

COMPLEX SEMISIMPLE QUANTUM GROUPS AND REPRESENTATION THEORY

CHRISTIAN VOIGT AND ROBERT YUNCKEN

2010 *Mathematics Subject Classification.* 16T05, 17B37, 46L65, 81R50.

The first author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Operator Algebras: Subfactors and their applications, where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1. The first author was supported by the Polish National Science Centre grant no. 2012/06/M/ST1/00169. This paper was partially supported by the grant H2020-MSCA-RISE-2015-691246-QUANTUM DYNAMICS. The second author was supported by the project SINGSTAR of the Agence Nationale de la Recherche, ANR-14-CE25-0012-01 and by the CNRS PICS project OpPsi. Both authors would also like to thank the Erwin Schrödinger Institute in Vienna, where some of the present research was undertaken.

INTRODUCTION

The theory of quantum groups, originating from the study of integrable systems, has seen a rapid development from the mid 1980's with far-reaching connections to various branches of mathematics, including knot theory, representation theory and operator algebras, see [25], [57], [17]. While the term *quantum group* itself has no precise definition, it is used to denote a number of related constructions, including in particular quantized universal enveloping algebras of semisimple Lie algebras and, dually, deformations of the algebras of polynomial functions on the corresponding semisimple groups. In operator algebras, the theory of locally compact quantum groups [54] is a powerful framework which allows one to extend Pontrjagin duality to a fully noncommutative setting.

In both the algebraic and the analytic theory of quantum groups, an important role is played by the Drinfeld double, also known as the quantum double, which is designed to produce solutions to the quantum Yang-Baxter equation. While the algebraic version of this construction, due to Drinfeld, appears already in [25], the analytical analogue of the Drinfeld double was defined and studied first by Podleś and Woronowicz in [63].

If one applies the Drinfeld double construction to the Hopf algebra of functions on the q -deformation of a compact semisimple Lie group then in accordance with the quantum duality principle [25], [30], the resulting quantum group can be viewed as a quantization of the corresponding complex semisimple group. This fact, which was exhibited in [63] in the case of the quantum Lorentz group $SL_q(2, \mathbb{C})$, is key to understanding the structure of these Drinfeld doubles. More precisely, one can transport techniques from the representation theory of classical complex semisimple groups to the quantum situation, and it turns out that the main structural results carry over, albeit with sometimes quite different proofs.

These notes contain an introduction to the theory of complex semisimple quantum groups, that is, Drinfeld doubles of compact quantum groups arising from q -deformations. Our main aim is to present the classification of irreducible Harish-Chandra modules for these quantum groups, or equivalently the irreducible Yetter-Drinfeld modules of q -deformations of compact semisimple Lie groups. We also treat some operator algebraic aspects of these constructions, but we put our main emphasis on the algebraic considerations based on quantized universal enveloping algebras.

The main reason for going into a considerable amount of detail on the algebraic side is that the existing literature does not quite contain the results in the form needed for our purposes. Many authors work over the field $\mathbb{Q}(q)$ of rational functions in q , while we are mainly interested in the case that $q \in \mathbb{C}^\times$ is not a root of unity. Although it is folklore that this does not affect the general theory in a serious way, there are subtle differences which are easily overlooked, in particular when it comes to Verma modules and Harish-Chandra bimodules. In addition, different conventions are used in the literature, which can make it cumbersome to combine results from different sources.

In the first part of these notes, consisting of Chapters 1 and 2, we work over a general ground field \mathbb{K} and a deformation parameter $q \in \mathbb{K}^\times$ which is not a root of unity. Technically, this means that one has to start from an element $s \in \mathbb{K}$ such that $s^L = q$ for a certain number $L \in \mathbb{N}$ depending on the type of the underlying semisimple Lie algebra. No assumptions on the characteristic of \mathbb{K} are made, and in particular we shall not rely on specialization at $q = 1$ in order to transport results from the classical situation to the quantum case. However, we shall freely use general constructions and facts from classical Lie theory as can be found, for instance, in [35].

In the second part of the notes, consisting of Chapters 3, 4 and 5, we restrict ourselves to the case $\mathbb{K} = \mathbb{C}$ and assume that $q = e^h \neq 1$ is a positive real number. Some parts of the material could be developed in greater generality, but for certain arguments the specific properties of the ground field \mathbb{C} and its exponential map do indeed play a role. This is the case, for instance, for the characterization of dominant and antidominant weights in Chapter 4.

We have used a number of textbooks as the basis of our presentation, let us mention in particular the books by Lusztig [57], Jantzen [39], Klimyk and Schmüdgen [48], Chari and Pressley [17], Brown and Goodearl [15], and the book by Humphreys on category \mathcal{O} in the classical setting [37]. The key source for the second half of these notes is the book by Joseph [41], which in turn builds to a large extent on work of Joseph and Letzter. Our account is largely self-contained, the only notable exception being the section on separation of variables in Chapter 2, which relies on the theory of canonical bases. Nonetheless, we assume that the reader has some previous acquaintance with quantum groups, and we have kept short those arguments which can be easily found elsewhere. Let us also point out that our bibliography is far from complete, and the reader should consult the above mentioned sources for a historically accurate attribution of the results covered here, along with much more background and motivation.

Let us now explain in detail how this text is organized.

In Chapter 1 we review the theory of Hopf algebras and multiplier Hopf algebras. While the basics of Hopf algebras can be found in numerous textbooks, the non-unital version, developed by Van Daele [73], [74] under the name of multiplier Hopf algebras, is not as widely known. Multiplier Hopf algebras provide a natural setting for the study of some aspects of quantized enveloping algebras, like the universal R -matrix. Moreover, they are a very useful tool for studying the link between the algebraic and analytic theory of quantum groups [51]. Since only a limited amount of the general theory of multiplier Hopf algebras is needed for our purposes, we refer to the original sources for most of the proofs.

Chapter 2 contains an exposition of the basic theory of quantized universal enveloping algebras. Most of the material is standard, but we have made an effort to establish uniform conventions and notation. We work throughout with what is often called the *simply connected* version of $U_q(\mathfrak{g})$. This is crucial for some of the more advanced parts of the theory, notably in connection with the l -functionals. Our discussion of the braid group action on $U_q(\mathfrak{g})$ and its modules is significantly more detailed than what can be found in the standard textbooks. We have also tried to simplify and streamline various arguments in the literature, in particular we avoid some delicate filtration arguments in [41] in the proof of Noetherianity of the locally finite part of $U_q(\mathfrak{g})$.

In Chapter 3 we introduce our main object of study, namely complex semisimple quantum groups. These quantum groups can be viewed as quantizations of complex semisimple Lie groups viewed as *real* groups. As indicated above, they are obtained by applying the quantum double construction to compact semisimple quantum groups. We discuss the structure of complex quantum groups as locally compact quantum groups, including their Haar weights and dual Haar weights.

The remaining two chapters are devoted to representation theory. Chapter 4 contains a discussion of category \mathcal{O} for quantized universal enveloping algebras. This is parallel to the theory for classical universal enveloping algebras, but some peculiar new features arise in the quantum situation due to the periodicity in the space of purely imaginary weights.

Finally, Chapter 5, covers the representation theory of complex semisimple quantum groups and their associated Harish-Chandra modules. We also provide a detailed account of intertwining operators, by taking advantage of the relation between the category of Harish-Chandra modules and category \mathcal{O} . This both simplifies and extends the work by Pusz and Woronowicz [66], [64] on the quantum Lorentz group $SL_q(2, \mathbb{C})$.

Let us conclude with some remarks on notation and conventions. By default, an algebra is a unital associative algebra over a commutative ground ring, which will typically be a field \mathbb{K} . In some situations we have to deal with non-unital algebras and their multiplier algebras, but it should always be clear from the context when a non-unital algebra appears. Unadorned tensor products are over the ground field \mathbb{K} . In some situations we also use \otimes to denote the tensor product of Hilbert spaces, the spatial tensor product of von Neumann algebras, or the minimal tensor product of C^* -algebras. Again, the meaning should be clear from the context.

Let us note that we have aimed to make the transition between the algebraic and analytic points of view as convenient as possible. This results in some slightly unconventional choices for the algebraically-minded reader, in particular in terms of the pairing between quantized enveloping algebras and their dual function algebras, which we define to be skew-pairings by default.

Last but not least, it is a pleasure to thank a number of people with whom we have discussed aspects of quantum groups and representation theory over the past few years. Let us mention in particular Y. Arano, P. Baumann, K. A. Brown, I. Heckenberger, N. Higson, U. Krähmer, S. Neshveyev, S. Riche and D. Vogan; we are grateful to all of them for sharing their insight with us.

CONTENTS

Introduction	2
1. Multiplier Hopf algebras	7
1.1. Hopf algebras	7
1.2. Multiplier Hopf algebras	9
1.3. Integrals	12
1.4. The Drinfeld double - algebraic level	16
2. Quantized universal enveloping algebras	20
2.1. q -calculus	20
2.2. The definition of $U_q(\mathfrak{g})$	21
2.3. Verma modules	41
2.4. Characters of $U_q(\mathfrak{g})$	44
2.5. Finite dimensional representations of $U_q(\mathfrak{sl}(2, \mathbb{K}))$	45
2.6. Finite dimensional representations of $U_q(\mathfrak{g})$	48
2.7. Braid group action and PBW basis	51
2.8. The Drinfeld pairing and the quantum Killing form	79
2.9. The quantum Casimir element and simple modules	99
2.10. Quantized algebras of functions	102
2.11. The universal R -matrix	104
2.12. The locally finite part of $U_q(\mathfrak{g})$	110
2.13. The centre of $U_q(\mathfrak{g})$ and the Harish-Chandra homomorphism	115
2.14. Noetherianity	120
2.15. Canonical bases	128
2.16. Separation of Variables	133
3. Complex semisimple quantum groups	139
3.1. Locally compact quantum groups	139
3.2. Algebraic quantum groups	141
3.3. Compact semisimple quantum groups	150
3.4. Complex semisimple quantum groups	154
3.5. Polynomial functions	158
3.6. The quantized universal enveloping algebra of a complex group	161
3.7. Parabolic quantum subgroups	164
4. Category \mathcal{O}	166
4.1. The definition of category \mathcal{O}	166
4.2. Submodules of Verma modules	172
4.3. The Shapovalov determinant	176
4.4. Jantzen filtration and the BGG Theorem	180
4.5. The PRV determinant	183
4.6. Annihilators of Verma modules	200
5. Representation theory of complex semisimple quantum groups	202
5.1. Verma modules for $U_q^{\mathbb{R}}(\mathfrak{g})$	202
5.2. Representations of G_q	206
5.3. Action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on K_q -types	209
5.4. Principal series representations	211
5.5. An equivalence of categories	219
5.6. Irreducible Harish-Chandra modules	227
5.7. The principal series for $SL_q(2, \mathbb{C})$	232
5.8. Intertwining operators in higher rank	235
5.9. Submodules and quotient modules of the principal series	246
5.10. Unitary representations	248

References

1. MULTIPLIER HOPF ALGEBRAS

In this chapter we collect definitions and basic results regarding Hopf algebras and multiplier Hopf algebras. Algebras are not assumed to have identities in general. Throughout we shall work over an arbitrary base field \mathbb{K} , and all tensor products are over \mathbb{K} .

1.1. Hopf algebras. The language of Hopf algebras is the starting point for the study of noncommutative symmetries. In this section we collect some basic definitions and facts from the theory. There is a large variety of textbooks devoted to Hopf algebras and quantum groups where more information can be found, let us mention in particular [17], [39], [47], [48].

An algebra is a vector space H together with a linear map $m : H \otimes H \rightarrow H$ such that the diagram

$$\begin{array}{ccc} H \otimes H \otimes H & \xrightarrow{m \otimes \text{id}} & H \otimes H \\ \downarrow \text{id} \otimes m & & \downarrow m \\ H \otimes H & \xrightarrow{m} & H \end{array}$$

is commutative. An algebra H is called unital if there exists a linear map $u : \mathbb{K} \rightarrow H$ such that the diagram

$$\begin{array}{ccccc} \mathbb{K} \otimes H & \xrightarrow{u \otimes \text{id}} & H \otimes H & \xleftarrow{\text{id} \otimes u} & H \otimes \mathbb{K} \\ & \cong \searrow & \downarrow m & \swarrow \cong & \\ & & H & & \end{array}$$

is commutative. Here the isomorphisms are induced by scalar multiplication. As usual, we write fg for $m(f \otimes g)$, and 1_H or simply 1 for $u(1)$ in the unital case.

A (unital) algebra homomorphism between (unital) algebras A and B is a linear map $\varphi : A \rightarrow B$ such that $\varphi m_A = m_B(\varphi \otimes \varphi)$ (and $\varphi u_A = u_B$ in the unital case). Here m_A, m_B denote the multiplication maps of A and B , respectively, and u_A, u_B the unit maps in the unital case. In other words, φ is an algebra homomorphism iff $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in A$, and additionally $\varphi(1_A) = 1_B$ in the unital case.

By definition, a coalgebra is a vector space H together with a linear map $\Delta : H \rightarrow H \otimes H$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H \otimes H \\ \downarrow \Delta & & \downarrow \text{id} \otimes \Delta \\ H \otimes H & \xrightarrow{\Delta \otimes \text{id}} & H \otimes H \otimes H \end{array}$$

is commutative. A coalgebra H is called counital if there exists a linear map $\epsilon : H \rightarrow \mathbb{K}$ such that the diagram

$$\begin{array}{ccccc} \mathbb{K} \otimes H & \xleftarrow{\epsilon \otimes \text{id}} & H \otimes H & \xrightarrow{\text{id} \otimes \epsilon} & H \otimes \mathbb{K} \\ & \cong \searrow & \uparrow \Delta & \swarrow \cong & \\ & & H & & \end{array}$$

is commutative.

