Chapter 5

Line and surface integrals: Solutions

Example 5.1 Find the work done by the force $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$ in moving a particle along the curve which runs from $(1, 0)$ to $(0, 1)$ along the unit circle and then from $(0, 1)$ to $(0, 0)$ along the y-axis (see Figure 5.1).

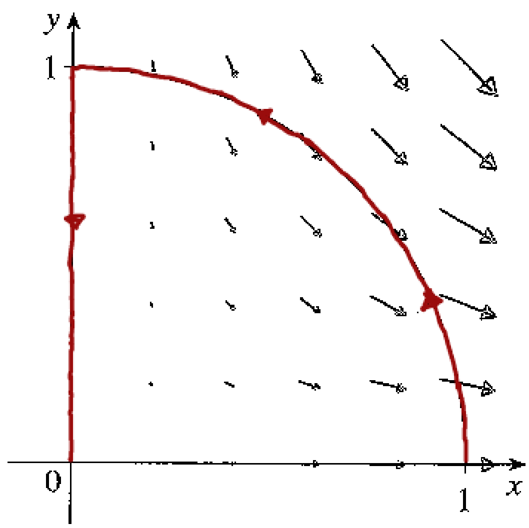

Figure 5.1: Shows the force field $\mathbf{F}$ and the curve C. The work done is negative because the field impedes the movement along the curve.

Solution Split the curve C into two sections, the curve C_1 and the line that runs along the y-axis C_2. Then,

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \, .$$

Curve C_1: Parameterise C_1 by $\mathbf{r}(t) = (x(t), y(t) = (\cos t, \sin t)$, where $0 \leq t \leq \pi/2$ and $\mathbf{F} = (x^2, -xy)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} x^2 dx - xy dy = \int_0^{\pi/2} \cos^2 t \frac{dx}{dt} dt - \int_0^{\pi/2} \cos t \sin t \frac{dy}{dt} dt = - \int_0^{\pi/2} 2\cos^2 t \sin t dt = -2/3,$$

by applying Beta functions to solve the integral where $m = 2$, $n = 1$ and $K = 1$.

Curve C_2: Parameterise C_2 by $\mathbf{r}(t) = (x(t), y(t) = (0, t)$, where $0 \leq t \leq 1$. Hence,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} 0\frac{dx}{dt}dt - \int_0^{\pi/2} 0t\frac{dy}{dt}dt = 0.$$

So the work done, $W = -2/3 + 0 = -2/3$. □

Example 5.2 Evaluate the line integral $\int_C (y^2)dx + (x)dy$, where C is the is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$

Solution Parameterise C by $\mathbf{r}(t) = (x(t), y(t) = (4 - t^2, t)$, where $-3 \leq t \leq 2$, since $-3 \leq y \leq 2$. $\mathbf{F} = (y^2, x)$ and $d\mathbf{r} = (dx, dy)$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C y^2 dx + xdy = \int_{-3}^{2} t^2 \frac{dx}{dt}dt - \int_{-3}^{2} (4 - t^2)\frac{dy}{dt}dt = \int_{-3}^{2} -2t^3 + (4 - t^2)dt = 245/6.$$

□

Example 5.3 Evaluate the line integral, $\int_C (x^2 + y^2)dx + (4x + y^2)dy$, where C is the straight line segment from $(6, 3)$ to $(6, 0)$.

Solution : We can do this question without parameterising C since C does not change in the x-direction. So $dx = 0$ and $x = 6$ with $0 \leq y \leq 3$ on the curve. Hence

$$I = \int_C (x^2 + y^2)0 + (4x + y^2)dy = \int_3^0 24 + y^2 dy = -81.$$

□

Example 5.4 Use Green's Theorem to evaluate $\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy$, where C is the circle $x^2 + y^2 = 9$.

