# Cylindric Macdonald functions and a deformation of the Verlinde algebra

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arxiv: 1110.6356

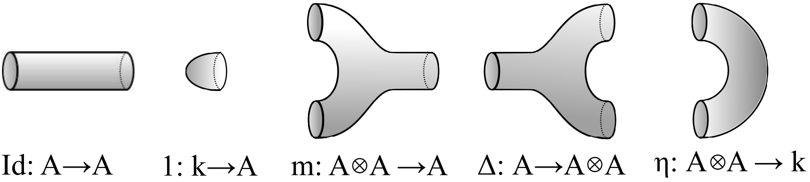
December 6, 2011

# Outline

1. q-Heisenberg algebra and exactly solvable lattice models
2. partition functions: cylindric Macdonald functions
3. transfer matrices and noncommutative Macdonald polynomials
4. Frobenius structures
5. Summary

# The Verlinde algebra as 2D topological field theory

A Frobenius algebra $A$ is a finite-dimn'l, unital, assoc algebra with non-degenerate bilinear form $\eta(a \cdot b, c) = \eta(a, b \cdot c), \ a, b, c \in A$.




**Fact**

Commutative Frobenius algebras are categorically equivalent to 2D topological quantum field theories.

**Proposition**

*The Verlinde algebra $\mathcal{V}_k$ is a commutative Frobenius algebra over $\mathbb{C}$ with (non-degenerate) bilinear form $\eta(\lambda, \mu) = \delta_{\lambda^* \mu}$.*

# Reminder: the Verlinde algebra

$\mathrm{Rep}\,\mathfrak{g} = \bigoplus_{\lambda\in\mathcal{P}^+}\mathbb{Z}[\pi_\lambda]$ Grothendieck ring with product $\otimes$

$$\mathfrak{I}_k : \qquad [\pi_\lambda]-(-1)^{\ell(w)}[\pi_{w\circ\lambda}], \qquad k\in\mathbb{Z}_{\geq 0},\ w\in\tilde{W}$$

with $w\circ\lambda := w(\lambda+\rho)-\rho$ shifted level-$k$ action of $\tilde{W}$, e.g. for $\mathfrak{g}=\mathfrak{sl}(n)$ one has $\sigma_0\circ\lambda=(\lambda_n+k-n,\lambda_2,\ldots,\lambda_{r-1},\lambda_1-k+n)$ and $\sigma_i\circ\lambda=(\lambda_1,\ldots,\lambda_{i+1}-1,\lambda_i+1,\ldots,\lambda_n),\quad i=1,\ldots,n-1.$

**Definition**

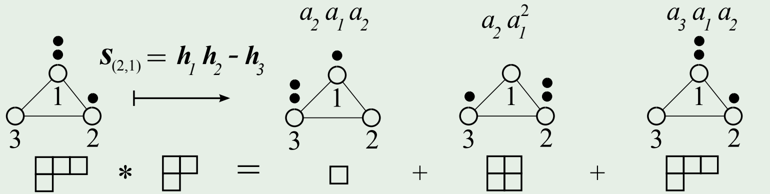
The Verlinde ring is the quotient $\mathcal{V}_k^{\mathbb{Z}} := \mathrm{Rep}\,\mathfrak{g}/\mathfrak{I}_k$.

$$[\pi_\lambda]*[\pi_\mu] := [\pi_\lambda]\otimes[\pi_\mu] \mod \mathfrak{I}_k = \sum_{\nu\in\mathcal{P}_k^+} N_{\lambda\mu}^{\nu}[\pi_\nu]$$

The fusion coefficients $N_{\lambda\mu}^{\nu}\in\mathbb{Z}_{\geq 0}$ appear in vertex operator algebras (CFT), algebraic geometry (conformal blocks), representation theory (tilting modules of QG at roots of 1).

## Example

Set $\mathfrak{g} = \mathfrak{sl}(n)$ and $n = k = 3$.