We shall use the Sweedler notation

$$\Delta(c) = c_{(1)} \otimes c_{(2)}$$

when performing computations in coalgebras, where we are suppressing a summation on the right-hand side. For instance, the counit axiom reads $\epsilon(c_{(1)})c_{(2)} = c = c_{(1)}\epsilon(c_{(2)})$ in this notation.

In analogy to the case of algebras, a (counital) coalgebra homomorphism between (counital) coalgebras C and D is a linear map $\varphi : C \rightarrow D$ such that $(\varphi \otimes \varphi)\Delta_C = \Delta_D\varphi$ (and $\epsilon_D\varphi = \epsilon_C$ in the counital case).

If C is a (counital) coalgebra then the linear dual space $C^* = \text{Hom}(C, \mathbb{K})$ is a (unital) algebra with multiplication

$$(fg, c) = (f, c_{(1)})(g, c_{(2)})$$

(and unit element ϵ_C). Here $(,)$ denotes the canonical pairing between C and C^* .

Conversely, if A is a finite dimensional (unital) algebra, then the linear dual space A^* becomes a (counital) coalgebra using the transpose $m^* : A^* \rightarrow (A \otimes A)^* \cong A^* \otimes A^*$ of the multiplication map $m : A \otimes A \rightarrow A$ (and counit $\epsilon(f) = f(1)$). We point out that if A is infinite dimensional this will typically break down since then $(A \otimes A)^* \neq A^* \otimes A^*$.

However, at least in the finite dimensional situation, we have a complete duality between algebras and coalgebras. This can be used to transport concepts from algebras to coalgebras and vice versa. For instance, a coalgebra C is called cosimple if it does not admit any proper subcoalgebras. Here a subcoalgebra of C is a linear subspace $D \subset C$ such that $\Delta(D) \subset D \otimes D$. This notion corresponds to simplicity for the dual algebra A . A coalgebra is called cosemisimple if it is a direct sum of its simple subcoalgebras.

A basic example of a cosimple coalgebra is the dual coalgebra $C = M_n(\mathbb{K})^*$ of the algebra of $n \times n$ -matrices over \mathbb{K} . Explicitly, the coproduct of C is given by

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$$

for $1 \leq i, j \leq n$, where the elements $u_{ij} \in C$ are dual to the basis of standard matrix units for $M_n(\mathbb{K})$. All cosemisimple coalgebras we will encounter later on are direct sums of such cosimple matrix coalgebras.

A Hopf algebra structure is the combination of the structures of a unital algebra and a counital coalgebra as follows.

Definition 1.1. A bialgebra is a unital algebra H which is at the same time a counital coalgebra such that the comultiplication $\Delta : H \rightarrow H \otimes H$ and the counit $\epsilon : H \rightarrow \mathbb{K}$ are algebra homomorphisms.

A bialgebra is a Hopf algebra if there exists a linear map $S : H \rightarrow H$, called the antipode, such that the diagrams

$$\begin{array}{ccc} H & \xrightarrow{u\epsilon} & H \\ \downarrow \Delta & & \uparrow m \\ H \otimes H & \xrightarrow{\text{id} \otimes S} & H \otimes H \end{array} \quad \begin{array}{ccc} H & \xrightarrow{u\epsilon} & H \\ \downarrow \Delta & & \uparrow m \\ H \otimes H & \xrightarrow{S \otimes \text{id}} & H \otimes H \end{array}$$

are commutative.

In the definition of a bialgebra, one can equivalently require that the multiplication map m and the unit map u are coalgebra homomorphisms.

For many examples of Hopf algebras the antipode S is an invertible linear map. We write S^{-1} for the inverse of S in this situation. We remark that S is always invertible if H is a finite dimensional Hopf algebra.

If H is a finite dimensional Hopf algebra then the dual space $\hat{H} = H^* = \text{Hom}(H, \mathbb{K})$ can be naturally equipped with a Hopf algebra structure with comultiplication $\hat{\Delta} : \hat{H} \rightarrow \hat{H} \otimes \hat{H}$ and counit $\hat{\epsilon}$ such that

$$\begin{aligned} (xy, f) &= (x \otimes y, \Delta(f)) \\ (\hat{\Delta}(x), f \otimes g) &= (x, gf) \\ \hat{\epsilon}(x) &= (x, 1) \\ (\hat{S}(x), f) &= (x, S^{-1}(f)) \end{aligned}$$

for all $x, y \in \hat{H}$ and $f, g \in H$. Here $(,)$ denotes the canonical pairing between H and \hat{H} , or $H \otimes H$ and $\hat{H} \otimes \hat{H}$ respectively, given by evaluation. Note that we are choosing the convention whereby \hat{H} and H are skew-paired, in the sense that the comultiplication $\hat{\Delta}$ is dual to the opposite multiplication $f \otimes g \mapsto gf$.

In order to generalize this construction to certain infinite dimensional examples we shall work with the theory of multiplier Hopf algebras discussed in the next section.

1.2. Multiplier Hopf algebras. The theory of multiplier Hopf algebras developed by van Daele and his coauthors [74], [23], [24] is an extension of the theory of Hopf algebras to the case where the underlying algebras do not necessarily have identity elements.

1.2.1. Essential algebras. In order to obtain a reasonable theory, it is necessary to impose some conditions on the multiplication of an algebra. We will work with algebras that are essential in the following sense.

Definition 1.2. An algebra H is called essential if $H \neq 0$ and the multiplication map induces an isomorphism $H \otimes_H H \cong H$.

Clearly, every unital algebra is essential.

More generally, assume that H has local units in the sense that for every finite set of elements $h_1, \dots, h_n \in H$ there exists $u, v \in H$ such that $uh_j = h_j$ and $h_jv = h_j$ for all j . Then H is essential. Regular multiplier Hopf algebras, to be defined below, automatically have local units, see [24].

The prototypical example of a non-unital essential algebra to keep in mind is an algebra of the form

$$H = \bigoplus_{i \in I} A_i,$$

where $(A_i)_{i \in I}$ is a family of unital algebras and multiplication is componentwise. In fact, all non-unital essential algebras that we will encounter later on are of this form.

Let H be an algebra. A left H -module V is called essential if the canonical map $H \otimes_H V \rightarrow V$ is an isomorphism. An analogous definition can be given for right modules. In particular, an essential algebra H is an essential left and right module over itself.

1.2.2. Algebraic multiplier algebras. To proceed further we need to discuss multipliers. A left multiplier for an algebra H is a linear map $L : H \rightarrow H$ such that $L(fg) = L(f)g$ for all $f, g \in H$. Similarly, a right multiplier is a linear map $R : H \rightarrow H$ such that $R(fg) = fR(g)$ for all $f, g \in H$. We let $\mathcal{M}_l(H)$ and $\mathcal{M}_r(H)$ be the spaces of left and right multipliers, respectively. These spaces become algebras with multiplication given by composition of maps.

Definition 1.3. The algebraic multiplier algebra $\mathcal{M}(H)$ of an algebra H is the space of all pairs (L, R) where L is a left multiplier and R is a right multiplier for H such that $fL(g) = R(f)g$ for all $f, g \in H$.

The algebra structure of $\mathcal{M}(H)$ is inherited from $\mathcal{M}_l(H) \oplus \mathcal{M}_r(H)$. There is a natural homomorphism $\iota : H \rightarrow \mathcal{M}(H)$. By construction, H is a left and right $\mathcal{M}(H)$ -module in a natural way.

In the case that $H = \bigoplus_{i \in I} A_i$ is a direct sum of unital algebras A_i for $i \in I$, it is straightforward to check that

$$\mathcal{M}(H) \cong \prod_{i \in I} A_i$$

is the direct product of the algebras A_i .

Let H and K be algebras and let $\varphi : H \rightarrow \mathcal{M}(K)$ be a homomorphism. Then K is a left and right H -module in an obvious way. We say that the homomorphism $\varphi : H \rightarrow \mathcal{M}(K)$ is essential if it turns K into an essential left and right H -module, that is, we require $H \otimes_H K \cong K \cong K \otimes_H H$. Note that the identity map $\text{id} : H \rightarrow H$ defines an essential homomorphism $H \rightarrow \mathcal{M}(H)$ iff the algebra H is essential.

Lemma 1.4. *Let H be an algebra and let $\varphi : H \rightarrow \mathcal{M}(K)$ be an essential homomorphism into the multiplier algebra of an essential algebra K . Then there exists a unique unital homomorphism $\Phi : \mathcal{M}(H) \rightarrow \mathcal{M}(K)$ such that $\Phi \iota = \varphi$ where $\iota : H \rightarrow \mathcal{M}(H)$ is the canonical map.*

Proof. We obtain a linear map $\Phi_l : \mathcal{M}_l(H) \rightarrow \mathcal{M}_l(K)$ by

$$\mathcal{M}_l(H) \otimes K \cong \mathcal{M}_l(H) \otimes H \otimes_H K \xrightarrow{m \otimes \text{id}} H \otimes_H K \cong K$$

and accordingly a linear map $\Phi_r : \mathcal{M}_r(H) \rightarrow \mathcal{M}_r(K)$ by

$$K \otimes \mathcal{M}_r(H) \cong K \otimes_H H \otimes \mathcal{M}_r(H) \xrightarrow{\text{id} \otimes m} K \otimes_H H \cong K.$$

It is straightforward to check that $\Phi((L, R)) = (\Phi_l(L), \Phi_r(R))$ defines a unital homomorphism $\Phi : \mathcal{M}(H) \rightarrow \mathcal{M}(K)$ such that $\Phi \iota = \varphi$. Uniqueness of Φ follows from the fact that $\varphi(H) \cdot K = K$ and $K \cdot \varphi(H) = K$. \square

We note that essential homomorphisms behave well under tensor products. More precisely, assume that H_1, H_2 are essential algebras and let $\varphi_1 : H_1 \rightarrow \mathcal{M}(K_1)$ and $\varphi_2 : H_2 \rightarrow \mathcal{M}(K_2)$ be essential homomorphisms into the multiplier algebras of algebras K_1 and K_2 . Then the induced homomorphism $\varphi_1 \otimes \varphi_2 : H_1 \otimes H_2 \rightarrow \mathcal{M}(K_1 \otimes K_2)$ is essential.

Following the terminology of van Daele, we say that an algebra H is nondegenerate if the multiplication map $H \times H \rightarrow H$ defines a nondegenerate bilinear pairing. That is, H is nondegenerate iff $fg = 0$ for all $g \in H$ implies $f = 0$ and $fg = 0$ for all f implies $g = 0$. These conditions can be reformulated by saying that the natural maps

$$H \rightarrow \mathcal{M}_l(H), \quad H \rightarrow \mathcal{M}_r(H)$$

are injective. In particular, for a nondegenerate algebra the canonical map $H \rightarrow \mathcal{M}(H)$ is injective.

Nondegeneracy of an algebra is a consequence of the existence of a faithful linear functional in the following sense.

Definition 1.5. Let H be an algebra. A linear functional $\omega : H \rightarrow \mathbb{K}$ is called faithful if $\omega(fg) = 0$ for all g implies $f = 0$ and $\omega(fg) = 0$ for all f implies $g = 0$.

1.2.3. *Multiplier Hopf algebras.* In this subsection we introduce the notion of a multiplier Hopf algebra.

Let H be an essential algebra and let $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ be a homomorphism. The left Galois maps $\gamma_l, \gamma_r : H \otimes H \rightarrow \mathcal{M}(H \otimes H)$ for Δ are defined by

$$\gamma_l(f \otimes g) = \Delta(f)(g \otimes 1), \quad \gamma_r(f \otimes g) = \Delta(f)(1 \otimes g).$$

Similarly, the right Galois maps $\rho_l, \rho_r : H \otimes H \rightarrow \mathcal{M}(H \otimes H)$ for Δ are defined by

$$\rho_l(f \otimes g) = (f \otimes 1)\Delta(g), \quad \rho_r(f \otimes g) = (1 \otimes f)\Delta(g).$$

These maps, or rather their appropriate analogues on the Hilbert space level, play an important role in the analytical study of quantum groups [5], [54]. Our terminology originates from the theory of Hopf-Galois extensions, see for instance [60].

We say that an essential homomorphism $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ is a comultiplication if it is coassociative, that is, if

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$$

holds. Here both sides are viewed as maps from H to $\mathcal{M}(H \otimes H \otimes H)$. An essential algebra homomorphism $\varphi : H \rightarrow \mathcal{M}(K)$ between algebras with comultiplications is called a coalgebra homomorphism if $\Delta\varphi = (\varphi \otimes \varphi)\Delta$.

We need some more terminology. The opposite algebra H^{opp} of H is the space H equipped with the opposite multiplication. That is, the multiplication m^{opp} in H^{opp} is defined by $m^{\text{opp}} = m\sigma$ where $m : H \otimes H \rightarrow H$ is the multiplication in H and $\sigma : H \otimes H \rightarrow H \otimes H$ is the flip map given by $\sigma(f \otimes g) = g \otimes f$. An algebra antihomomorphism between H and K is an algebra homomorphism from H to K^{opp} . Equivalently, an algebra antihomomorphism can be viewed as an algebra homomorphism $H^{\text{opp}} \rightarrow K$. If $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ is a comultiplication then Δ also defines a comultiplication $H^{\text{opp}} \rightarrow \mathcal{M}(H^{\text{opp}} \otimes H^{\text{opp}})$.

