We shall distinguish between two different notions of integrability:

- **Classical physics**: A system is called (Liouville) integrable if there are as many independent *integrals of motion* as degrees of freedom.

- **Quantum physics**: A system is called integrable if it possesses an infinite number of *conserved charges*.

Non-abelian algebras in integrable systems mainly occur in two contexts:

- **Symmetry**: the algebra commutes with the Hamiltonian or Lagrangian of the theory.
  
  *Example: conformal field theory & the Virasoro algebra*

- **Generating structure**: the algebra provides the underlying mathematical structure for the computation of the integrals of motion, the spectrum of the Hamiltonian or correlation functions.
  
  *Example: the Ising model & Onsager's algebra*


**Example**: Probability to find two spins aligned: 
\[
\langle \sigma_i^z \sigma_{i+n}^z \rangle = \frac{\text{Tr}[\sigma_i^z \sigma_{i+n}^z e^{-\beta H}]}{\text{Tr} e^{-\beta H}}
\]

- encode electric, magnetic and transport properties

**Critical Phenomena**: systems undergoing a continuous phase transition

- exponential decay: 
  \[
  T > T_c : \quad \langle \sigma_i^z \sigma_{i+n}^z \rangle \sim \exp(-n/\xi), \quad \xi(T) \sim \frac{1}{|T - T_c|}
  \]

- massive QFT: \( \xi \sim 1/m \) is the Compton wavelength

- algebraic decay:  
  \[
  T = T_c : \quad \langle \sigma_i^z \sigma_{i+n}^z \rangle \sim \frac{1}{n^{2\Delta}}, \quad \Delta = h + \bar{h}
  \]

- massless QFT = CFT: primary fields  
  \[
  L_0 \phi(z) |0\rangle = h \phi(z) |0\rangle, \quad \bar{L}_0 \phi(\bar{z}) |0\rangle = \bar{h} \phi(\bar{z}) |0\rangle
  \]

**Comparison with quantum field theory in the continuum limit**:  
- vacuum expectation values of local operators  
  **NPB 636 (2002) 435** [K. Seaton, La Trobe]  
- long distance asymptotics of correlation functions
Example: the planar Ising model

Lattice model:

\[ H_{\text{Ising}} = \sum_n \sigma_n^z \sigma_{n+1}^z + h \sigma_n^x \]

Conformal Field Theory \( c = \frac{1}{2} \)

\[ [L_m, L_n] = (m - n) L_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \]

**Virasoro algebra**

\[ S = S_{\text{CFT}} + \lambda \int \phi_{(1,2)}(x) d^2 x, \quad \Delta_{(1,2)} = 1/16 \]

Integrable QFT \((h > 0)\)
Lorentz spins \( s = 1, 7, 11, 13, 17, 19, 23, 29 \)

GKO coset \((h = 0)\)

\[ \left( \hat{E}_8 \right)_1 \otimes \left( \hat{E}_8 \right)_1 / \left( \hat{E}_8 \right)_2 \]

GKO coset \((T = T_c)\)

\[ \left( \hat{A}_1 \right)_1 \otimes \left( \hat{A}_1 \right)_1 / \left( \hat{A}_1 \right)_2 \]

Massive free fermion \((T > T_c)\)
Lorentz spin \( s = 1 \)

\[ S = S_{\text{CFT}} + \lambda \int \phi_{(2,1)}(x) d^2 x, \quad \Delta_{(2,1)} = 1/2 \]

\[ \lambda \sim m^{2(1-\Delta)} \]
Vector space $V = \text{linear span of statistical variables}$

 Partition function:

$$Z = \sum_{\{\alpha_i,\beta_i,\gamma_i,\delta_i\}} \prod_{\text{vertices } i} R_{\alpha_i\beta_i}^{\gamma_i\delta_i} = \text{Tr } T^M$$

 Sum over all vertex configurations.

 Transfer matrix:

$$T = \text{Tr} R_0 R_{0N} R_{0N-1} \cdots R_{01}$$

 Discrete evolution operator.
**The Heisenberg Spin-Chain**

Baxter’s concept of commuting transfer matrices: \[ [T(u), T(\nu)] = 0 \]

*find solutions of the quantum Yang-Baxter equation*

\[ R_{12}(u)R_{13}(u + \nu)R_{23}(\nu) = R_{23}(\nu)R_{13}(u + \nu)R_{12}(u) \quad R(u) \in \mathcal{A} \otimes \mathcal{A} \]

*construct set of commuting transfer matrices by evaluating solutions in particular representations*

\[ T^{(s)}(u) = (\text{Tr}_{\pi^{(s)}} \otimes \pi_{\tilde{\mathcal{S}}})R(u), \quad s \in \frac{1}{2} \mathbb{Z} \]

