Small Quantum Cohomology as Quantum Integrable System

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Outline

- Reminder: (small) quantum cohomology
- Quantum Kostka numbers and toric Schur functions
- Vicious and osculating walkers on the cylinder
- Free fermion description

Related literature (http://www.maths.gla.ac.uk/~ck/publications.html)

quantum cohomology/WZNW fusion rings as integrable models:

- C. K. and C. Stroppel, Adv Math 225 (2010) 200-68; arXiv:0909.2347 (type A: $\mathfrak{g} = \hat{\mathfrak{u}}(n)_k$ and $\widehat{\mathfrak{su}}(n)_k$)
- C. K. A combinatorial derivation of the Racah-Speiser algorithm for Gromov-Witten invariants; arXiv:0910.3395
- C. K. J Phys A 43 (2010) 434021; arxiv:1006.4710
- C. K. QC via vicious and osculating walkers; arxiv:1204.4109

Bethe vectors as idempotents: discrete quantum NLS model

- C. K. RIMS Kokyuroku Bessatsu B28 (2011) pp. 121-53; arxiv:1106.5342 (Section 5, Cor 5.3 and Section 7)
- C. K. Cylindric Macdonald functions and a deformation of the Verlinde algebra. preprint; arxiv:1110.6356 (Section 7, Prop 7.15)

$\mathrm{Gr}_{n,n+k}$ Grassmannian of n-planes in $\mathbb{C}^{n+k}$

- small quantum cohomology ring [Siebert-Tian 1997]

$$qH^*(\mathrm{Gr}_{n,n+k}) \cong \mathbb{Z}[q][e_1,\ldots,e_n]/\langle h_{k+1},\ldots,h_{n+k-1},h_{n+k}+q(-1)^n\rangle$$

where $h_r = \det(e_{1-i+j})_{1\le i,j\le r}$ and a vector space basis is given by $\{s_\lambda := \det(e_{\lambda'_i-i+j})_{1\le i,j\le k}\}$ with

$$\lambda \in \{\text{partitions with Young diagram in } n\times k \text{ box }\}$$

- Fusion ring of $\hat{\mathfrak{u}}(n)_k$ Wess-Zumino-Novikov-Witten model [Gepner, Intriligator, Vafa, Witten] and [Agnihotri]:

$$\mathcal{F}^{\mathbb{Z}}_{n,k} \cong qH^*(\mathrm{Gr}_{n,n+k})/\langle q-1\rangle$$

$\mathcal{F}_{n,k} := \mathcal{F}^{\mathbb{Z}}_{n,k}\otimes_{\mathbb{Z}}\mathbb{C}$ is called *Verlinde algebra*.

Schubert varieties and rational maps

Given a flag $F_1 \subset F_2 \subset \cdots \subset F_{n+k} = \mathbb{C}^{n+k}$ the *Schubert variety* $\Omega_\lambda(F)$ is defined as

$$\Omega_\lambda(F) = \{V \in \mathrm{Gr}_{n,n+k} \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i,\ i = 1, \ldots n\}.$$

Definition of 3-point Gromov-Witten invariants

$C^{\nu,d}_{\lambda,\mu} = \#$ of rational $f : \mathbb{P}^1 \to \mathrm{Gr}_{n,N}$ of degree d which meet $\Omega_\lambda(F)$, $\Omega_\mu(F')$, $\Omega_{\nu^\vee}(F'')$ for general flags F, F', F'' modulo automorphisms in $\mathbb{P}^1$. If there is an ∞ number of such maps, set $C^{\nu,d}_{\lambda,\mu} = 0$.

Poincaré duality: $\nu^\vee = (k - \nu_n, \ldots, k - \nu_1)$
Schubert class: $[\Omega_\lambda] \mapsto s_\lambda$

Quantum Kostka numbers and Gromov-Witten invariants

[Bertram, Ciocan-Fontanine, Fulton]:

$$s_\mu \star s_{\lambda_1} \star \cdots \star s_{\lambda_r} = \sum_{d\geq 0, \nu\in(n,k)} q^d s_\nu K_{\nu/d/\mu,\lambda}$$

$$s_\mu \star s_{(1^{\lambda_1})} \star \cdots \star s_{(1^{\lambda_r})} = \sum_{d\geq 0, \nu\in(n,k)} q^d s_\nu K_{\nu'/d/\mu',\lambda}$$

Quantum Giambelli formula [Bertram]: $s_\lambda = \det(s_{\lambda_i - i + j})$

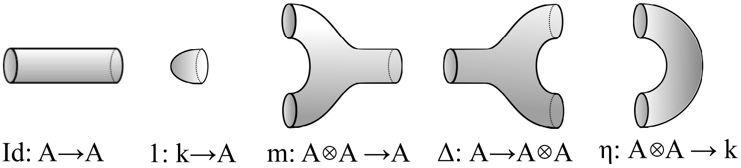
$$s_\mu \star s_\lambda = \sum_{d\geq 0, \nu\in(n,k)} q^d C^{\nu,d}_{\lambda\mu} s_\nu, \qquad d = \frac{|\lambda| + |\mu| - |\nu|}{n+k}$$

3-point, genus 0 Gromov-Witten invariants

Proposition (intersection pairing)

$\mathcal{F}_{n,k}$ is a commutative Frobenius algebra with $\eta(s_\lambda, s_\mu) = \delta_{\lambda^\vee \mu}$.

A Frobenius algebra A is a finite-dimn'l, unital, assoc algebra with non-degenerate bilinear form $\eta(a \star b, c) = \eta(a, b \star c)$, $a, b, c \in A$.