Apart from changing the order of multiplication we may also reverse the order of a comultiplication. If $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ is a comultiplication then the opposite comultiplication Δ^{cop} is the essential homomorphism from H to $\mathcal{M}(H \otimes H)$ defined by $\Delta^{\text{cop}} = \sigma\Delta$. Here $\sigma : \mathcal{M}(H \otimes H) \rightarrow \mathcal{M}(H \otimes H)$ is the extension of the flip map to multipliers. We write H^{cop} for H equipped with the opposite comultiplication. Using opposite comultiplications we obtain the notion of a coalgebra antihomomorphism. That is, a coalgebra antihomomorphism between H and K is a coalgebra homomorphism from H to K^{cop} , or equivalently, from H^{cop} to K .

Let us now give the definition of a multiplier Hopf algebra [73].

Definition 1.6. A multiplier Hopf algebra is an essential algebra H together with a comultiplication $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ such that the Galois maps γ_r, ρ_l are isomorphisms from $H \otimes H$ to $H \otimes H \subset \mathcal{M}(H \otimes H)$.

A regular multiplier Hopf algebra is an essential algebra H together with a comultiplication $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ such that all Galois maps $\gamma_l, \gamma_r, \rho_l, \rho_r$ are isomorphisms from $H \otimes H$ to $H \otimes H$.

A morphism between multiplier Hopf algebras H and K is an essential algebra homomorphism $\alpha : H \rightarrow \mathcal{M}(K)$ such that $(\alpha \otimes \alpha)\Delta = \Delta\alpha$.

Note that a multiplier Hopf algebra H is regular iff H^{opp} is a multiplier Hopf algebra, or equivalently, iff H^{cop} is a multiplier Hopf algebra.

We have the following fundamental result due to van Daele [73].

Theorem 1.7. *Let H be a multiplier Hopf algebra. Then there exists an essential algebra homomorphism $\epsilon : H \rightarrow \mathbb{K}$ and an algebra antihomomorphism $S : H \rightarrow \mathcal{M}(H)$ such that*

$$(\epsilon \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \epsilon)\Delta$$

and

$$m(S \otimes \text{id})\gamma_r = \epsilon \otimes \text{id}, \quad m(\text{id} \otimes S)\rho_l = \text{id} \otimes \epsilon.$$

If H is a regular multiplier Hopf algebra, then S is a linear isomorphism from H to H .

It follows from Theorem 1.7 that Definition 1.6 reduces to the definition of Hopf algebras in the unital case. A unital regular multiplier Hopf algebra is the same thing as a Hopf algebra with invertible antipode.

We will be exclusively interested in regular multiplier Hopf algebras. For a regular multiplier Hopf algebra, the antipode S is a coalgebra antihomomorphism, that is, it satisfies $(S \otimes S)\Delta = \Delta^{\text{cop}}S$, see [73].

Given a multiplier Hopf algebra we will again use the Sweedler notation

$$\Delta(f) = f_{(1)} \otimes f_{(2)}$$

for the comultiplication of H . This is useful for writing calculations formally, although in principle one always has to reduce everything to manipulations with Galois maps.

1.3. Integrals. In this section we discuss integrals for multiplier Hopf algebras. For a detailed treatment we refer to [74]. The case of ordinary Hopf algebras can be found in [69].

1.3.1. The definition of integrals. Let H be a regular multiplier Hopf algebra. For technical reasons we shall suppose throughout that H admits a faithful linear functional, see Definition 1.5. As noted before that definition, we may therefore view H as a subalgebra of the algebraic multiplier algebra $\mathcal{M}(H)$. An analogous statement applies to tensor powers of H .

Assume that ω is a linear functional on H . Then we define for any $f \in H$ a multiplier $(\text{id} \otimes \omega)\Delta(f) \in \mathcal{M}(H)$ by

$$\begin{aligned} (\text{id} \otimes \omega)\Delta(f) \cdot g &= (\text{id} \otimes \omega)\gamma_l(f \otimes g) \\ g \cdot (\text{id} \otimes \omega)\Delta(f) &= (\text{id} \otimes \omega)\rho_l(g \otimes f). \end{aligned}$$

To check that this is indeed a two-sided multiplier observe that

$$(f \otimes 1)\gamma_l(g \otimes h) = \rho_l(f \otimes g)(h \otimes 1)$$

for all $f, g, h \in H$. In a similar way we define $(\omega \otimes \text{id})\Delta(f) \in \mathcal{M}(H)$ by

$$\begin{aligned} (\omega \otimes \text{id})\Delta(f) \cdot g &= (\omega \otimes \text{id})\gamma_r(f \otimes g) \\ g \cdot (\omega \otimes \text{id})\Delta(f) &= (\omega \otimes \text{id})\rho_r(g \otimes f). \end{aligned}$$

Definition 1.8. Let H be a regular multiplier Hopf algebra. A linear functional $\phi : H \rightarrow \mathbb{K}$ is called a left invariant integral if

$$(\text{id} \otimes \phi)\Delta(f) = \phi(f)1$$

for all $f \in H$. Similarly, a linear functional $\psi : H \rightarrow \mathbb{K}$ is called a right invariant integral if

$$(\psi \otimes \text{id})\Delta(f) = \psi(f)1$$

for all $f \in H$.

Definition 1.9. A regular multiplier Hopf algebra with integrals is a regular multiplier Hopf algebra H together with a faithful left invariant functional $\phi : H \rightarrow \mathbb{K}$ and a faithful right invariant functional $\psi : H \rightarrow \mathbb{K}$.

Note that the existence of one of ϕ or ψ is sufficient. More precisely, one obtains a faithful right/left invariant functional by taking a faithful left/right invariant functional and precomposing with the antipode.

It can be shown that left/right invariant integrals are always unique up to a scalar, see Section 3 in [74]. Moreover, they are related by a modular element, that is, there exists an invertible multiplier $\delta \in M(H)$ such that

$$(\phi \otimes \text{id})\Delta(f) = \phi(f)\delta$$

for all $f \in H$. The multiplier δ is a group-like element in the sense that

$$\Delta(\delta) = \delta \otimes \delta, \quad \epsilon(\delta) = 1, \quad S(\delta) = \delta^{-1}.$$

It follows that a right invariant integral ψ is defined by

$$\psi(f) = \phi(f\delta) = \phi(\delta f).$$

We say that H is unimodular if $\delta = 1$, or equivalently if we can choose $\psi = \phi$.

For ordinary (unital) Hopf algebras, the existence of integrals is closely related to cosemisimplicity.

Proposition 1.10. *Let H be a Hopf algebra. Then the following conditions are equivalent.*

- a) H is cosemisimple.
- b) H admits a left and right invariant integral ϕ such that $\phi(1) = 1$.

Proof. a) \Rightarrow b) Assume that $H = \bigoplus_{\lambda \in \Lambda} C_\lambda$ is written as a direct sum of its simple coalgebras, and without loss of generality let us write $C_0 \subset H$ for the one-dimensional simple subcoalgebra spanned by $1 \in H$. Then we can define $\phi : H \rightarrow \mathbb{K}$ to be the projection onto $C_0 \cong \mathbb{K}$. Since $\Delta(C_\lambda) \subset C_\lambda \otimes C_\lambda$ it is straightforward to check that ϕ is a left and right invariant integral such that $\phi(1) = 1$.

b) \Rightarrow a). Let us only sketch the argument. Firstly, using the invariant integral ϕ one proves that every comodule for H is a direct sum of simple comodules. This is then shown to be equivalent to H being cosemisimple. We refer to chapter 14 of [69] for the details. \square

1.3.2. *The dual multiplier Hopf algebra.* Given a regular multiplier Hopf algebra with integrals we shall introduce the dual multiplier Hopf algebra and discuss the Biduality Theorem. These constructions and results are due to van Daele [74].

We define \hat{H} as the linear subspace of the dual space $H^* = \text{Hom}(H, \mathbb{K})$ given by all functionals of the form $\mathcal{F}(f)$ for $f \in H$ where

$$(\mathcal{F}(f), h) = \mathcal{F}(f)(h) = \phi(hf).$$

It can be shown that one obtains the same space of linear functions upon replacing ϕ by ψ , or reversing the order of multiplication under the integral in the above formula, see [74]. Note that these other choices correspond to reversing the multiplication or comultiplication of H , respectively.

Using the evaluation of linear functionals, we obtain the canonical pairing $\hat{H} \times H \rightarrow \mathbb{K}$ by

$$(x, f) = x(f)$$

for $x \in \hat{H}$ and $f \in H$. Note that the formula for $\mathcal{F}(f)$ above can be used to extend this to a pairing $\hat{H} \times \mathcal{M}(H) \rightarrow \mathbb{K}$. In a similar way one obtains pairings between tensor powers of H and \hat{H} .

Let us now explain how the space \hat{H} can be turned into a regular multiplier Hopf algebra with integrals. We point out, however, that we work with the *opposite comultiplication* on \hat{H} compared to the conventions in [74].

Theorem 1.11. *Let H be a regular multiplier Hopf algebra with integrals. The vector space \hat{H} becomes a regular multiplier Hopf algebra with comultiplication $\hat{\Delta} : \hat{H} \rightarrow \mathcal{M}(\hat{H} \otimes \hat{H})$, counit $\hat{\epsilon} : \hat{H} \rightarrow \mathbb{K}$, and antipode $\hat{S} : \hat{H} \rightarrow \hat{H}$, such that*

$$\begin{aligned} (xy, f) &= (x \otimes y, \Delta(f)) \\ (\hat{\Delta}(x), f \otimes g) &= (x, gf) \\ \hat{\epsilon}(x) &= (x, 1) \\ (\hat{S}(x), f) &= (x, S^{-1}(f)) \end{aligned}$$

for $x, y \in \hat{H}$ and $f, g \in H$. A left invariant integral for \hat{H} is given by

$$\hat{\phi}(\mathcal{F}(f)) = \epsilon(f)$$

for $f \in H$.

Proof. We shall only sketch two parts of the argument; a central part of the proof consists in showing that the above formulas are well-defined.

Firstly, for $f, g \in H$ the product $\mathcal{F}(f)\mathcal{F}(g)$ in \hat{H} is defined by

$$\mathcal{F}(f)\mathcal{F}(g) = \phi(S^{-1}(g_{(1)})f)\mathcal{F}(g_{(2)}) = (\mathcal{F} \otimes \phi)\gamma_l^{-1}(g \otimes f).$$

Then formally using

$$\begin{aligned} (\mathcal{F}(f)\mathcal{F}(g))(h) &= (\mathcal{F}(f) \otimes \mathcal{F}(g), \Delta(h)) \\ &= \phi(h_{(1)}f)\phi(h_{(2)}g) \\ &= \phi(h_{(1)}g_{(2)}S^{-1}(g_{(1)})f)\phi(h_{(2)}g_{(3)}) \\ &= \phi(S^{-1}(g_{(1)})f)\phi(hg_{(2)}) \\ &= \phi(S^{-1}(g_{(1)})f)\mathcal{F}(g_{(2)})(h) \end{aligned}$$

we see that this agrees with the transposition of the coproduct of H . From the fact that the product can be described in this way it follows easily that multiplication in \hat{H} is associative.

Secondly, assuming that $\hat{\Delta}$ is well-defined let us check that $\hat{\phi}$ is left invariant. For this we compute

$$((\text{id} \otimes \hat{\phi})\hat{\Delta}(\mathcal{F}(f)), h) = \hat{\phi}(\mathcal{F}(hf)) = \epsilon(f)\epsilon(h) = (1\hat{\phi}(\mathcal{F}(f)), h)$$

for any $h \in H$.

For the construction of the remaining structure maps and the verification of the axioms see [74]. \square

We shall again use the Sweedler notation

$$\hat{\Delta}(x) = x_{(1)} \otimes x_{(2)}$$

for the comultiplication of \hat{H} . The compatibility between H and \hat{H} can be summarized as follows.

Proposition 1.12. *Let H be a regular multiplier Hopf algebra with integrals. Then the canonical evaluation pairing between \hat{H} and H satisfies*

$$(xy, f) = (x, f_{(1)})(y, f_{(2)}), \quad (x, fg) = (x_{(2)}, f)(x_{(1)}, g)$$

and

$$(\hat{S}(x), f) = (x, S^{-1}(f))$$

for all $x, y \in \hat{H}$ and $f, g \in H$.

Note that the final relation implies

$$(\hat{S}^{-1}(x), f) = (x, S(f))$$

for all $x \in \hat{H}$ and $f \in H$.

The next major result is the Biduality Theorem, which states that $\hat{\hat{H}} \cong H$ as multiplier Hopf algebras. It is important to note, however, that under this isomorphism the canonical pairing $\hat{\hat{H}} \times \hat{H} \rightarrow \mathbb{K}$ will not correspond to evaluation between H and \hat{H} . This is due to our choice to work with the coopposite comultiplication in the definition of the dual. We therefore introduce the following convention, which will apply throughout this work.

Definition 1.13. If H is a regular multiplier Hopf algebra, we define the reverse pairing $H \times \hat{H} \rightarrow \mathbb{K}$ by

$$(f, x) = (x, S^{-1}(f)),$$

for all $f \in H, x \in \hat{H}$.