Hamiltonian is “recovered” from

\[ H = \left. \frac{d}{du} \ln T^{(s=1/2)}(u) \right|_{u=0} \]

<table>
<thead>
<tr>
<th>Model</th>
<th>Restriction</th>
<th>Algebra $\mathcal{A}$</th>
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</thead>
<tbody>
<tr>
<td>XYZ</td>
<td>$-$</td>
<td>Sklyanin (elliptic) algebra</td>
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<tr>
<td>XXZ</td>
<td>$g_x = g_y$</td>
<td>Quantum group $U_q(s\tilde{l}_2)$</td>
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<tr>
<td>XXX</td>
<td>$g_x = g_y = g_z$</td>
<td>Yangian $Y(sl_2)$</td>
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</tbody>
</table>

\[ H = J \sum_{n} \left\{ g_x \sigma_n^x \sigma_{n+1}^x + g_y \sigma_n^y \sigma_{n+1}^y + g_z \sigma_n^z \sigma_{n+1}^z \right\} \]
Bethe’s Ansatz

Solution of the XXX Heisenberg spin-chain by Hans Bethe [1931]

- superposition of plane waves: \( |k_1, \ldots, k_n \rangle = \sum_{0 \leq x_1 < \ldots < x_n \leq L} \psi_k(x_1, \ldots, x_n) \sigma_{x_1}^{-} \cdots \sigma_{x_n}^{-} \uparrow \uparrow \cdots \uparrow \rangle \)

- Bethe’s wavefunction:
  \[ \psi_k(x_1, \ldots, x_n) = \sum_{p \in S_n} A(p_1, \ldots, p_n) e^{i(k_{p_1}x_1 + \cdots + k_{p_n}x_n)} \]

- boundary & eigenvector conditions:
  \[ s_{p_i p_{i+1}} A(p_1, \ldots, p_n) = -s_{p_{i+1} p_i} A(p_1, \ldots, p_{i+1}, p_i, \ldots, p_n) \]
  \[ e^{iLk_{p_1}} A(p_2, \ldots, p_n, p_1) = A(p_1, \ldots, p_n) \]

- Bethe’s equations:
  Bethe roots = quasi momenta
  \[ b_\ell^M = (-1)^{n-1} \prod_{j=1}^{n} \frac{1 - 2g_z b_\ell + b_j b_\ell}{1 - 2g_z b_j + b_j b_\ell} \]
  \[ b_\ell = e^{ik_\ell} \]

- “exact” energy spectrum:
  \[ E = - \sum_{j=1}^{n} (b_j + b_j^{-1} - 2g_z) \]
**Difference equations from representation theory**

**Question:** How do you solve the BAE? Can you connect the solutions (Bethe roots) to the representation theory of the underlying algebraic structure?

**Answer:** Yes. Construct Baxter’s Q-operator!

Consider “transfer matrix” of an $\infty$-dimensional representation [Verma module]:

\[
Q(u; x) = (\text{Tr}_{\pi(x)} \otimes \pi_{\delta}) R(u) = Q^+(u)Q^-(u + s), \quad x \in \mathbb{C}
\]


**Functional equations from rep theory**

\[
0 \rightarrow \pi(x') \rightarrow \pi(x) \otimes \pi^{(1/2)} \rightarrow \pi(x'') \rightarrow 0
\]

**Analytic continuation of the fusion hierarchy**

\[
T(u; x) = Q^+(u - x\lambda)Q^-(u + x\lambda) - Q^+(u + x\lambda)Q^-(u - x\lambda)
\]

**“Quantum Wronskian”**

\[
XXX : \quad u^M = Q^+(u + 1)Q^-(u) - Q^+(u)Q^-(u + 1), \quad Q^{\pm}(u) = \prod_{i=1}^{d_{\pm}} (u - u^\pm_i)
\]

This identity leads to a system of **quadratic** equations for “solving” the model. This supersedes the Bethe ansatz equations whose polynomial order = number of sites.
vector space $V = \text{particle types, e.g. solitons, breathers, ...}$

observable $O = S$ scattering matrix

Integrability $\rightarrow$ factorization of scattering into 2 particle events

$V_1 \otimes V_2 \otimes V_3 : S_{12}(p_{12})S_{13}(p_{13})S_{23}(p_{23}) = S_{23}(p_{23})S_{13}(p_{13})S_{12}(p_{12})$

$S : V \otimes V \rightarrow V \otimes V$ Yang-Baxter equation $\rightarrow$ quantum groups
The Bootstrap Programme

**Idea:** Exact construction of the 2-particle S-matrix/corr functions from a set of axioms.

- **unitarity & crossing:**
  \[ S_{12}(\theta)S_{21}(-\theta) = 1 \]
  \[ S(i\pi - \theta)_{ab}^{cd} = S(\theta)_{da}^{cb} = S(\theta)_{bd}^{ac} \]

