University of Glasgow

Quantum Integrable Models:

Prospectives & Perspectives

Christian Korff

www.maths.gla.ac.uk/~ck

Introduction: integrable models & algebras

We shall distinguish between two different notions of integrability:

- **Classical physics**: A system is called (Liouville) integrable if there are as many independent *integrals of motion* as degrees of freedom.
- **Quantum physics**: A system is called integrable if it possesses an infinite number of *conserved charges*.

Non-abelian algebras in integrable systems mainly occur in two contexts:

- **Symmetry**: the algebra commutes with the Hamiltonian or Lagrangian of the theory.
  Example: conformal field theory & the Virasoro algebra
- **Generating structure**: the algebra provides the underlying mathematical structure for the computation of the integrals of motion, the spectrum of the Hamiltonian or correlation functions.
  Example: the Ising model & Onsager's algebra

Long term goal: Correlation functions

Example: Probability to find two spins aligned: $\langle \sigma_i^z \sigma_{i+n}^z \rangle = \dfrac{\mathrm{Tr}[\sigma_i^z \sigma_{i+n}^z e^{-\beta H}]}{\mathrm{Tr}\, e^{-\beta H}}$

- *encode electric, magnetic and transport properties*

- **Critical Phenomena**: systems undergoing a continuous phase transition

  - *exponential decay:* $\boxed{T > T_c \;:\; \langle \sigma_i^z \sigma_{i+n}^z \rangle \sim \exp(-n/\xi), \qquad \xi(T) \sim \dfrac{1}{|T - T_c|}}$

  massive QFT: $\xi \sim 1/m$ is the Compton wavelength

  - *algebraic decay:* $\boxed{T = T_c \;:\; \langle \sigma_i^z \sigma_{i+n}^z \rangle \sim \dfrac{1}{n^{2\Delta}}, \qquad \Delta = h + \bar{h}}$

  massless QFT = CFT: primary fields $\begin{aligned} L_0 \varphi(z)|0\rangle &= h \varphi(z)|0\rangle \\ \bar{L}_0 \varphi(\bar{z})|0\rangle &= \bar{h} \varphi(\bar{z})|0\rangle \end{aligned}$

Comparison with quantum field theory in the continuum limit:

- vacuum expectation values of local operators NPB 636 (2002) 435 [K. Seaton, La Trobe]
- long distance asymptotics of correlation functions

Example: the planar Ising model

Lattice model:

$$H_{\text{Ising}} = \sum_n \sigma_n^z \sigma_{n+1}^z + h\sigma_n^x$$

Conformal Field Theory $c=½$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}\, m(m^2-1)\delta_{m+n,0}$$

Virasoro algebra

h magnetic field

$$S = S_{CFT} + \lambda \int \phi_{(1,2)}(x) d^2x, \qquad \Delta_{(1,2)} = 1/16$$

Integrable QFT ($h > 0$)

Lorentz spins $s = 1, 7, 11, 13, 17, 19, 23, 29$

$$\left(\hat{E}_8\right)_1 \otimes \left(\hat{E}_8\right)_1 / \left(\hat{E}_8\right)_2$$

GKO coset ($h = 0$)

T temperature

$$\left(\hat{A}_1\right)_1 \otimes \left(\hat{A}_1\right)_1 / \left(\hat{A}_1\right)_2$$

GKO coset ($T = T_c$)

$$S = S_{CFT} + \lambda \int \phi_{(2,1)}(x) d^2x, \qquad \Delta_{(2,1)} = 1/2$$

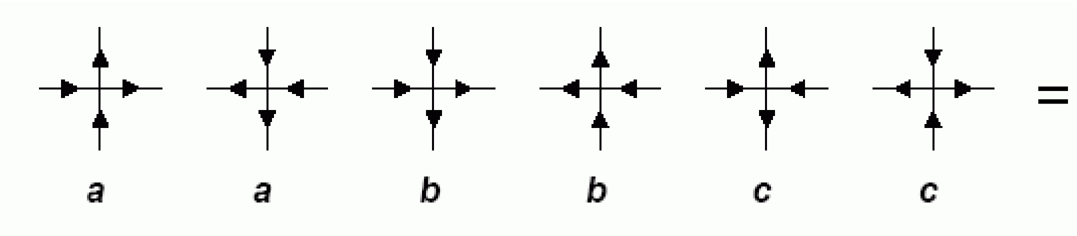
Massive free fermion ($T > T_c$)