Let us stress that with these conventions, we have $(f, x) \neq (x, f)$ in general.

Since \hat{H} is again a regular multiplier Hopf algebra with integrals, we can construct its dual. As in the case of finite dimensional Hopf algebras one has a duality result, see [74].

Theorem 1.14 (Biduality Theorem). *Let H be a regular multiplier Hopf algebra with integrals. Then the dual of \hat{H} is isomorphic to H . The isomorphism is given by*

$$B : H \rightarrow \hat{\hat{H}}; \quad B(f)(x) = (f, x),$$

for $f \in H, x \in \hat{H}$.

Proof. Again we shall only give a brief sketch of the argument. The key step in the proof—which we will not carry out—is to show that B is well-defined. Injectivity then follows from the nondegeneracy of the pairing $(\ , \)$. The fact that B is compatible with multiplication and comultiplication is a consequence of Proposition 1.12. \square

Let us conclude this section by writing down a formula for the inverse of the Fourier transform \mathcal{F} .

Proposition 1.15. *Let H be a regular multiplier Hopf algebra. For any $x, y \in \hat{H}$ we have*

$$(y, \mathcal{F}^{-1}(x)) = \hat{\phi}(\hat{S}^{-1}(y)x).$$

Proof. Let us first observe that for any $x \in \hat{H}, f \in H$ we have the formula

$$x\mathcal{F}(f) = (x, S^{-1}(f_{(1)}))\mathcal{F}(f_{(2)}),$$

which can be confirmed by pairing each side with some $g \in H$ and using the left invariance of ϕ . Then, using the proposed formula for \mathcal{F}^{-1} from the proposition, we compute

$$(y, \mathcal{F}^{-1}\mathcal{F}(f)) = \hat{\phi}(\hat{S}^{-1}(y)\mathcal{F}(f)) = \hat{\phi}(\mathcal{F}(f_{(2)}))(y, f_{(1)}) = (y, f),$$

for all $f \in H, y \in \hat{H}$. Since \mathcal{F} is surjective by definition, we see that \mathcal{F} is invertible with inverse \mathcal{F}^{-1} as proposed. \square

If, as above, we identify the dual of \hat{H} with H via the reverse pairing $H \times \hat{H} \rightarrow \mathbb{K}$, then we can reinterpret the relation $\mathcal{F}^{-1}\mathcal{F} = \text{id}$ by saying that $\hat{\mathcal{F}} = \mathcal{F}^{-1}$. Indeed, notice that

$$(\hat{\mathcal{F}}(x), y) = \hat{\phi}(yx) = (\hat{S}(y), \mathcal{F}^{-1}(x)) = (\mathcal{F}^{-1}(x), y)$$

for all $x, y \in \hat{H}$.

This simple relation between \mathcal{F} and $\hat{\mathcal{F}}$ is a feature of our conventions to work with the coopposite comultiplication on the dual.

1.4. The Drinfeld double - algebraic level. In this section we discuss the algebraic version of the Drinfeld double construction. This construction produces a regular multiplier Hopf algebra $L \bowtie K$ out of two regular multiplier Hopf algebras K, L equipped with an invertible skew-pairing. For ordinary Hopf algebras—that is, in the situation that K and L are unital—this is covered in all standard textbooks, see for instance [48]. For a more detailed account in the non-unital setting we refer to [23].

This construction will be crucial for the later chapters, since the quantized convolution algebra of a complex semisimple group is defined as the Drinfeld double of the quantized convolution algebra of the associated compact form, see Section 3.4 for a precise statement.

Let us first discuss skew-pairings in the unital case. If K and L are Hopf algebras then a skew-pairing between L and K is simply a bilinear map $\tau : L \times K \rightarrow \mathbb{K}$ satisfying

$$\begin{aligned}\tau(xy, f) &= \tau(x, f_{(1)})\tau(y, f_{(2)}) \\ \tau(x, fg) &= \tau(x_{(1)}, g)\tau(x_{(2)}, f) \\ \tau(1, f) &= \epsilon_K(f) \\ \tau(x, 1) &= \epsilon_L(x)\end{aligned}$$

for all $f, g \in K$ and $x, y \in L$.

In the non-unital case this needs to be phrased more carefully. Assume that K, L are regular multiplier Hopf algebras and let $\tau : L \times K \rightarrow \mathbb{K}$ be a bilinear map. For $x \in L$ and $f \in K$ we define linear maps $\tau_x : K \rightarrow \mathbb{K}$ and ${}_f\tau : L \rightarrow \mathbb{K}$ by

$$\tau_x(g) = \tau(x, g), \quad {}_f\tau(y) = \tau(y, f).$$

We can then define multipliers $\tau(x_{(1)}, f)x_{(2)} = ({}_f\tau \otimes \text{id})\Delta_L(x)$, $\tau(x_{(2)}, f)x_{(1)} = (\text{id} \otimes {}_f\tau)\Delta_L(x) \in \mathcal{M}(L)$ by first multiplying the outer leg of the coproduct of x with elements of L , and then applying ${}_f\tau$ to the first leg. In the same way one obtains multipliers $\tau(x, f_{(1)})f_{(2)} = (\tau_x \otimes \text{id})\Delta_K(f)$, $\tau(x, f_{(2)})f_{(1)} = (\text{id} \otimes \tau_x)\Delta_K(f) \in \mathcal{M}(K)$.

We will say that the pairing τ is regular if the following two conditions are satisfied. Firstly, we require that all the multipliers defined above are in fact contained in L and K , respectively, and not just in $\mathcal{M}(L)$ and $\mathcal{M}(K)$. Secondly, we require that the linear span of all $\tau(x_{(1)}, f)x_{(2)}$ is equal to L , and the linear span of all $\tau(x, f_{(1)})f_{(2)}$ is equal to K , as well as the analogous conditions for the flipped comultiplications. It is shown in [23] that this implies that the multiplier

$$\tau(x_{(1)}, f_{(1)})x_{(2)} \otimes f_{(2)}$$

of $\mathcal{M}(L \otimes K)$ is in fact contained in $L \otimes K$. The same holds for the multipliers obtained by flipping the comultiplications in this formula in one or both factors.

Definition 1.16. Let K, L be regular multiplier Hopf algebras. A skew-pairing between L and K is a regular bilinear map $\tau : L \times K \rightarrow \mathbb{K}$ which satisfies

- a) $\tau_{xy}(f) = \tau_x(\text{id} \otimes \tau_y)\Delta_K(f) = \tau_y(\tau_x \otimes \text{id})\Delta_K(f)$
- b) ${}_f{}_g\tau(x) = {}_g\tau(\text{id} \otimes {}_f\tau)\Delta_L(x) = {}_f\tau({}_g\tau \otimes \text{id})\Delta_L(x)$

for all $f, g \in K$ and $x, y \in L$.

Occasionally we will identify τ with its associated linear map $L \otimes K \rightarrow \mathbb{K}$ and write $\tau(x \otimes f)$ instead of $\tau(x, f)$.

Note that the conditions in Definition 1.16 are well-defined by the regularity assumption on τ . Informally, these conditions can be written as

$$\tau(xy, f) = \tau(x, f_{(1)})\tau(y, f_{(2)}), \quad \tau(x, fg) = \tau(x_{(1)}, g)\tau(x_{(2)}, f)$$

for $f, g \in K, x, y \in L$, as in the unital case explained above.

Regularity implies that the skew-pairing τ induces essential module structures of K acting on L and vice versa. More precisely, K becomes an essential left L -module and an essential right L -module by

$$x \triangleright f = \tau(x, f_{(2)})f_{(1)}, \quad f \triangleleft x = \tau(x, f_{(1)})f_{(2)}$$

and L becomes an essential left and right K -module by

$$f \triangleright x = \tau(x_{(1)}, f)x_{(2)}, \quad x \triangleleft f = \tau(x_{(2)}, f)x_{(1)}.$$

This means in particular that τ extends canonically to bilinear pairings $\mathcal{M}(L) \times K \rightarrow \mathbb{K}$ and $L \times \mathcal{M}(K) \rightarrow \mathbb{K}$ by setting

$$\tau(x, f) = \epsilon_K(x \triangleright f) = \epsilon_K(f \triangleleft x), \quad \tau(y, g) = \epsilon_L(g \triangleright y) = \epsilon_L(y \triangleleft g)$$

for $x \in \mathcal{M}(L), f \in K$ and $y \in L, g \in \mathcal{M}(K)$, respectively. With this notation, we get in particular

$$\tau(x, 1) = \epsilon_L(x), \quad \tau(1, f) = \epsilon_K(f)$$

for all $x \in L, f \in K$, again reproducing the formulas in the unital case.

The convolution product $\tau\eta$ of two regular pairings $\tau, \eta : L \otimes K \rightarrow \mathbb{K}$ is defined by

$$(\tau\eta)(x \otimes f) = \eta(x_{(1)} \otimes f_{(1)})\tau(x_{(2)} \otimes f_{(2)}).$$

This defines an associative multiplication on the linear space of all regular pairings $L \otimes K \rightarrow \mathbb{K}$, with unit element $\epsilon_L \otimes \epsilon_K$. Any skew-pairing $\tau : L \otimes K \rightarrow \mathbb{K}$ between regular multiplier Hopf algebras is convolution invertible, with inverse given by

$$\tau^{-1}(x, f) = \tau(S_L(x), f) = \tau(x, S_K^{-1}(f)).$$

Let us now introduce the Drinfeld double of skew-paired regular multiplier Hopf algebras.

Definition 1.17. Let K, L be regular multiplier Hopf algebras and let $\tau : L \otimes K \rightarrow \mathbb{K}$ be a skew-pairing between them. The Drinfeld double $L \bowtie K$ is the regular multiplier Hopf algebra with underlying vector space $L \otimes K$, equipped with the multiplication

$$(x \bowtie f)(y \bowtie g) = x\tau(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau^{-1}(y_{(3)}, f_{(3)})g,$$

the coproduct

$$\Delta_{L \bowtie K}(x \bowtie f) = (x_{(1)} \bowtie f_{(1)}) \otimes (x_{(2)} \bowtie f_{(2)}),$$

antipode

$$S_{L \bowtie K}(x \bowtie f) = \tau^{-1}(x_{(1)}, f_{(1)})S_L(x_{(2)}) \bowtie S_K(f_{(2)})\tau(x_{(3)}, f_{(3)}),$$

and the counit

$$\epsilon_{L \bowtie K}(x \bowtie f) = \epsilon_L(x)\epsilon_K(f).$$

It is proved in [23] that these structures indeed turn $L \bowtie K$ into a regular multiplier Hopf algebra such that $\mathcal{M}(L \bowtie K)$ contains both L and K as multiplier Hopf subalgebras in a natural way. Note that the formula for the antipode is forced by requiring that it be compatible with the antipodes S_L and S_K on the subalgebras L and K in $\mathcal{M}(L \bowtie K)$, thanks to the formula

$$S_{L \bowtie K}(x \bowtie f) = S_{L \bowtie K}((x \bowtie 1)(1 \bowtie f)) = (1 \bowtie S_K(f))(S_L(x) \bowtie 1),$$

for $x \in L, f \in K$.

Formally, associativity of the multiplication in $L \bowtie K$ follows from

$$\begin{aligned}
((x \bowtie f)(y \bowtie g))(z \bowtie h) &= (x\tau(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau^{-1}(y_{(3)}, f_{(3)})g)(z \bowtie h) \\
&= x\tau(y_{(1)}, f_{(1)})y_{(2)}\tau(z_{(1)}, f_{(2)}g_{(1)})z_{(2)} \bowtie f_{(3)}g_{(2)}\tau^{-1}(z_{(3)}, f_{(4)}g_{(3)})\tau^{-1}(y_{(3)}, f_{(5)})h \\
&= x\tau(y_{(1)}z_{(2)}, f_{(1)})\tau(z_{(1)}, g_{(1)})y_{(2)}z_{(3)} \bowtie f_{(2)}\tau^{-1}(y_{(3)}z_{(4)}, f_{(3)})g_{(2)}\tau^{-1}(z_{(5)}, g_{(3)})h \\
&= (x \bowtie f)(y\tau(z_{(1)}, g_{(1)})z_{(2)} \bowtie g_{(2)}\tau^{-1}(z_{(3)}, g_{(3)})h) \\
&= (x \bowtie f)((y \bowtie g)(z \bowtie h))
\end{aligned}$$

for $x \bowtie f, y \bowtie g, z \bowtie h \in L \bowtie K$. To check that $\Delta_{L \bowtie K}$ is an algebra homomorphism one may formally compute

$$\begin{aligned}
\Delta_{L \bowtie K}((x \bowtie f)(y \bowtie g)) &= \Delta_{L \bowtie K}(x\tau(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau^{-1}(y_{(3)}, f_{(3)})g) \\
&= x_{(1)}\tau(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}g_{(1)} \otimes x_{(2)}y_{(3)} \bowtie \tau^{-1}(y_{(4)}, f_{(4)})f_{(3)}g_{(2)} \\
&= (x_{(1)}\tau(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau^{-1}(y_{(3)}, f_{(3)})g_{(1)}) \\
&\quad \otimes (x_{(2)}\tau(y_{(4)}, f_{(4)})y_{(5)} \bowtie f_{(5)}\tau^{-1}(y_{(6)}, f_{(6)})g_{(2)}) \\
&= (x_{(1)} \bowtie f_{(1)} \otimes x_{(2)} \bowtie f_{(2)})(y_{(1)} \bowtie g_{(1)} \otimes y_{(2)} \bowtie g_{(2)}) \\
&= \Delta_{L \bowtie K}(x \bowtie f)\Delta_{L \bowtie K}(y \bowtie g)
\end{aligned}$$

