Chapter 1

Partial differentiation

Example 1.1 What is the maximal domain of the real function $g$ defined by $g(x) = \sqrt{x^2 + 3x + 2}$?

Solution: The key point is that the square root only gives a real result if the argument is non-negative. Hence we need to identify those $x$ for which $x^2 + 3x + 2 \geq 0$. Factorizing this quadratic gives $x^2 + 3x + 2 = (x + 2)(x + 1) \geq 0 \iff x \leq -2$ or $x \geq -1$.

(To see this, draw the graph of $(x + 2)(x + 1)$). Hence the maximal domain is $(-\infty, -2] \cup [-1, \infty)$. □

Example 1.2 Determine the maximal domain of the function $f$ defined by $f(x, y) = \sqrt{1 - x^2 - y^2}$.

Solution: Since the square root is defined if and only if $1 - x^2 - y^2 \geq 0$, $f$ has maximal domain $D = \{(x, y) : x^2 + y^2 \leq 1\}$, which is the unit disk centre $(0, 0)$. □

Example 1.3 Sketch the graph of $f(x, y) = -\sqrt{1 - 2x - x^2 - y^2}$.

Solution: Let $z = f(x, y)$. Completing the square, we have

\[
z^2 = 1 - 2x - x^2 - y^2 = 2 - (x + 1)^2 - y^2,
\]

i.e. $(x + 1)^2 + y^2 + z^2 = 2$. This is the sphere with centre $(-1, 0, 0)$ and radius $\sqrt{2}$. The part given by $z = -\sqrt{1 - 2x - x^2 - y^2} (\leq 0)$ is the hemisphere below the $x, y$-plane. See Figure 1.1. □

Example 1.4 By considering the level curves and the cross-sections $x = 0$ and $y = 0$, obtain a sketch of $z = \sqrt{x^2 + y^2}$.

Solution: The level curves are defined by

\[
L_c = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = c\}.
\]

For $c < 0$, $L_c = \emptyset$ (since $\sqrt{\cdot} \geq 0$), $L_0 = \{(0, 0)\}$ (since $x^2 + y^2 = 0 \iff x = y = 0$) and for $c > 0$, $L_c = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = c^2\}$, the circle of radius $c$, centre $(0, 0)$.

Fixing $x = 0$ we get $z = \sqrt{y^2} = |y|$ and fixing $y = 0$ we get $z = |x|$. These cross-sections are illustrated in Figure 1.2.

Putting this information together, we see that the surface defined by $z = \sqrt{x^2 + y^2}$ is a (circular) cone with vertex at $(0, 0)$ (Figure 1.3). □
Example 1.5 Sketch the part of the surface

\[ 2x + y + 4z = 1, \]

where \( x, y, z \geq 0. \)

Solution : We consider the cross-section with the coordinate planes \( x = 0 \) (\( y, z \)-plane), \( y = 0 \) (\( x, z \)-plane) and \( z = 0 \) (\( x, y \)-plane).

The cross-section of \( 2x + y + 4z = 1 \) with \( x = 0 \) is the line \( y + 4z = 1 \) (lying in the \( y, z \)-plane). This passed through the points \((0, 0, \frac{1}{4})\) and \((0, 1, 0)\). In a similar way we obtain the cross-section with the other coordinate planes; \( 2x + 4z = 1 \) in the \( x, z \)-plane, passing through \((0, 0, \frac{1}{4})\) and \((\frac{1}{2}, 0, 0)\) and \( 2x + y = 1 \) in the \( x, y \)-plane, passing through \((0, 1, 0)\) and \((\frac{1}{2}, 0, 0)\).

A sketch of the plane is shown in Figure 1.4.