- Particle configuration = Young diagram $\in B_{k\omega_1}$ (KR crystal)
- $\mathbf{s}_\lambda$ = Schur polynomial in Kashiwara's crystal operators $a_i$

## Theorem (Korff-Stroppel)

*$\mathbb{Z}\mathcal{P}_k^+$ with product $\lambda \circledast \mu := \mathbf{s}_\lambda \mu$ is canonically isomorphic to $\mathcal{V}_k^{\mathbb{Z}}$.*

# q-bosons and Fock space

## Definition ($q$-deformed boson algebra)

$\mathcal{H}_q$ unital, associative $\mathbb{C}(q)$-algebra generated by $\{q^{\pm N}, \beta, \beta^*\}$ and

$$\begin{aligned} q^N q^{-N} &= q^{-N} q^N = 1, \quad q^N \beta = \beta q^{N-1}, \quad q^N \beta^* = \beta^* q^{N+1}, \\ \beta\beta^* - \beta^*\beta &= (1-q^2) q^{2N}, \quad \beta\beta^* - q^2 \beta^* \beta = 1 - q^2. \end{aligned}$$

## Proposition (Fock space)

*Set $\mathcal{F} = \mathcal{H}_q / \mathcal{I}$ with $\mathcal{I}$ two-sided ideal generated by $\beta$ and $q^N - 1$.*
*(i) There exists a basis $\{|m\rangle : m \geq 0\}$ in $\mathcal{F}$ such that $|0\rangle := 1 + \mathcal{I}$,*

$$q^N |m\rangle = q^m |m\rangle, \quad \beta^* |m\rangle = (1 - q^{2m+2}) |m+1\rangle, \quad \beta |m\rangle = |m-1\rangle$$

*(ii) $\mathcal{F}$ is simple ($q^r \neq 1$).*

## Physical interpretation

$\beta$, $\beta^*$ destroy/create a q-boson, $|m\rangle$ is a state with $m$ q-bosons.

# Solution to the Yang-Baxter equation

Define $L(x) : \mathcal{F}[\![x]\!] \otimes \mathcal{H}_q \to \mathcal{F}[\![x]\!] \otimes \mathcal{H}_q$, $\mathcal{F}[\![x]\!] = \mathcal{F} \otimes \mathbb{C}[\![x]\!]$ by

$$L(x) = (L(x)_{mm'}) \text{ with } L(x)_{mm'} = x^m (\beta^*)^m \beta^{m'} / \prod_{r=1}^{m} (1 - q^{2r})\,.$$

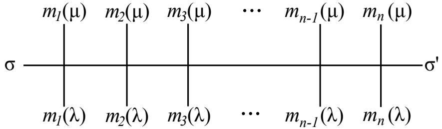
Set $T(x) := L_n(x) \cdots L_1(x) \in \mathrm{End}(\mathcal{F}[\![x]\!]) \otimes \mathcal{H}_q^{\otimes n}$.

**Proposition (Yang-Baxter equation)**

*There exists an invertible $R(x, y) : \mathcal{F}[\![x]\!] \otimes \mathcal{F}[\![y]\!] \to \mathcal{F}[\![x]\!] \otimes \mathcal{F}[\![y]\!]$ such that $R_{12}(x, y) T_1(x) T_2(y) = T_2(y) T_1(x) R_{12}(x, y)$.*

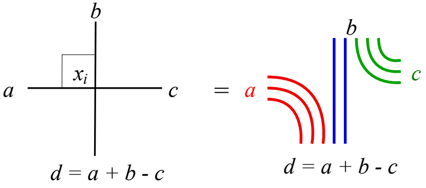
Let $\mathcal{P}_k^+ := \{\lambda : k = \lambda_1 \geq \cdots \geq \lambda_n\}$. Define $m_j(\lambda) = \lambda_j - \lambda_{j+1}$ and $|\lambda\rangle := |m_1(\lambda)\rangle \otimes \cdots \otimes |m_n(\lambda)\rangle \in \mathcal{F}^{\otimes n}$. Row-partition function:

$\langle \lambda | T(x)_{\sigma\sigma'} | \mu \rangle =$



# $\infty$-friendly walkers – non-intersecting paths

Define exactly solvable vertex model (statistical mechanics):




$$x_i^a \frac{(1\text{-}q^{2d})\cdots(1\text{-}q^{2d-a+1})}{(1\text{-}q^2)\ \cdots\ (1\text{-}q^{2a})}$$

Assign each lattice configuration $\gamma$ a (Boltzmann) weight:

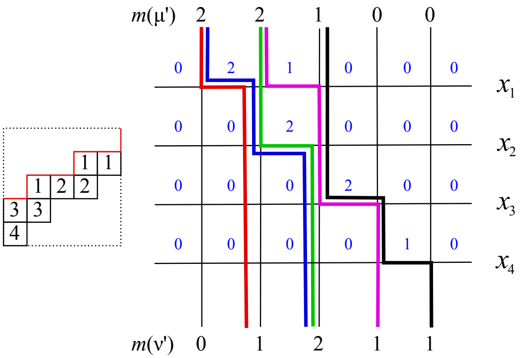
$$\mathrm{wt}(\gamma) := \prod_{\langle i,j\rangle \in \dot{\mathbb{L}}} \mathrm{wt}(\gamma_{\langle i,j\rangle}), \quad \mathrm{wt}(\gamma_{\langle i,j\rangle}) := \langle d|L(x_i)_{a,c}|b\rangle$$

## Definition (partition function)

$$Z_{\lambda,\mu}(x_1,\ldots,x_{n-1};t=q^2) := \sum_{\gamma\in\Gamma_{\lambda,\mu}} \mathrm{wt}(\gamma), \qquad \lambda,\mu \in \mathcal{P}_k^+$$

# $\infty$-friendly walkers on the finite strip

Fix values of bottom and top vertical edges through multiplicities $m_j(\lambda), m_j(\mu)$ with $\lambda, \mu \in \mathcal{P}_k^+$.




**Proposition (skew Macdonald functions)**

$$Z_{\lambda,\mu}(x_1,\ldots,x_{n-1};q) = P_{\lambda/\mu}(x_1,\ldots,x_{n-1};q^2,0)$$

# Skew Macdonald functions

Let $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots \in \mathbb{C}(q,t)[x_1,\ldots,x_{n-1}]^{S_{n-1}}$ be the power sums and set

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \delta_{\lambda\mu} z_\lambda \prod_i \left(\frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}\right), \qquad \langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu} \,.$$

Macdonald functions are the orthogonal basis determined by $P_\lambda(q,t) = m_\lambda + \sum_{\mu<\lambda} c_\mu(q,t) m_\mu$ with $m_\lambda$ monomial function.

## Bi-algebra structure

$$\text{Product:} \quad P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(q,t) P_\lambda, \quad f^\lambda_{\mu\nu}(q,t) \in \mathbb{Z}[q,t]$$

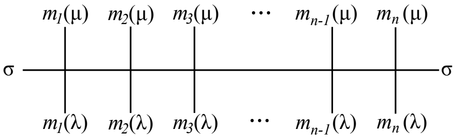
$$\text{coproduct:} \quad \Delta P_\lambda = \sum_\mu P_{\lambda/\mu} \otimes P_\mu, \quad P_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu}(q,t) P_\nu$$

- $P_\lambda(0,t)$ – Hall-Littlewood functions
- $P_\lambda(q,0)$ – Demazure characters, q-Witthaker functions

# $\infty$-friendly walkers on the cylinder

Impose periodic boundary conditions in the horizontal direction, i.e.

$$\langle\lambda|\boldsymbol{G}(x)|\mu\rangle = \sum_{\sigma} z^{\sigma}$$




## Lemma (partition function of the cylinder)

$$Z^{cyl}_{\lambda,\mu}(x_1,\ldots,x_{n-1};q) = \langle\lambda|\mathbf{G}(x_1)\cdots\mathbf{G}(x_{n-1})|\mu\rangle$$

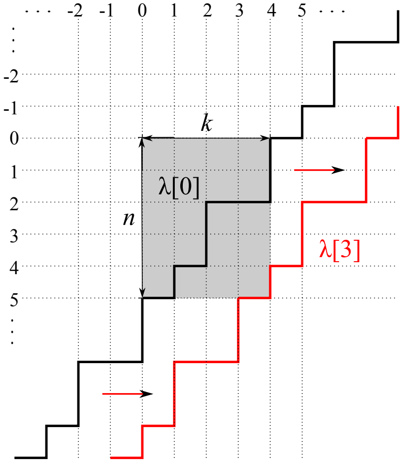
## Definition (cylindric skew Macdonald functions)

$$Z^{cyl}_{\lambda,\mu}(x_1,\ldots,x_{n-1};q) =: \sum_{d\geq 0} z^d P_{\lambda/d/\mu}(x_1,\ldots,x_{n-1};q^2,0)$$

