Statistical Mechanics & Enumerative Geometry:
Clifford Algebras and Quantum Cohomology

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Adv Math 225 (2010) 200-268
arXiv:0909.2347; arXiv:0910.3395; arXiv:1006.4710

July 2010

Outline

Part 1: Quantum Cohomology

- Motivation
- Vicious walkers on the cylinder
- Gromov-Witten invariants
- Fermions hopping on Dynkin diagrams
- nil-Temperley-Lieb algebra and nc Schur polynomials
- New recursion formulae for Gromov-Witten invariants

Part 2: $\widehat{\mathfrak{sl}}(n)_k$ Verlinde algebra/WZNW fusion ring

- Relating fusion coefficients and Gromov-Witten invariants

Main result

New algorithm for computing Gromov-Witten invariants.

Motivation

Quantum cohomology originated in the works of Gepner, Vafa, Intriligator and Witten (topological field & string theory).

- Witten's 1995 paper *The Verlinde algebra and the cohomology of the Grassmannian*:
  The fusion coefficients of $\widehat{\mathfrak{gl}}(n)_k$ WZNW theory can be defined in geometric terms using Gromov's pseudoholomorphic curves.
- Kontsevich's formula ("big" quantum cohomology)
  How many curves of degree d pass through $3d-1$ points in the complex projective plane?

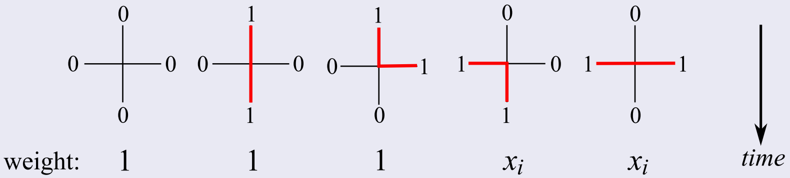
For more on big and small quantum cohomology see e.g. notes by Fulton and Pandharipande (alg-geom/960811v2).

Vicious walkers on the cylinder: the 5-vertex model

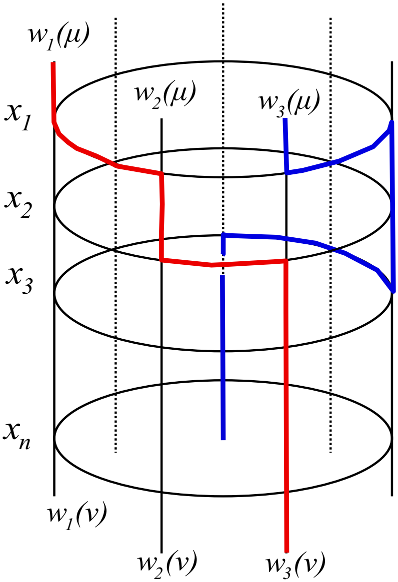
Define statistical model whose partition function generates Gromov-Witten invariants.

Consider an $n \times N$ square lattice ($0 \leq n \leq N$) with quasi-periodic boundary conditions (twist parameter q) in the horizontal direction.

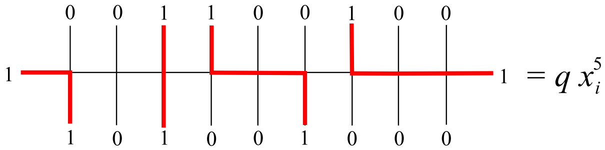
Allowed vertex configurations and their weights



Here x_i is the spectral parameter in the i^{th} lattice row.
(Vicious walker model, c.f. [Fisher][Forrester][Guttmann et al])


$w_1(\mu)$
$w_2(\mu)$
$w_3(\mu)$
x_1
x_2
x_3
x_n
$w_1(\nu)$
$w_2(\nu)$
$w_3(\nu)$

Example of an i^{th} lattice row configuration ($n = 3$ and $N = 9$):



The variable x_i counts the number of horizontal edges, while the boundary variable q counts the outer horizontal edges divided by 2.