Topological quantum field theories [Witten, Segal, Atiyah]

Commutative Frobenius algebras are categorically equivalent to 2D topological quantum field theories. [Dijkgraaf]

Computation of the coproduct

Frobenius isomorphism $\Phi : s_\lambda \mapsto \eta(s_\lambda, \circ)$

$$\begin{array}{ccc} \mathcal{F}_{n,k} & \xrightarrow{\Delta} & \mathcal{F}_{n,k} \otimes \mathcal{F}_{n,k} \\ \downarrow \Phi & & \downarrow \Phi\otimes\Phi \\ \mathcal{F}^*_{n,k} & \xrightarrow{\mathfrak{m}^*} & \mathcal{F}^*_{n,k} \otimes \mathcal{F}^*_{n,k} \end{array}$$

Proposition (generalised skew Schur function)

$$\Delta s_\nu = \sum_{d,\mu} s_{\nu/d/\mu} \otimes s_\mu, \quad s_{\nu/d/\mu} := \sum_\lambda C^{\nu,d}_{\lambda\mu} s_\lambda \ .$$

Toric Schur function [Postnikov]:

$$\begin{aligned} s_{\nu/d/\mu}(x_1,\dots,x_n) &= \sum_{\lambda} C^{\nu,d}_{\lambda\mu} s_\lambda(x_1,\dots,x_n) \\ &= \sum_{\lambda} K_{\nu/d/\mu,\lambda} m_\lambda(x_1,\dots,x_n), \end{aligned}$$

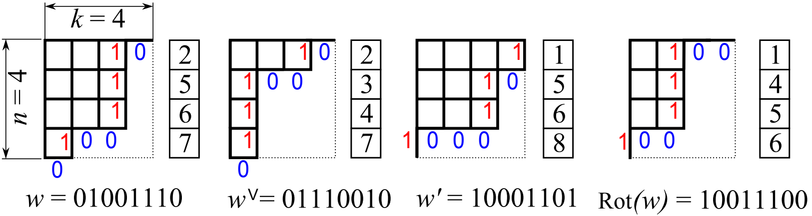
∞-many variables: cylindric Schur functions
[Gessel-Krattenthaler] [McNamara] [Lapointe-Morse] [Lam]

Fusion ring as quantum integrable model (Korff-Stroppel 2010)

Identify the toric Schur functions as partition functions and the fusion ring as the quantum integrals of motion.

Other example: XXX Bethe algebra [Varchenko et al]

Reminder: correspondence between 01-words and partitions



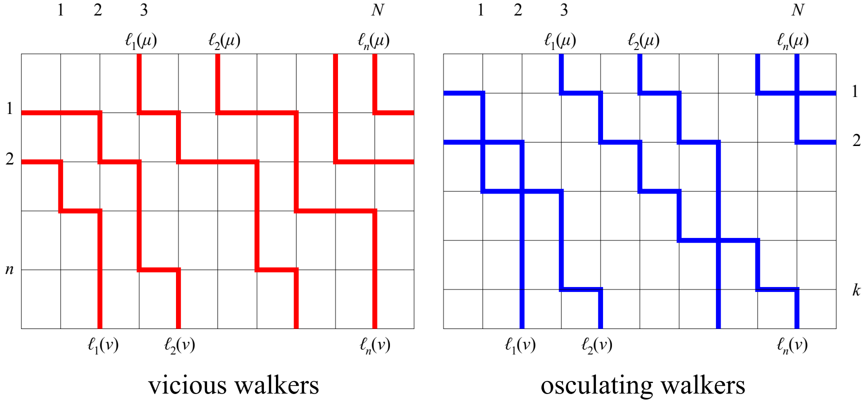
The following bijections induce symmetries of GW invariants:

$$\begin{aligned} w &\mapsto w^{\vee} = w_N \dots w_2 w_1 \\ w &\mapsto w' = (1-w_N)\cdots(1-w_2)(1-w_1) \\ w &\mapsto \mathrm{Rot}(w) := w_2 w_3 \dots w_N w_1 \end{aligned}$$

Position of 1-letters: $\ell_i(\lambda) = \lambda_{n+1-i} + i$

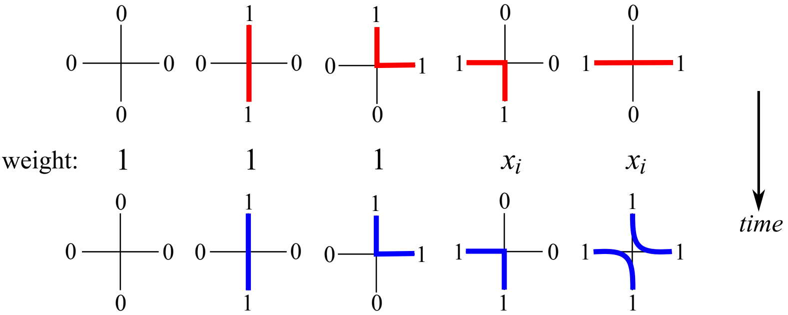
Vicious and osculating walkers on the cylinder

Statistical models on $n \times N$ and $k \times N$ square lattice with periodic boundary conditions in the horizontal direction ($N = n + k$).



Allowed vertex configurations and their weights

x_i indeterminate assigned to the i^{th} lattice row.



(percolation c.f. [Brak][Fisher][Forrester][Guttmann et al][Wu])

Row partition functions

Fix start/end positions via 01-words $w(\mu), w(\nu)$ of length N.

