Chapter 5

Line and surface integrals

5.1 Line integrals in two dimensions

Instead of integrating over an interval $[a, b]$ we can integrate over a curve C. Such integrals are called *line integrals*. They were invented in the early 19th century to solve problems involving forces, fluid flow and magnetism.

Work Done We begin by recalling some basic ideas about *work done*. The *work done* W, by a variable force $f(x)$ in moving a particle from a point a to a point b along the x-axis is

$$W = \int_a^b f(x)dx = \sum f(x)\delta x = \text{Force} \times \text{distance=Work.}$$

We now generalise this idea to a particle moving a long a general curve C and this gives a line integral.

Suppose that the force is given by the vector $\mathbf{F}$ in the direction $\overrightarrow{PR}$ pointing as shown in Figure 5.1. If the force moves the object from P to Q, then the *displacement vector* is $\mathbf{D} = \overrightarrow{PQ}$. The *work done* done by this force is defined to be the product of the component of the force along $\mathbf{D}$ and the distance moved:

$$W = |\mathbf{D}||\mathbf{F}| \cos\theta = \mathbf{F} \cdot \mathbf{D}\ ,$$

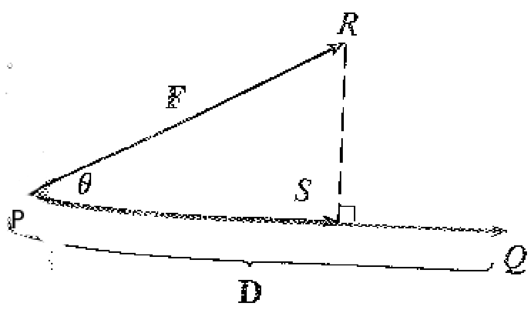

Figure 5.1: The force acting in the $\overrightarrow{PQ}$ direction is $|\mathbf{F}| \cos\theta$

So then if $\mathbf{r}(t) = (x(t), y(t))$ describes the parameterised curve C, it follows that $d\mathbf{r}$ is small step along that curve and hence

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \text{Force} \times \text{distance} \; .$$

The notation is as follows, $d\mathbf{r} = (dx, dy)$ and $\mathbf{F} = (P(x, y), Q(x, y))$. For the purposes of this section we only consider two dimensions, but this can easily be extended to higher dimensions. So in two dimensions the work done by moving a particle along the curve C is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy \; ,$$

where $P(x, y)$ is the force in the x direction and $Q(x, y)$ is the force in the y direction.

It is usually helpful to parameterise the curve C using a parameter t, say. Starting with a plane curve C the parametric equations are given by

$$x = x(t), \quad y = y(t) \quad a \leq t \leq b$$

thus, $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. So we can change variables on the line integral by writing $d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt$. This gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \frac{d\mathbf{r}}{dt} dt = \int_a^b P(x(t), y(t)) \frac{dx}{dt} dt + \int_a^b Q(x(t), y(t)) \frac{dy}{dt} dt \; .$$

Example 5.1 Find the work done by the force $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the curve which runs from $(1, 0)$ to $(0, 1)$ along the unit circle and then from $(0, 1)$ to $(0, 0)$ along the y-axis (see Figure 5.2).

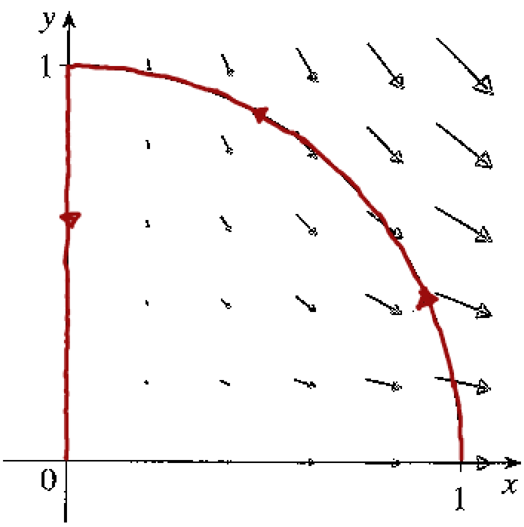

Figure 5.2: Shows the force field $\mathbf{F}$ and the curve C. The work done is negative because the field impedes the movement along the curve.

