A rep theoretic construction of Baxter's Q operator and solutions to the discrete Liouville equation

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Outline

1. Recap: quantum IS, transfer matrices, fusion hierarchy
2. Baxter's TQ equation, Bethe ansatz, quantum Wronskian
3. Rep theoretic construction of Q
4. trace functional and continuum limit
5. Outlook

Quantum Yang-Baxter equation and algebras

Quantum integrable lattice models are constructed from solns of the quantum YBE,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$$

Solns are obtained from the rep theory of a "quantum algebra" $\mathcal{A}$ (i.e. Yangian, q-deformed env algebra, elliptic algebra).

Definition

Let V, W be $\mathcal{A}$-modules and R_{VW} the intertwiner of the tensor product

$$R_{VW} : V \otimes W \to V \otimes W \,.$$

Then the corresponding transfer matrix is defined as

$$T_V = \operatorname*{Tr}_V R_{VW} \in \mathrm{End}\, W,$$

where V is called "auxiliary space" and W is the "quantum" (physical state) space.

Quantum Integrability and Hirota-Miwa equation

"Quantum Integrability" now follows from construction:

Theorem

Let V, V' be two $\mathcal{A}$-modules. If $R_{VV'}$ exists then it solves the QYBE and one has $[T_V, T_{V'}] = 0$.

The set of all transfer matrices constitutes the fusion hierarchy.

sl_k fusion relation

Let $V \to V_s^a(u)$ be labelled by rectang Young diagram:

$$T_s^a(u+1)T_s^a(u-1) - T_{s+1}^a(u)T_{s-1}^a(u) = T_s^{a+1}(u)T_s^{a-1}(u),$$

3D integrable system: Hirota-Miwa (discrete KP) equation!

c.f. [Kuniba et al 1994], [Krichever et al 1997]

$k = 2$: Truncation of the Hirota-Miwa equation

Specialize to sl_2. Consider evaluation modules $V \to V_s(u)$ of $\mathcal{A} = Y(sl_2)$, $U_q(\widehat{sl}_2)$ with $u \in \mathbb{C}$ spectral parameter such that $V_s = V_s(u=0)$ is an sl_2 (resp. $U_q(sl_2)$) module of "spin" $s/2$.

H-M equation $\longrightarrow$ discrete Liouville equation

Boundary conditions: $T_s^a(u) = 0$ for $a < 0$ and $a > 2$
Remaining non-trivial relation:

$$T_s^{a=1}(u+1)\,T_s^{a=1}(u-1) - T_{s+1}^{a=1}(u)\,T_{s-1}^{a=1}(u) = T_s^0(u)\,T_s^2(u).$$

Identify

$$T_s^{a=1}(u) = T_{V_s(u)}, \quad T_s^{a=0}(u) = \varphi(u-s), \quad T_s^{a=2}(u) = \varphi(u+2+s)$$

and

$$\varphi(u) = T_{s=0}^{a=1}(u-1)$$

is a scalar function associated with the trivial representation $s = 0$ depending on the quantum/physical space.

Bäcklund transformation: auxiliary linear problems

[Krichever et al 1997]: Given a solution $T_s^a(u)$ consider the linear set of equations determining $Q_s^a(u)$,

$$\begin{aligned} T_{s+1}^{a+1}(u)Q_s^a(u) - T_s^{a+1}(u+1)Q_{s+1}^a(u-1) &= T_s^a(u)Q_{s+1}^{a+1}(u), \\ T_{s+1}^a(u+1)Q_s^a(u) - T_s^a(u)Q_{s+1}^a(u+1) &= T_s^{a+1}(u+1)Q_{s+1}^{a-1}(u) \end{aligned}$$

Auto-Bäcklund transformation: $T_s^a(u) \to Q_s^a(u)$

New solution satisfies different b.c.:

$$Q_s^a(u) = 0 \;\text{ for }\; a < 0 \;\text{ and }\; a > 1\,.$$

For $k = 2$ these b.c. lead to Baxter's TQ-equation. If we assume $Q_s^{a=0,1}(u)$ to be analytic in the spectral parameter, then

$$\rightsquigarrow\; Q_s^0(u) = Q(u-s) \;\text{ and }\; Q_s^1(u) = \bar{Q}(u+s)$$

