Statistical Mechanics & Enumerative Geometry II:
Combinatorial Construction of WZNW Fusion Rings

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Outline

In the first talk we saw the combinatorial construction of the (small) quantum cohomology ring. In this talk we see that analogous structures appear in the $\widehat{\mathfrak{sl}}(n)_k$ WZNW fusion ring, albeit with some important differences.

1. reminder: Sugawara construction of WZNW CFT
2. ∞-friendly walkers on the cylinder
3. affine plactic algebra and crystals
4. affine plactic Schur polynomials and combinatorial fusion ring
5. recursion identities for the fusion ring
6. Summary

Reminder: Sugawara construction of WZNW CFT

$$T(z)=\sum_{n\in\mathbb{Z}} z^{-n-2}L_n,\ [L_m,L_n]=(m-n)L_{m+n}+\tfrac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Sugawara construction for (quantum) WZNW models:

$$T(z)=\frac{1}{2(h+k)}\sum_a :J^a(z)J^a(z):\ ,\ J^a(z)=\sum_{n\in\mathbb{Z}} z^{-n-1}J^a_n$$

$$\hat{\mathfrak{g}}_k:\ [J^a_n,J^b_m]=i\sum_c f_{abc}J^c_{n+m}+kn\delta_{ab}\delta_{n+m,0},\ \text{level } k\in\mathbb{Z}_{\geq 0}$$

Primary fields $\equiv$ highest weight vectors ($\hat\lambda\in P^+_k$)

$$J^a_0\phi_{\hat\lambda}=-t^a_{\hat\lambda}\phi_{\hat\lambda},\qquad J^a_n\phi_{\hat\lambda}=0,\quad n>0\ \Rightarrow\ L_n\phi_{\hat\lambda}=0,\ n>0$$

OPE and fusion rules: $\boxed{\phi_{\hat\lambda}*\phi_{\hat\mu}=\sum_{\hat\nu\in P^+_k}\mathcal{N}^{(k)\hat\nu}_{\hat\lambda\hat\mu}\phi_{\hat\nu}}$

Result

Combinatorial and recursive computation of $\mathcal{N}^{(k)\hat\nu}_{\hat\lambda\hat\mu}$ for $\hat{\mathfrak{g}}=\widehat{\mathfrak{sl}}(n)$.

Alternative interpretations of fusion coefficients

Representation theory and algebraic geometry:

- fusion coefficient of tilting modules of quantum groups at roots of 1
- dimensions of moduli spaces for generalized θ-functions

Weights and partitions

$\hat{\mathfrak{sl}}(n) = \mathfrak{sl}(n) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{C}$

Define *integral dominant weights of level* k,

$$P_k^+ = \left\{ \hat{\lambda} = \sum_{i=1}^{n} m_i \hat{\omega}_i \;\middle|\; \sum_{i=1}^{n} m_i = k \right\},$$

where $\hat{\lambda} = \lambda + k\hat{\omega}_n$.

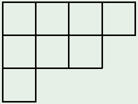
The n-tuple $m = (m_1, \ldots, m_n)$ are called **Dynkin labels**.

Given $\hat{\lambda} \in P_k^+$ we identify weights with partitions:

$$\hat{\lambda}^t = (1^{m_1(\hat{\lambda})} \ldots n^{m_n(\hat{\lambda})}), \quad m_i(\hat{\lambda}) = \#\text{ of } i\text{-columns}$$

Note that $\hat{\lambda}_1 = k$ and $\hat{\lambda}$ has at most n nonzero parts.

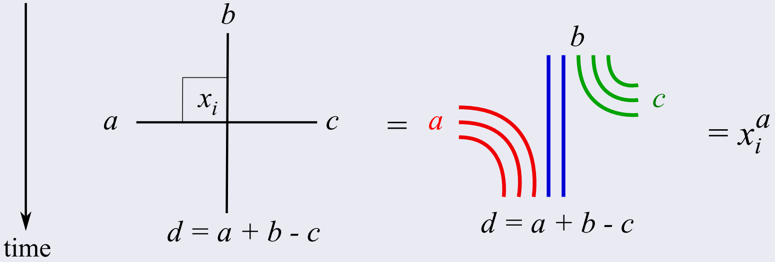
Example

Set $n = 3$, $k = 4$ and $m(\hat{\lambda}) = (1, 2, 1)$. Then $\hat{\lambda} =$  .