Definition (transfer matrices)

Weighted sums over row configurations:

$$E(x_i)_{\nu,\mu} := \sum_{\text{osc row config}} q^{\frac{\#\text{ of outer edges}}{2}} x_i^{\#\text{ of horizontal edges}}$$

$$H(x_i)_{\nu,\mu} := \sum_{\text{vicious row config}} q^{\frac{\#\text{ of outer edges}}{2}} x_i^{\#\text{ of horizontal edges}}$$

Proposition (integrability $\equiv$ commuting transfer matrices)

$$E(x)E(y) = E(y)E(x),\ H(x)H(y) = H(y)H(x),\ E(x)H(y) = H(y)E(x)$$

Theorem (Generating function for Gromov-Witten invariants)

The partition functions have the following expansions,

$$\begin{aligned}(H(x_n)\cdots H(x_2)\cdot H(x_1))_{\nu,\mu} &= \sum_{d\geq 0} q^d s_{\nu/d/\mu}(x_1,\ldots,x_n)\\ (E(x_k)\cdots E(x_2)\cdot E(x_1))_{\nu,\mu} &= \sum_{d\geq 0} q^d s_{\nu'/d/\mu'}(x_1,\ldots,x_k)\end{aligned}$$

Let $h(s), c(s)$ be hook length and content of $s \in \lambda$.

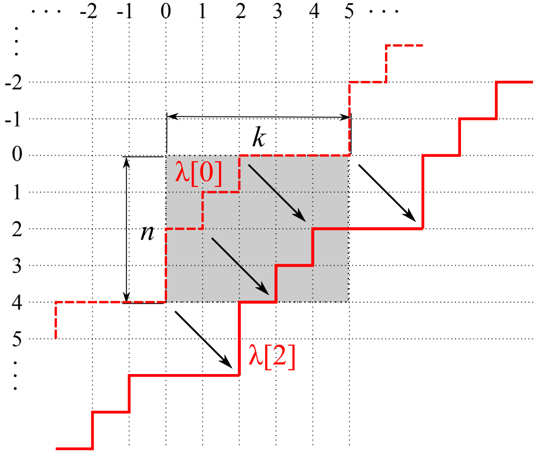
Corollary (Sum rule for Gromov-Witten invariants)

Set $x_i = q = 1$ *for all* $1 \leq i \leq n$. *Then*

$$H^n_{\nu,\mu} = \sum_{d,\lambda} C^{\nu,d}_{\lambda\mu} \prod_{s\in\lambda} \frac{n+c(s)}{h(s)} = E^n_{\nu',\mu'}$$

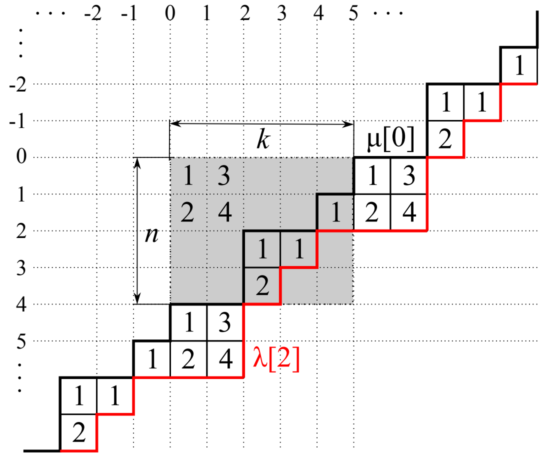
Cylindric loops

$$\lambda[r] := (\ldots, \underset{r}{\lambda_n + r} + k, \underset{r+1}{\lambda_1 + r}, \ldots, \underset{r+n}{\lambda_n + r}, \underset{r+n+1}{\lambda_1 + r} - k, \ldots)$$



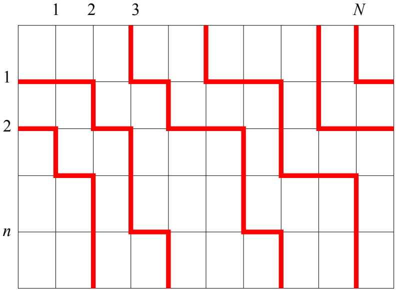
Cylindric skew tableaux

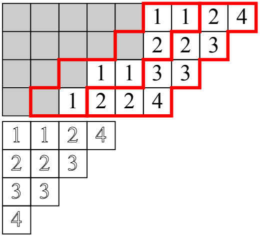
$$\lambda/d/\mu := \{\langle i,j\rangle \in \mathbb{Z}\times\mathbb{Z}/(n,-k)\mathbb{Z} \mid \lambda[d]_i \geq j > \mu[0]_i\} \; .$$



Proposition

Vicious/osculating paths are in bijection with cylindric tableaux.





Level-rank duality: $\tau \circ H = E \circ \tau$ with $\tau : \lambda \mapsto \lambda'$

$K_{\nu/d/\mu,\lambda} = \#$ of cylindric tableaux of weight λ [BCF][Postnikov]

Quantum integrals of motion

Define matrices $S_{(a|b)}$ via the expansion

$$H(x)E(y) = 1 + (x+y)\sum_{a,b\geq 0} x^a y^b S_{(a|b)}$$

Definition (Fusion matrices)

Let $\lambda = (\alpha_1, \ldots, \alpha_r | \beta_1, \ldots, \beta_r)$ with $\lambda_1, \lambda_1' < N$.

$$S_\lambda := \det(S_{(\alpha_i|\beta_j)})_{1\leq i,j\leq r}$$

Proposition (Functional relation)

$H(x)E(-x) = 1 + (-1)^n q x^{n+k}$

Theorem (Korff-Stroppel 2010)

There exists an orthogonal basis $\{\mathfrak{e}_\lambda\}_{\lambda\in(n,k)}$ *such that*

1. *the matrices* H, E *and the* S_λ*'s are diagonal.*
2. *mapping the* $\mathfrak{e}_\lambda$*'s onto the idempotents of* $\mathcal{F}_{n,k}$ *yields an algebra isomorphism, in particular*

$$S_\lambda S_\mu = \sum_{d\geq 0,\nu\in(n,d)} q^d\, C^{\nu,k}_{\lambda\mu} S_\nu \ .$$

XX-Heisenberg spin chain

The transfer matrices H, E commute with the Hamiltonian of the so-called quantum XX-Heisenberg spin chain.