Example 1.6 Find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) where

(a) \( f(x, y) = x^3y^2 + x, \) \quad (b) \( f(x, y) = \sin(x + xy). \)

Solution :

(a) To calculate the partial \( x \) derivative, we think of \( y \) as a constant and differentiate in the usual way with respect to \( x \). Hence, we have

\[
\frac{\partial f}{\partial x} = y^2 \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial x}(x) = 3x^2y^2 + 1.
\]

For the \( y \) derivative, we think of \( x \) as a constant and differentiate with respect to \( y \);

\[
\frac{\partial f}{\partial y} = x^3 \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(x) = 2x^3y.
\]

(b) Let \( u = x + xy \). We have \( f(x, y) = \sin u \). Then by the chain rule (for a function of one variable),

\[
\frac{\partial f}{\partial x} = \frac{d}{du}(\sin u) \frac{\partial u}{\partial x} = \cos u \times (1 + y) = (1 + y) \cos(x + xy).
\]
It was only for the purpose of explanation that the substitution \( u \) was made. When familiar with the process, the chain rule may be used directly

\[
\frac{\partial f}{\partial y} = \cos(x + xy) \frac{\partial}{\partial y}(x + xy) = x \cos(x + xy).
\]

\[\square\]

**Example 1.7** Find \( \frac{\partial z}{\partial x} \) where \( z = \sin^{-1}\left(\frac{x}{x+y}\right) \) and \( x, y > 0 \).

[Note that \( \sin^{-1} u \) is the inverse sine function (sometimes written as \( \arcsin u \)), and not the reciprocal \( 1/\sin u \). The domain of \( \sin^{-1} \) is \([-1, 1]\) and, since \( x, y > 0 \), \( x/(x + y) \) lies in this domain.]

**Solution** : Let \( u = x/(x+y) \). So, by the chain rule

\[
\frac{\partial z}{\partial x} = \frac{d}{du}(\sin^{-1} u) \frac{\partial u}{\partial x}.
\]

We have

\[
\frac{d}{du}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}} = \frac{1}{\sqrt{1 - \left(\frac{x}{x+y}\right)^2}} = \frac{|x+y|}{\sqrt{(x+y)^2-x^2}} = \frac{x+y}{\sqrt{2xy+y^2}},
\]
Figure 1.3: The cone $z = \sqrt{x^2 + y^2}$

Figure 1.4: The plane $2x + y + 4z = 1$

since $x, y > 0$. Also, by the quotient rule,

\[
\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \left( (x + y) \right) - x \frac{\partial}{\partial x} (x + y) \frac{1}{(x + y)^2} = \frac{y}{(x + y)^2}.
\]

Hence

\[
\frac{\partial z}{\partial x} = \frac{y}{x + y} \frac{1}{\sqrt{2xy + y^2}}.
\]

□

**Example 1.8** Let $u = f(r)$ where $r^2 = x^2 + y^2 + z^2$. Show that

\[x u_x + y u_y + z u_z = r f'(r).\]

**Solution** : By the chain rule,

\[u_x = \frac{\partial u}{\partial x} = \frac{df}{dr} \frac{\partial r}{\partial x} = f'(r) \frac{\partial r}{\partial x},\]
and similarly,
\[ u_y = f'(r) \frac{\partial r}{\partial y}, \quad u_z = f'(r) \frac{\partial r}{\partial z}. \]

Now
\[ \frac{\partial}{\partial x} (r^2) = \frac{\partial}{\partial x} (x^2 + y^2 + z^2) \quad \text{i.e.,} \quad 2r \frac{\partial r}{\partial x} = 2x. \]

Therefore
\[ \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}. \]

and similarly,
\[ \frac{\partial r}{\partial y} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{x}{r}. \]

Thus we have
\[ xu_x + yu_y + zu_z = \frac{x^2}{r} f'(r) + \frac{y^2}{r} f'(r) + \frac{z^2}{r} f'(r) = \frac{(x^2 + y^2 + z^2)}{r} f'(r) = r f'(r), \]
as required.

\[ \square \]