Solution $P(x, y) = 3y - e^{\sin x}$ and $Q(x, y) = 7x + \sqrt{y^4 + 1}$. Hence, $\frac{\partial Q}{\partial x} = 7$ and $\frac{\partial P}{\partial y} = 3$. Applying Green's Theorem where D is given by the interior of C, i.e. D is the disc such that $x^2 + y^2 \leq 9$.

$$\int_C (3y - e^{\sin x})dx + (7x + \sqrt{y^4 + 1})dy = \int\int_D (7 - 3)dxdy = \int_0^{2\pi} \int_0^3 4rdrd\theta = \int_0^{2\pi} 18d\theta = 36\pi$$

The D integral is solved by using polar coordinates to describe D. □

Example 5.5 Evaluate $\int_C (3x - 5y)dx + (x - 6y)dy$, where C is the ellipse $\frac{x^2}{4} + y^2 = 1$ in the anticlockwise direction. Evaluate the integral by (i) Green's Theorem, (ii) directly.

Solution (i) **Green's Theorem:** $P(x,y) = 3x - 5y$ and $Q(x,y) = x - 6y$. Hence, $\frac{\partial Q}{\partial x} = 1$ and $\frac{\partial P}{\partial y} = -5$. Applying Green's Theorem where D is given by the interior of C, i.e. D is the ellipse such that $x^2/4+y^2 \leq 1$.

$$\int_C (3x - 5y)dx + (x + 6y)dy = \int\int_D (1 - (-5))dxdy = 6\int\int_D 1dxdy = 6 \times (\text{Area of the ellipse}) = 6 \times 2\pi.$$

See chapter 2 for calculating the area of an ellipse by change of variables for a double integral.

(i) **Directly:** Parameterise C by $x(t) = 2\cos t$, $y(t) = \sin t$, where $0 \leq t \leq 2\pi$.

$$\begin{aligned} I = & \int_0^{2\pi} (6\cos t - 5\sin t)\tfrac{dx}{dt}dt + (2\cos t - 6\sin t)\tfrac{dy}{dt}dt \\ = & \int_0^{2\pi} 18\cos t \sin t + 10\sin^2 t + 2\cos^2 t dt \\ = & \ 0 + 40\int_0^{\pi/2} \sin^2 t dt + 8\int_0^{\pi/2} \cos^2 t dt \\ = & \ 0 + 40\tfrac{\pi}{2}(1/2) + 8\tfrac{\pi}{2}(1/2) = 12\pi. \end{aligned}$$

The integrals are calculated using symmetry properties of $\cos t$ and $\sin t$ and beta functions. Using the table of signs below we see that $\int_0^{2\pi} \sin^2 t = 4\int_0^{\pi/2} \sin^t dt$ etc.

Quadrant	1	2	3	4	Total
$\cos t$	+	−	−	+	
$\sin t$	+	+	−	−	
$\cos t \sin t$	+	−	+	−	0
$\sin^2 t$	+	+	+	+	4
$\cos^2 t$	+	+	+	+	4

□

Example 5.6 Evaluate

$$\int\int_S z^2 \, dS$$

where S is the hemisphere given by $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

Solution We first find $\frac{\partial z}{\partial x}$ etc. These terms arise because $dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}dxdy$. Since this change of variables relates to the surface S we find these derivatives by differentiating both sides of the surface $x^2 + y^2 + z^2 = 1$ with respect to x, giving $2x + 2z\frac{\partial z}{\partial x} = 0$. Hence, $\frac{\partial z}{\partial x} = -x/z$. Similarly, $\frac{\partial z}{\partial y} = -y/z$. Hence,

$$\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}} = 1/z.$$

Then the integrals becomes the following, where D is the projection of the surface, S, onto the $x-y$-plane. i.e. $D = \{(x, y) : x^2 + y^2 \leq 1\}$.

$$\begin{aligned}
\int\int_S z^2 dS &= \int\int_D z^2 \frac{1}{z} dxdy \\
&= \int\int_D \sqrt{1 - x^2 - y^2} dxdy \\
&= \int_0^{2\pi} d\theta \int_0^1 \sqrt{1 - r^2} r dr \\
&= -\int_0^{2\pi} d\theta \int_1^0 \frac{1}{2}\sqrt{u} du \\
&= \int_0^{2\pi} \frac{1}{3} d\theta \\
&= 2\pi/3.
\end{aligned}$$

□

Example 5.7 Find the area of the ellipse cut on the plane $2x + 3y + 6z = 60$ by the circular cylinder $x^2 = y^2 = 2x$.