Fermion creation and annihilation

Fix $N = n + k$ and consider the vector space (Fock space)

$$\mathcal{F} = \bigoplus_{n=0}^{N} \mathcal{F}_{n,k}, \qquad \mathcal{F}_{n,k} = \mathbb{C}W_{n,k},$$

where $\mathcal{F}_{0,N} = \mathbb{C}\{0\cdots 0\} = \mathbb{C}$ and $w = 0\cdots 0$ is the *vacuum* $\varnothing$.

Let $n_i(w) = w_1 + \cdots + w_i$ be the number of 1-letters in $[1, i]$.

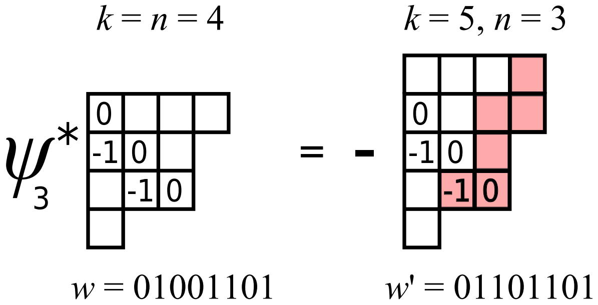
For $1 \leq i \leq N$ define the (linear) maps $\psi_i^*, \psi_i : \mathcal{F}_{n,k} \to \mathcal{F}_{n\pm 1,k\mp 1}$,

$$\psi_i^*(w) := \begin{cases} (-1)^{n_{i-1}(w)} w', & w_i = 0 \text{ and } w_j' = w_j + \delta_{i,j} \\ 0, & w_i = 1 \end{cases}$$

$$\psi_i(w) := \begin{cases} (-1)^{n_{i-1}(w)} w', & w_i = 1 \text{ and } w_j' = w_j - \delta_{i,j} \\ 0, & w_i = 0. \end{cases}$$

Example

Take $n = k = 4$ and $\mu = (4, 3, 3, 1)$.



The boundary ribbon (shaded boxes) starts in the $(3 - n) = -1$ diagonal. Below the diagram the respective 01-words $w(\mu)$ and $w(\psi_3^* \mu)$ are displayed.

Proposition (Clifford algebra)

The maps $\psi_i, \psi_i^* : \mathcal{F}_n \to \mathcal{F}_{n\mp1,k\pm1}$ *yield an irred rep of the Clifford algebra, i.e. one has the relations* $(i, j = 1, \ldots, N)$

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0, \qquad \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij} \ .$$

Introducing $\langle w, w' \rangle = \prod_i \delta_{w_i, w_i'}$ *one has* $\langle \psi_i^* w, w' \rangle = \langle w, \psi_i w' \rangle$ *for any* $w, w' \in \mathcal{F}$*.*

Bijections: partitions – 01-words – words in the Clifford algebra

$$\lambda \ \longleftrightarrow \ w(\lambda) \ \longleftrightarrow \ \psi^*_{\ell_1(\lambda)} \cdots \psi^*_{\ell_n(\lambda)} \varnothing$$

The Clifford algebra is the fundamental object in the description of quantum cohomology.

Connection with vicious walkers

Expansion of the transfer matrix $H(x) : \mathcal{F}_{n,k} \to \mathcal{F}_{n,k}$,

$$H(x) = \sum_{r=0}^{N} x^r H_r, \quad H_r = \sum_{|\alpha|=r} \psi_N^{\alpha_N} u_{N-1}^{\alpha_{N-1}} \cdots u_1^{\alpha_1} ((-1)^{r-1} q \psi_1^*)^{\alpha_N}$$

where $u_i = \psi_{i+1}^* \psi_i$ shifts one particle from site i to site $i+1$.

Proposition (commutation relation with fusion matrices)

$$S_\lambda(q)\psi_i^* = \psi_i^* S_\lambda(-q) + \sum_{r=1}^{\ell(\lambda)} \psi_{i+r}^* \sum_{\lambda/\mu=(r)} S_\mu(-q)$$

where $\psi_{j+N}^* = (-1)^{\mathrm{n}+1} q \psi_j^*$ *and* $\mathrm{n} =$ *particle number operator.*

Fermion creation of quantum cohomology rings

Corollary (Korff-Stroppel 2010)

The last commutation relation implies the product formula

$$\lambda \star \psi_i^*(\mu) = S_\lambda(q)\psi_i^*(\mu) = \sum_{r=0}^{\lambda_1} \sum_{\lambda/\nu=(r)} \psi_{i+r}^*(\nu \,\bar{\star}\, \mu)$$

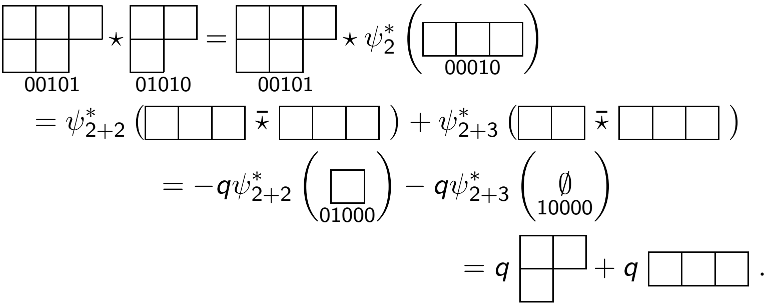
where $\bar{\star}$ *denotes the product with* q *replaced by* $-q$*.*

Inductive algorithm

One can successively generate the entire ring hierarchy $\{qH^*(\mathrm{Gr}_{n,n+k})\}_{n=0}^{n+k}$ starting from $n = 0$.