Solution :

Answer: Workdone=-2/3. Notice the order of limits must reflect the direction along the curve. Work done is negative because the force field impedes the movement along the cure. □

Example 5.2 Evaluate the line integral $\int_C (y^2)dx+(x)dy$, where C is the is the arc of the parabola $x = 4-y^2$ from $(-5, -3)$ to $(0, 2)$

Solution :

Answer: $\frac{245}{6}$. $\square$

Remark When the curve C is something simple like a straight line then it is often easier to not parameterise the curve and instead use $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y)dx + Q(x, y)dy$ as it stands, as we shall see in the following example.

Example 5.3 Evaluate the line integral, $\int_C (x^2 + y^2)dx + (4x + y^2)dy$, where C is the straight line segment from $(6, 3)$ to $(6, 0)$.

Solution :

Answer: -81. $\square$

5.2 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane D bounded by C. (See Figure 5.4. We assume that D consists of all points inside C as well as all points on C). In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to moving round C in an **anitclockwise** direction. The region D is always on the left as we move round C. (Warning: if you move round C in the clockwise direction you get negative the integral you get when you go round in the anticlockwise direction).

Green's Theorem in two dimensions

Theorem *Let C be a positively oriented simple closed curve in the plane and let D be the region bounded by C. If $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region that contains D, then*

$$\int_C P(x, y)dx + Q(x, y)dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \ .$$

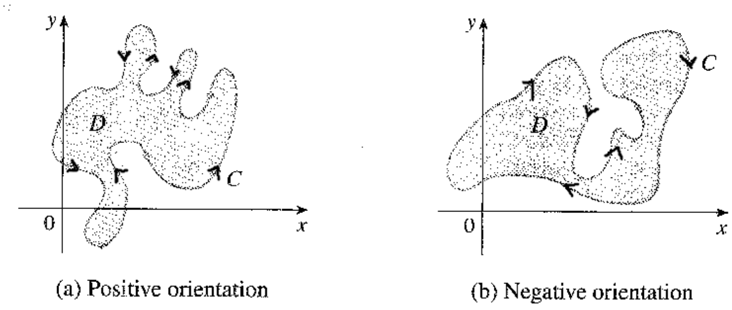

Figure 5.3: Closed curves C.

Example 5.4 Use Green's Theorem to evaluate $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$, where C is the circle $x^2 + y^2 = 9$.

Solution :

Answer:36π $\square$

Example 5.5 Evaluate $\int_C (3x - 5y)dx + (x - 6y)dy$, where C is the ellipse $\frac{x^2}{4} + y^2 = 1$ in the anticlockwise direction. Evaluate the integral by (i) Green's Theorem, (ii) directly.

Solution :

$\square$

5.3 Surface integrals

Consider a crop growing on a hillside S, Suppose that the crop yeild per unit surface area varies across the surface of the hillside and that it has the value $f(x, y, z)$ at the point (x, y, z). We may then ask what is the total yield of the crop over the whole surface of the hillside, a surface integrals will give the answer to this question.

Consider a small element of surface δS containing the point (x, y, z). Then assuming that f is well behaved the contribution to the total crop from this small element of surface is $f(x, y, z)\delta S$. Summing over all elements of surfave amd taking the limit as $\delta S \to 0$ we obtain the *surface integral of f over the surface S*.