Dependence on "light cone coordinates" implies the following invariant of the s-dynamics:

$$A(u) = \frac{\varphi(u+2)T_{s+1}(u+s) + \varphi(u)T_{s-1}(u+2+s)}{T_s(u+1+s)}$$

which satisfies the equations

$$Q(u)A(u) = \varphi(u+2)Q(u-2) + \varphi(u)Q(u+2)$$

Exploiting the initial conditions $T_{-1}(u) = 0$ and $T_0(u) = \varphi(u+1)$ the quantity $A(u)$ specializes at $s = 0$ to $A(u) = T_1(u)$ implying

- alternative version of the fusion relation

$$T_1(u)T_s(u+1+s) = \varphi(u+2)T_{s+1}(u+s) + \varphi(u)T_{s-1}(u+2+s)$$

- Baxter's TQ-equation [1972] $\Rightarrow$ Bethe ansatz eqns [1933]

$$Q(u)T_1(u) = \varphi(u+2)Q(u-2) + \varphi(u)Q(u+2) \; .$$

The XXZ Heisenberg quantum spin-chain

Prototype of a quantum integrable model:

$$H_{\text{XXZ}} = \frac{1}{2}\sum_{m=1}^{M}\left\{\sigma^x_m\sigma^x_{m+1} + \sigma^y_m\sigma^y_{m+1} + \frac{q+q^{-1}}{2}\left(\sigma^z_m\sigma^z_{m+1} - 1\right)\right\}$$

quasi-periodic b.c. : $\sigma^{\pm}_{M+1} \equiv q^{\pm 2\alpha}\sigma^{\pm}_1, \quad \sigma^z_{M+1} \equiv \sigma^z_1$

Quantum space: $W = \bigotimes_{i=1}^{M} V_1(z_i)$

In the homogeneous case $z_i = 1$ we have

$$H_{\text{XXZ}} \propto z\frac{d}{dz}\ln T_1(z), \qquad T_1(z) = \operatorname*{Tr}_{V_1(z)} q^{\alpha h\otimes 1} R_{V_1(z),W}$$

Note: All transfer matrices commute with Hamiltonian.

Quantum group reminder

Quantum algebra $\mathcal{A} = U_q(\widehat{sl}_2)$ and evaluation modules $V_s(z = q^u)$. For $z = 0$ these reduce to the $U_q(sl_2)$-modules V_s.

Algebra relations:

$$q^h e = e q^{h+2}, \quad q^h f = f q^{h-2}, \quad [e, f] = [h]_q$$

Modules V_s:

$$\begin{aligned} (e, v_k) &\mapsto [s-k+1]_q [k]_q v_{k-1}, \qquad (e, v_0) \mapsto 0, \\ (f, v_k) &\mapsto v_{k+1}, \qquad (q^h, v_k) \mapsto q^{s-2k} v_k, \qquad k = 0, 1, 2, 3, \ldots \end{aligned}$$

If $s \in \mathbb{N}_{\geq 0}$ then $V_s \cong \mathbb{C}^{s+1}$ and the module truncates

$$(f, v_s) \mapsto 0$$

otherwise the module is infinite-dimensional.

Fusion hierarchy and relation

Consider first $s \in \mathbb{N}_{\geq 0}$.

Theorem (Chari,Pressley 1990)

Decomposition of tensor module:

$$0 \to V_{s-1}(q^{u+s+2}) \hookrightarrow V_s(q^{u+s+1}) \otimes V_1(q^u) \to V_{s+1}(q^{u+s}) \to 0$$

Setting

$$T_s(u) \to T_s(z = q^u) = \operatorname*{Tr}_{V_s(z)} q^{\alpha h \otimes 1} R_{V_s(z),W}, \qquad s \in \mathbb{N}_{\geq 0}$$

we obtain the fusion relation

$$T_s(q^{u+s+1})T_1(q^u) = T_0(q^{u+1})T_{s+1}(q^{u+s}) + T_0(q^{u-1})T_{s-1}(q^{u+s+2})$$

with "quantum determinant" (spin-1/2 chain)

$$T_0(q^{u-1}) = \varphi(u) = \prod_{m=1}^{M}(1 - q^u z_m)$$

Let s be generic.