Example

Consider the ring $qH^*(\mathrm{Gr}_{2,5})$. Via $\psi_i^* : qH^*(\mathrm{Gr}_{1,5}) \to qH^*(\mathrm{Gr}_{2,5})$ one can compute the product in $qH^*(\mathrm{Gr}_{2,5})$ through the product in $qH^*(\mathrm{Gr}_{1,5})$:



Fermionic product formula

Let $\lambda, \mu \in (n,k)$. Then

$$w(\lambda)\star w(\mu) := \sum_T \psi^*_{\ell_1(\mu)+t_n}\bar{\psi}^*_{\ell_2(\mu)+t_{n-1}}\psi^*_{\ell_3(\mu)+t_{n-2}}\bar{\psi}^*_{\ell_4(\mu)+t_{n-3}}\cdots\varnothing,$$

where

- $\ell_i(\mu)$ positions of 1-letters in $w(\mu)$
- $T =$ (semistandard) tableau of shape λ
- $t_i =$ number of entries $1 \le i \le n$ in T
- $\bar{\psi}^*_i = \psi^*_i$ for $i = 1, \ldots, N$ and

$$\psi^*_{i+N} := (-1)^{\mathrm{n}+1} q\psi^*_i, \quad \bar{\psi}^*_{i+N} := (-1)^{\mathrm{n}} q\bar{\psi}^*_i, \quad \mathrm{n} := \sum_{i=1}^{N}\psi^*_i\psi_i \ .$$

Example

Set $N = 7$, $n = N - k = 4$ and $\lambda = (2,2,1,0)$, $\mu = (3,3,2,1)$.

Step 1. Positions of 1-letters: $\ell(\mu) = (\ell_1, \ldots, \ell_4) = (2,4,6,7)$.

Step 2. Write down all tableaux of shape λ such that

$\ell' = (\ell_1 + t_n, \ldots, \ell_n + t_1)$ mod N with $\ell'_i \neq \ell'_j$ for $i \neq j$.

$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&3\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 2&2\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&1\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline 3\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&3\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}$,	$\begin{array}{\|c\|c\|}\hline 2&2\\\hline 3&4\\\hline 4\\\cline{1-1}\end{array}$
(3,5,7,9)	(3,5,7,9)	(3,4,8,9)	(4,5,6,9)	(3,6,7,8)	(3,6,7,8)	(3,6,8,7)	(4,5,7,8)	(4,5,7,8)	(4,5,8,7)

Step 3. For each $\ell'_i > N$ make the replacement

$$\psi^*_{\ell'_1} \cdots \psi^*_{\ell'_n} \varnothing \to (-1)^{n+1} q \psi^*_{\ell'_1} \cdots \psi^*_{\ell'_i - N} \cdots \psi^*_{\ell'_n} \varnothing$$

Step 4. Let ℓ'' be the reduced positions in $[1, N]$. Choose permutation $\pi \in S_n$ s.t. $\ell''_1 < \cdots < \ell''_n$ and multiply with $(-1)^{\ell(\pi)}$.

The three tableaux

$$\underset{(4,5,7,8)}{\ytableaushort{13,24,4}}\;,\quad \underset{(4,5,7,8)}{\ytableaushort{12,34,4}}\;,\quad \underset{(4,5,8,7)}{\ytableaushort{22,34,4}}$$

yield the same 01-word $w = 1001101$, $\lambda(w) = (3,2,2,0)$ but with changing sign,

$$\psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+1}\bar{\psi}^*_{\ell_4+1}\varnothing = \psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+1}\bar{\psi}^*_{\ell_4+1}\varnothing = \\ -\psi^*_{\ell_1+2}\bar{\psi}^*_{\ell_2+1}\psi^*_{\ell_3+2}\bar{\psi}^*_{\ell_4}\varnothing = q\;\psi^*_1\psi^*_4\psi^*_5\psi^*_7\varnothing\;.$$

We obtain the product expansion

$$\ydiagram{2,2,1} \star \ydiagram{3,3,2,1} = q\,\ydiagram{2,2,2,1} + 2q\,\ydiagram{3,2,1,1} + q\,\ydiagram{3,3,2} + q\,\ydiagram{3,3,1} + q^2\,\emptyset\;.$$

Corollary (Quantum Racah-Speiser Algorithm, C.K. 2009)

Let $\lambda,\mu,\nu\in\mathfrak{P}_{n,k}$. *Given a permutation* $\pi\in S_n$ *set*

$$\alpha_i(\pi)=(\ell_i(\nu)-\ell_{\pi(i)}(\mu))\,\mathrm{mod}\,N\geq 0$$
$$d(\pi)=\#\{i\mid \ell_i(\nu)-\ell_{\pi(i)}(\mu)<0\}\ .$$

Then one has the following identity for Gromov-Witten invariants,

$$C_{\lambda\mu}^{\nu,d}=\sum_{\pi\in S_n,\,d(\pi)=d}(-1)^{\ell(\pi)+(n-1)d}K_{\lambda,\alpha(\pi)}\ ,$$

where $K_{\lambda\mu}$ *are the* Kostka numbers*.*

Setting $q=0$ the formula specializes to the known Racah-Speiser algorithm for Littlewood-Richardson coefficients.