Definition of the transfer matrix

Given a pair of 01-words $w = 010 \cdots 10$, $w' = 011 \cdots 01$ of length N, the transfer matrix $Q(x_i)$ is defined as

$$Q(x_i)_{w,w'} := \sum_{\textit{allowed row configuration}} q^{\frac{\#\text{ of outer edges}}{2}} x_i^{\#\text{ of horizontal edges}} .$$

Interlude: 01-words and Young diagrams

Row configurations are described through 01-words w in the set

$$W_n = \left\{ w = w_1 w_2 \cdots w_N \;\middle|\; |w| = \sum_i w_i = n\,,\; w_i \in \{0,1\} \right\}.$$

Let $\ell_1 < ... < \ell_n$ with $1 \leq \ell_i \leq N$ be the positions of 1-letters in a word w. Then

$$w = 0 \cdots 0\underset{\ell_1}{1}0 \cdots 0\underset{\ell_n}{1}0 \cdots 0 \mapsto \lambda = (\lambda_1, \ldots, \lambda_n),\; \lambda_i = \ell_{n+1-i} - i$$

defines a bijection from W_n into the set

$$\mathfrak{P}_{\leq n,k} := \{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n)\; |\lambda_1 \leq k \text{ and } \lambda_n \geq 0\}$$

which are the partitions whose Young diagram fits into a $n \times k$ bounding box with $k = N - n$.

Vicious walkers on the cylinder: partition function

Given $\mu, \nu \in \mathfrak{P}_{\leq n,k}$, let $w(\mu)$, $w(\nu)$ be the corresponding 01-words.

Boundary conditions: fix the values of the edges on the top and bottom to be $w(\mu)$ and $w(\nu)$, respectively.

The **partition function** is the weighted sum over all allowed lattice configurations and is given by

$$Z_\nu^\mu(x_1, \ldots, x_n; q) = (Q(x_n) \cdot Q(x_{n-1}) \cdots Q(x_1))_{w(\nu), w(\mu)} \ .$$

Theorem (Generating function for Gromov-Witten invariants)

The partition function has the following expansion in terms of Schur functions s_λ,

$$Z_\nu^\mu(x_1, \ldots, x_n; q) = \sum_{\lambda \in (n,k)} q^d \, C_{\lambda\mu}^{\nu,d} s_\lambda(x_1, \ldots, x_n),$$

where $C_{\lambda\mu}^{\nu,d}$ are 3-point Gromov-Witten invariants ($d = \frac{|\lambda|+|\mu|-|\nu|}{N}$).

Reminder: Schur functions

The ring of symmetric function Λ plays an important role in representation theory, combinatorics and enumerative geometry.

$$\Lambda = \lim_{\leftarrow} \Lambda_n, \qquad \Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$$

An important bases are Schur functions which we define as

$$s_\lambda(x) = \sum_{|T|=\lambda} x^T, \qquad x^T \equiv x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $\alpha(T) = (\alpha_1, \ldots, \alpha_n)$ is the weight of a (semi-standard) tableau T of shape λ.

Example

Let $n = 3$ and $\lambda = (2, 1)$. Then

$$T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Special cases of Schur functions are the

- elementary symmetric functions $\lambda = (1^r) = (\underbrace{1, \dots, 1}_{r})$

$$e_r(x) = \sum_{p \vdash r,\ p_i = 0,1} x_1^{p_1} \cdots x_n^{p_n} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}$$

- complete symmetric functions, $\lambda = (r, 0, 0, \dots)$

$$h_r(x) = \sum_{p \vdash r} x_1^{p_1} \cdots x_n^{p_n} = \sum_{i_1 \leq \cdots \leq i_r} x_{i_1} \cdots x_{i_r}$$

Product of Schur functions via Littlewood-Richardson coefficients:

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu \,.$$

Schubert varieties and rational maps

Let $\mathrm{Gr}_{n,N}$ be the *Grassmannian* of n-dimn'l subspaces V in $\mathbb{C}^N$.