∞-friendly walkers on the cylinder: the defector model

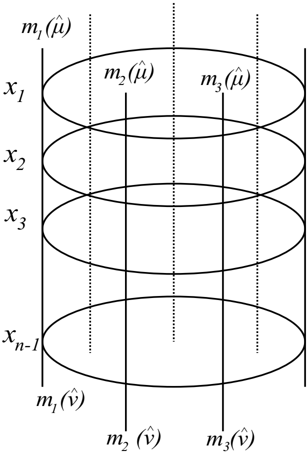
Consider an $(n-1) \times n$ square lattice ($n \geq 3$) with quasi-periodic boundary conditions (twist parameter z) in the horizontal direction.

Allowed vertex configurations and their weights ($\mathcal{R}$-matrix)



Here $a, b, c, d \in \mathbb{Z}_{\geq 0}$ and x_i is the spectral parameter in the i^{th} lattice row. (∞-friendly walkers $\rightarrow$ [Guttmann et al].)

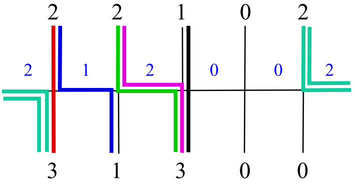
Given $\hat{\mu}, \hat{\nu} \in P_k^+$, let $m(\hat{\mu})$, $m(\hat{\nu})$ be the Dynkin labels.



Boundary conditions: fix the values of the outer edges on the top and bottom to be $m(\hat{\mu})$ and $m(\hat{\nu})$, respectively.

∞-friendly walkers on the cylinder: transfer matrix

Example of an i^{th} lattice row configuration $(n = 5,\ k = 7)$:



$= z^2 x_i^5$

The variable x_i counts the number of horizontal edges, while the variable z counts the outer horizontal edges divided by 2.

Definition of the transfer matrix

Given $m = (m_1, \ldots, m_n), m' = (m'_1, \ldots, m'_n) \in \mathbb{Z}^n_{\geq 0}$, the transfer matrix $Q(x_i)$ is defined as

$$Q(x_i)_{m,m'} := \sum_{\textit{allowed row configuration}} z^{\frac{\#\text{ of outer edges}}{2}} x_i^{\#\text{ of horizontal edges}}.$$

∞-friendly walkers on the cylinder: partition function

The partition function is the weighted sum over all allowed lattice configurations and is given by

$$Z_{\hat{\nu}}^{\hat{\mu}}(x_1,\ldots,x_{n-1};z) = \left(Q(x_{n-1})\cdot Q(x_{n-2})\cdots Q(x_1)\right)_{m(\hat{\nu}),m(\hat{\mu})} \;.$$

Theorem (Generating function for fusion coefficients)

The partition function has the following expansion in terms of Schur functions s_λ,

$$Z_{\hat{\nu}}^{\hat{\mu}}(x_1,\ldots,x_{n-1};z) = \sum_{\hat{\lambda}\in P_k^+} z^d \mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}} s_\lambda(x_1,\ldots,x_{n-1}),$$

where $\mathcal{N}_{\hat{\lambda}\hat{\mu}}^{(k),\hat{\nu}}$ *are* the Fusion coefficients *and* $d\,n = |\lambda| + |\hat{\mu}| - |\hat{\nu}|$.

Here λ is the partition obtained from deleting all n-columns in $\hat{\lambda}$.

The phase algebra

Consider the quantum space $\mathcal{H} := \mathbb{C}P^+ = \bigoplus_{k=0}^{\infty} \mathcal{H}_k, \quad \mathcal{H}_k := \mathbb{C}P^+_k.$

Define $N_j,\ \varphi_j^*,\ \varphi_j \in \mathrm{End}\,\mathcal{H}$ as $N_j m = m_j m$ and

$$\varphi_j^* m = (\ldots, m_j + 1, \ldots), \quad \varphi_j m = \begin{cases} (\ldots, m_j - 1, \ldots), & m_j > 0 \\ 0, & m_j = 0 \end{cases}$$