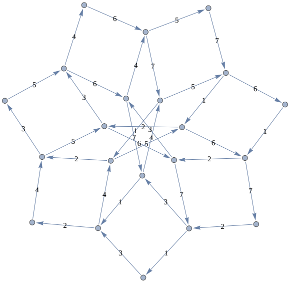
Thank you for your attention!

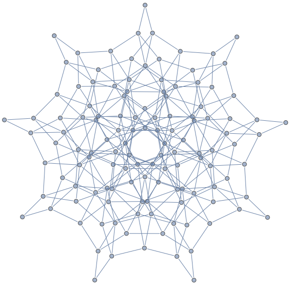
Example

KR crystal: $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$, $B_r = \{$ 01-words with r 1-letters$\}$

$$e_i w = \begin{cases} (w_1, \ldots, w_{i-1} = 1, w_i = 0, \ldots, w_N), & w_{i-1} = 0,\ w_i = 1 \\ \emptyset, & \text{else} \end{cases}$$

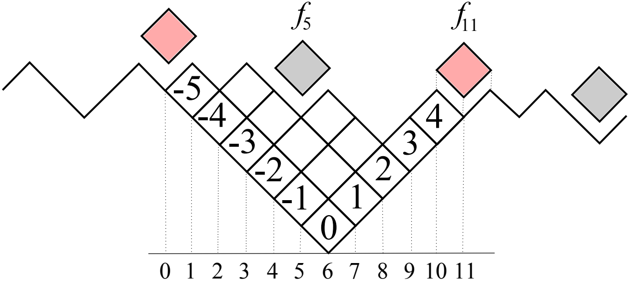
$$f_i w = \begin{cases} (w_1, \ldots, w_i = 0, w_{i+1} = 1, \ldots, w_N), & w_{i+1} = 0,\ w_i = 1 \\ \emptyset, & \text{else} \end{cases}$$



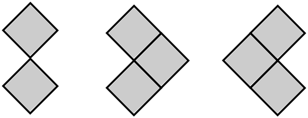


Example (cont'd)

Affine nil Temperley-Lieb algebra



$$f_i^2 = f_i f_{i+1} f_i = f_{i+1} f_i f_{i+1} = 0$$



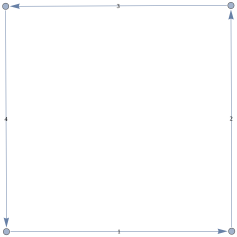
Tensor products of crystals

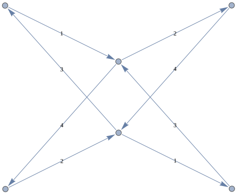
$B \otimes B'$ is the set $B \times B'$ together with the maps,

$$e_i(b \otimes b') = \begin{cases} e_i(b) \otimes b', & \varepsilon_i(b) > \varphi_i(b') \\ b \otimes e_i(b'), & \text{else} \end{cases}$$

$$f_i(b \otimes b') = \begin{cases} f_i(b) \otimes b', & \varepsilon_i(b) \geq \varphi_i(b') \\ b \otimes f_i(b'), & \text{else} \end{cases}$$

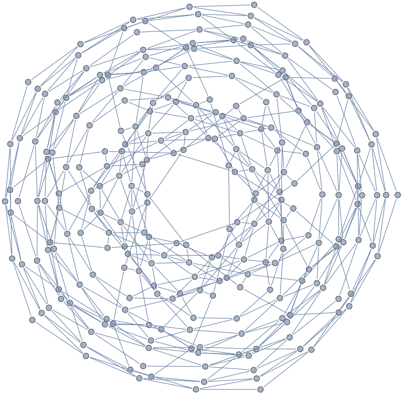
where one sets $b \otimes \emptyset = \emptyset$ and $\emptyset \otimes b' = \emptyset$.





Things can get more complicated ...

Set $N = 5$ and consider the KR crystal $B = B_2 \otimes B_1 \otimes B_1$:



KR crystals are *perfect* crystals: they and their tensor products are always connected.

The combinatorial R-matrix

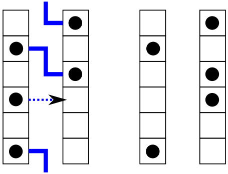
Theorem (Kashiwara et al)

There exists a unique graph isomorphism $R_{r,s} : B_r \otimes B_s \to B_s \otimes B_r$ *which preserves the crystal structure.*

Example (c.f. Nakayashiki-Yamada)

Let $N = 6$ and $r = 3, s = 2$. Then we find

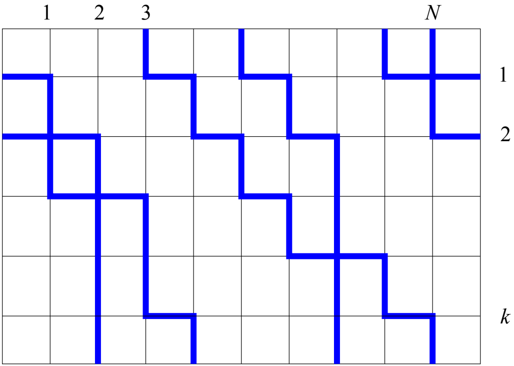
$$R_{3,2}\left(\begin{array}{|c|}\hline 2\\\hline 4\\\hline 6\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)=\begin{array}{|c|}\hline 2\\\hline 6\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}$$



Lattice paths as crystal vertices

$\mathcal{T}_{\nu',\mu'}(\lambda')$ cylindric tableaux of shape $\nu'/d/\mu'$ and weight λ'.

