Chapter 4

Differentiation of vectors: solutions

Example 4.1 A particle moves with constant angular speed (i.e. rate of change of angle) ω around a circle of radius a and centre $(0,0)$ and the particle is initially at $(a,0)$. Show that the position of the particle is $\mathbf{r}(t) = a(\cos\omega t, \sin\omega t)$.

Determine the velocity and speed of the particle at time t and prove that the acceleration of the particle is always directed towards the centre of the circle.

Solution : Let θ be the standard polar angle. Therefore the position of the particle on the circle can be written as

$$\mathbf{r} = (a\cos\theta, a\sin\theta) = a(\cos\theta, \sin\theta).$$

Also, $\dot{\theta}(t) = \omega$ and so $\theta(t) = \omega t + C$, where C is a constant. Since $\mathbf{r}(0) = (a,0)$ we want to have $\theta(0) = 0$ and so we take $C = 0$. Hence $\mathbf{r}(t) = a(\cos\omega t, \sin\omega t)$ as required.

Differentiating with respect to t, we get

$$\mathbf{v} = \dot{\mathbf{r}} = a(-\omega\sin\omega t, \omega\cos\omega t) = a\omega(-\sin\omega t, \cos\omega t),$$

so that the speed of the particle is

$$v = |\mathbf{v}| = a\omega\sqrt{(-\sin\omega t)^2 + (\cos\omega t)^2} = a\omega.$$

Further,

$$\mathbf{a} = \dot{\mathbf{v}} = -a\omega^2(\cos\omega t, \sin\omega t) = -\omega^2\mathbf{r}.$$

The position vector $\mathbf{r}$ is directed outwards from the centre of the circle to the particle and so the acceleration $\mathbf{a} = -\omega^2\mathbf{r}$ points in the opposite direction, that is, toward the centre as required. □

Example 4.2 Find the velocity of a particle with position vector $\mathbf{r}(t) = (\cos^2 t, \sin^2 t, \cos 2t)$. Describe the motion of the particle.

Solution : We have

$$\mathbf{v} = (-2\cos t\sin t, 2\sin t\cos t, -2\sin 2t) = \sin 2t(-1, 1, -2)$$

and

$$\mathbf{a} = 2\cos 2t(-1, 1, -2).$$

Since the velocity is always parallel to vector $(-1,1,-2)$, this means that the motion is in a straight line, also parallel to this vector. In fact, using the identities $\cos^2 t = 1 - \sin^2 t$ and $\cos 2t = 1 - 2\sin^2 t$, we can rewrite the position of the particle as

$$\mathbf{r} = (1, 0, 1) + \sin^2 t(-1, 1, -2).$$

Further, since $\sin^2 t$ has values in $[0,1]$ only, the particle moves to and fro along the line segment joining $(1,0,1)$ and $(1,0,1)+(-1,1,-2)=(0,1,-1)$. $\square$

Example 4.3 Find the tangent vector and the equation of the tangent to the helix

$$x=\cos\theta,\ y=\sin\theta,\ z=\theta,\quad \theta\in[0,2\pi),$$

at the point where it crosses the xy-plane.

Solution : The tangent vector to the helix is

$$\mathbf{T}=\frac{d}{d\theta}(\cos\theta,\sin\theta,\theta)=(-\sin\theta,\cos\theta,1).$$

and the unit tangent vector is

$$\hat{\mathbf{T}}=\frac{(-\sin\theta,\cos\theta,1)}{|(-\sin\theta,\cos\theta,1)|}=\frac{(-\sin\theta,\cos\theta,1)}{\sqrt{\sin^2\theta+\cos^2\theta+1}}=\frac{1}{\sqrt{2}}(-\sin\theta,\cos\theta,1),$$

The helix crosses the xy-plane when $z=\theta=0$, that is at $(1,0,0)$, and the tangent vector at this point is $(0,1,1)$.

The equation of the tangent to the helix at this point is hence

$$\mathbf{r}=(1,0,0)+t(0,1,1),\quad t\in\mathbb{R}.$$

$\square$

Example 4.4 Find the gradient of the scalar field $f(x,y,z)=x^2y+x\cosh yz$.