Phase algebra relations, compare with [Bogoliubov et al]:

$$\varphi_i \varphi_j = \varphi_j \varphi_i, \quad \varphi_i^* \varphi_j^* = \varphi_j^* \varphi_i^*, \quad N_i N_j = N_j N_i \tag{1}$$

$$N_i \varphi_j - \varphi_j N_i = -\delta_{ij} \varphi_i, \quad N_i \varphi_j^* - \varphi_j^* N_i = \delta_{ij} \varphi_i^*, \tag{2}$$

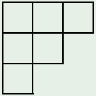
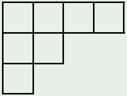
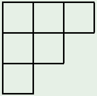
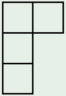
$$\varphi_i \varphi_i^* = 1, \quad \varphi_i \varphi_j^* = \varphi_j^* \varphi_i, \tag{3}$$

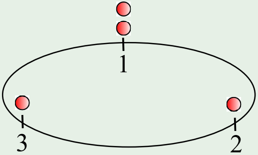
$$N_i (1 - \varphi_i^* \varphi_i) = 0 = (1 - \varphi_i^* \varphi_i) N_i \tag{4}$$

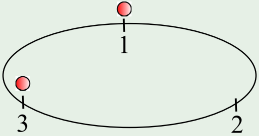
$$\langle \varphi_i^* \hat{\lambda}, \hat{\mu} \rangle = \langle \hat{\lambda}, \varphi_i \hat{\mu} \rangle, \qquad \langle \hat{\lambda}, \hat{\mu} \rangle = \delta_{\hat{\lambda}, \hat{\mu}}. \tag{5}$$

Example

Set $n = k = 3$ then

φ_1^*  $=$  and φ_2  $=$ 





$m = (2,1,1)$ $m = (1,0,1)$

Local affine plactic algebra

We define the local *affine* plactic algebra as the free algebra generated by $\{a_1, \ldots, a_n\}$ modulo the relations

$$\begin{cases} a_{i+1}a_i^2 = a_i a_{i+1} a_i \\ a_{i+1}^2 a_i = a_{i+1} a_i a_{i+1} \end{cases}, \quad a_i a_j = a_j a_i, \ |i-j| \bmod n > 1,$$

where all indices are understood to be in $\mathbb{Z}_n$. [CK, Stroppel]
Restricting to $\{a_1, ..., a_{n-1}\}$ one obtains the relations of the local (finite) plactic algebra [Lascoux, Schützenberger][Fomin, Greene].

Proposition (CK, Stroppel)

A faithful representation is given by the maps $\mathcal{H}_k \to \mathcal{H}_k$,

$$a_i = \varphi_i \varphi_{i+1}^* \qquad \textit{and} \qquad a_n = z\varphi_n \varphi_1^* \,.$$

Physical interpretation: hopping of particles in clockwise direction on the $\widehat{\mathfrak{sl}}(n)$ Dynkin diagram.

Crystal structure: Kirillov-Reshetikhin modules

$$\text{Let} \quad \mathrm{Aff}(\mathfrak{B}_{1,k}) := \left\{ z^p \otimes (m_1, ..., m_n) \;\middle|\; \sum_{i=1}^{n} m_i = k,\; p \in \mathbb{Z} \right\}$$

be the set of all k-particle configurations. This set gives rise to a directed, coloured graph by connecting two elements

$$\mathfrak{b} \xrightarrow{i} \mathfrak{b}', \quad \mathfrak{b}, \mathfrak{b}' \in \mathrm{Aff}(\mathfrak{B}_{1,k}) \qquad \text{if} \qquad \mathfrak{b}' = a_i \mathfrak{b} \;.$$

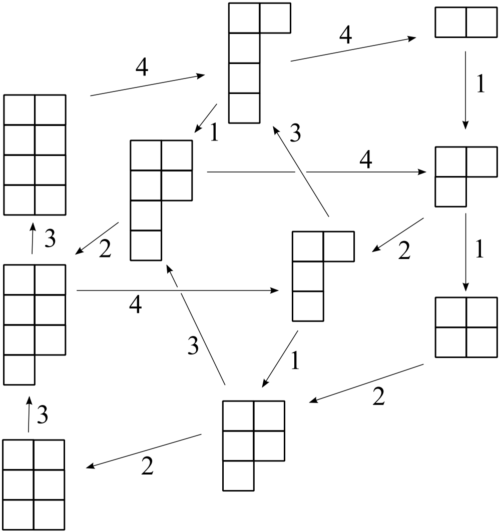
Proposition

$\mathrm{Aff}(\mathfrak{B}_{1,k})$ is the affinization of the Kashiwara crystal graph of the k-fold q-symmetric tensor product in the $U_q\hat{\mathfrak{sl}}(n)$-module