Lorentz spin s = 1

$$\lambda \sim m^{2(1-\Delta)}$$

Statistical Mechanics: the 6-vertex model

Vector space V = linear span of statistical variables



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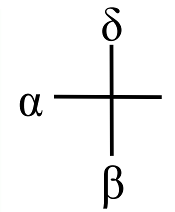
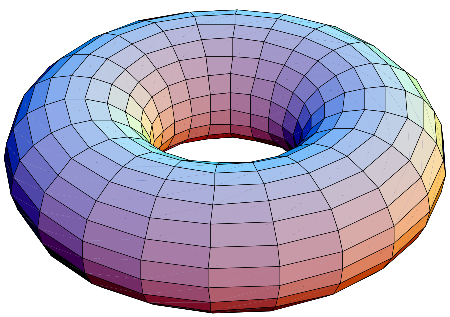
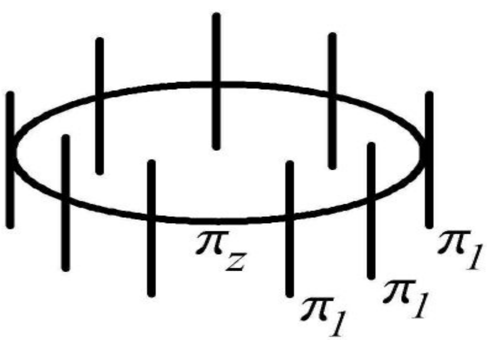


$= R(a,b,c)^{\gamma\delta}_{\alpha\beta}$





<u>Partition function</u>:

$$Z = \sum_{\{\alpha_i,\beta_i,\gamma_i,\delta_i\}} \prod_{\text{vertices } i} R^{\gamma_i\delta_i}_{\alpha_i\beta_i} = \mathrm{Tr}\; T^M$$

Sum over all vertex configurations.

<u>Transfer matrix</u>:

$$T = \mathrm{Tr}_0\, R_{0N} R_{0N-1} \cdots R_{01}$$

Discrete evolution operator.

The Heisenberg Spin-Chain

Baxter's concept of commuting transfer matrices: $[T(u),T(v)] = 0$

- *find solutions of the quantum Yang-Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u) \qquad R(u) \in \mathfrak{A} \otimes \mathfrak{A}$$

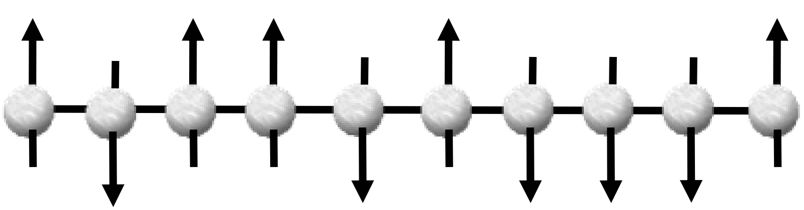
- *construct set of commuting transfer matrices by evaluating solutions in particular representations*

$$\pi^{(s)} : \mathfrak{A} \to \operatorname{End} V^{(s)} \qquad T^{(s)}(u) = (\mathrm{Tr}_{\pi^{(s)}} \otimes \pi_{\mathfrak{H}})R(u), \quad s \in \tfrac{1}{2}\mathbb{Z}$$

Hamiltonian is "recovered" from

$$H = \frac{d}{du} \ln T^{(s=1/2)}(u)\Big|_{u=0}$$

Model	Restriction	Algebra $\mathfrak{A}$
XYZ	–	Sklyanin (elliptic) algebra
XXZ	$g_x = g_y$	Quantum group $U_q(\tilde{sl}_2)$
XXX	$g_x = g_y = g_z$	Yangian $Y(sl_2)$



$$H = J\sum_n \{g_x\sigma_n^x\sigma_{n+1}^x + g_y\sigma_n^y\sigma_{n+1}^y + g_z\sigma_n^z\sigma_{n+1}^z\}$$