Solution The surface S lies in the plane $2x+3y+6z = 60$ so we use this to calculate $dS = \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} dxdy$. Differentiating the equation for the plane with respect to x gives,

$$2 + 6\frac{\partial z}{\partial x} = 0 \quad \text{thus,} \ \frac{\partial z}{\partial x} = -1/3.$$

Differentiating the equation for the plane with respect to y gives,

$$3 + 6\frac{\partial z}{\partial y} = 0 \quad \text{thus,} \ \frac{\partial z}{\partial y} = -1/2.$$

Hence,

$$\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2} = \sqrt{1 + \frac{1}{9} + \frac{1}{4}} = 7/6.$$

Then the area of S is found be calculating the suface integral over S for the function $f(x, y, z) = 1$. The the projection of the surface, S, onto the $x-y$-plane is given by $D = \{(x, y) : x^2 - 2x + y^2 = (x-1)^2 + y^2 \leq 1\}$. Hence the area of S is given by

$$\begin{aligned}
\int\int_S 1 dS &= \int\int_D 1\frac{7}{6} dxdy \\
&= \frac{7}{6}\int\int_D 1 dxdy \\
&= \frac{7}{6} \times \ \text{Area of D} = \frac{7}{6}\pi.
\end{aligned}$$

Note, since D is a cricle or radius 1 centred at $(1, 0)$ the area of D is the area of a unit circle which is π. □

Example 5.8 Use Gauss' Divergence Theorem to evaluate

$$I = \int\int_S x^4 y + y^2 z^2 + xz^2 \ dS,$$

where S is the entire surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution In order to apply Gauss' Divergence Theorem we first need to determine $\mathbf{F}$ and the unit normal $\mathbf{n}$ to the surface S. The normal is $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, 2z)$, where $f(z, y, z) = x^2 + y^2 + z^2 - 1 = 0$. We require the unit normal, so $\mathbf{n} = (2x, 2y, 2z)/|(2x, 2y, 2z)| = (2x, 2y, 2z)/2 = (x, y, z)$. To find $\mathbf{F} = (F_1, F_2, F_3)$ we note that

$$\begin{aligned}\mathbf{F}\cdot\mathbf{n} &= x^4y + y^2z62 + xz^2 \\ &= F_1x + F_2y + F_3z\end{aligned}$$

Hence, comparing terms we have $F_1 = x^3y$, $F_2 = yz^2$ and $F_3 = xz$. Applying the Divergence Theorem noting that V is the volume enclosed by the sphere S gives

$$\begin{aligned}I = \int\int_S \mathbf{F}\cdot\mathbf{n}dS &= \int\int\int_V \operatorname{div}\mathbf{F}dxdydz \\ &= \int\int\int_V 3x^2y + z^2 + xdxdydz \\ &= 0 + \int\int\int_V z^2dxdydz + 0 \\ &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^1 r^2\cos^2\theta r^2\sin\theta dr \\ &= 2\pi \int\int_0^{\pi} \cos^2\theta\sin\theta d\theta \int_0^1 r^4dr \\ &= 2\pi \times 2 \times \frac{1\cdot 1}{3\cdot 1} \times 1 = \frac{4\pi}{15}.\end{aligned}$$

Remarks

1. As V is a sphere it is natural to use spherical polar coordinates to solve the integral. Thus, $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\theta$ and $dxdydz = r^2\sin\theta$.

2. $\int\int\int_V 3x^2ydxdydz = 0$ and $\int\int\int_V xdxdydz = 0$ from the symmetry of the cosine and sine functions. We look at the signs in each quadrant as ϕ changes. Think about a fixed θ. $\cos\phi$ and $\sin\phi$ terms in x^2y and x then have the following signs

Quadrant	1	2	3	4	Total
$\cos\phi$	+	−	−	+	
$\sin\phi$	+	+	−	−	
x^2y	+	+	−	−	0
x	+	+	−	−	0

The positive and negative contribution from the integral cancel out in these two cases so the integrals are zero.

□

Example 5.9 Find $I = \int\int_S \mathbf{F}\cdot\mathbf{n}\, dS$ where $\mathbf{F} = (2x, 2y, 1)$ and where S is the entire surface consisting of S_1=the part of the paraboloid $z = 1 - x^2 - y^2$ with $z = 0$ together with S_2=disc $\{(x, y) : x^2 + y^2 \leq 1\}$. Here $\mathbf{n}$ is the outward pointing unit normal.