$$\int\int_S f(x, y, z) dS \ .$$

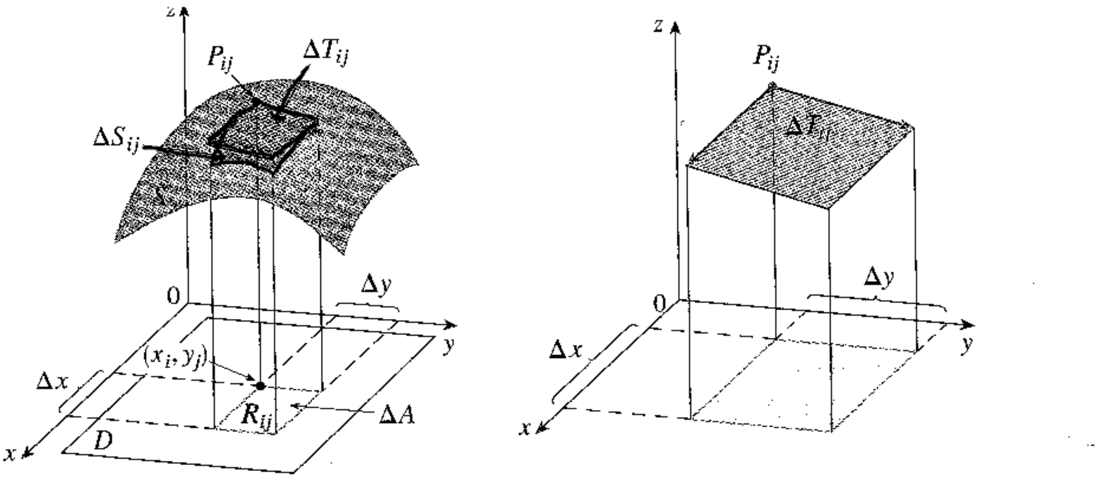

Figure 5.4: Shows the surface S and the tangent plane.

Evaluating a surface integral

We need to relate δS to the area of an element at the base $\delta x \delta y$ as shown in Figure **??**. For a curved surface this relationship changes with x and y. In the special case where the surface S can be expressed as $z = z(x, y)$, or $\mathbf{r} = (x, y, z(x, y))$ the plane tangent to the surface at a point approximates a small piece of surface very well. The vector tangent to the surface in the x-directions is $\mathbf{r}_x = (1, 0, \frac{\partial z}{\partial x})$. The vector tangent to the surface in the y-direction is $\mathbf{r}_y = (0, 1, \frac{\partial z}{\partial y})$. The area of the tangent plane is then the area of the plane with sides of length δx and δy and direction given by the two tangent vectors. The area is the area of a parallelogram, with sides $\delta x \mathbf{r}_x$ and $\delta y \mathbf{r}_y$. The area of a parallelogram is given by the magnitude of the cross product which is given by

$$\delta S \approx |\mathbf{r}_x \delta x \times \mathbf{r}_y \delta y| = |\mathbf{r}_x \times \mathbf{r}_y| \delta x \delta y = \left|(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)\right| \delta x \delta y = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \delta x \delta y$$

Rule for evaluating surface integrals

Using the above explanation we can replace dS by $\sqrt{1 + (\frac{\partial z}{\partial x})^2 + \left(\frac{\partial z}{\partial y}\right)^2} \delta x \delta y$ in the surface integral

$$\int\int_S f(x, y, z) dS = \int\int_D f(x, y, z) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx\, dy \ ,$$

where D is the projection of S onto the xy-plane.

Example 5.6 Evaluate

$$\int\int_S z^2 \, dS$$

where S is the hemisphere given by $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

Solution :

Answer: $2\pi/3$. $\square$

Remark A surface integral can also be used to calculate the area of a surface S.

$$\int\int_S 1\ dS = \text{Area of surface S}$$

An intuition for this can be obtained be thinking about the crop analogy again. If the crop density is 1kg/square metre ($f = 1$), and the total crop is 65kg ($\int\int_S 1\ dS = 65$), then the area of the crop is 65 square metres (Area of S=65).

Example 5.7 Find the area of the ellipse cut on the plane $2x + 3y + 6z = 60$ by the circular cylinder $x^2 = y^2 = 2x$.