Bethe's Ansatz

Solution of the XXX Heisenberg spin-chain by Hans Bethe [1931]

- *superposition of plane waves:* $$|k_1,\ldots,k_n\rangle = \sum_{0\le x_1<\ldots<x_n\le L} \psi_k(x_1,\ldots,x_n)\sigma^-_{x_1}\cdots\sigma^-_{x_n}|\uparrow\uparrow\ldots\uparrow\rangle$$
- *Bethe's wavefunction:* $$\psi_k(x_1,\ldots,x_n) = \sum_{p\in S_n} A(p_1,\ldots,p_n)e^{i(k_{p_1}x_1+\cdots+k_{p_n}x_n)}$$
- *boundary & eigenvector conditions:* $$s_{p_i p_{i+1}}A(p_1,\ldots,p_n) = -s_{p_{i+1}p_i}A(p_1,\ldots,p_{i+1},p_i,\ldots,p_n)$$ $$e^{iLk_{p_1}}A(p_2,\ldots,p_n,p_1) = A(p_1,\ldots,p_n)\ .$$
- *Bethe's equations:* *Bethe roots = quasi momenta* $$b_\ell^M = (-1)^{n-1}\prod_{j=1}^{n}\frac{1-2g_z b_\ell + b_j b_\ell}{1-2g_z b_j + b_j b_\ell},\quad b_\ell = e^{ik_\ell}$$
- *"exact" energy spectrum:* $$E = -\sum_{j=1}^{n}(b_j + b_j^{-1} - 2g_z)$$

Difference equations from representation theory

Question: How do you solve the BAE? Can you connect the solutions (Bethe roots) to the representation theory of the underlying algebraic structure?
Answer: Yes. Construct Baxter's Q-operator!

Consider "transfer matrix" of an ∞-dimensional representation [Verma module]:

$$\boxed{Q(u;x) = (\mathrm{Tr}_{\pi^{(x)}} \otimes \pi_{\mathfrak{H}})R(u) = Q^{+}(u)Q^{-}(u+s), \quad x \in \mathbb{C}}$$

Result: JPA 36 (2003) 5229; JPA 37 (2004) 385, 7227; JPA 38 (2005) 47

- Functional equations from rep theory

$$0 \to \pi^{(x')} \to \pi^{(x)} \otimes \pi^{(1/2)} \to \pi^{(x'')} \to 0$$

- Analytic continuation of the fusion hierarchy

$$T(u;x) = Q^{+}(u - x\lambda)Q^{-}(u + x\lambda) - Q^{+}(u + x\lambda)Q^{-}(u - x\lambda)$$

- "Quantum Wronskian"

$$\text{XXX}: \quad u^{M} = Q^{+}(u+1)Q^{-}(u) - Q^{+}(u)Q^{-}(u+1), \quad Q^{\pm}(u) = \prod_{i=1}^{d_{\pm}}(u - u_i^{\pm})$$