Solution : We have

$$\frac{\partial f}{\partial x}=2xy+\cosh yz,\quad \frac{\partial f}{\partial y}=x^2+xz\sinh yz,\quad \frac{\partial f}{\partial x}=xy\sinh yz.$$

Therefore,

$$\text{grad}\, f=(2xy+\cosh yz, x^2+xz\sinh yz, xy\sinh yz).$$

$\square$

Example 4.5 Let $\mathbf{r}=(x,y,z)$ so that $r=|\mathbf{r}|=\sqrt{x^2+y^2+z^2}$. Show that

$$\nabla(r^n)=nr^{n-2}\mathbf{r},$$

for any integer n and deduce the values of grad(r), grad(r^2) and grad$(1/r)$.

Solution : We have

$$\begin{aligned}\frac{\partial}{\partial x} r^n &= \frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} \\ &= 2x\,\frac{n}{2}(x^2+y^2+z^2)^{n/2-1} \\ &= nxr^{n-2}.\end{aligned}$$

Then, using the symmetry of r with respect to x, y and z, we get

$$\frac{\partial}{\partial y} r^n = nyr^{n-2}, \quad \frac{\partial}{\partial z} r^n = nzr^{n-2},$$

and thus

$$\nabla(r^n) = \left(\frac{\partial}{\partial x}(r^n), \frac{\partial}{\partial y}(r^n), \frac{\partial}{\partial z}(r^n)\right) = (nxr^{n-2}, nyr^{n-2}, nzr^{n-2}) = nr^{n-2}\mathbf{r}.$$

Hence

$$\begin{aligned}\mathrm{grad}(r) &= \nabla(r) = 1r^{1-2}\mathbf{r} = \frac{\mathbf{r}}{r}, \\ \mathrm{grad}(r^2) &= 2r^{2-2}\mathbf{r} = 2\mathbf{r},\end{aligned}$$

and

$$\mathrm{grad}(1/r) = \nabla(r^{-1}) = (-1)r^{-1-2}\mathbf{r} = -\mathbf{r}/r^3.$$

□

Example 4.6 Determine grad$(\mathbf{c}\cdot\mathbf{r})$, when c is a constant (vector).

Solution : Let $\mathbf{c} = (c_1, c_2, c_3)$ so that

$$\begin{aligned}\mathrm{grad}(\mathbf{c}\cdot\mathbf{r}) &= \mathrm{grad}(c_1x + c_2y + c_2z) \\ &= \left(\frac{\partial(c_1x+c_2y+c_2z)}{\partial x}, \frac{\partial(c_1x+c_2y+c_2z)}{\partial y}, \frac{\partial(c_1x+c_2y+c_2z)}{\partial z}\right) \\ &= (c_1, c_2, c_3) = \mathbf{c}.\end{aligned}$$

□

Example 4.7 Find the directional derivative of $f = x^2yz^3$ at the point $P(3, -2, -1)$ in the direction of the vector $(1, 2, 2)$.

Solution : The *unit* vector with the same direction as $(1, 2, 2)$ is

$$\mathbf{u} = \frac{(1,2,2)}{\sqrt{1^2+2^2+2^2}} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right).$$

Hence the required directional derivative is

$$\begin{aligned}\mathbf{u}\cdot\nabla f &= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\cdot(2xyz^3, x^2z^3, 3x^2yz^2) \\ &= \frac{1}{3}(2xyz^3 + 2x^2z^3 + 6x^2yz^2).\end{aligned}$$

At the point P, this gives

$$\frac{\partial f}{\partial \mathbf{u}}(3, -2, -1) = \frac{1}{3}(12 - 18 - 108) = -38.$$

□

Example 4.8 Show that the divergence of $\mathbf{F} = (x - y^2, z, z^3)$ is positive at all points in $\mathbb{R}^3$.

Solution : We have

$$\operatorname{div} \mathbf{F} = \frac{\partial(x - y^2)}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial(z^3)}{\partial z} = 1 + 0 + 3z^2 = 1 + 3z^2.$$

Hence for every (x, y, z), $\operatorname{div} \mathbf{F} \geq 1 > 0$. $\square$

Example 4.9 Find the values of n for which $\nabla^2(r^n) = 0$.