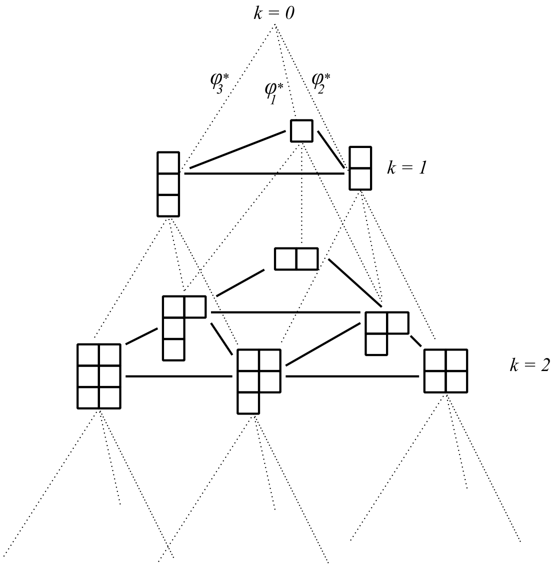
$$V(zq^{-k+1}) \otimes V(zq^{-k+3}) \otimes \cdots \otimes V(zq^{k-1}),$$

where $V(a)$ is the n-dimn'l vector evaluation module of $U_q\hat{\mathfrak{sl}}(n)$.

Example: $n = 4$ and $k = 2$



Example: $n = 3$



Noncommutative symmetric polynomials

Define generating functions of elementary and complete symmetric polynomials in the noncommutative alphabet $\{a_1, \ldots, a_n\}$,

$$T(x) = \prod_{1\le i\le n}^{\circlearrowleft} (1 + xa_i) := \sum_{r=0}^{n} e_r(a)x^r,$$

$$Q(x) = \prod_{1\le i\le n}^{\circlearrowright} (1 - xa_i)^{-1} := \sum_{r\ge 0} h_r(a)x^r,$$

where $e_0(a) = h_0(a) = 1$, $e_n(a) = a_n a_{n-1} \cdots a_1 = z$ and

$$e_r(a) = \sum_{I=\{i_1,\ldots,i_r\}} \prod_{i\in I}^{\circlearrowleft} a_i, \qquad h_r(a) = \sum_{J=\{j_1,\ldots,j_r\}} \prod_{j\in J}^{\circlearrowleft} a_j.$$

The elements in I are mutually distinct, while those in J are not.

Proposition (Integrability)

The elements in the sets $\{e_r(a)\}$ and $\{h_r(a)\}$ commute pairwise.

Cyclic Ordering

Example

Set $n = 4$ then

$$e_2(a) = a_2 a_1 + a_3 a_1 + a_1 a_4 + a_3 a_2 + a_4 a_2 + a_4 a_3$$

$$\begin{aligned} h_3(a) = \sum_{i=1}^{4} a_i^3 + a_1^2 a_2 + a_1 a_2^2 + a_1^2 a_3 + a_1 a_3^2 + a_4^2 a_1 + a_4 a_1^2 \\ + a_2^2 a_3 + a_2 a_3^2 + a_2^2 a_4 + a_4^2 a_2 + a_3^2 a_4 + a_3 a_4^2 \\ + a_1 a_2 a_3 + a_4 a_1 a_2 + a_2 a_3 a_4 \ . \end{aligned}$$

For $r > n$ we define

$$h_r(a) = \sum_{p \vdash r} (z\varphi_1^*)^{p_0} a_1^{p_1} \cdots a_{n-1}^{p_{n-1}} \varphi_n^{p_0} \ .$$

R-matrix and transfer matrix revisited

Set $\mathcal{M}_x := \bigoplus_{m=0}^{\infty} \mathbb{C}(x) v_a$ and $\mathcal{R}(x/y) : \mathcal{M}_x \otimes \mathcal{M}_y \to \mathcal{M}_x \otimes \mathcal{M}_y$

$$\mathcal{R}(x) = \mathcal{P} \left[\sum_{\alpha \in \mathbb{Z}_{\geq 0}} (\varphi^*)^\alpha \otimes \varphi^\alpha \right] (x^N \otimes 1) \,,$$

where $\mathcal{P}(v_m \otimes v_n) = v_n \otimes v_m$, $N v_m = m\, v_m$, $\varphi^* v_m = v_{m+1}$, $\varphi v_m = v_{m-1}$ and $\varphi v_0 = 0$.