for $x \bowtie f, y \bowtie g \in L \bowtie K$. It is clear from the definition that $\Delta_{L \bowtie K}$ is coassociative and that $\epsilon_{L \bowtie K}$ is a counit for $\Delta_{L \bowtie K}$. For the antipode axiom we compute

$$\begin{aligned}
m_{L \bowtie K}(S_{L \bowtie K} \otimes \text{id})\Delta_{L \bowtie K}(x \bowtie f) &= m_{L \bowtie K}(S_{L \bowtie K} \otimes \text{id})(x_{(1)} \bowtie f_{(1)} \otimes x_{(2)} \bowtie f_{(2)}) \\
&= m_{L \bowtie K}(\tau^{-1}(x_{(1)}, f_{(1)})S_L(x_{(2)}) \bowtie S_K(f_{(2)})\tau(x_{(3)}, f_{(3)}) \otimes x_{(4)} \bowtie f_{(4)}) \\
&= \tau^{-1}(x_{(1)}, f_{(1)})S_L(x_{(2)})\tau(x_{(4)}, S_K(f_{(4)}))x_{(5)} \\
&\quad \bowtie S_K(f_{(3)}) \times \tau^{-1}(x_{(6)}, S_K(f_{(2)}))\tau(x_{(3)}, f_{(5)})f_{(6)} \\
&= \tau^{-1}(x_{(1)}, f_{(1)})S_L(x_{(2)})x_{(3)} \bowtie \tau^{-1}(x_{(4)}, S_K(f_{(2)}))S_K(f_{(3)})f_{(4)} \\
&= \tau^{-1}(x_{(1)}, f_{(1)})1 \bowtie \tau^{-1}(x_{(2)}, S_K(f_{(2)}))1 \\
&= \epsilon_{L \bowtie K}(x \bowtie f)1,
\end{aligned}$$

where $m_{L \bowtie K}$ denotes the multiplication of $L \bowtie K$. In a similar way one verifies the other antipode condition.

Let K, L be regular multiplier Hopf algebras and let $\rho : L \otimes K \rightarrow \mathbb{K}$ be an invertible skew-pairing. Then $\tau : K \otimes L \rightarrow \mathbb{K}$ given by

$$\tau(f, x) = \rho(x, S_K^{-1}(f)) = \rho(S_L(x), f)$$

is also an invertible skew-pairing.

Definition 1.18. With the above notation, the Rosso form is the bilinear form κ on the Drinfeld double $L \bowtie K$ defined by

$$\kappa(x \bowtie f, y \bowtie g) = \rho(y, f)\rho(S_L(x), S_K^{-1}(g)) = \tau(S_K(f), y)\tau(g, S_L(x)).$$

The left adjoint action of a regular multiplier Hopf algebra H on itself is defined by

$$\text{ad}(f)(g) = f \rightarrow g = f_{(1)}gS(f_{(2)}).$$

A bilinear form κ on H is called ad-invariant if

$$\kappa(f \rightarrow g, h) = \kappa(g, S(f) \rightarrow h)$$

for all $f, g, h \in H$.

For the following result compare Section 8.2.3 in [48].

Proposition 1.19. *Let K, L be regular multiplier Hopf algebras equipped with an invertible skew-pairing $\rho : L \times K \rightarrow \mathbb{K}$. Then the Rosso form on the Drinfeld double $L \bowtie K$ is ad-invariant.*

Proof. Since $L \bowtie K$ is generated as an algebra by L and K it suffices to consider the adjoint action of $x \in L$ and $f \in K$ on an element $y \bowtie g$ of the double. We compute

$$\begin{aligned} \text{ad}(x)(y \bowtie g) &= (x_{(1)} \bowtie 1)(y \bowtie g)(S_L(x_{(2)}) \bowtie 1) \\ &= x_{(1)}y S_L(x_{(3)})\rho(S_L(x_{(4)}), g_{(1)}) \bowtie \rho^{-1}(S_L(x_{(2)}), g_{(3)})g_{(2)}. \end{aligned}$$

Noting that $\rho(S_L(x), S_K(f)) = \rho(x, f)$ we obtain

$$\begin{aligned} \kappa(x \rightarrow (y \bowtie g), z \bowtie h) &= \kappa(x_{(1)}y S_L(x_{(3)})\rho^{-1}(x_{(4)}, g_{(1)}) \bowtie \rho^{-1}(S_L(x_{(2)}), g_{(3)})g_{(2)}, z \bowtie h) \\ &= \rho^{-1}(S_L(x_{(1)}y S_L(x_{(3)}), h)\rho^{-1}(x_{(4)}, g_{(1)})\rho^{-1}(S_L(x_{(2)}), g_{(3)})\rho(z, g_{(2)})) \\ &= \rho^{-1}(S_L(x_{(1)}), h_{(1)})\rho^{-1}(S_L(y), h_{(2)})\rho^{-1}(S_L^2(x_{(3)}), h_{(3)}) \\ &\quad \times \rho^{-1}(x_{(4)}, g_{(1)})\rho^{-1}(S_L(x_{(2)}), g_{(3)})\rho(z, g_{(2)}) \\ &= \rho^{-1}(S_L(x_{(1)}), h_{(1)})\rho^{-1}(S_L(y), h_{(2)})\rho^{-1}(S_L^2(x_{(3)}), h_{(3)})\rho(S_L(x_{(4)})z S_L^2(x_{(2)}), g) \\ &= \kappa(y \bowtie g, S_L(x_{(4)})z S_L^2(x_{(2)})\rho^{-1}(S_L(x_{(1)}), h_{(1)}) \bowtie \rho^{-1}(S_L^2(x_{(3)}), h_{(3)})h_{(2)}) \\ &= \kappa(y \bowtie g, S_L(x) \rightarrow (z \bowtie h)). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{ad}(f)(y \bowtie g) &= (1 \bowtie f_{(1)})(y \bowtie g)(1 \bowtie S_K(f_{(2)})) \\ &= \rho(y_{(1)}, f_{(1)})y_{(2)} \bowtie \rho^{-1}(y_{(3)}, f_{(3)})f_{(2)}g S_K(f_{(4)}) \end{aligned}$$

and so

$$\begin{aligned} \kappa(f \rightarrow (y \bowtie g), z \bowtie h) &= \kappa(\rho(y_{(1)}, f_{(1)})y_{(2)} \bowtie \rho^{-1}(y_{(3)}, f_{(3)})f_{(2)}g S_K(f_{(4)}), z \bowtie h) \\ &= \rho(y_{(1)}, f_{(1)})\rho^{-1}(S_L(y_{(2)}), h)\rho^{-1}(y_{(3)}, f_{(3)})\rho(z, f_{(2)}g S_K(f_{(4)})) \\ &= \rho(y_{(1)}, f_{(1)})\rho(y_{(2)}, S_K^{-2}(h))\rho(y_{(3)}, S_K^{-1}(f_{(3)}))\rho(z_{(1)}, S_K(f_{(4)}))\rho(z_{(2)}, g)\rho(z_{(3)}, f_{(2)}) \\ &= \rho(y, S_K^{-1}(f_{(3)})S_K^{-2}(h)f_{(1)})\rho(z_{(1)}, S_K(f_{(4)}))\rho(z_{(2)}, g)\rho^{-1}(z_{(3)}, S_K(f_{(2)})) \\ &= \kappa(y \bowtie g, \rho(z_{(1)}, S_K(f_{(4)}))z_{(2)} \bowtie \rho^{-1}(z_{(3)}, S_K(f_{(2)}))S_K(f_{(3)})h S_K^2(f_{(1)})) \\ &= \kappa(y \bowtie g, S_K(f) \rightarrow (z \bowtie h)). \end{aligned}$$

This yields the claim. \square

2. QUANTIZED UNIVERSAL ENVELOPING ALGEBRAS

In this chapter we collect background material on quantized universal enveloping algebras. We give in particular a detailed account of the construction of the braid group action and PBW-bases, and discuss the finite dimensional representation theory in the setting that the base field \mathbb{K} is an arbitrary field and the deformation parameter $q \in \mathbb{K}^\times$ is not a root of unity. Our presentation mainly follows the textbooks [39], [41] and [57].

2.1. q -calculus. Let \mathbb{K} be a field. We will assume $q \in \mathbb{K}^\times$ is not a root of unity.

For $n \in \mathbb{Z}$ we write

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{-n+1} + q^{-n+3} + \dots + q^{n-1}$$

for the corresponding q -number. For $n \in \mathbb{N}$ we set

$$[n]_q! = \prod_{k=1}^n [k]_q,$$

and in addition we define $[0]_q! = 1$.

The q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q [n-1]_q \cdots [n-m+1]_q}{[m]_q!}$$

for $n \in \mathbb{Z}$ and $m \in \mathbb{N}$. In addition we declare $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ for all $n \in \mathbb{Z}$ and $\begin{bmatrix} n \\ m \end{bmatrix}_q = 0$ for $m < 0$. Note that if $n \in \mathbb{N}_0$ we have

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n-m]_q! [m]_q!}$$

for $0 \leq m \leq n$. We shall often omit the subscripts q if no confusion is likely.

Lemma 2.1. *We have*

$$q^m \begin{bmatrix} n \\ m \end{bmatrix} + q^{m-n-1} \begin{bmatrix} n \\ m-1 \end{bmatrix} = \begin{bmatrix} n+1 \\ m \end{bmatrix}$$

and

$$q^{-m} \begin{bmatrix} n \\ m \end{bmatrix} + q^{n+1-m} \begin{bmatrix} n \\ m-1 \end{bmatrix} = \begin{bmatrix} n+1 \\ m \end{bmatrix}$$

for any $n \in \mathbb{Z}$ and $m \in \mathbb{N}$.

Proof. We calculate

$$\begin{aligned} & \left(q^m \begin{bmatrix} n \\ m \end{bmatrix} + q^{m-n-1} \begin{bmatrix} n \\ m-1 \end{bmatrix} \right) \times [m] \\ &= q^m [n] \dots [n-m+2] [n-m+1] + q^{m-n-1} [n] \dots [n-m+2] [m] \\ &= (q^m [n-m+1] + q^{m-n-1} [m]) [n] \dots [n-m+2] \\ &= \left(\frac{q^{n+1} - q^{2m-n-1} + q^{2m-n-1} - q^{-n-1}}{q - q^{-1}} \right) [n] \dots [n-m+2] \\ &= [n+1] [n] \dots [n-m+2] \\ &= \begin{bmatrix} n+1 \\ m \end{bmatrix} \times [m]. \end{aligned}$$

This yields the first claim. The second follows by replacing q by q^{-1} . \square

Lemma 2.2. For any $n \in \mathbb{N}_0$ we have

$$\sum_{m=0}^n (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} q^{m(n-1)} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0 \end{cases}$$

and

$$\sum_{m=0}^n (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} q^{-m(n-1)} = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases}$$

Proof. For $n = 0$ or 1 the claim can be checked directly. For $n \geq 2$ we compute, using Lemma 2.1,

$$\begin{aligned} & \sum_{m=0}^n (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} q^{m(n-1)} \\ &= \sum_{m=0}^{n-1} (-1)^m q^m \begin{bmatrix} n-1 \\ m \end{bmatrix} q^{nm-m} + \sum_{m=1}^n (-1)^m q^{m-n} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} q^{nm-m} \\ &= \sum_{m=0}^{n-1} (-1)^m q^{nm} \begin{bmatrix} n-1 \\ m \end{bmatrix} + \sum_{m=1}^n (-1)^m q^{n(m-1)} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} = 0. \end{aligned}$$

For $n = 1$ the claim can be checked directly. The second equation follows by replacing q by q^{-1} . \square

Consider the special case $\mathbb{K} = \mathbb{Q}(q)$. It is clear by definition that $[n] \in \mathbb{Z}[q, q^{-1}] \subset \mathbb{Q}(q)$. Moreover, from Lemma 2.1 one obtains by induction that all q -binomial coefficients are contained in $\mathbb{Z}[q, q^{-1}]$. That is, the above formulas, suitably interpreted, are valid in $\mathbb{Z}[q, q^{-1}]$. With this in mind, one can dispense of our initial assumption that q is not a root of unity.

2.2. The definition of $U_q(\mathfrak{g})$. In this section we explain the construction of the quantized universal enveloping algebra of a semisimple Lie algebra.

2.2.1. Semisimple Lie algebras. Let \mathfrak{g} a semisimple Lie algebra over \mathbb{C} of rank N . We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a set $\Sigma = \{\alpha_1, \dots, \alpha_N\}$ of simple roots. We write $(\ , \)$ for the bilinear form on \mathfrak{h}^* obtained by rescaling the Killing form such that the shortest root α of \mathfrak{g} satisfies $(\alpha, \alpha) = 2$. Moreover we set

$$d_i = (\alpha_i, \alpha_i)/2$$

for all $i = 1, \dots, N$ and let

$$\alpha_i^\vee = d_i^{-1} \alpha_i$$

be the simple coroot corresponding to α_i .