- **fusing (“bound states”):**
  \[ S_{lk}(\theta) = S_{li}(\theta + i\eta^i_{ik})S_{lj}(\theta - i\eta^i_{jk}) \]

- **Form Factor Expansion:**
  \[ \langle O(x)O(0) \rangle = \sum_n \int \prod_{k=1}^n \frac{d\theta_k}{2\pi k} e^{ix\cdot p_k(\theta_k)} |F^n_\theta(\theta_1, \ldots, \theta_n)|^2 \]
  \[ F^n_\theta(\theta_1, \ldots, \theta_n) := \langle 0|O(0)|\theta_1, \ldots, \theta_n \rangle \]

- **exchange of particles:**
  \[ F^n_\theta(\ldots, \theta_i, \theta_{i+1}, \ldots) = F^n_\theta(\ldots, \theta_{i+1}, \theta_i, \ldots)S_{ii+1}(\theta_{ii+1}) \]

- **recurrence relations:**
  - **fusing & crossing**
  \[ F^n_{\theta_{i2}}(\theta_3, \ldots, \theta_n) \rightarrow \text{Res} \quad F^n_\theta(\theta_1, \ldots, \theta_n) \]
  \[ F^{n-1}_\theta(\theta_b, \theta_3, \ldots, \theta_n) \rightarrow \text{Res} \quad F^n_\theta(\theta_1, \ldots, \theta_n) \]
Affine Toda field theory

\[ \mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{m^2}{\beta^2} \sum_{i=0}^{\ell} n_i \exp(\beta \alpha_i \cdot \varphi) \]

The \( \alpha_i \) are simple roots of a Lie algebra \( g \).

\[ \alpha_0 = -\sum_{i=0}^{\ell} n_i \alpha_i \]

The quantum two-particle scattering amplitude can be calculated exactly:

\[ S_{ij}(\theta, B) = \exp(4 \int_0^\infty \frac{dt}{t} \Phi_{ij}(t) \sinh(\frac{\theta t}{i \pi})) \]

\[ \Phi_{ij}(t) = (q - q^{-1})(q^{\sqrt{t_i}} - q^{-\sqrt{t_i}})K(q, q^{\sqrt{)})^{-1}_{ij} \]

\[ K(q, q^{\sqrt{)})_{ij} = (q q^{\sqrt{t_i}} + q^{-1} q^{-\sqrt{t_i}})\delta_{ij} - [I_{ij}]_q^{\sqrt{}} \]

\[ K_{ij} = 2 \frac{\alpha_i \cdot \alpha_j}{\alpha_j \cdot \alpha_j} \]

\[ q = \exp \frac{(2 - B)t}{2h}, \quad q^{\sqrt{}} = \exp \frac{B t}{2\ell h^{\sqrt{}}} \]

\[ t_i K_{ij} = K_{ij} t_j \quad I = 2 - K \]


Renormalization flow between \( g \) and its Langlands dual \( g^{\sqrt{}} (0 \leq B \leq 2) \).
The UV Limit: the Thermodynamic Bethe Ansatz

Place particles on a (big) circle: compactify space

Take thermodynamic limit \((L,N \to \infty)\) to obtain system of coupled nonlinear integral equations:

\[
e^{iLm_k \sinh \theta_k} \prod_{l \neq k}^{N} S_{kl}(\theta_k - \theta_l) = 1
\]

\[
\varepsilon_i(\theta) = rm_i \cosh \theta - \sum_{j=1}^{n} \int \varphi_{ij}(\theta - \theta') \ln \left(1 + e^{-\varepsilon_j(\theta')}\right) \quad \varphi_{ij} = -i \frac{d}{d\theta} \ln S_{ij}(\theta)
\]

Compute scaling function:

\[
c(r) = \frac{3}{\pi^2} \sum_{i=1}^{n} m_i' r \int d\theta \cosh \theta \ln \left(1 + e^{-\varepsilon_i(\theta)}\right)
\]

In the UV limit \((r \to 0)\) obtain central charge:

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n,0}
\]
Outlook

**Integrable lattice models/spin-chains:**
- Extension to more complicated models (XYZ, boundaries, high-energy)
- Numerical computations (entanglement, quantum information ?)
- Difference equations and roots of unity
- Correlation functions at finite & infinite volume

**Integrable field theory:**
- Long distance asymptotics and comparison with CFT/QFT
- Connection with the bootstrap program

**Non-local degrees of freedom & Log CFT:**
- Temperley-Lieb algebra: dense polymers, percolation, Potts models …
- Non self-adjoint representations: PT-symmetry, Jordan blocks …
- “Log Quantum Field Theory”?