Define $\iota : \mathcal{T}_{\nu',\mu'}(\lambda') \to B_{\lambda'_1} \otimes B_{\lambda'_2} \cdots \otimes B_{\lambda'_k}$ as follows:



3	1	1	6	3
5	4	2	7	8
8	6	5		
9	9			

$$\lambda = 5,5,3,2$$

For simplicity assume $d > d_{min} := \max_\ell\{\sum_{i=1}^{\ell}(w_i(\nu) - w_i(\mu))\}$.

Theorem (Osculating walkers as crystal vertices)

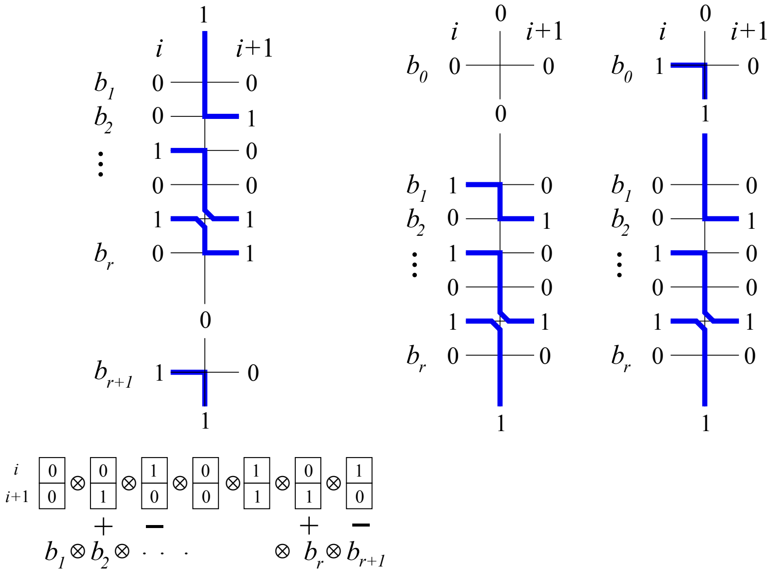
Let $b \in B_\lambda := B_{\lambda'_1} \otimes \cdots \otimes B_{\lambda'_k}$. The following are equivalent.

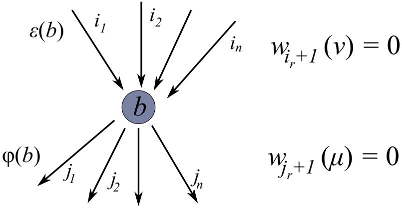
(i) $b \in \iota(\mathcal{T}_{\nu',\mu'}(\lambda'))$

(ii) $\varphi(b) = \sum_{i\in I}(1 - w_{i+1}(\mu))\omega_i, \quad \varepsilon(b) = \sum_{i\in I}(1 - w_{i+1}(\nu))\omega_i.$

(iii) $R_\lambda(b_\mu \otimes \mathrm{Rot}^{-1} b) = b \otimes b_\nu$, *where* Rot *is the* $\widehat{\mathfrak{sl}}_N$ *Dynkin diagram automorphism.*

Here $R_\lambda := R_{n,\lambda'_r} \cdots R_{n,\lambda'_2} R_{n,\lambda'_1}$ is the unique crystal graph isomorphism $R_\lambda : B_n \otimes B_\lambda \to B_\lambda \otimes B_n$.


i
i+1
b1
b2
br
br+1
b0
b1 ⊗ b2 ⊗ . . . ⊗ br ⊗ br+1



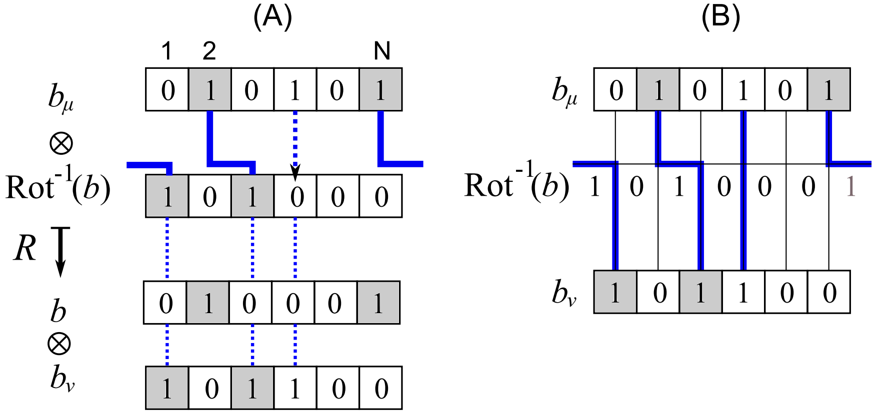
Claim

It follows from (ii) that

$$\varepsilon(b_\mu \otimes \mathrm{Rot}^{-1} b) = \varepsilon(b \otimes b_\nu) \text{ and } \varphi(b_\mu \otimes \mathrm{Rot}^{-1} b) = \varphi(b \otimes b_\nu) \ .$$

Claim + uniqueness of combinatorial R $\Rightarrow$ (iii)

Osculating walks via the combinatorial R-matrix



The case of minimal degree

$d = d_{min} := \max_\ell\{\sum_{i=1}^{\ell}(w_i(\nu) - w_i(\mu))\}$.