Denote by $\varpi_1, \dots, \varpi_N$ the fundamental weights of \mathfrak{g} , satisfying the relations $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$. We write

$$\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z} \varpi_j, \quad \mathbf{Q} = \bigoplus_{j=1}^N \mathbb{Z} \alpha_j, \quad \mathbf{Q}^\vee = \bigoplus_{j=1}^N \mathbb{Z} \alpha_j^\vee,$$

for the weight, root and coroot lattices of \mathfrak{g} , respectively. Note that $\mathbf{Q} \subset \mathbf{P} \subset \mathfrak{h}^*$ and that \mathbf{Q}^\vee identifies with the \mathbb{Z} -dual of \mathbf{P} under the pairing.

The set \mathbf{P}^+ of dominant integral weights is the set of all non-negative integer combinations of the fundamental weights. We also write \mathbf{Q}^+ for the non-negative integer combinations of the simple roots.

The Cartan matrix for \mathfrak{g} is the matrix $(a_{ij})_{1 \leq i, j \leq N}$ with coefficients

$$a_{ij} = (\alpha_i^\vee, \alpha_j) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}.$$

We write W for the Weyl group, that is, the finite group of automorphisms of \mathbf{P} generated by the reflections s_i in the hyperplanes orthogonal to the simple roots $\alpha_i \in \mathbf{\Delta}^+$, namely

$$s_i(\mu) = \mu - (\alpha_i^\vee, \mu)\alpha_i$$

for all $\mu \in \mathbf{P}$.

Throughout we fix the smallest positive integer L such that the numbers (ϖ_i, ϖ_j) take values in $\frac{1}{L}\mathbb{Z}$ for all $1 \leq i, j \leq N$, see Section 1 in [68] for the explicit values of L in all types and more information.

In the sequel we shall also work over more general ground fields \mathbb{K} . We will always keep the notions of weights and roots from the classical case as above. Remark that associated to the complex semisimple Lie algebra \mathfrak{g} we can construct a Lie algebra over \mathbb{K} using the Serre presentation of \mathfrak{g} . We will sometimes work implicitly with this Lie algebra and, by slight abuse of notation, denote it by \mathfrak{g} again. This apparent ambiguity is resolved by observing that the starting point for all constructions in the sequel is in fact a finite Cartan matrix, rather than a semisimple Lie algebra.

2.2.2. The quantized universal enveloping algebra without Serre relations. For practical purposes it is convenient to construct the quantized universal enveloping algebra in two steps, mimicking the standard approach to defining Kac-Moody algebras. In this subsection we present the first step, namely the definition of a Hopf algebra which will admit the quantized universal enveloping algebra as a quotient.

Let \mathfrak{g} be a semisimple Lie algebra. We keep the notation introduced above.

Definition 2.3. Let \mathbb{K} be a field and $q = s^L \in \mathbb{K}^\times$ be an invertible element such that $q_i \neq \pm 1$ for all $1 \leq i \leq N$ where $q_i = q^{d_i}$. The algebra $\tilde{U}_q(\mathfrak{g})$ over \mathbb{K} has generators K_λ for $\lambda \in \mathbf{P}$, and E_i, F_i for $i = 1, \dots, N$, and the defining relations

$$\begin{aligned} K_0 &= 1 \\ K_\lambda K_\mu &= K_{\lambda+\mu} \\ K_\lambda E_j K_\lambda^{-1} &= q^{(\lambda, \alpha_j)} E_j \\ K_\lambda F_j K_\lambda^{-1} &= q^{-(\lambda, \alpha_j)} F_j \\ [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \end{aligned}$$

for all $\lambda, \mu \in \mathbf{P}$ and all $1 \leq i, j \leq N$. Here we abbreviate $K_i = K_{\alpha_i}$ for all simple roots.

Our hypothesis on q mean that we always require $q^2 \neq 1$, and in fact $q^4 \neq 1$ or $q^6 \neq 1$ in the case that \mathfrak{g} contains a component of type B_n, C_n, F_4 or G_2 , respectively.

Notice that $q^{(\alpha_i, \alpha_j)} = q_i^{\alpha_{ij}}$. Therefore we have

$$K_i E_j K_i^{-1} = q_i^{\alpha_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-\alpha_{ij}} F_j$$

for all $1 \leq i, j \leq N$. In particular

$$K_i E_i K_i^{-1} = q_i^2 E_i, \quad K_i F_i K_i^{-1} = q_i^{-2} F_i.$$

The algebra $\tilde{U}_q(\mathfrak{g})$ admits a Hopf algebra structure as follows.

Lemma 2.4. *The algebra $\tilde{U}_q(\mathfrak{g})$ is a Hopf algebra with comultiplication $\hat{\Delta} : \tilde{U}_q(\mathfrak{g}) \rightarrow \tilde{U}_q(\mathfrak{g}) \otimes \tilde{U}_q(\mathfrak{g})$ given by*

$$\begin{aligned} \hat{\Delta}(K_\mu) &= K_\mu \otimes K_\mu, \\ \hat{\Delta}(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\ \hat{\Delta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \end{aligned}$$

count $\hat{\epsilon} : \tilde{U}_q(\mathfrak{g}) \rightarrow \mathbb{K}$ given by

$$\hat{\epsilon}(K_\lambda) = 1, \quad \hat{\epsilon}(E_j) = 0, \quad \hat{\epsilon}(F_j) = 0,$$

and antipode $\hat{S} : \tilde{U}_q(\mathfrak{g}) \rightarrow \tilde{U}_q(\mathfrak{g})$ given by

$$\hat{S}(K_\lambda) = K_{-\lambda}, \quad \hat{S}(E_j) = -E_j K_j^{-1}, \quad \hat{S}(F_j) = -K_j F_j.$$

Proof. These are straightforward calculations. In order to verify that $\hat{\Delta}$ extends to a homomorphism as stated we compute, for instance,

$$\begin{aligned} [\hat{\Delta}(E_i), \hat{\Delta}(F_j)] &= [E_i \otimes K_i + 1 \otimes E_i, F_j \otimes 1 + K_j^{-1} \otimes F_j] \\ &= K_j^{-1} \otimes [E_i, F_j] + [E_i, F_j] \otimes K_i \\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}} \hat{\Delta}(K_i - K_i^{-1}). \end{aligned}$$

To see that \hat{S} defines an antihomomorphism we check for instance

$$\begin{aligned} \hat{S}([E_i, F_j]) &= \frac{\delta_{ij}}{q_i - q_i^{-1}} \hat{S}(K_i - K_i^{-1}) \\ &= \frac{\delta_{ij}}{q_i - q_i^{-1}} (K_i^{-1} - K_i) \\ &= K_j F_j E_i K_i^{-1} - K_j E_i F_j K_i^{-1} \\ &= K_j F_j E_i K_i^{-1} - E_i K_i^{-1} K_j F_j \\ &= [-K_j F_j, -E_i K_i^{-1}] = [\hat{S}(F_j), \hat{S}(E_i)]. \end{aligned}$$

The Hopf algebra axioms are easily verified on generators. \square

Throughout we will use the Sweedler notation

$$\hat{\Delta}(X) = X_{(1)} \otimes X_{(2)}$$

for the coproduct of $X \in \tilde{U}_q(\mathfrak{g})$.

The following fact is frequently useful.

Lemma 2.5. *There exists an algebra automorphism $\omega : \tilde{U}_q(\mathfrak{g}) \rightarrow \tilde{U}_q(\mathfrak{g})$ given by*

$$\omega(K_\lambda) = K_{-\lambda}, \quad \omega(E_j) = F_j, \quad \omega(F_j) = E_j$$

on generators. Moreover ω is a coalgebra antihomomorphism.

Proof. It is straightforward to verify that the defining relations of $\tilde{U}_q(\mathfrak{g})$ are preserved by the above assignment. For the coalgebra antihomomorphism property notice that

$$\sigma \hat{\Delta}(\omega(E_j)) = F_j \otimes K_j^{-1} + 1 \otimes F_j = (\omega \otimes \omega)(\hat{\Delta}(E_j))$$

for all $j = 1, \dots, N$. Here σ denotes the flip map. \square

Note that, as a consequence of Lemma 2.5, we have $\omega \circ \hat{S} = \hat{S}^{-1} \circ \omega$.

Let us record the following explicit formulas for certain coproducts in $\tilde{U}_q(\mathfrak{g})$.

Lemma 2.6. *We have*

$$\begin{aligned} \hat{\Delta}(E_j^r) &= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r-i} K_j^i \\ \hat{\Delta}(F_j^r) &= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} F_j^{r-i} K_j^{-i} \otimes F_j^i \end{aligned}$$

for all $r \in \mathbb{N}$ and $j = 1, \dots, N$.

Proof. We use induction on r . The claim for $\hat{\Delta}(E_j^1) = \hat{\Delta}(E_j)$ is clear. Assume the formula for $\hat{\Delta}(E_j^r)$ is proved for $r \in \mathbb{N}$. Using Lemma 2.1, we compute

$$\begin{aligned}
\hat{\Delta}(E_j^{r+1}) &= \hat{\Delta}(E_j)\hat{\Delta}(E_j^r) \\
&= (E_j \otimes K_j + 1 \otimes E_j) \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r-i} K_j^i \\
&= \sum_{i=0}^r \left(q_j^{2(r-i)} q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^{i+1} \otimes E_j^{r-i} K_j^{i+1} + q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r+1-i} K_j^i \right) \\
&= \sum_{i=1}^{r+1} q_j^{2(r+1-i)} q_j^{(i-1)(r+1-i)} \begin{bmatrix} r \\ i-1 \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r+1-i} K_j^i \\
&\quad + \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r+1-i} K_j^i \\
&= \sum_{i=0}^{r+1} q_j^{i(r+1-i)} \begin{bmatrix} r+1 \\ i \end{bmatrix}_{q_j} E_j^i \otimes E_j^{r+1-i} K_j^i.
\end{aligned}$$

The formula for $\hat{\Delta}(F_j^r)$ is proved in a similar way, or by applying the automorphism ω from Lemma 2.5 to the first formula. \square

Let

$$\rho = \sum_{i=1}^N \varpi_i = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

be the half-sum of all positive roots. Note that for all $1 \leq i \leq N$, we have

$$(\rho, \alpha_i^\vee) = 1, \quad (\rho, \alpha_i) = d_i.$$

Lemma 2.7. *For all $X \in U_q(\mathfrak{g})$ we have*

$$\hat{S}^2(X) = K_{2\rho} X K_{-2\rho}.$$

Proof. This is easily checked on generators. \square

In particular, the antipode \hat{S} is invertible, so that $\tilde{U}_q(\mathfrak{g})$ is a regular Hopf algebra, compare the remarks after Theorem 1.7. The inverse of \hat{S} is given on generators by

$$\hat{S}^{-1}(K_\lambda) = K_{-\lambda}, \quad \hat{S}^{-1}(E_j) = -K_j^{-1} E_j, \quad \hat{S}^{-1}(F_j) = -F_j K_j.$$

Let $\tilde{U}_q(\mathfrak{n}_+)$ be the subalgebra of $\tilde{U}_q(\mathfrak{g})$ generated by the elements E_1, \dots, E_N , and let $\tilde{U}_q(\mathfrak{n}_-)$ be the subalgebra generated by F_1, \dots, F_N . Moreover we let $\tilde{U}_q(\mathfrak{h})$ be the subalgebra generated by the elements K_λ for $\lambda \in \mathbf{P}$. We write $\tilde{U}_q(\mathfrak{b}_+)$ for the subalgebra of $\tilde{U}_q(\mathfrak{g})$ generated by E_1, \dots, E_N and all K_λ for $\lambda \in \mathbf{P}$, and similarly we write $\tilde{U}_q(\mathfrak{b}_-)$ for the subalgebra generated by the elements $F_1, \dots, F_N, K_\lambda$ for $\lambda \in \mathbf{P}$. The algebras $\tilde{U}_q(\mathfrak{h})$ and $\tilde{U}_q(\mathfrak{b}_\pm)$ are Hopf subalgebras. It is often convenient to use the automorphism ω from Lemma 2.5 to transport results for $\tilde{U}_q(\mathfrak{n}_+)$ and $\tilde{U}_q(\mathfrak{b}_+)$ to $\tilde{U}_q(\mathfrak{n}_-)$ and $\tilde{U}_q(\mathfrak{b}_-)$, and vice versa.

Proposition 2.8. *Multiplication in $\tilde{U}_q(\mathfrak{g})$ induces a linear isomorphism*

$$\tilde{U}_q(\mathfrak{n}_-) \otimes \tilde{U}_q(\mathfrak{h}) \otimes \tilde{U}_q(\mathfrak{n}_+) \cong \tilde{U}_q(\mathfrak{g}).$$

Proof. In order to prove the claim it suffices to show that the elements $F_I K_\mu E_J$, for $F_I = F_{i_1} \cdots F_{i_k}, E_J = E_{j_1} \cdots E_{j_l}$ finite sequences of simple root vectors and $\mu \in \mathbf{P}$, form a linear basis of $\tilde{U}_q(\mathfrak{g})$. This in turn is an easy consequence of the Diamond Lemma [12]. Indeed, from the definition of $\tilde{U}_q(\mathfrak{g})$ we see that there are only overlap ambiguities, and all of these turn out to be resolvable. \square

In a similar way we obtain linear isomorphisms

$$\tilde{U}_q(\mathfrak{n}_\pm) \otimes \tilde{U}_q(\mathfrak{h}) \cong \tilde{U}_q(\mathfrak{b}_\pm);$$

moreover $\tilde{U}_q(\mathfrak{h})$ identifies with the group algebra of \mathbf{P} , whereas $\tilde{U}_q(\mathfrak{n}_\pm)$ is isomorphic to the free algebra generated by the E_i or F_j , respectively.