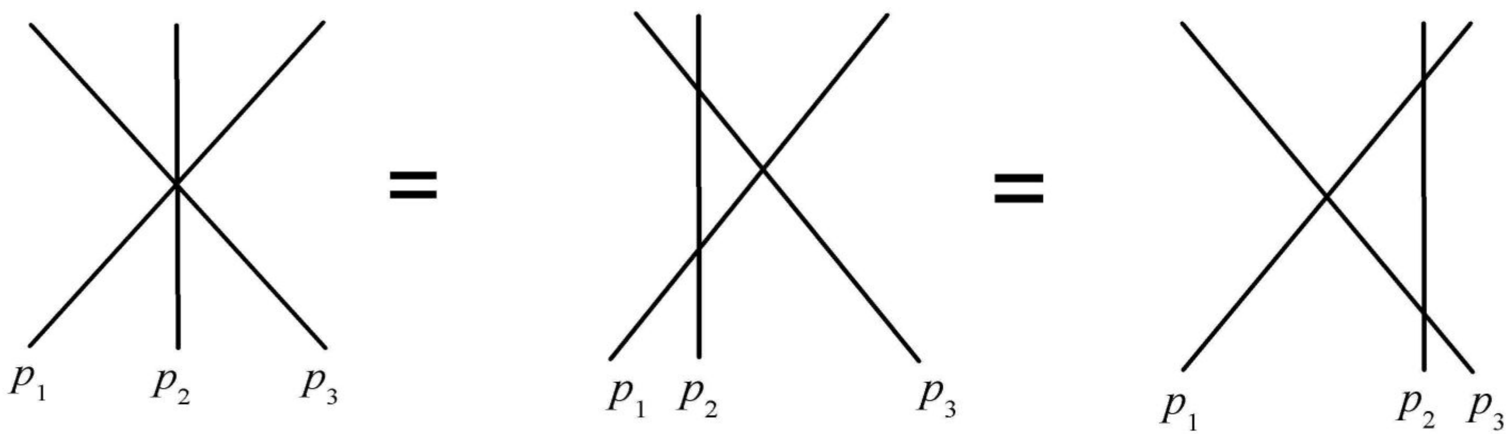
This identity leads to a system of ***quadratic*** *equations for "solving" the model. This supersedes the Bethe ansatz equations whose polynomial order = number of sites.*

1+1 Dimensional Quantum Field Theory

vector space V = particle types, e.g. solitons, breathers, ...

observable $O = S$ scattering matrix

Integrability → factorization of scattering into 2 particle events



$$V_1 \otimes V_2 \otimes V_3 \; : \; S_{12}(p_{12})S_{13}(p_{13})S_{23}(p_{23}) = S_{23}(p_{23})S_{13}(p_{13})S_{12}(p_{12})$$

$S : V \otimes V \to V \otimes V$ Yang-Baxter equation → quantum groups

The Bootstrap Programme

Idea: Exact construction of the 2-particle S-matrix/corr functions from a set of axioms.

- *unitarity & crossing:* $$S_{12}(\theta)S_{21}(-\theta) = 1 \qquad S(i\pi-\theta)^{cd}_{ab} = S(\theta)^{c\bar{b}}_{da} = S(\theta)^{\bar{a}d}_{b\bar{c}}$$

- *fusing ("bound states"):* $$S_{lk}(\theta) = S_{li}(\theta + i\eta^{j}_{ik})S_{lj}(\theta - i\eta^{i}_{jk})$$

- *Form Factor Expansion:* $$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \sum_n \int \prod_{k=1}^{n} \frac{d\theta_k}{2\pi k} e^{ix\cdot p_k(\theta_k)} |F^n_{\mathcal{O}}(\theta_1,\ldots,\theta_n)|^2$$

$$F^n_{\mathcal{O}}(\theta_1,\ldots,\theta_n) := \langle 0|\mathcal{O}(0)|\theta_1,\ldots,\theta_n\rangle$$

- *exchange of particles:* $$F^n_{\mathcal{O}}(\ldots,\theta_i,\theta_{i+1},\ldots) = F^n_{\mathcal{O}}(\ldots,\theta_{i+1},\theta_i,\ldots)S_{ii+1}(\theta_{ii+1})$$

- *recurrence relations: fusing & crossing* $$F^{n-2}_{\mathcal{O}}(\theta_3,\ldots,\theta_n) \to \operatorname*{Res}_{\theta_{12}=i\pi} F^n_{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$ $$F^{n-1}_{\mathcal{O}}(\theta_b,\theta_3,\ldots,\theta_n) \to \operatorname*{Res}_{\theta_{12}=\theta_b} F^n_{\mathcal{O}}(\theta_1,\ldots,\theta_n)$$

Affine Toda field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^2}{\beta^2}\sum_{i=0}^{\ell} n_i \exp(\beta\alpha_i\cdot\varphi)$$