The corresponding "transfer matrix" is NOT the analytic continuation w.r.t. the continuum limit of the discrete Liouville equation. Instead "generating matrix" for 2 lin indep solutions to Baxter's TQ equation.

Definition

$$Q(z;s) = \operatorname*{Tr}_{V_s(z)} q^{\alpha h\otimes 1} R_{V_s(z),W}, \qquad s \in \mathbb{C}/\mathbb{N}_{\geq 0}$$

Note: $V_s(z)$ is now infinite-dimensional. Trace is defined through analytic continuation in α. For instance,

$$Q(0;s) = \operatorname*{Tr}_{V_s(z)} q^{\alpha h\otimes 1 - h\otimes S^z} := \frac{(-1)^M q^{\alpha - S^z}}{1-q^{2(S^z-\alpha)}}, \qquad S^z = \sum_m \frac{\sigma^z_m}{2}$$

Theorem (Bazhanov et al 1999, CK 2008)

We have the decomposition $Q(z;s)=Q(0)Q^{+}(z)Q^{-}(zq^{s+1})$ *with*

$$Q^{+}(z)=\lim_{q^{s+1}\to 0}Q(z;s)/Q(0),\quad Q^{-}(z)=\lim_{q^{s+1}\to\infty}Q(z;s)/Q(0)$$

Proof [CK 2008].

$$0\to V_{s+t}(q^{u-2s})\hookrightarrow V_{s}(q^{u-2s})\otimes V_{t}(q^{u})\to V_{s+t}(q^{u+s})\otimes V_{-1}(q^{u+2})\to 0$$

Solutions to the TQ equation:

$$T_1(q^u)Q^{\pm}(q^u)=T_0(q^{u+1})Q^{\pm}(q^{u-2})+T_0(q^{u+1})Q^{\pm}(q^{u+2})$$

with eigenvalues

$$Q^{\pm}(q^u)=q^{\mp\frac{\alpha-S^z}{2}u}\prod_{i=1}^{n_\pm}(1-q^u/x_i^{\pm}),\quad n_\pm=\frac{M}{2}\mp S^z\,.$$

The Quantum Wronskian

The TQ-equation is a 2nd order Δ equation. The two lin indep solutions $Q^{\pm}$ satisfy non-trivial Wronskian type relation ($s = 1$):

$$\left(q^{S^z-\alpha} - q^{\alpha-S^z}\right) T_{s-1}(u) = \begin{vmatrix} Q^+(q^{u+s}) & Q^-(q^{u+s}) \\ Q^+(q^{u-s}) & Q^-(q^{u-s}) \end{vmatrix}$$

Conjecture

Provided $\alpha \neq 0$ is generic, there exist precisely $\dim W$ *solutions.*
$\Rightarrow$ *Bethe ansatz is complete.*
For $\alpha = 0$ and M odd, the above continues to hold true.
For $\alpha = 0$ and M even, there do not exist solutions with the required analyticity requirements.

Decomposition at periodic b.c. [CK 2005]:

$$M \in 2\mathbb{N} : \quad \lim_{\alpha \to 0} T_{s-1}(u) = f(q^u, q^s) + s\, g(q^u, q^s).$$

Summary

- One can realize complete ($=\dim W$) set of solutions to the discrete Liouville equation (subject to analyticity) as spectrum of explicitly constructed Q-operator.
- Representation theory yields functional relations.

Note: If q is root of 1 and $\alpha \in \mathbb{Z}$, then number of solns $\leq \dim W$.

$\tilde{sl}_2$-symmetry [Deguchi et al '00][CK,McCoy '01][CK '04]. The number is obtained by counting paths on restricted Bratelli diagrams (combinatorial problem).

Degeneracies also occur for XXX: sl_2-symmetry.

