Integrating factors

We derive the formula for the integrating factor used to solve linear ODEs of the form,

$$\frac{dy}{dx} + a(x)y = b(x). \tag{1}$$

Let μ be the integrating factor. Consider equation (1) $\times\ \mu$,

$$\mu\frac{dy}{dx} + \mu a(x)y = \mu b(x). \tag{2}$$

We would like to use the product rule to simplify the LHS of (2). The product rule states:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Equating terms in the product rule with those in the LHS of equation (2) gives:

$$u = \mu \quad dv/dx = dy/dx, \quad v = y \quad \text{and} \quad du/dx = \mu a(x).$$

The first and fourth expressions give $du/dx = d\mu/dx = \mu a(x)$. Solving this equation using separation of variables gives:

$$\int \frac{1}{\mu} du = \int a(x)dx \Rightarrow \mu = \exp\left(\int a(x)dx\right),$$

the expression for the integrating factor of a linear ODE.

Now the LHS of equation (2) can be re-written using the product rule,

$$\frac{d}{dx}(\mu y) = b(x)\mu.$$

Now we know μ we can integrate this equation w.r.t. x and solve to find y.