The α_i are simple roots of a Lie algebra g.

$$\alpha_0 = -\sum_{i=0}^{\ell} n_i\alpha_i$$

The quantum two-particle scattering amplitude can be calculated **exactly**:

$$S_{ij}(\theta,B) = \exp 4\int_0^\infty \frac{dt}{t}\,\Phi_{ij}(t)\sinh\left(\frac{\theta t}{i\pi}\right)$$

$$\Phi_{ij}(t) = (q-q^{-1})(q^{\vee t_i} - q^{\vee -t_i})K(q,q^\vee)^{-1}_{ij}$$

Result: NPB 567 (2000) 409 [with A.Fring, B.Schultz]

$$K(q,q^\vee)_{ij} = (q\,q^{\vee t_i} + q^{-1}q^{\vee -t_i})\delta_{ij} - [I_{ij}]_{q^\vee}$$

$$K_{ij} = 2\frac{\alpha_i\cdot\alpha_j}{\alpha_j\cdot\alpha_j}$$

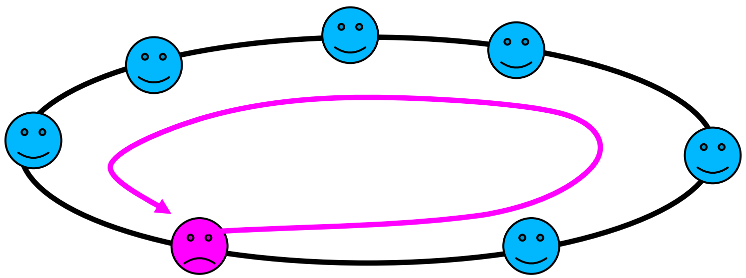
$$q = \exp\frac{(2-B)t}{2h}\,, \qquad q^\vee = \exp\frac{B\,t}{2\ell h^\vee}$$

$$t_i K_{ij} = K_{ij}t_j \qquad I = 2-K$$

Renormalization flow between g and its Langlands dual $g^\vee$ $(0 \le B \le 2)$.

The UV Limit: the Thermodynamic Bethe Ansatz

- *Place particles on a (big) circle: compactify space*



$$e^{iLm_k \sinh\theta_k} \prod_{l\neq k}^{N} S_{kl}(\theta_k - \theta_l) = 1$$

- *Take thermodynamic limit (L,N → ∞) to obtain system of coupled nonlinear integral equations:*