Note: for $z=0$ (finite strip) one has $Z_{\lambda,\mu} = P_{\lambda/\mu}(q^2,0)$

# Cylindric loops

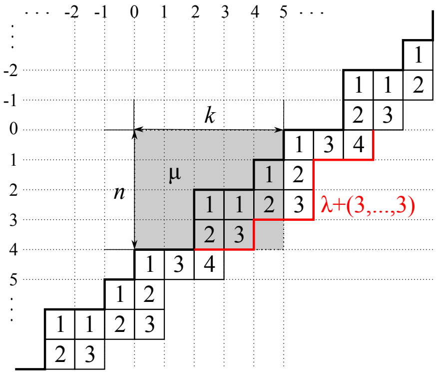
Set $\lambda[0]_i := \lambda_i$ for $1 \leq i \leq n$ and $\lambda[0]_{i+n} := \lambda[0]_i - ik$ else.




Shift in direction $(0,1)$: $\lambda[r]_i := \lambda[0]_i + r$

# Cylindric skew diagrams and tableaux

$$\lambda/d/\mu := \{\langle i,j\rangle \in \mathbb{Z}\times\mathbb{Z} \mid \lambda[d]_i \geq j > \mu[0]_i\}$$

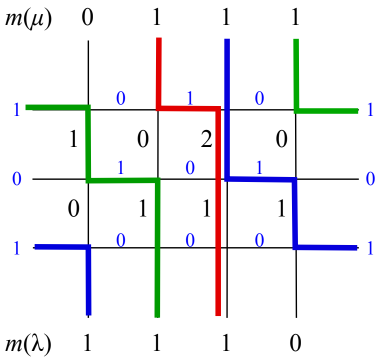


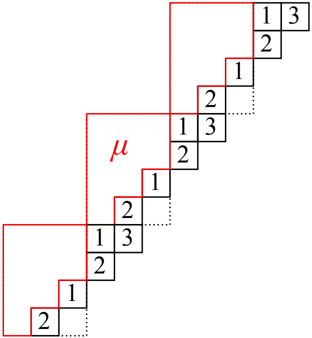

A cylindric tableau is a map $T : \lambda/d/\mu \to \mathbb{N}$ such that
$T(i,j) = T(i+n, j-k),\ T(i,j) < T(i+1,j),\ T(i,j) \leq T(i,j+1).$

# Cylindric tableaux bijection

**Lemma**

*Lattice configurations are in bijection with cylindric tableaux.*



# q-bosons and the quantum plactic algebra

Denote by $\hat{\mathcal{A}}_n \subset \mathcal{H}_q^{\otimes n}$ the subalgebra generated by the letters

$$a_i = \beta^*_{i+1}\beta_i, \quad 0 < i < n \qquad \text{and} \qquad a_n = z\beta^*_1\beta_n \ .$$

Then we have non-local commutativity,

$$a_i a_j = a_j a_i \quad \text{for} \quad |i-j| \bmod n > 1 \ ,$$

and the "quantum Knuth relations" $(t = q^2)$

$$\begin{aligned} a_{i+1}a_i^2 + t a_i^2 a_{i+1} &= (1+t) a_i a_{i+1} a_i, \\ a_{i+1}^2 a_i + t a_i a_{i+1}^2 &= (1+t) a_{i+1} a_i a_{i+1} \ , \end{aligned}$$

where all indices are understood modulo $n$.

**Generalisations of Lascoux-Schützenberger's plactic algebra**

For $t = 0$ this is the local affine plactic algebra (Korff-Stroppel) and for $t = 0$ and $z = 0$ the local plactic algebra (Fomin-Greene).

# Quantum plactic Rogers-Szegö polynomials

Multivariate Rogers-Szegö polynomials $g_r(x;q,0)$ :

$$(x;q)_\infty := \prod_{r=0}^{\infty}(1-xq^r),\quad (x;q)_r := \prod_{s=0}^{r-1}(1-xq^s) = \frac{(x;q)_\infty}{(xq^r;q)_\infty}$$

$$G(x) = \prod_{i\geq 1}\frac{(t\ y_i x;q)_\infty}{(y_i x;q)_\infty} = \sum_{r\geq 0} g_r(y;q,t)x^r,\ g_r(q,t) = \sum_{|\mu|=r}\frac{(t;q)_\mu}{(q;q)_\mu}\, m_\mu$$