Solution : We have $r = \sqrt{x^2 + y^2 + z^2}$ and so from Example 4.5,

$$\frac{\partial(r^n)}{\partial x} = nxr^{n-2}.$$

Therefore,

$$\begin{aligned}\frac{\partial^2(r^n)}{\partial x^2} &= nr^{n-2} + nx\,(n-2)xr^{n-4} \\ &= nr^{n-4}\big(r^2 + (n-2)x^2\big),\end{aligned}$$

and because of the symmetry in r with respect to x, y and z, we also have

$$\frac{\partial^2(r^n)}{\partial y^2} = nr^{n-4}\big(r^2 + (n-2)y^2\big), \quad \frac{\partial^2(r^n)}{\partial z^2} = nr^{n-4}\big(r^2 + (n-2)z^2\big).$$

Taking the sum of these we get

$$\begin{aligned}\nabla^2(r^n) &= nr^{n-4}\big(3r^2 + (n-2)(x^2 + y^2 + z^2)\big) \\ &= n(n+1)r^{n-2}.\end{aligned}$$

Hence $\nabla^2(r^n) = 0$ if and only if $n = 0$ or $n = -1$. $\square$

Example 4.10 Determine $\operatorname{curl} \mathbf{F}$ when $\mathbf{F} = (x^2y, xy^2 + z, xy)$.

Solution : We have

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & xy^2 + z & xy \end{vmatrix} \\ &= (x-1)\mathbf{i} + (0-y)\mathbf{j} + (y^2 - x^2)\mathbf{k} \\ &= (x-1, -y, y^2 - x^2).\end{aligned}$$

$\square$

Example 4.11 If $\mathbf{c}$ is a constant vector, find $\operatorname{curl}(\mathbf{c} \times \mathbf{r})$.

Solution : We have $\mathbf{r} = (x, y, z)$ and let $\mathbf{c} = (c_1, c_2, c_3)$. First, we calculate

$$\mathbf{c} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2 z - c_3 y, c_3 x - c_1 z, c_1 y - c_2 x).$$

Then,

$$\begin{aligned} \operatorname{curl}(\mathbf{c} \times \mathbf{r}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ c_2 z - c_3 y & c_3 x - c_1 z & c_1 y - c_2 x \end{vmatrix} \\ &= \big(c_1 - (-c_1)\big)\mathbf{i} + \big(c_2 - (-c_2)\big)\mathbf{j} + \big(c_3 - (-c_3)\big)\mathbf{k} \\ &= 2\mathbf{c}. \end{aligned}$$

□

Example 4.12 Prove the identities

$$\text{(i)}\ \operatorname{curl}\operatorname{grad} f = 0, \quad \text{(ii)}\ \operatorname{curl}(f\mathbf{F}) = f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}, \quad \text{(iii)}\ \operatorname{div}(f\mathbf{F}) = f \operatorname{div}\mathbf{F} + (\operatorname{grad} f)\cdot\mathbf{F}$$

Solution : We have (i)

$$\begin{aligned} \operatorname{curl}\operatorname{grad} f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} \\ &= \big((f_z)_y - (f_y)_z\big)\mathbf{i} + \big((f_x)_z - (f_z)_x\big)\mathbf{j} + \big((f_y)_x - (f_x)_y\big)\mathbf{k} \\ &= (0, 0, 0) = \mathbf{0}, \end{aligned}$$

and (ii),

$$\begin{aligned} \operatorname{curl}(f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} \\ &= \big((fF_3)_y - (fF_2)_z\big)\mathbf{i} + \big((fF_1)_z - (fF_3)_x\big)\mathbf{j} + \big((fF_2)_x - (fF_1)_y\big)\mathbf{k} \\ &= f\left[\big((F_3)_y - (F_2)_z\big)\mathbf{i} + \big((F_1)_z - (F_3)_x\big)\mathbf{j} + \big((F_2)_x - (F_1)_y\big)\mathbf{k}\right] \\ &\qquad + (f_y F_3 - f_z F_2)\mathbf{i} + (f_z F_1 - f_x F_3)\mathbf{j} + (f_x F_2 - f_y F_1)\mathbf{k} \\ &= f \operatorname{curl}\mathbf{F} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f_x & f_y & f_z \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= f \operatorname{curl}\mathbf{F} + \operatorname{grad} f \times \mathbf{F}, \end{aligned}$$