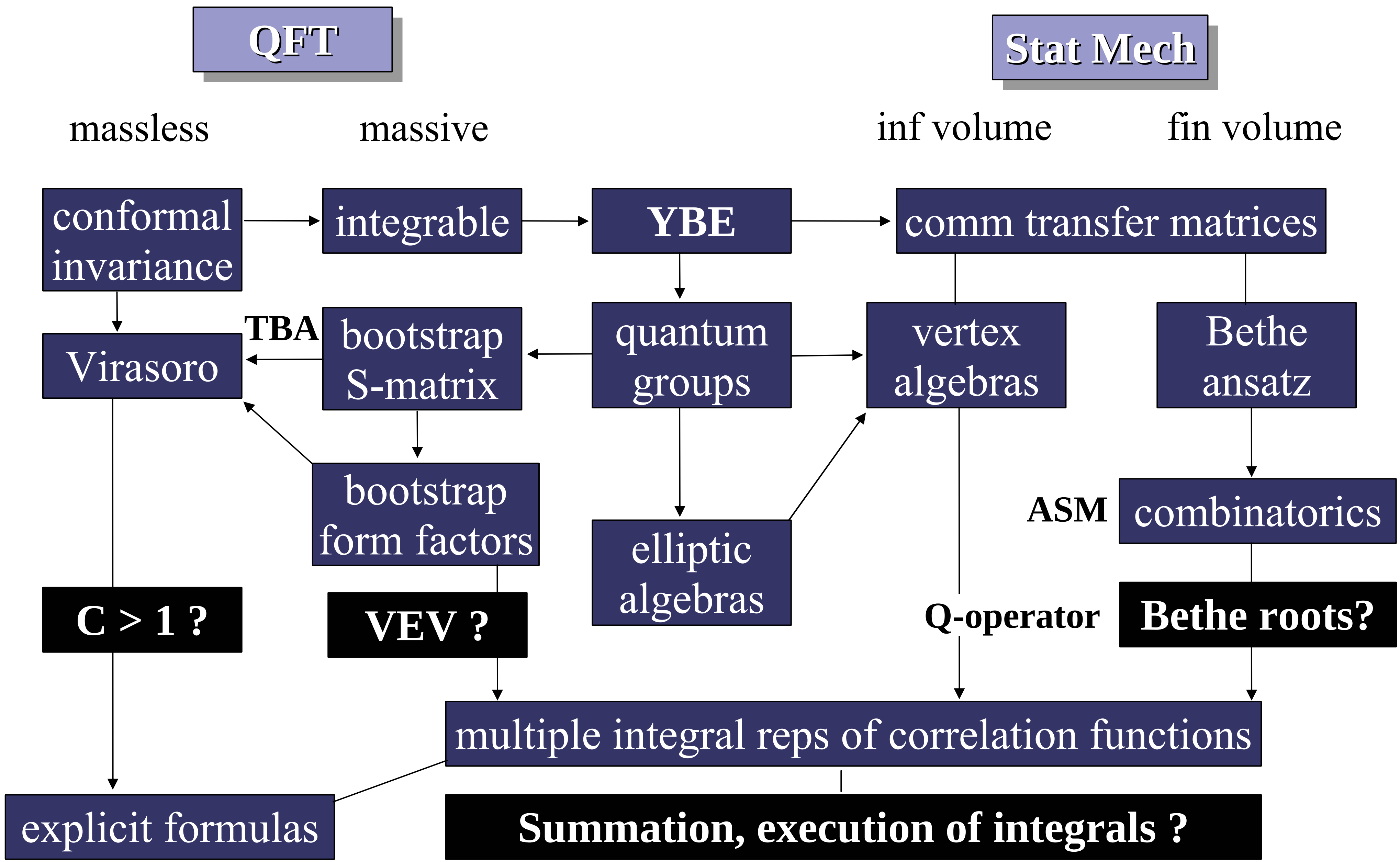
$$\varepsilon_i(\theta) = r\, m_i \cosh\theta - \sum_{j=1}^{n} \int \varphi_{ij}(\theta-\theta')\ln\left(1+e^{-\varepsilon_j(\theta')}\right) \qquad \varphi_{ij} = -i\frac{d}{d\theta}\ln S_{ij}(\theta)$$

- *Compute scaling function:* $c(r) = \dfrac{3}{\pi^2}\displaystyle\sum_{i=1}^{n} m_i' r \int d\theta \cosh\theta \ln(1+e^{-\varepsilon_i(\theta)})$

- *In the UV limit (r → 0) obtain central charge:*

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

Overview


QFT
Stat Mech
massless
massive
inf volume
fin volume
conformal invariance
integrable
YBE
comm transfer matrices
Virasoro
TBA
bootstrap S-matrix
quantum groups
vertex algebras
Bethe ansatz
bootstrap form factors
elliptic algebras
ASM
combinatorics
C > 1 ?
VEV ?
Q-operator
Bethe roots?
multiple integral reps of correlation functions
explicit formulas
Summation, execution of integrals ?

Outlook

Integrable lattice models/spin-chains:

- Extension to more complicated models (*XYZ, boundaries, high-energy*)
- Numerical computations (*entanglement, quantum information* ?)
- Difference equations and roots of unity
- Correlation functions at finite & infinite volume

Integrable field theory:

- Long distance asymptotics and comparison with CFT/QFT
- Connection with the bootstrap program

Non-local degrees of freedom & Log CFT:

- Temperley-Lieb algebra: dense polymers, percolation, Potts models …
- non self-adjoint representations: PT-symmetry, Jordan blocks …
- "Log Quantum Field Theory"?