Proposition (CK)

Define $\mathcal{S}(x) = (1-x)\mathcal{R}(x) + \mathcal{P}(x^{N+1} \otimes 1)$ *then*

$$\mathcal{S}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{S}_{12}(x) \,.$$

Moreover, $\mathcal{S}$ *is invertible,* $\mathcal{S}^{-1}(x) = \mathcal{P}\mathcal{S}(x^{-1})\mathcal{P}$.

Transfer matrix Q of the ∞-friendly walker model

$$Q(x) = \mathrm{Tr}_0 \left[z^{N \otimes 1} \mathcal{R}_{0n}(x) \cdots \mathcal{R}_{01}(x) \right] \in \mathrm{End}\, \mathcal{H}, \quad \mathcal{H} \cong \mathcal{M}^{\otimes n}.$$

Baxter's TQ equation in the crystal limit

What about the T-operator? T coincides with the transfer matrix of the phase model of Bogoliubov, Izergin, Kitanine; see Prop 5.13 in [CK, Stroppel, AIM 2010].

Proposition (TQ-equation at $q = 0$)

Let π_k be the (orthogonal) projector onto $\mathcal{H}_k \subset \mathcal{H}$.

$$T(-u)Q(u) = 1 + (-1)^n z \sum_{k\geq 0} u^{n+k} h_k(a)\pi_k$$
$$\Rightarrow \quad e_r(a) = \det[h_{1-i+j}(a)]_{1\leq i,j\leq r} \,,$$

Note

Both models are obtained as a special crystal limit ($q \to 0$) of the XXZ model with ∞ spin, $\mathcal{H} \cong \mathcal{M}^{\otimes n}$ [CK, to appear in JPA].

Noncommutative Schur polynomials

Proposition (Noncommutative Cauchy identities)

$$T(x_1)\cdots T(x_k) = \sum_\lambda s_{\lambda^t}(x_1,\ldots,x_k)s_\lambda(a)$$

$$Q(x_1)\cdots Q(x_{n-1}) = \sum_\lambda s_\lambda(x_1,\ldots,x_{n-1})s_\lambda(a)\;.$$

Here the noncommutative Schur polynomials

$$s_\lambda(a) = \det(e_{\lambda_i^t-i+j}(a))_{1\le i,j\le n} = \det(h_{\lambda_i-i+j}(a))_{1\le i,j\le n}$$

form a **commutative subalgebra** of the affine plactic algebra,

$$s_\lambda(a)s_\mu(a) = s_\mu(a)s_\lambda(a)\;.$$

Combinatorial construction of the Verlinde algebra

Theorem (CK, Stroppel)

Introduce on the set of basis elements in $\mathcal{H}_k$ the following product,

$$\hat{\lambda} * \hat{\mu} := s_\lambda(a)\hat{\mu}, \qquad \hat{\lambda}, \hat{\mu} \in P_k^+ .$$

This defines a unital, commutative, associative algebra V_k whose structure constants are given by the Verlinde formula ($z = 1$),

$$\langle \hat{\nu}, s_\lambda(a)\hat{\mu} \rangle = N_{\hat{\lambda}\hat{\mu}}^{(k)\hat{\nu}}, \quad N_{\hat{\lambda}\hat{\mu}}^{(k)\hat{\nu}} = \sum_{\hat{\sigma} \in P_k^+} \frac{\mathcal{S}_{\hat{\lambda}\hat{\sigma}} \mathcal{S}_{\hat{\mu}\hat{\sigma}} \bar{\mathcal{S}}_{\hat{\nu}\hat{\sigma}}}{\mathcal{S}_{0\hat{\sigma}}} \in \mathbb{Z}_{\geq 0} \,.$$

Here $\mathcal{S}_{\hat{\lambda}\hat{\sigma}}$ is the modular S-matrix (Kac-Peterson formula).

Partition function of ∞-friendly walker model

The proof of the initial theorem now follows from the 2^{nd} nc Cauchy identity involving the Q-matrix (and not T).