Solution Applying the Divergence Theorem noting that V is the volume enclosed by S_1 and S_2 and $\operatorname{div}\mathbf{F} = 2+2+0$ gives

$$\begin{aligned} I = \int\int_S \mathbf{F}\cdot\mathbf{n}dS &= \int\int\int_V \operatorname{div}\mathbf{F}dxdydz \\ &= \int\int\int_V 4dxdydz \\ &= 4\int\int_{\{(x,y:)x^2+y^2\le 1\}} dxdy \int_0^{1-x^2-y^2} 1dz \\ &= 4\int\int_{\{(x,y:)x^2+y^2\le 1\}} 1-x^2-y^2dxdy \\ &= 4\int_0^{2\pi} d\theta \int_0^1 (1-r^2)r\ dr \\ &= 4\times 2\pi(1/2-1/4) = 2\pi. \end{aligned}$$

□

Example 5.10 Vector fields $\mathbf{V}$ and $\mathbf{W}$ are defined by

$$\mathbf{V} = (2x-3y+z, -3x-y+4z, 4y+z)$$

$$\mathbf{W} = (2x-4y-5z, -4x+2y, -5x+6z)\ .$$

One of these is conservative while the other is not. Determine which is conservative and denote it by $\mathbf{F}$. Find a potential function ϕ for $\mathbf{F}$ and evaluate

$$\int_C \mathbf{F}\cdot d\mathbf{r}\ ,$$

where C is the curve from A(1,0,0) to B(0,0,1) in which the plane $x+z=1$ cuts the hemisphere given by $x^2+y^2+z^2=1$, $y\ge 0$.

Solution We have

$$\begin{aligned} \operatorname{curl}\mathbf{V} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-3y+z & -3x-y+4z & 4y+z \end{vmatrix} \\ &= (0,1,0) \ne \mathbf{0}. \end{aligned}$$

Since $\operatorname{curl}\mathbf{V}\ne\mathbf{0}$, $\mathbf{F}$ is **NOT** conservative.

We have

$$\begin{aligned} \operatorname{curl}\mathbf{W} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-4y-5z & -4x+2y & -5x+6z \end{vmatrix} \\ &= (0,0,0) = \mathbf{0}. \end{aligned}$$

Since curl $\mathbf{V} = \mathbf{0}$, **F is** conservative.

Suppose that $\operatorname{grad} \phi = \mathbf{W}$. Then

$$\frac{\partial \phi}{\partial x} = 2x - 4y - 5z, \tag{1}$$

$$\frac{\partial \phi}{\partial y} = -4x + 2y, \tag{2}$$

$$\frac{\partial \phi}{\partial z} = -5x + 6z. \tag{3}$$

Integrating (1) with respect to x, holding the other variables constant, we get

$$\phi = \int_{y,z \text{ fixed} 2x-4y-5z} dx = x^2 - 4yx - 5zx + A(y, z),$$

where A is an arbitrary function. Substituting this expression into (2) gives,

$$-4x + \frac{\partial A}{\partial y} = -4x + 2y, \quad \text{i.e. } \frac{\partial A}{\partial y} = 2y,$$

and therefore

$$A(y, z) = \int_{z \text{ fixed}} (2y)\, dy = y^2 + B(z),$$

where B is an arbitrary function, giving

$$\phi = x^2 - 4yx - 5zx + y^2 + B(z).$$

Finally, substituting this into (3) gives

$$-5x + \frac{dB}{dz} = -5x + 6z, \quad \text{i.e. } \frac{dB}{dz} = 6z,$$

so that $B = 3z^2 + C$, where C is a constant. Hence, by taking $C = 0$ we obtain a potential

$$\phi = x^2 - 4yx - 5zx + y^2x + 3z^2.$$

□

Remark Notice that the potential function is not unique; we may always add an arbitrary constant to a potential and it remains a potential.

So the line integral is:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \operatorname{div} \phi \cdot d\mathbf{r} = \phi(0, 0, 1) - \phi(1, 0, 0) = 3 - 1 = 2.$$