## Proposition

*Set* $\mathbf{m}_\lambda := \sum_{w\in S_n^\lambda}(z\beta_1^*)^{\lambda_{w_n}} a_1^{\lambda_{w_1}}\cdots a_{n-1}^{\lambda_{w_{n-1}}}\beta_n^{\lambda_{w_n}}$ *then*

$$\mathbf{G}(x) := \sum_{m\geq 0} z^m T(x)_{mm} = \sum_{r\geq 0} x^r \mathbf{g}_r,\quad \mathbf{g}_r := \sum_{\lambda\vdash r}\frac{\mathbf{m}_\lambda}{(q^2;q^2)_\lambda}$$

## Corollary (Integrability)

*YBE* $\Rightarrow \mathbf{G}(x)\mathbf{G}(y) = \mathbf{G}(y)\mathbf{G}(x) \Rightarrow \mathbf{g}_r\mathbf{g}_{r'} = \mathbf{g}_{r'}\mathbf{g}_r$ *for all* $r, r'$.

# Noncommutative Cauchy identities

- $\mathbf{S}_\nu := \det(\mathbf{g}_{\nu_i - i + j})$ q-plactic (dual) Schur polynomial
- $\mathbf{Q}_\nu := \sum_\sigma \mathbf{S}_\sigma K(q^2)^{-1}_{\sigma'\nu'}$ q-plactic Macdonald polynomial

**Proposition**

$$\mathbf{G}(x_1)\cdots\mathbf{G}(x_{n-1}) = \sum_\lambda s_\lambda(x)\mathbf{S}_\lambda = \sum_\lambda P_\lambda(x;q^2,0)\mathbf{Q}_\lambda$$

**Corollary (expansions of cylindric Macdonald polynomials)**

$$P_{\lambda/d/\mu}(x;q^2,0) = \sum_\nu \langle\lambda|\mathbf{S}_\nu|\mu\rangle s_\nu(x) = \sum_\nu \langle\lambda|\mathbf{Q}_\nu|\mu\rangle P_\nu(x;q^2,0)$$

**Conjecture**

$\langle\lambda|\mathbf{S}_\nu|\mu\rangle$ are polynomials in $t = q^2$ with non-negative coefficients.

For $\mu = (k,\ldots,k)$ these are Kostka-Foulkes polynomials.

# Frobenius structures

**Proposition**

*There exists $U_q\hat{\mathfrak{sl}}(n)$-module isomorphism $\mathcal{F}^{\otimes n} \cong \bigoplus_{k\geq 0} W(k\omega_1)$, where $W(k\omega_1)$ is the Kirillov-Reshetikhin module of weight $k\omega_1$.*

**Theorem (deformed Verlinde algebra, $z = 1$)**

*For $\lambda, \mu \in \mathcal{P}_k^+$ define the product $|\lambda\rangle * |\mu\rangle := \mathbf{Q}_\lambda |\mu\rangle$. Then*

(i) $\mathcal{F}_{n,k} = (W(k\omega_1), *, \eta)$ *with* $\eta(\lambda, \mu) = \delta_{\lambda^*,\mu} / \prod_i (t)_{m_i(\lambda)}$ *is a commutative Frobenius algebra.*

(ii) $\{|\lambda\rangle = \mathbf{Q}_\lambda |k^n\rangle\} \subset W(k\omega_1)$ *is Lusztig's canonical basis.*

(iii) *Set* $t = 0$ $(q = 0)$ *then* $\mathcal{F}_{n,k}$ *is the Verlinde algebra.*

**NC Macdonald polynomials = deformed fusion matrices**

Let $\mathbf{Q}_\lambda^{(k)} := \mathbf{Q}_\lambda|_{W(k\omega_1)}$. Then $\mathbf{Q}_\lambda^{(k)} \mathbf{Q}_\mu^{(k)} = \sum_{\nu \in \mathcal{P}_k^+} N_{\lambda\mu}^{\nu}(t) \mathbf{Q}_\nu^{(k)}$, where $N_{\mu\nu}^{\lambda}(0) = N_{\mu\nu}^{\lambda}$ are the WZNW fusion coefficients.