The algebra $\tilde{U}_q(\mathfrak{g})$ is equipped with a \mathbf{Q} -grading such that the generators K_λ have degree 0, the elements E_i degree α_i , and the elements F_i degree $-\alpha_i$. We shall also refer to the degree of a homogeneous element with respect to this grading as its weight, and denote by $\tilde{U}_q(\mathfrak{g})_\beta \subset \tilde{U}_q(\mathfrak{g})$ the subspace of all elements of weight β . The weight grading induces a direct sum decomposition

$$\tilde{U}_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbf{Q}} \tilde{U}_q(\mathfrak{g})_\beta$$

of $\tilde{U}_q(\mathfrak{g})$. Notice that $\tilde{U}_q(\mathfrak{g})_\alpha \tilde{U}_q(\mathfrak{g})_\beta \subset \tilde{U}_q(\mathfrak{g})_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbf{Q}$.

If $q \in \mathbb{K}^\times$ is not a root of unity we can describe the weight grading equivalently in terms of the adjoint action. More precisely, we have that $X \in \tilde{U}_q(\mathfrak{g})$ is of weight λ iff

$$\text{ad}(K_\mu)(X) = K_\mu X K_\mu^{-1} = q^{(\mu, \lambda)} X$$

for all $\mu \in \mathbf{P}$.

As for any Hopf algebra, we may consider the adjoint action of $\tilde{U}_q(\mathfrak{g})$ on itself, given by

$$\text{ad}(X)(Y) = X \rightarrow Y = X_{(1)} Y \hat{S}(X_{(2)}).$$

Explicitly, we obtain

$$\begin{aligned} E_j \rightarrow Y &= Y \hat{S}(E_j) + E_j Y \hat{S}(K_j) = -Y E_j K_j^{-1} + E_j Y K_j^{-1} = [E_j, Y] K_j^{-1} \\ F_j \rightarrow Y &= K_j^{-1} Y \hat{S}(F_j) + F_j Y = -K_j^{-1} Y K_j F_j + F_j Y \\ K_\lambda \rightarrow Y &= K_\lambda Y \hat{S}(K_\lambda) = K_\lambda Y K_\lambda^{-1} \end{aligned}$$

for all $1 \leq j \leq N$ and $\lambda \in \mathbf{P}$.

We shall occasionally also consider the right adjoint action of $\tilde{U}_q(\mathfrak{g})$ on itself, given by

$$Y \leftarrow X = \hat{S}(X_{(1)}) Y X_{(2)}.$$

These actions are linked via

$$\omega(X \rightarrow Y) = \omega(Y) \leftarrow \hat{S}^{-1}(\omega(X)),$$

where ω is the automorphism from Lemma 2.5.

Lemma 2.9. *We have*

$$\begin{aligned} E_j^r \rightarrow Y &= \sum_{i=0}^r (-1)^{r-i} q_j^{-(r-i)(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i Y E_j^{r-i} K_j^{-r} \\ F_j^r \rightarrow Y &= \sum_{i=0}^r (-1)^i q_j^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} F_j^{r-i} K_j^{-i} Y K_j^i F_j^i \end{aligned}$$

for all $r \in \mathbb{N}$.

Proof. We apply Lemma 2.6 to obtain

$$\begin{aligned}
E_j^r \rightarrow Y &= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i Y \hat{S}(E_j^{r-i} K_j^i) \\
&= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} (-1)^{r-i} E_j^i Y K_j^{-i} (E_j K_j^{-1})^{r-i} \\
&= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} (-1)^{r-i} q_j^{-(r-i)(r-i-1)-2i(r-i)} E_j^i Y E_j^{r-i} K_j^{-r} \\
&= \sum_{i=0}^r (-1)^{r-i} q_j^{-(r-i)(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} E_j^i Y E_j^{r-i} K_j^{-r}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_j^r \rightarrow Y &= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} F_j^{r-i} K_j^{-i} Y \hat{S}(F_j)^i \\
&= \sum_{i=0}^r q_j^{i(r-i)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} (-1)^i q_j^{(i-1)i} F_j^{r-i} K_j^{-i} Y K_j^i F_j^i \\
&= \sum_{i=0}^r (-1)^i q_j^{i(r-1)} \begin{bmatrix} r \\ i \end{bmatrix}_{q_j} F_j^{r-i} K_j^{-i} Y K_j^i F_j^i.
\end{aligned}$$

This yields the claim. \square

2.2.3. *The Serre elements.* For the construction of $U_q(\mathfrak{g})$ we shall be interested in the *Serre elements* of $\tilde{U}_q(\mathfrak{g})$, defined by

$$\begin{aligned}
u_{ij}^+ &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k \\
u_{ij}^- &= \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k,
\end{aligned}$$

for $1 \leq i, j \leq N$ with $i \neq j$. We note that

$$u_{ij}^- = F_i^{1-a_{ij}} \rightarrow F_j, \quad u_{ij}^+ = E_j \leftarrow \hat{S}^{-1}(E_i^{1-a_{ij}})$$

due to Lemma 2.9. Note also that $\omega(u_{ij}^+) = u_{ij}^-$.

Proposition 2.10. *We have*

$$\begin{aligned}
\hat{\Delta}(u_{ij}^+) &= u_{ij}^+ \otimes K_i^{1-a_{ij}} K_j + 1 \otimes u_{ij}^+ \\
\hat{\Delta}(u_{ij}^-) &= u_{ij}^- \otimes 1 + K_i^{-(1-a_{ij})} K_j^{-1} \otimes u_{ij}^-
\end{aligned}$$

and

$$\begin{aligned}
\hat{S}(u_{ij}^+) &= -u_{ij}^+ K_i^{a_{ij}-1} K_j^{-1} \\
\hat{S}(u_{ij}^-) &= -K_i^{1-a_{ij}} K_j u_{ij}^-
\end{aligned}$$

for all $1 \leq i, j \leq N$ with $i \neq j$.

Proof. Let us prove the first claim for u_{ij}^+ , following Lemma 4.10 in [39]. We note that

$$\hat{\Delta}(u_{ij}^+) = \sum_{k=0}^{1-a_{ij}} (-1)^{1-a_{ij}-k} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \hat{\Delta}(E_i^k) \hat{\Delta}(E_j) \hat{\Delta}(E_i^{1-a_{ij}-k}),$$

and by Lemma 2.6 we have

$$\hat{\Delta}(E_i^r) = E_i^r \otimes K_i^r + 1 \otimes E_i^r + \sum_{m=1}^{r-1} q_i^{m(r-m)} \begin{bmatrix} r \\ m \end{bmatrix}_{q_i} E_i^m \otimes E_i^{r-m} K_i^m.$$

It follows that $\hat{\Delta}(u_{ij}^+)$ has the form

$$\hat{\Delta}(u_{ij}^+) = u_{ij}^+ \otimes K_i^{1-a_{ij}} K_j + 1 \otimes u_{ij}^+ + \sum_{m=1}^{1-a_{ij}} E_i^m \otimes X_m + \sum_{m,n} E_i^m E_j E_i^n \otimes Y_{mn}$$

for suitable elements X_k, Y_{kl} , where the second sum is over all $m, n \geq 0$ such that $m+n \leq 1-a_{ij}$.

We shall show that all X_k and Y_{kl} are zero. Collecting terms in the binomial expansions we obtain

$$\begin{aligned} Y_{mn} &= \sum_{k=m}^{1-a_{ij}-n} (-1)^{1-a_{ij}-k} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} q_i^{m(k-m)} \\ &\quad \times \begin{bmatrix} k \\ m \end{bmatrix}_{q_i} E_i^{k-m} K_i^m K_j q_i^{n(1-a_{ij}-k-n)} \begin{bmatrix} 1-a_{ij}-k \\ n \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k-n} K_i^n. \end{aligned}$$

We have

$$\begin{aligned} \begin{bmatrix} r \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ m \end{bmatrix}_{q_i} \begin{bmatrix} r-k \\ n \end{bmatrix}_{q_i} &= \frac{[r]![k]![r-k]!}{[k]![m]![n]![r-k]![k-m]![r-k-n]!} \\ &= \frac{[r]!}{[m]![n]![k-m]![r-k-n]!} \\ &= \frac{[r-m-n]![r-n]![r]!}{[k-m]![m]![n]![r-k-n]![r-m-n]![r-n]!} \\ &= \begin{bmatrix} r-m-n \\ k-m \end{bmatrix}_{q_i} \begin{bmatrix} r-n \\ m \end{bmatrix}_{q_i} \begin{bmatrix} r \\ n \end{bmatrix}_{q_i}. \end{aligned}$$

This shows

$$Y_{mn} = (-1)^{1-a_{ij}} y_{mn} \begin{bmatrix} 1-a_{ij}-n \\ m \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_{q_i} E_i^{1-a_{ij}-m-n} K_i^{m+n} K_j$$

where

$$\begin{aligned} y_{mn} &= \sum_{k=m}^{1-a_{ij}-n} (-1)^k \begin{bmatrix} 1-a_{ij}-m-n \\ k-m \end{bmatrix}_{q_i} q_i^{m(k-m)+(n+2m+a_{ij})(1-a_{ij}-k-n)} \\ &= (-1)^m q_i^{(n+2m+a_{ij})(1-a_{ij}-m-n)} \\ &\quad \times \sum_{l=0}^{1-a_{ij}-m-n} (-1)^l \begin{bmatrix} 1-a_{ij}-m-n \\ l \end{bmatrix}_{q_i} q_i^{(1-a_{ij}-m-n)l} q_i^{-l} = 0 \end{aligned}$$

according to Lemma 2.2.

For X_m we get

$$\begin{aligned} X_m &= \sum_{k=0}^{1-a_{ij}} (-1)^{1-a_{ij}-k} \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \sum_l q_i^{l(k-l)} \begin{bmatrix} k \\ l \end{bmatrix}_{q_i} E_i^{k-l} K_i^l E_j \\ &\quad \times q_i^{(m-l)(1-a_{ij}-k-m+l)} \begin{bmatrix} 1-a_{ij}-k \\ m-l \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k-m+l} K_i^{m-l}, \end{aligned}$$

where l runs over $\max(0, m + k + a_{ij} - 1) \leq l \leq \min(k, m)$. We get

$$X_m = (-1)^{1-a_{ij}} \sum_{s=0}^{1-a_{ij}-m} a_{ms} E_i^s E_j E_i^{1-a_{ij}-m-s} K_i^m,$$

where

$$\begin{aligned} a_{ms} &= \sum_{k=s}^{m+s} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ s \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-k \\ m+s-k \end{bmatrix}_{q_i} \\ &\quad \times q_i^{(k-s)s+(m+s-k)(1-a_{ij}-m-s)+a_{ij}(k-s)+2(k-s)(1-a_{ij}-m-s)}. \end{aligned}$$

Using

$$\begin{aligned} \begin{bmatrix} r \\ k \end{bmatrix}_{q_i} \begin{bmatrix} k \\ s \end{bmatrix}_{q_i} \begin{bmatrix} r-k \\ m+s-k \end{bmatrix}_{q_i} &= \frac{[r]![k]![r-k]!}{[k]![s]![m+s-k]![r-k]![k-s]![r-m-s]!} \\ &= \frac{[r]!}{[s]![m+s-k]![k-s]![r-m-s]!} \\ &= \frac{[r]![r-m]![m]!}{[m]![s]![k-s]![r-m]![r-m-s]![m+s-k]!} \\ &= \begin{bmatrix} r \\ m \end{bmatrix}_{q_i} \begin{bmatrix} r-m \\ s \end{bmatrix}_{q_i} \begin{bmatrix} m \\ k-s \end{bmatrix}_{q_i}, \end{aligned}$$

this simplifies to

$$\begin{aligned} a_{ms} &= \sum_{k=s}^{m+s} (-1)^k \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-m \\ s \end{bmatrix}_{q_i} \begin{bmatrix} m \\ k-s \end{bmatrix}_{q_i} \\ &\quad \times q_i^{(k-s)s+(m+s-k)(1-a_{ij}-m-s)+a_{ij}(k-s)+2(k-s)(1-a_{ij}-m-s)} \\ &= \sum_{k=s}^{m+s} (-1)^k \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-m \\ s \end{bmatrix}_{q_i} \begin{bmatrix} m \\ k-s \end{bmatrix}_{q_i} \\ &\quad \times q_i^{(k-s)s+(m+k-s)(1-a_{ij}-m-s)+a_{ij}(k-s)} \\ &= \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-m \\ s \end{bmatrix}_{q_i} \sum_{k=s}^{m+s} (-1)^k \begin{bmatrix} m \\ k-s \end{bmatrix}_{q_i} q_i^{m(1-a_{ij}-m-s)+(k-s)(1-m)} \\ &= (-1)^s q_i^{m(1-a_{ij}-m-s)} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} \begin{bmatrix} 1-a_{ij}-m \\ s \end{bmatrix}_{q_i} \sum_{l=0}^m (-1)^l \begin{bmatrix} m \\ l \end{bmatrix}_{q_i} q_i^{l(1-m)} = 0, \end{aligned}$$

taking into account Lemma 2.2.

The claim for the coproduct of u_{ij}^- is obtained by applying the automorphism ω from Lemma 2.5.