**Example 1.9** Determine all second order derivatives of \( u = \sin xy \) and verify that \( u_{xy} = u_{yx} \).

**Solution** : We have first derivatives
\[ u_x = y \cos xy, \quad u_y = x \cos xy. \]

Hence, the second derivatives are
\[ u_{xx} = \frac{\partial}{\partial x} (u_x) = y \frac{\partial}{\partial x} (\cos xy) = -y^2 \sin xy, \]
\[ u_{xy} = \frac{\partial}{\partial y} (u_x) = \frac{\partial}{\partial y} (\cos xy) + x \frac{\partial}{\partial y} (\cos xy) = \cos xy - yx \sin xy, \]
\[ u_{yx} = \frac{\partial}{\partial x} (u_y) = \frac{\partial}{\partial x} (\cos xy) + x \frac{\partial}{\partial x} (\cos xy) = \cos xy - xy \sin xy, \]
\[ u_{yy} = \frac{\partial}{\partial y} (u_y) = x \frac{\partial}{\partial y} (\cos xy) = -x^2 \sin xy. \]

Hence \( u_{xy} = u_{yx} = \cos xy - xy \sin xy \) as required.

\[ \square \]

**Example 1.10** Let \( u = f(x/y) \), where \( f \) is an arbitrary (twice differentiable, with continuous second derivative) function of one variable. Show that
\[ xu_x + yu_y = 0, \]
and deduce that
\[ x^2 u_{xx} + 2xy u_{xy} + y^2 u_{yy} = 0. \]
Solution: Using the chain rule, we have,

\[ u_x = f'(\frac{x}{y}) \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = \frac{1}{y} f' \left( \frac{x}{y} \right), \]

\[ u_y = f'(\frac{x}{y}) \frac{\partial}{\partial y} \left( \frac{x}{y} \right) = -\frac{x}{y^2} f' \left( \frac{x}{y} \right). \]

So,

\[ xu_x + yu_y = x \frac{1}{y} f' \left( \frac{x}{y} \right) - y \frac{x}{y^2} f' \left( \frac{x}{y} \right) = 0. \]

[Although we could proceed by calculating \( u_{xx} \), \( u_{xy} \) and \( u_{yy} \) and taking the appropriate combination, it is much less work to deduce the final part as indicated below.]

Since \( xu_x + yu_y = 0 \), its \( x \)- and \( y \)-derivatives must also equal 0. Hence

\[ xu_{xx} + u_x + yu_{yx} = 0, \hspace{1cm} (1) \]

and

\[ xu_{xy} + yu_{yy} + u_y = 0. \hspace{1cm} (2) \]

Taking \( x \times (1) + y \times (2) \) [the need to have the correct coefficient for \( u_{xx} \) and \( u_{yy} \) dictates the choice of this combination of (1) and (2)] we get

\[ x^2 u_{xx} + xu_x + xyu_{yx} + yu_{xy} + y^2 u_{yy} + yu_y = 0. \]

Since \( u_{xy} = u_{yx} \) and \( xu_x + yu_y = 0 \), we get

\[ x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0. \]

as required.

□

Example 1.11 Let \( w = u^2 + v^2 \) where \( u = \sin \theta \) and \( v = \cos \phi \). Use the chain rule to calculate \( w_\theta \) and \( w_\phi \) in terms of \( \theta \) and \( \phi \).

Solution: Using the chain rule, we have

\[ w_\theta = \frac{\partial u}{\partial \theta} \frac{\partial w}{\partial u} + \frac{\partial v}{\partial \theta} \frac{\partial w}{\partial v} = \cos \theta \cdot 2u + 0.2v = 2 \cos \theta \sin \theta = \sin 2\theta, \]

and

\[ w_\phi = \frac{\partial u}{\partial \phi} \frac{\partial w}{\partial u} + \frac{\partial v}{\partial \phi} \frac{\partial w}{\partial v} = 0.2u + (- \sin \phi) \cdot 2v = -2 \sin \phi \cos \phi = -2 \sin 2\phi. \]

□

Example 1.12 Find the general solution of the PDE,

\[ \frac{\partial f}{\partial x} = x^2 + y + 9, \]

where \( f \) is a function of two independent variables \( x \) and \( y \).
Solution: Integrating with respect to $x$ and treating $y$ as fixed gives

$$f = \int_y^{\text{fixed}} x^2 + y + 9dx = \frac{x^3}{3} + xy + 9x + A(y),$$

where $A$ is an arbitrary function depending on the fixed variable $y$. □

Example 1.13 Find the general solution of the PDE,

$$\frac{\partial^2 f}{\partial x \partial y} = 2x,$$

where $f$ is a function of two independent variables $x$ and $y$.