# Bethe ansatz and idempotents

- Construct a common eigenbasis of $\{\mathbf{Q}_\lambda : \lambda \in \mathcal{P}_k^+\}$,

$$\mathfrak{e}_y = \sum_{\lambda' \in \mathcal{P}_k^+} P_{\lambda'}(y_1, \ldots, y_k; 0, t)|\lambda\rangle, \quad y_i^n \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j} = \prod_{j \neq i} \frac{y_i t - y_j}{y_i - y_j}$$

- Define algebraic variety $V_k \subset \mathbb{C}\{\{t\}\}^k$ (Puiseux series),

$$\mathbb{C}\{\{t\}\} := \bigcup_{m=1}^{\infty} \mathbb{C}((t^{1/m}))$$

- Show that the coordinate ring of $V_k$ is $\mathfrak{H}_k \otimes \mathbb{C}\{\{t\}\}/\mathcal{I}_n$.

**Corollary (Peirce decomposition)**

*The $\mathfrak{e}_y$'s are the idempotents of $\mathcal{F}_{n,k} \otimes \mathbb{C}\{\{t\}\}$ and $1 = \sum_y \mathfrak{e}_y$.*

*Bernstein presentation*: $\hat{H}_k$ is generated by $\{T_1, \ldots, T_{k-1}\}$ and a set of commuting, invertible elements $\{Y_1, \ldots, Y_k\}$ obeying

$$(T_i - q^{-1})(T_i + q) = 0,$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad T_i T_j = T_j T_i, \quad |i-j| > 1, \quad i, j \in \mathbb{Z}_k$$
$$T_i Y_i T_i = Y_{i+1}, \qquad T_i Y_j = Y_j T_i \qquad \text{for} \quad j \neq i, i+1$$

Define $\mathbf{1} := (1/c_k) \sum_{w \in S_k} q^{-\ell(w)} T_w$ with $c_k = \sum_w q^{-\ell(w)}$.

**Satake isomorphism**

Let $\Phi : \mathcal{Z}(\hat{H}_k) \longrightarrow \mathfrak{H}_k := \mathbf{1}\hat{H}_k\mathbf{1}$ be the map

$$\Phi : \quad \frac{c_\lambda}{c_k} P_\lambda(Y_1, \ldots, Y_k; 0, t = q^2) \mapsto M_\lambda := \mathbf{1} Y^\lambda \mathbf{1}$$

where $\{M_\lambda : \lambda_1 \geq \ldots \geq \lambda_k \geq 0\}$ is a basis of $\mathfrak{H}_k$.

# Extended affine Weyl group symmetry

The Hall-Littlewood (spherical Macdonald) functions $P_\lambda(0,t) := \prod_{i\geq 0} \frac{(1-t)^k}{(t)_{m_i(\lambda)}} R_\lambda(t)$ obey the *straightening rules*

$$R_{\lambda.\sigma_i} = tR_\lambda - R_{(\ldots,\lambda_i-1,\lambda_{i+1}+1,\ldots)} + tR_{(\ldots,\lambda_{i+1}+1,\lambda_i-1,\ldots)} \tag{1}$$

Consider the ideal $\mathcal{I}_n$ generated by

$$\begin{gathered} R_{\lambda.\sigma_0} - tR_\lambda + R_{(\lambda_1+1,\ldots,\lambda_k-1)} - tR_{(\lambda_k-1+n,\ldots,\lambda_1+1-n)} \\ R_\lambda - R_{(\lambda_2,\ldots,\lambda_k,\lambda_1-n)} \end{gathered} \tag{2}$$

**Theorem**

*The map $|\lambda\rangle \mapsto P_{\lambda'}(0,t)$ defines an algebra isomorphism $\mathcal{F}_{n,k} \cong \mathfrak{H}_k/\mathcal{I}_n$, where $\lambda'$ is the conjugate partition and $t=q^2$.*

**Computation of deformed fusion coefficients**

The last isomorphism provides an algorithm to compute $N_{\mu\nu}^{\lambda}(t)$.

# From quantum integrable systems to 2D TFT

In the context of 4D SUSY $N = 2$ Yang-Mills theories Shatashvili and Nekrasov have conjectured that the infrared limit is described by a 2D TFT whose states are in one-to-one correspondence with the eigenstates of integrable quantum many body systems.

- commuting transfer matrices: generate Frobenius algebra
- Bethe ansatz eqns: moduli space of vacua
- Bethe vectors: idempotents
- discrete version of quantum nonlinear Schrödinger model