Given a flag $F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^N$ the *Schubert variety* $\Omega_\lambda(F)$ is defined as

$$\Omega_\lambda = \{V \in \mathrm{Gr}_{n,N} \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i,\ i = 1, 2, \ldots n\}.$$

Definition of 3-point Gromov-Witten invariants

$C^{\nu,d}_{\lambda,\mu} = \#$ of rational maps $f : \mathbb{P}^1 \to \mathrm{Gr}_{n,N}$ of degree d which meet the varieties $\Omega_\lambda(F)$, $\Omega_\mu(F')$, $\Omega_{\nu^\vee}(F'')$ for general flags F, F', F'' (up to automorphisms in $\mathbb{P}^1$).
If there is an ∞ number of such maps, set $C^{\nu,d}_{\lambda,\mu} = 0$.

$\nu^\vee = (k - \nu_n, \ldots, k - \nu_1)$ (complement $\to$ Poincaré dual).

Small Quantum (= q-deformed) Cohomology

Define $qH^*(\mathrm{Gr}_{n,k}) := \mathbb{Z}[q] \otimes_{\mathbb{Z}} H^*(\mathrm{Gr}_{n,k})$.

Small quantum cohomology ring (Gepner, Witten, Agnihotri,. . .)

The product $\sigma_\lambda \star \sigma_\mu := \sum_{d,\nu} q^d C_{\lambda\mu}^{\nu,d} \sigma_\nu$, with $\sigma_\lambda := 1 \otimes [\Omega_\lambda]$, turns $qH^*(\mathrm{Gr}_{n,k})$ into a commutative ring.

Theorem (Siebert-Tian 1997)

Set $\Lambda_n = \mathbb{Z}[e_1, \ldots, e_n]$ then $\sigma_\lambda \mapsto s_\lambda$ is a ring isomorphism

$$qH^*(\mathrm{Gr}_{n,N}) \cong (\mathbb{Z}[q] \otimes_{\mathbb{Z}} \Lambda_n)/\langle h_{k+1}, ..., h_{n+k-1}, h_{n+k} + (-1)^n q\rangle$$

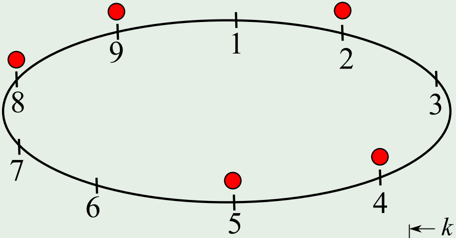
Specialisations

$q = 0$: cup product in cohomology ring $H^*(\mathrm{Gr}_{n,n+k})$.
$q = 1$: fusion product of the gauged $\widehat{\mathfrak{gl}}(n)_k$ WZNW model (TFT).

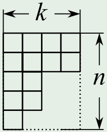
Fermions on a circle

Consider a circular lattice with N sites, $0 \leq n \leq N$ particles (called 'fermions') and $k = N - n$ 'holes'.

Example



$$w = \underbrace{010110011}_{n+k} \qquad \lambda = (4,4,2,2,1) =$$



Pauli's exclusion principle: at each site at most one particle is allowed (vicious walker constraint)!

Fermion creation and annihilation

Consider the finite-dimensional vector space

$$\mathfrak{F} = \bigoplus_{n=0}^{N} \mathfrak{F}_n, \qquad \mathfrak{F}_n = \mathbb{C}W_n \cong (\mathbb{C}^2)^{\otimes N},$$

where $\mathfrak{F}_0 = \mathbb{C}\{0\cdots 0\} = \mathbb{C}$ and $w = 0\cdots 0$ is the *vacuum* $\varnothing$.

Define $n_i(w) = w_1 + \cdots + w_i$, the number of 1-letters in $[1, i]$.