Solution The PDE can be expressed as

$$\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = 2x,$$

Integrating with respect to $x$ and treating $y$ as fixed gives

$$\frac{\partial f}{\partial y} = \int_y^{\text{fixed}} 2xdx = x^2 + A(y),$$

where $A$ is an arbitrary function depending on the fixed variable $y$. Integrating with respect to $y$, holding $x$ fixed then gives

$$f = \int_x^{\text{fixed}} x^2 + A(y)dy = x^2y + \int_x^{\text{fixed}} A(y)dy + B(x) = x^2y + C(y) + B(x).$$

$B$ is an arbitrary function of the fixed variable $x$. Since $A$ was an arbitrary function of $y$ its integral is also and arbitrary function of $y$ so let’s call this function $C(y) = \int_x^{\text{fixed}} A(y)dy$. □

Example 1.14 By changing variables from $(x, y)$ to $(u, v)$, where $u = xy$, $v = x/y$, solve the PDE

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2x^2 \sin(xy).$$

Solution By the chain rule,

$$\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial z}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial z}{\partial v} = y \frac{\partial z}{\partial u} + \frac{1}{y} \frac{\partial z}{\partial v},$$

and

$$\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial z}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial z}{\partial v} = \frac{x}{u} \frac{\partial z}{\partial u} - \frac{x}{y^2} \frac{\partial z}{\partial v}.$$ Therefore,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x \left( y \frac{\partial z}{\partial u} + \frac{1}{y} \frac{\partial z}{\partial v} \right) + y \left( \frac{x}{u} \frac{\partial z}{\partial u} - \frac{x}{y^2} \frac{\partial z}{\partial v} \right) = 2xy \frac{\partial z}{\partial u}.$$ Inverting the change of variables we have

$$x = \sqrt{uv}, \quad y = \frac{\sqrt{u}}{v}.$$
and so, after the change of variable the PDE becomes,
\[ 2u \frac{\partial z}{\partial u} = 2uv \sin u, \]
i.e.,
\[ \frac{\partial z}{\partial u} = v \sin u. \]
Then
\[ z = \int_{v \text{ fixed}} v \sin u \, du = -v \cos u + A(v), \]
and in terms of \( x \) and \( y \) this is \( z = -\frac{x}{y} \cos(xy) + A \left( \frac{z}{y} \right) \), where \( A \) is an arbitrary function. \( \square \)

**Example 1.15** By changing variables from \((x, y)\) to \((u, v)\), where \( u = x^3/y, \ v = x \), find \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) in terms of parital derivatives with respect to \( u \) and \( v \). Hence, solve the PDE
\[ x \frac{\partial f}{\partial x} + 3y \frac{\partial f}{\partial y} = \frac{6x^5}{y}. \]

**Solution** By the chain rule,
\[ \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial f}{\partial v} = \frac{3x^2}{y} \frac{\partial f}{\partial u} + 1 \frac{\partial f}{\partial v}, \]
and
\[ \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial f}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial f}{\partial v} = -\frac{x^3}{y^2} \frac{\partial f}{\partial u} - 0 \frac{\partial f}{\partial v}. \]
Therefore,
\[ x \frac{\partial f}{\partial x} + 3y \frac{\partial f}{\partial y} = x \left( \frac{3x^2}{y} \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} \right) + y \left( -\frac{x^3}{y^2} \frac{\partial f}{\partial u} \right) = x \frac{\partial f}{\partial v}. \]
Inverting the change of variables we have
\[ x = v, \quad y = \frac{v^3}{u}, \]
and so, after the change of variable the PDE becomes,
\[ v \frac{\partial f}{\partial v} = 6uv^2, \]
i.e.,
\[ \frac{\partial f}{\partial u} = 6uv. \]
Then
\[ f = \int_{u \text{ fixed}} 6uv \, dv = 3v^2 u + A(u), \]
and in terms of \( x \) and \( y \) this is \( f = 3\frac{x^2}{y} + A \left( \frac{z^2}{y} \right) \), where \( A \) is an arbitrary function. \( \square \)