Solution :

Answer: $7\pi/6$. □

Normal direction to a surface

In order to define *surface integrals of vector fields*, we need to consider *orientable surfaces* (2 -sided). The mobius strip is an example of a nonorientable surface (1-sided). We use the *normal* to the surface to give the surface orientation. The *normal* to the surface at a given point is the direction perpendicular to the tangent plane at that point. There are two possible possible normals, one points in the opposite direction to the other. So there are two possible orientations for any orientable surface (see Figure 5.5).

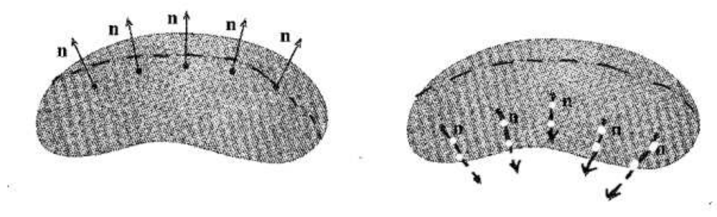

Figure 5.5: The two orientations of an orientable surface.

Remarks

1. For a surface in the form $f(x,y,z)=0$ the *normal vector* is given by

$$\mathbf{n}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

2. For a surface in the form $z=z(x,y)$ the *normal vector* is given by

$$\mathbf{n}=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},1\right)$$

This one follows from the fact that $\mathbf{r}_x \times \mathbf{r}_y$ is normal to the vectors $\mathbf{r}_x$ and $\mathbf{r}_y$ which lie in the tangent plane (see section 5.3).

Examples

1. For the plane $2x+7y+3z=50$ we have $f(x,y,z)=2x+7y+3z-50=0$, so the normal is,

$$\mathbf{n}=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)=(2,7,3)$$

as expected.

2. For the sphere $x^2+y^2+z^2-a^2=0$, the normal is, $(2x,2y,2z)$ or (x,y,z) or $(x/a,y/a,z/a)$ i.e. along the radius vector from the centre of the sphere.

5.3.1 Surface integral of a vector field

Imagine a fluid flowing through a surface S. (Think of S as an imaginary fishing net, so it doesn't impede the flow). $\mathbf{F}$ is the force field and it is related to the velocity and density of the fluid flowing through the surface. A measure of the total flux (flow) across the surface is given by

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{n}$ is the *unit normal.* $\mathbf{F}\cdot\mathbf{n}\,dS$ tells us the mass of fluid flowing across a region dS in the direction of $\mathbf{n}$.

Remark Some books use the alternative notation

$$\int\int \mathbf{F}\cdot\mathbf{dS}$$

for $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$. Notice in the alternative notation that $\mathbf{dS}$ is a vector.

5.4 Gauss' Divergence Theorem

Gauss' Divergence Theorem will help us calculate $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$ There are some similarities between Green's Theorem on $\mathbb{R}^2$ and Gauss' Divergence Theorem in $\mathbb{R}^3$ in the following respects:

Green's Theorem: Line integral round a boundary curve C of a *closed* region in $\mathbb{R}^2$=Double integral over the *enclosed* 2-dimensional region.

Gauss' Theorem: Surface integral over a boundary surface S of a *closed* region in $\mathbb{R}^3$=Triple integral over the *enclosed* 3-dimensional region.

Remark Note that for Green's Theorem the curve **must** be a *closed curve* and for Gauss' Theorem the surface **must** be a *closed surface*. We will not prove Gauss's theorem, but advanced books such as Stewart Calculus contain proofs. Gauss's Divergence Theorem is named after Gauss (1777-1855) who discovered it during his work on electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician who also discovered and published this result in 1826.

Result: Gauss' Divergence Theorem

Let V be a closed bounded volume on $\mathbb{R}^3$ with boundary surface S, given with positive (*outward*) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region containing V. Then

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\, dS = \int\int\int_V \operatorname{div}\mathbf{F}\; dx\; dy\; dz\ ,$$

where $\mathbf{n}$ denotes the outward pointing *unit normal* at each point on the surface S.