Modified quantum Wronskian [Pronko, Stroganov 1998]

$$u^M = Q^+(u-1)Q^-(u) - Q^+(u)Q^-(u-1) \quad \text{with}$$

$$Q^+(u) = \prod_{i=1}^{M/2-S^z}(u-v_i^+) \quad \text{and} \quad Q^-(u) = \frac{1}{2S^z+1}\prod_{i=1}^{M/2+S^z+1}(u-v_i^-)\;!$$

The continuum limit

Discrete Liouville equation can be recast into

$$Y_s(u+1)Y_s(u-1) = (1+Y_{s+1}(u))(1+Y_{s-1}(u)) \; .$$

with

$$Y_s(u) = \frac{T_{s+1}(u)T_{s-1}(u)}{\varphi(u-s)\varphi(u+2+s)}$$

The solution ϕ to the continuous Liouville equation

$$\phi_{tt} - \phi_{xx} = 2e^{\phi}$$

is then obtained by making the identification

$$e^{-\phi(x,t)} = \lim_{\delta\to 0} \delta^2 Y_{t/\delta}(x/\delta) \; .$$

Complex dimension and the trace functional

Consider scaling limit: $U_q(\hat{sl}_2) \to Y(sl_2)$. The corresponding model is the isotropic quantum Heisenberg spin-chain (XXX).

XXX: rational solutions [CK 2005]

$$\varphi(u) = \prod_{m=1}^{M} (u - u_m), \qquad Q^{\pm}(u) = \omega^{\mp u/2} \prod_{i=1}^{n_\pm} (u - v_j^{\pm}) \,.$$

[Boos,Jimbo,Miwa,Smirnov,Takeyama'02] Define trace functional

$$\operatorname*{Tr}_x : U(sl_2) \otimes \mathbb{C}[x] \to \mathbb{C}[x]$$

such that

- for $x \in \mathbb{N}$: ordinary trace, $\mathrm{Tr}_x(a) = \mathrm{Tr}_{V_{x-1}}\, a$, $\forall a \in U(sl_2)$
- action on the Cartan element $h \in sl_2$,

$$\operatorname*{Tr}_x e^{zh} = \frac{\sinh(zx)}{\sinh(x)} = x + \frac{x(x^2-1)}{6} z^2 + \frac{x(7-10x^2+3x^4)}{360} z^4 + \ldots$$

Definition

Define for complex x the transfer matrix (periodic b.c.)

$$T(u,x) := \operatorname*{Tr}_x R(u)_W, \quad R_W(u) \in U(sl_2) \otimes \mathrm{End}\, W.$$

Note: for $x = s+1 \in \mathbb{N}$, $T(u,s+1) = T_s(u) = \mathrm{Tr}_{V_s} R_{V_s(u),W}$.

Theorem (CK 2005)

$$T(u,x) = \lim_{\omega \to 1} \frac{\omega Q^+(u-x)Q^-(u+x) - \omega^{-1}Q^+(u+x)Q^-(u-x)}{\omega - \omega^{-1}}$$

Setting

$$Y_s(u) = \frac{T(u,s+2)T(u,s)}{T(u-s-1,1)T(u+s+1,1)}$$

yields single soliton solution in the continuum limit.

Solutions in the Continuum

Example

Homogeneous spin-1/2 XXX chain. Choose $M = 4$ then one eigenvalue of $T(u, s+1)$ gives rise to the solution

$$\phi(x,t) = -\lim_{\delta\to 0} \delta^2 Y_{t/\delta}(x/\delta) = -\log \frac{(t^5 + 10t^3x^2 + 5tx^4)^2}{25(t^2 - x^2)^4}$$

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Outlook

- n soliton solutions? 2D Toda lattice?
- Construction for $k > 2$? H-M eqn has solutions as Casoratian determinants [Ohta et al 1993][Nimmo 1997]
- Classification of solns to H-M eqn in terms of (nested) Bethe ansatz and vice versa? String hypothesis and thermodynamic Bethe ansatz?
- What about other, "continuum" quantum integrable models, e.g. QNLS, Liouville CFT?
- Ultra-discrete and crystal limit?
- Elliptic case, Baxter's 8-vertex and Belavin's model? Bethe roots = discrete integrable model? [Krichever et al 1997]