as required. □

Example 4.13 Let $\mathbf{F} = (x^2 y, yz, x + z)$. Find

$$\text{(i)}\ \operatorname{curl}\operatorname{curl}\mathbf{F}, \quad \text{(ii)}\ \operatorname{grad}\operatorname{div}\mathbf{F}.$$

Solution : We have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2y & yz & x+z \end{vmatrix} = (0-y, 0-1, 0-x^2) = -(y, 1, x^2),$$

and

$$\operatorname{div}\mathbf{F} = \frac{\partial(x^2y)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(x+z)}{\partial z} = 2xy + z + 1.$$

Hence (i),

$$\operatorname{curl}\operatorname{curl}\mathbf{F} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & 1 & x^2 \end{vmatrix} = -(0-0, 0-2x, 0-1) = (0, 2x, 1),$$

and (ii),

$$\operatorname{grad}\operatorname{div}\mathbf{F} = \left(\frac{\partial(2xy+z+1)}{\partial x}, \frac{\partial(2xy+z+1)}{\partial y}, \frac{\partial(2xy+z+1)}{\partial z}\right) = (2y, 2x, 1).$$

□

Example 4.14 Let $\mathbf{r} = (x, y, z)$ denote a position vector with length $r = \sqrt{x^2+y^2+z^2}$ and $\mathbf{c}$ is a constant (vector). Determine

$$\text{(i)}\ \operatorname{div}(r^n(\mathbf{c}\times\mathbf{r})), \quad \text{(ii)}\ \operatorname{curl}(r^n(\mathbf{c}\times\mathbf{r})).$$

Solution :

$$\mathbf{c}\times\mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c_1 & c_2 & c_3 \\ x & y & z \end{vmatrix} = (c_2z - c_3y, c_3x - c_1z, c_1y - c_2x).$$

(i) Using the identity $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad}(f)\cdot\mathbf{F}$, and setting $f = r^n$ and $\mathbf{F} = \mathbf{c}\times\mathbf{r}$ gives

$$\begin{aligned} \operatorname{div}(r^n(\mathbf{c}\times\mathbf{r})) &= r^n\operatorname{div}(\mathbf{c}\times\mathbf{r}) + \operatorname{grad}(r^n)\cdot(\mathbf{c}\times\mathbf{r}) \\ &= 0 + nr^{n-2}\mathbf{r}\cdot(\mathbf{c}\times\mathbf{r}) \\ &= 0. \end{aligned}$$

This uses the result from Example 4.5, and the fact that $\mathbf{c}\times\mathbf{r})$ is perpendicular to $\mathbf{r}$.

(ii), using the identity $\operatorname{curl}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \operatorname{grad}(f)\times\mathbf{F}$ gives

$$\begin{aligned} \operatorname{curl}(r^n(\mathbf{c}\times\mathbf{r})) &= r^n\operatorname{curl}((\mathbf{c}\times\mathbf{r})) + \operatorname{grad}(r^n)\times(\mathbf{c}\times\mathbf{r}) \\ &= r^n 2\mathbf{c} + nr^{n-2}\mathbf{r}\times(\mathbf{c}\times\mathbf{r}) \qquad \text{(by Ex 4.10 and 4.5)} \\ &= 2r^n\mathbf{c} + nr^{n-2}((\mathbf{r}\cdot\mathbf{r})\mathbf{c} - (\mathbf{r}\cdot\mathbf{c})\mathbf{r}) \quad \text{(by the vector tripple product)} \\ &= (2+n)r^n\mathbf{c} - nr^{n-2}(\mathbf{r}\cdot\mathbf{c})\mathbf{r} \end{aligned}$$

□