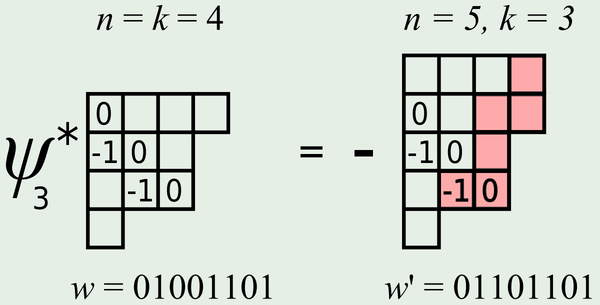
For $1 \le i \le N$ define the (linear) maps $\psi_i^*, \psi_i : \mathfrak{F}_n \to \mathfrak{F}_{n\pm 1}$,

$$\psi_i^*(w) := \begin{cases} (-1)^{n_{i-1}(w)} w', & w_i = 0 \text{ and } w'_j = w_j + \delta_{i,j} \\ 0, & w_i = 1 \end{cases}$$

$$\psi_i(w) := \begin{cases} (-1)^{n_{i-1}(w)} w', & w_i = 1 \text{ and } w'_j = w_j - \delta_{i,j} \\ 0, & w_i = 0. \end{cases}$$

Example

Take $n = k = 4$ and $\mu = (4, 3, 3, 1)$.



The boundary ribbon (shaded boxes) starts in the $(3 - n) = -1$ diagonal. Below the diagram the respective 01-words $w(\mu)$ and $w(\psi_3^*\mu)$ are displayed.

Clifford algebra

Proposition

The maps $\psi_i, \psi_i^* : \mathfrak{F}_n \to \mathfrak{F}_{n\mp 1}$ *yield a Clifford algebra on* $\mathfrak{F} = \bigoplus_{0\leq n\leq N} \mathfrak{F}_n$, *i.e. one has the relations*

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij} \;.$$

Introducing the scalar product

$$\langle w, w' \rangle = \prod_i \delta_{w_i, w_i'}$$

one has $\langle \psi_i^* w, w' \rangle = \langle w, \psi_i w' \rangle$ for any pair $w, w' \in \mathfrak{F}$.

Remark

The Clifford algebra turns out to be the fundamental object in the description of quantum cohomology.

Nil affine Temperley-Lieb algebra

Proposition (hopping operators)

The map $u_i \mapsto \psi^*_{i+1}\psi_i,\ i = 1, \ldots, N-1,\ u_N \mapsto (-1)^{n-1} q \psi^*_1 \psi_N$ *yields a faithful rep of the nil affine TL algebra in* $End(\mathfrak{F}_n)$,

$$\begin{aligned} u_i^2 &= u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} = 0, \quad i \in \mathbb{Z}_N \\ u_i u_j &= u_j u_i, \qquad |i-j| \bmod N > 1 \end{aligned}$$

Proposition (nc complete symmetric polynomials)

The transfer matrix of the vicious walker model is given by

$$Q(x_i) = \sum_{0 \le r < N} x_i^r h_r(u) \quad \textit{with}$$

$$h_r(u) = \sum_{p \vdash r} (q(-1)^{n-1}\psi_N)^{p_0} u_{N-1}^{p_{N-1}} \cdots u_1^{p_1} (\psi_1^*)^{p_0} \ .$$

Q *possesses a complete eigenbasis independent of* x_i*, hence* $[Q(x_i), Q(x_j)] = 0$ *for any pair* x_i, x_j.

Noncommutative Schur functions

Define $s_\lambda(u) := \det(h_{\lambda_i - i + j}(u))_{1\leq i,j\leq N}$, note $[h_r(u), h_{r'}(u)] = 0$.

Theorem (Postnikov 2005)

Combinatorial product formula for the quantum cohomology ring:

$$\lambda \star \mu = \sum_{d,\nu} q^d C^{\nu,d}_{\lambda\mu} \nu = s_\lambda(u)\mu \,.$$

Proposition (CK, Stroppel: noncommutative Cauchy identity)

Let Q be the transfer matrix of the vicious walker model, then

$$Q(x_1)\cdot Q(x_2)\cdots Q(x_n) = \sum_\lambda s_\lambda(u) s_\lambda(x_1,\ldots,x_n)\,.$$

Partition function of the vicious walker model

Taking the scalar product $\langle \nu, \ldots \mu\rangle$ in the nc Cauchy identity now proves the earlier stated expansion of the partition function.