Recursion identities for fusion coefficients

Recursion relation for complete symmetric functions:

$$h_r(a_1,\ldots,a_n)=h_r(a_1,\ldots,a_{n-1})+z\varphi_1^* h_{r-1}(a_1,\ldots,a_n)\varphi_n$$

Taking scalar products $\hat\mu,\hat\nu\in P_k^+$ on both sides yields

$$\mathcal{N}_{(r)\hat\mu}^{(k)\hat\nu}=c_{(r)\hat\mu}^{\hat\nu}+\mathcal{N}_{(r)\varphi_n\hat\mu}^{(k-1)\varphi_1\hat\nu}\;.$$

General fusion coefficients are obtained as follows:

$$Q(x_1)\cdots Q(x_{n-1})=\sum_\lambda s_\lambda(x_1,\ldots,x_{n-1})s_\lambda(a)$$

$$\begin{aligned} h_\alpha(a)_k &= h_{\alpha_1}(a)_k\cdots h_{\alpha_r}(a)_k=\sum_{\hat\lambda\in P_k^+}K_{\lambda\alpha}s_\lambda(a)_k \\ \Rightarrow &\quad \sum_{\hat\rho^{(i)}}\mathcal{N}_{(\alpha_1)\hat\rho^{(1)}}^{(k)\hat\nu}\mathcal{N}_{(\alpha_2)\hat\rho^{(2)}}^{(k)\hat\rho^{(1)}}\cdots\mathcal{N}_{(\alpha_r)\hat\mu}^{(k)\hat\rho^{(r-1)}}=\sum_{\hat\lambda\in P_k^+}K_{\lambda\alpha}\mathcal{N}_{\hat\lambda\hat\mu}^{(k)\hat\nu}\end{aligned}$$

where $K_{\lambda\mu}$ are the (classical) Kostka numbers.

Summary

Quantum cohomology	Verlinde algebra
Clifford algebra $\{\psi_i, \psi_i^*\}_{i=1}^{n+k=N}$	Phase algebra $\{\varphi_i, \varphi_i^*\}_{i=1}^{n}$
affine nil-Temperley-Lieb algebra $u_i = \psi_{i+1}^*\psi_i,\ u_N = (-1)^{k-1} q \psi_1^* \psi_N$	local affine plactic algebra $a_i = \varphi_{i+1}^*\varphi_i, \quad a_n = z\varphi_1^*\varphi_n$
k-particle space = $\widehat{\mathfrak{sl}}(N)$ evaluation module	k-particle space = affine $U_v\widehat{\mathfrak{sl}}(n)$ crystal
transfer matrices = nc polynomials in u_i's	transfer matrix, Baxter's Q = nc polynomials in a_i's
combinatorial product: $\lambda \star \mu = s_\lambda(u)\mu$	combinatorial product: $\hat{\lambda} * \hat{\mu} = s_\lambda(a)\hat{\mu}$
Bethe ansatz ⇒ Bertram-Vafa-Intrilligator formula	Bethe ansatz ⇒ Verlinde formula
Λ_k quotient w.r.t. $h_{n+1} = \cdots = h_{n+k-1} = 0,$ $h_{n+k} = (-1)^{k-1} q$	Λ_k quotient w.r.t. $h_{n+1} = \cdots = h_{n+k-1} = 0,$ $h_n = 1,\ h_{n+k} = (-1)^{k-1} e_k$
nc Schur polynomial expansion: $s_\lambda(u)s_\mu(u) = \sum_\nu C_{\lambda,\mu}^{\nu,d} s_\nu(u)$	nc Schur polynomial expansion: $s_\lambda(a)s_\mu(a) = \sum_\nu N_{\lambda,\mu}^{(k)\nu} s_\nu(a)$

Novel description in terms of integrable systems

Simplified, combinatorial approach using the physical picture of quantum particles hopping on Dynkin diagrams:

- integrability $\Rightarrow$ simple proof of associativity
- Bethe ansatz $\Rightarrow$ ring isomorphism, Verlinde formulae
- particle creation/annihilation operators $\Rightarrow$ new identities
- Connection with small quantum cohomology and Gromov-Witten invariants: $\hat{\mathfrak{g}} = \widehat{\mathfrak{gl}}(n)$ (TFT)
- generalizations to other algebras + deformations (in preparation)

Thank you for your attention!