle. *Rational homology spheres and the four-ball genus of knots*, Advances in Mathematics **200** (2006), 196–216.

[10] B. Owens & S. Strle. *A characterisation of the $n\langle 1\rangle \oplus \langle 3\rangle$ form and applications to rational homology spheres*, Math. Res. Lett. **13** (2006) 259–271.

[11] B. Owens & S. Strle. *Dehn surgery and negative-definite manifolds*, Selecta Math. (N.S.) (2012), DOI: 10.1007/s00029-012-0086-2.

[12] P. Ozsváth & Z. Szabó. *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Advances in Mathematics **173** (2003), 179–261.

[13] P. Ozsváth & Z. Szabó. *On the Floer homology of plumbed three-manifolds*, Geometry and Topology **7** (2003), 225–254.

[14] P. Ozsváth & Z. Szabó. *On knot Floer homology and lens space surgeries*, Topology **44** (2005), 1281–1300.

[15] P. Ozsváth & Z. Szabó. *Knot Floer homology and integer surgeries*, Algebr. Geom. Topol. **8** (2008), 101–153.

[16] P. Ozsváth & Z. Szabó. *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), 1–68.

[17] J.-P. Serre. *A course in arithmetic*, Springer, 1973.

[18] L. R. Taylor. *Relative Rochlin invariants*, Topology Appl. **18** (1984), 259–280.

[19] C. T. C. Wall. *Quadratic forms on finite groups, and related topics*, Topology **2** (1963), 281–298.

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