Explicit construction of eigenbasis

Construct common eigenbasis $\{b(y)\}$ of $s_\lambda(u_1,\ldots,u_N)$'s using Clifford algebra (Jordan-Wigner transformation):

$$s_\lambda(u)b(y) = s_\lambda(y)b(y), \quad b(y) := \hat{\psi}^*(y_1)\cdots\hat{\psi}^*(y_n)\varnothing,$$

where $\hat{\psi}^*(y_i) = \sum_a y_i^a \psi_a^*$ and $y_1^N = \cdots = y_n^N = (-1)^{n-1}q$.

$\rightsquigarrow$ Siebert-Tian presentation of $qH^*(\mathrm{Gr}_{n,N})$

$$h_{k+1} = \cdots = h_{n+k-1} = 0, \quad h_{n+k} = (-1)^{n-1}q$$

$\rightsquigarrow$ Bertram-Vafa-Intrilligator formula for $C_{\lambda\mu}^{\nu,d}$

$$C_{\lambda\mu}^{\nu,d} = \sum_y \frac{\langle \nu, s_\lambda(u)b(y)\rangle\langle b(y),\mu\rangle}{\langle b(y), b(y)\rangle}$$

Fermion fields and nc Schur functions

Proposition

The following commutation relation holds true,

$$s_\lambda(u,q)\psi_i^* = \psi_i^* s_\lambda(u,-q) + \sum_{r=1}^{\ell(\lambda)} \psi_{i+r}^* \sum_{\lambda/\mu=(r)} s_\mu(u,-q)$$

where we set $\psi_{j+N}^* = (-1)^{\mathrm{n}+1} q\psi_j^*$, $\mathrm{n}=$ *particle number operator.*

Fermion creation of quantum cohomology rings

ψ_i^*, ψ_i induce maps $qH^*(\mathrm{Gr}_{n,N}) \to qH^*(\mathrm{Gr}_{n\pm1,N})$.

Corollary (Recursive product formula)

One has the following relation for the product in $qH^*(\mathrm{Gr}_{n,N})$,

$$\lambda \star \psi_i^*(\mu) = s_\lambda(u,q)\psi_i^*(\mu) = \sum_{r=0}^{\lambda_1} \sum_{\lambda/\nu=(r)} \psi_{i+r}^*(\nu \,\bar{\star}\, \mu)$$

where $\bar{\star}$ denotes the product in $qH^*(\mathrm{Gr}_{n-1,N})$ with $q \to -q$.

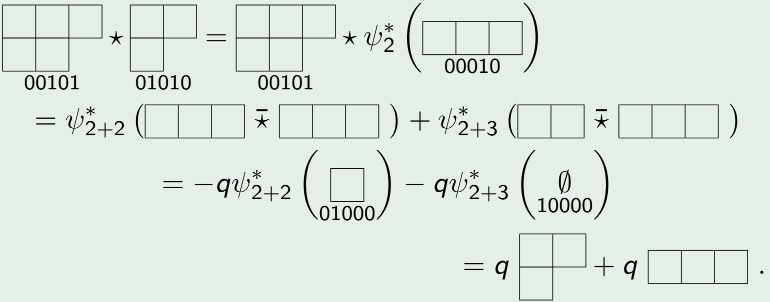
Corollary (Recursion formula for Gromov-Witten invariants)

Suppose $\psi_j^*\mu \neq 0$ for some $1 \leq j \leq N$. Then

$$C_{\lambda\mu}^{\nu,d}(n,N) = \sum_{r=0}^{\lambda_1} (-1)^{d_j(\mu,\nu)} \sum_{\lambda/\rho=(r)} C_{\rho,\psi_j\mu}^{\psi_{j+r}\nu,d_r}(n-1,N)\,.$$

Example

Consider the ring $qH^*(\mathrm{Gr}_{2,5})$. Via $\psi_i^* : qH^*(\mathrm{Gr}_{1,5}) \to qH^*(\mathrm{Gr}_{2,5})$ one can compute the product in $qH^*(\mathrm{Gr}_{2,5})$ through the product in $qH^*(\mathrm{Gr}_{1,5})$:



Inductive algorithm

One can successively generate the entire ring hierarchy $\{qH^*(\mathrm{Gr}_{n,N})\}_{n=0}^{N}$ starting from $n = 0$.