Example 5.8 Use Gauss' Divergence Theorem to evaluate

$$I = \int\int_S x^4y + y^2z^2 + xz^2\; dS,$$

where S is the entire surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution :

Answer: $4\pi/15$. □

Example 5.9 Find $I = \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{F} = (2x, 2y, 1)$ and where S is the entire surface consisting of S_1=the part of the paraboloid $z = 1 - x^2 - y^2$ with $z = 0$ together with S_2=disc $\{(x, y) : x^2 + y^2 \leq 1\}$. Here $\mathbf{n}$ is the outward pointing unit normal.

Solution :

Answer: 2π.

$\square$

5.5 Curvilinear line integrals in $\mathbb{R}^3$

In section 5.1 we considered line integrals of the form $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C was a curve in $\mathbb{R}^2$ this formula works equally well in three dimensions. Take $\mathbf{F} = (P(x,y,z), Q(x,y,z), R(x,y,z))$ and $d\mathbf{r} = (dx, dy, dz)$ and now consider C as a curve in $\mathbb{R}^3$. This integral now represents the work done to move a particle along a cure in $\mathbb{R}^3$.

Independence of path

If we consider two curves C_1 and C_2 (which are called *paths*) with the same initial point A and the same end point B. We know that in general $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$. However when $\mathbf{F} = \nabla\phi$ for some continuous scalar-valued function ϕ then we have $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ and we say that the line integral is **path independent**.

When we can find a scalar-valued function ϕ such that $\mathbf{F} = \nabla\phi$ we say that $\mathbf{F}$ is a **conservative vector field** and we denote ϕ as the **potential function**. The fact that $\mathbf{F}$ is conservative ensures the independence of path and gives an integral that is related to the Fundamental Theorem of Calculus which states $\int_a^b F'(x)dx = F(b) - F(a)$. In fact, $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla\phi\ d\mathbf{r} = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a))$, where $\mathbf{r}(t)$ is the parameterised curve and the parameter t satisfies $a \leq t \leq b$.

Conservative vector fields

The grad of every smooth scalar field is a vector field. It is natural to ask whether *all* vector fields are the grad of some scalar field. In general, the answer is "no", but we can characterise those vector fields $\mathbf{F}$ for which this is the case. If there exists a scalar field ϕ such that the vector field $\mathbf{F} = \operatorname{grad}\phi = \nabla\phi$, we say that $\mathbf{F}$ is *conservative* and ϕ is called a *potential* for $\mathbf{F}$.

These names reflect an application of this notion in physics; a force (vector field) that does not expend energy is said to be conservative and can be written as the gradient of a potential energy (scalar field). Gravitational force is a conservative force, whereas friction is not.

We have already seen (See Example 4.12) that $\operatorname{curl}\operatorname{grad}\phi = \mathbf{0}$ for all smooth scalar fields ϕ. This means that if $\mathbf{F} = \operatorname{grad}\phi$ for some ϕ then $\operatorname{curl}\mathbf{F} = \mathbf{0}$. This is a *necessary* condition for $\mathbf{F}$ to be conservative (i.e. if $\mathbf{F}$ is to be conservative then we must have $\operatorname{curl}\mathbf{F} = \mathbf{0}$). For a vector field that is defined everywhere then it is also *sufficient* (i.e. if $\mathbf{F}$ is defined everywhere and $\operatorname{curl}\mathbf{F} = \mathbf{0}$ then $\mathbf{F}$ is conservative).

Example 5.10 Vector fields $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$\mathbf{V} = (2x - 3y + z, -3x - y + 4z, 4y + z)$$

$$\mathbf{W} = (2x - 4y - 5z, -4x + 2y, -5x + 6z)\ .$$

One of these is conservative while the other is not. Determine which is conservative and denote it by $\mathbf{F}$. Find a potential function ϕ for $\mathbf{F}$ and evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}\ ,$$

where C is the curve from A(1,0,0) to B(0,0,1) in which the plane $x + z = 1$ cuts the hemisphere given by $x^2 + y^2 + z^2 = 1$, $y \geq 0$.

Solution :

Answer: $\square$