Proposition (Fulton-Woodward, Yong, Postnikov)

There exists $1 \le a \le N$ *such that*

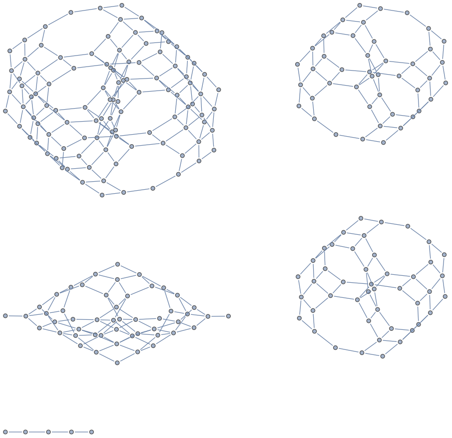
$$K_{\nu/d/\mu,\lambda} = K_{\mathrm{Rot}^a(\nu)/\mathrm{Rot}^a(\mu),\lambda} \quad \textit{and} \quad C_{\lambda,\mu}^{\nu,d} = C_{\lambda\mathrm{Rot}^a(\mu)}^{\mathrm{Rot}^a(\nu),0}$$

Robinson-Schensted-Knuth correspondence:

$$B_\lambda \cong \bigoplus_{\alpha \le \lambda} B(\alpha) \times \mathrm{SST}(\alpha', \lambda') ,$$

where $B(\alpha)$ is the irred $\mathfrak{sl}_N$-crystal of highest weight α.

Previous example with $N = 5$ and $B_{2,1,1} = B_2 \otimes B_1 \otimes B_1$:



http://demonstrations.wolfram.com/KirillovReshetikhinCrystals/

Vicious walks

$\tau : B_n \to B_{N-n},\ b_\lambda \mapsto b_{\lambda'}$ swaps zero and one-letters in 01-word b and then reverses its order.

Define $R'_{r,s} := (1 \otimes \tau) R_{N-r,s} (\tau \otimes 1)$ and $R'_\lambda := R'_{n,\lambda_r} \cdots R'_{n,\lambda_1}$.

Corollary

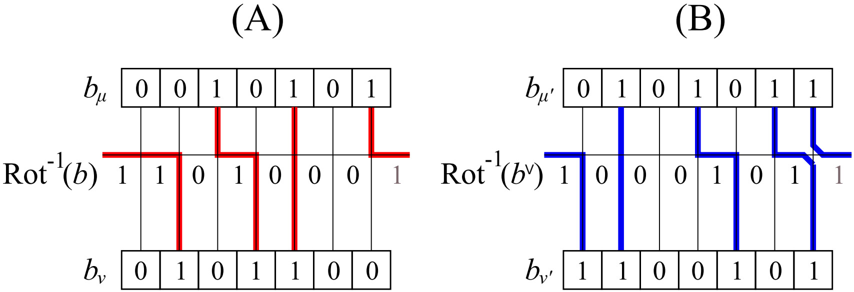
Let $b \in B_{\lambda'}$. The following statements are equivalent.

(i) $b \in \iota(\mathcal{T}_{\nu,\mu}(\lambda))$

(ii) $\varphi(b) = \sum_{i \in I} w_{i+1}(\mu^\vee)\omega_i,\ \varepsilon(b) = \sum_{i \in I} w_{i+1}(\nu^\vee)\omega_i$

(iii) $R'_\lambda(b_{\mu^\vee} \otimes \mathrm{Rot}^{-1} b) = b \otimes b_{\nu^\vee}$

Constructing vicious from osculating walks



Corollary

We have the following symmetries of quantum Kostka numbers,

$$K_{\nu/d/\mu,\lambda} = K_{\nu/d/\mu,s_i\lambda} = K_{\mathrm{Rot}(\nu)/d^R/\mathrm{Rot}(\mu),\lambda} = K_{\mu^\vee/d/\nu^\vee,\lambda},$$

where $s_i\lambda = (\ldots,\lambda_{i+1},\lambda_i,\ldots)$ *and* $d^R = d + w_1(\mu) - w_1(\nu)$.

Proof

- Yang-Baxter equation: $R_{23}\circ R_{12}\circ R_{23} = R_{12}\circ R_{23}\circ R_{12}$
- Rotation (Dynkin diagram automorphism) $\mathrm{Rot}(b_1\otimes b_2) := \mathrm{Rot}(b_1)\otimes\mathrm{Rot}(b_2)$, $\mathrm{Rot}\circ R = R\circ\mathrm{Rot}$.
- reversing 01-words (Lusztig involution) $\vee : B_r\otimes B_s \to B_s\otimes B_r$ with $b_1\otimes b_2 \mapsto b_2^\vee\otimes b_1^\vee$,

$$\vee\circ f_i = e_{N-i}\circ\vee, \qquad \vee\circ e_i = f_{N-i}\circ\vee, \qquad \vee\circ R_{r,s} = R_{s,r}\circ\vee\ .$$

Conclusions

Additional results

- Eigenvectors of the transfer matrices = idempotents.
- Yang-Baxter algebras, affine nil Temperley-Lieb algebra and Schur polynomials.
- Algorithms to generate vicious/osculating walks.
- The $\widehat{\mathfrak{sl}}_N$-Verlinde algebra and cylindric Macdonald functions.

Outlook

- Combinatorial definition of GW invariants and positivity.
- Other Lie algebras.
- Quantum Horn conjecture.
- Categorification of integrable systems.