Corollary (Fermionic product formula)

One has the following alternative product formula in $qH^*(\mathrm{Gr}_{n,N})$,

$$\lambda \star \mu = \sum_{|T|=\lambda} \psi^*_{\ell_1(\mu)+\alpha_n} \bar{\psi}^*_{\ell_2(\mu)+\alpha_{n-1}} \psi^*_{\ell_3(\mu)+\alpha_{n-2}} \bar{\psi}^*_{\ell_4(\mu)+\alpha_{n-3}} \cdots \varnothing,$$

where ℓ_i *are the particle positions in* μ, α *is the weight of* T,
$\bar{\psi}^*_i = \psi^*_i$ *for* $i = 1, \ldots, N$ *and* $\bar{\psi}^*_{i+N} = (-1)^{\mathrm{n}} q \bar{\psi}^*_i$.

Corollary (Quantum Racah-Speiser Algorithm)

For $\pi \in S_n$ *set* $\alpha_i(\pi) = (\ell_i(\nu) - \ell_{\pi(i)}(\mu)) \bmod N \geq 0$ *and* $d(\pi) = \#\{i \mid \ell_i(\nu) - \ell_{\pi(i)}(\mu) < 0\}$. *Then*

$$C^{\nu,d}_{\lambda\mu} = \sum_{\pi \in S_n,\ d(\pi)=d} (-1)^{\ell(\pi)+(n-1)d} K_{\lambda,\alpha(\pi)} \ ,$$

where $K_{\lambda,\alpha}$ *are the Kostka numbers.*

Example

Set $N=7,\ n=N-k=4$ and $\lambda=(2,2,1,0),\ \mu=(3,3,2,1)$.

Step 1. Positions of 1-letters: $\ell(\mu)=(\ell_1,\ldots,\ell_4)=(2,4,6,7)$.

Step 2. Write down all tableaux of shape λ such that
$\ell'=(\ell_1+\alpha_n,\ldots,\ell_n+\alpha_1)$ mod N with $\ell'_i\neq\ell'_j$ for $i\neq j$.

$$\begin{array}{cccccccccc}
\begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}, &
\begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array} \\
(3,5,7,9) & (3,5,7,9) & (3,4,8,9) & (4,5,6,9) & (3,6,7,8) & (3,6,7,8) & (3,6,8,7) & (4,5,7,8) & (4,5,7,8) & (4,5,8,7)
\end{array}$$

Step 3. Impose quasi-periodic boundary conditions, i.e. for each $\ell'_i>N$ make the replacement

$$\psi^*_{\ell'_1}\cdots\psi^*_{\ell'_n}\varnothing\to(-1)^{n+1}q\psi^*_{\ell'_1}\cdots\psi^*_{\ell'_i-N}\cdots\psi^*_{\ell'_n}\varnothing$$

Step 4. Let ℓ'' be the reduced positions in $[1,N]$. Choose permutation $\pi\in S_n$ s.t. $\ell''_1<\cdots<\ell''_n$ and multiply with $(-1)^{\ell(\pi)}$. Done.

The three tableaux

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 4 & \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 4 & \\ \cline{1-1} \end{array}\,, \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & 4 \\ \hline 4 & \\ \cline{1-1} \end{array}$$

(4,5,7,8) (4,5,7,8) (4,5,8,7)

yield the same 01-word $w = 1001101$, $\lambda(w) = (3,2,2,0)$ but with changing sign,

$$\psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+1}\bar{\psi}^*_{\ell_4+1}\varnothing = \psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+1}\bar{\psi}^*_{\ell_4+1}\varnothing = \\ -\psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+2}\bar{\psi}^*_{\ell_4}\varnothing = q\ \psi^*_1\psi^*_4\psi^*_5\psi^*_7\varnothing\ .$$

We obtain the product expansion