The remaining assertions follow from the formulas for $\hat{\Delta}(u_{ij}^\pm)$ and the antipode relations. \square

Lemma 2.11. *For any $1 \leq k \leq N$ and $r \in \mathbb{N}$ we have*

$$[E_k, F_k^r] = [r]_{q_k} F_k^{r-1} \frac{1}{q_k - q_k^{-1}} (q_k^{-(r-1)} K_k - q_k^{r-1} K_k^{-1}).$$

Proof. For $r = 1$ the formula clearly holds. Assuming the claim for r , we compute

$$\begin{aligned}
 [E_k, F_k^{r+1}] &= [E_k, F_k^r]F_k + F_k^r[E_k, F_k] \\
 &= [r]_{q_k} F_k^{r-1} \frac{1}{q_k - q_k^{-1}} (q_k^{-(r-1)} K_k - q_k^{r-1} K_k^{-1}) F_k + F_k^r \frac{1}{q_k - q_k^{-1}} (K_k - K_k^{-1}) \\
 &= \frac{1}{q_k - q_k^{-1}} [r]_{q_k} F_k^r (q_k^{-(r+1)} K_k - q_k^{r+1} K_k^{-1}) + F_k^r \frac{1}{q_k - q_k^{-1}} (K_k - K_k^{-1}) \\
 &= \frac{1}{q_k - q_k^{-1}} (q_k^{r-1} + q_k^{r-3} + \dots + q_k^{-r+1}) F_k^r (q_k^{-(r+1)} K_k - q_k^{r+1} K_k^{-1}) \\
 &\quad + F_k^r \frac{1}{q_k - q_k^{-1}} (K_k - K_k^{-1}) \\
 &= (q_k^r + q_k^{r-2} + \dots + q_k^{-r}) F_k^r \frac{1}{q_k - q_k^{-1}} (q_k^{-r} K_k - q_k^r K_k^{-1}) \\
 &= [r+1]_{q_k} F_k^r \frac{1}{q_k - q_k^{-1}} (q_k^{-r} K_k - q_k^r K_k^{-1}).
 \end{aligned}$$

This finishes the proof. \square

Lemma 2.12. For $1 \leq i, j \leq N$ and $i \neq j$ we have

$$[E_k, u_{ij}^-] = 0 = [F_k, u_{ij}^+]$$

for all $k = 1, \dots, N$.

Proof. Let us verify $[E_k, u_{ij}^-] = 0$. If $k \neq i, j$ then the elements E_k commute with F_i, F_j , so that the assertion is obvious. Assume now $k = i$. Note that

$$[E_k, u_{ij}^-] = \text{ad}(E_k)(u_{ij}^-)K_k,$$

so it suffices to show $\text{ad}(E_k)(u_{ij}^-) = 0$. As observed at the beginning of this subsection, we have $\text{ad}(F_k^{1-a_{kj}})(F_j) = u_{kj}^-$. Combining this relation with Lemma 2.11 we obtain

$$\begin{aligned}
 \text{ad}(E_k)(u_{kj}^-) &= \text{ad}(E_k)\text{ad}(F_k^{1-a_{kj}})(F_j) \\
 &= \text{ad}(F_k^{1-a_{kj}})\text{ad}(E_k)(F_j) + \text{ad}([E_k, F_k^{1-a_{kj}}])(F_j) \\
 &= \text{ad}(F_k^{1-a_{kj}})\text{ad}(E_k)(F_j) \\
 &\quad + [1 - a_{kj}]_{q_k} \text{ad}(F_k^{-a_{kj}}) \frac{1}{q_k - q_k^{-1}} (q_k^{a_{kj}} \text{ad}(K_k) - q_k^{-a_{kj}} \text{ad}(K_k^{-1}))(F_j) \\
 &= 0,
 \end{aligned}$$

by using $[E_k, F_j] = 0$ and $\text{ad}(K_k)(F_j) = q_k^{-a_{kj}} F_j$.

Now let $k = j$. Then $[E_k, F_i] = 0$ and $K_k F_i = q_i^{a_{ik}} F_i K_k$, so that

$$\begin{aligned}
 [E_k, u_{ik}^-] &= \sum_{l=0}^{1-a_{ik}} (-1)^l \left[\begin{matrix} 1-a_{ik} \\ l \end{matrix} \right]_{q_i} F_i^{1-a_{ik}-l} [E_k, F_k] F_i^l \\
 &= \frac{1}{q_k - q_k^{-1}} \sum_{l=0}^{1-a_{ik}} (-1)^l \left[\begin{matrix} 1-a_{ik} \\ l \end{matrix} \right]_{q_i} F_i^{1-a_{ik}-l} (K_k - K_k^{-1}) F_i^l \\
 &= \frac{1}{q_k - q_k^{-1}} \sum_{l=0}^{1-a_{ik}} (-1)^l \left[\begin{matrix} 1-a_{ik} \\ l \end{matrix} \right]_{q_i} F_i^{1-a_{ik}} (q_i^{la_{ik}} K_k - q_i^{l(-a_{ik})} K_k^{-1}) = 0
 \end{aligned}$$

by Lemma 2.2.

Finally, the equality $[F_k, u_{ij}^+] = 0$ follows by applying the automorphism ω from Lemma 2.5. \square

2.2.4. *The quantized universal enveloping algebra.* Let us now give the definition of the quantized universal enveloping algebra of \mathfrak{g} . We fix a ground field \mathbb{K} .

Definition 2.13. Let \mathbb{K} be a field and $q = s^L \in \mathbb{K}^\times$ be an invertible element such that $q_i \neq \pm 1$ for all i . The algebra $U_q(\mathfrak{g})$ over \mathbb{K} has generators K_λ for $\lambda \in \mathbf{P}$, and E_i, F_i for $i = 1, \dots, N$, and the defining relations for $U_q(\mathfrak{g})$ are

$$\begin{aligned} K_0 &= 1 \\ K_\lambda K_\mu &= K_{\lambda+\mu} \\ K_\lambda E_j K_\lambda^{-1} &= q^{(\lambda, \alpha_j)} E_j \\ K_\lambda F_j K_\lambda^{-1} &= q^{-(\lambda, \alpha_j)} F_j \\ [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \end{aligned}$$

for all $\lambda, \mu \in \mathbf{P}$ and all i, j , together with the quantum Serre relations

$$\begin{aligned} \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^{1-a_{ij}-k} E_j E_i^k &= 0 \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^{1-a_{ij}-k} F_j F_i^k &= 0. \end{aligned}$$

In the above formulas we abbreviate $K_i = K_{\alpha_i}$ for all simple roots, and we use the notation $q_i = q^{d_i}$.

One often finds a slightly different version of the quantized universal enveloping algebra in the literature, only containing elements K_λ for $\lambda \in \mathbf{Q}$. For our purposes it will be crucial to work with the algebra as in Definition 2.13.

It follows from Lemma 2.4 and Proposition 2.10 that $U_q(\mathfrak{g})$ is a Hopf algebra with comultiplication $\hat{\Delta} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by

$$\begin{aligned} \hat{\Delta}(K_\lambda) &= K_\lambda \otimes K_\lambda, \\ \hat{\Delta}(E_i) &= E_i \otimes K_i + 1 \otimes E_i \\ \hat{\Delta}(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \end{aligned}$$

counit $\hat{\epsilon} : U_q(\mathfrak{g}) \rightarrow \mathbb{K}$ given by

$$\hat{\epsilon}(K_\lambda) = 1, \quad \hat{\epsilon}(E_j) = 0, \quad \hat{\epsilon}(F_j) = 0,$$

and antipode $\hat{S} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by

$$\hat{S}(K_\lambda) = K_{-\lambda}, \quad \hat{S}(E_j) = -E_j K_j^{-1}, \quad \hat{S}(F_j) = -K_j F_j.$$

Throughout the text we will use the Sweedler notation

$$\hat{\Delta}(X) = X_{(1)} \otimes X_{(2)}$$

for the coproduct of $X \in U_q(\mathfrak{g})$.

Let $U_q(\mathfrak{n}_+)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_1, \dots, E_N , and let $U_q(\mathfrak{n}_-)$ be the subalgebra generated by F_1, \dots, F_N . Moreover we let $U_q(\mathfrak{h})$ be the subalgebra generated by the elements K_λ for $\lambda \in \mathbf{P}$.

Proposition 2.14. *Multiplication in $U_q(\mathfrak{g})$ induces a linear isomorphism*

$$U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \cong U_q(\mathfrak{g}),$$

the quantum analogue of the triangular decomposition.

Proof. Let us write I^+ for the ideal in $\tilde{U}_q(\mathfrak{n}_+)$ generated by the elements u_{ij}^+ . We claim that under the isomorphism of Proposition 2.8 the ideal in $\tilde{U}_q(\mathfrak{g})$ generated by the u_{ij}^+ identifies with $\tilde{U}_q(\mathfrak{n}_-) \otimes \tilde{U}_q(\mathfrak{h}) \otimes I^+$. The left ideal property is obvious, and the right ideal property follows using Lemma 2.12. An analogous claim holds for the ideal I^- in $\tilde{U}_q(\mathfrak{n}_-)$ generated by the elements u_{ij}^- .

It follows that

$$I = I^- \otimes \tilde{U}_q(\mathfrak{h}) \otimes \tilde{U}_q(\mathfrak{n}_+) + \tilde{U}_q(\mathfrak{n}_-) \otimes \tilde{U}_q(\mathfrak{h}) \otimes I^+$$

is an ideal in $\tilde{U}_q(\mathfrak{g})$, and the quotient algebra $\tilde{U}_q(\mathfrak{g})/I$ is canonically isomorphic to $U_q(\mathfrak{g})$. The assertion now follows from Proposition 2.8. Note in particular that we obtain a canonical isomorphism $U_q(\mathfrak{h}) \cong \tilde{U}_q(\mathfrak{h})$. \square

We write $U_q(\mathfrak{b}_+)$ for the subalgebra of $U_q(\mathfrak{g})$ generated by E_1, \dots, E_N and all K_λ for $\lambda \in \mathbf{P}$, and similarly we write $U_q(\mathfrak{b}_-)$ for the subalgebra generated by the elements $F_1, \dots, F_N, K_\lambda$ for $\lambda \in \mathbf{P}$. These algebras are Hopf subalgebras. It follows from Proposition 2.14 that these algebras, as well as the algebras $U_q(\mathfrak{n}_\pm)$ and $U_q(\mathfrak{h})$, are canonically isomorphic to the universal algebras with the appropriate generators and relations from Definition 2.13.

As a consequence of Proposition 2.14 we obtain linear isomorphisms

$$U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_\pm) \cong U_q(\mathfrak{b}_\pm).$$

Also note that $U_q(\mathfrak{h})$ is isomorphic to the group algebra of \mathbf{P} .

We observe that the automorphism $\omega : \tilde{U}_q(\mathfrak{g}) \rightarrow \tilde{U}_q(\mathfrak{g})$ from Lemma 2.5 induces an algebra automorphism of $U_q(\mathfrak{g})$, which we will again denote by ω . This automorphism allows us to interchange the upper and lower triangular parts of $U_q(\mathfrak{g})$. We state this explicitly in the following Lemma.

Lemma 2.15. *There exists an algebra automorphism $\omega : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by*

$$\omega(K_\lambda) = K_{-\lambda}, \quad \omega(E_j) = F_j, \quad \omega(F_j) = E_j$$

on generators. Moreover ω is a coalgebra antihomomorphism.

Similarly, there exists an algebra anti-automorphism $\Omega : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ given by

$$\Omega(K_\lambda) = K_\lambda, \quad \Omega(E_j) = F_j, \quad \Omega(F_j) = E_j$$

on generators.

Proof. It is straightforward to verify that the defining relations of $U_q(\mathfrak{g})$ are preserved by these assignments. \square

We note that the automorphism ω can be used to translate formulas which depend on our convention for the comultiplication of $U_q(\mathfrak{g})$ to the convention using the coopposite comultiplication.

Let us introduce another symmetry of $U_q(\mathfrak{g})$.

Lemma 2.16. *There is a unique algebra antiautomorphism $\tau : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ such that*

$$\tau(E_j) = K_j F_j, \quad \tau(F_j) = E_j K_j^{-1}, \quad \tau(K_\lambda) = K_\lambda$$

for $j = 1, \dots, N$ and $\lambda \in \mathbf{P}$. Moreover τ is involutive, that is $\tau^2 = \text{id}$, and a coalgebra homomorphism.

Proof. Observe first that all of the Hopf algebra relations of $U_q(\mathfrak{g})$ are preserved by the transformation

$$E_i \mapsto -E_i, \quad F_i \mapsto -F_i, \quad K_\lambda \mapsto K_\lambda,$$

so that these generate an involutive Hopf algebra automorphism of $U_q(\mathfrak{g})$. Now τ agrees on generators with the composition of this map with the involution $\hat{S} \circ \omega$,

which is an algebra anti-automorphism and coalgebra automorphism. The result follows. \square

In the same way as for $\tilde{U}_q(\mathfrak{g})$ one obtains a \mathbf{Q} -grading on $U_q(\mathfrak{g})$ such that the generators K_λ have degree 0, the elements E_i degree α_i , and the elements F_i degree $-\alpha_i$. The degree of a homogeneous element with respect to this grading will again be referred to as its weight, and we write $U_q(\mathfrak{g})_\beta \subset U_q(\mathfrak{g})$ for the subspace of all elements of weight β . The weight grading induces a direct sum decomposition

$$U_q(\mathfrak{g}) = \bigoplus_{\beta \in \mathbf{Q}} U_q(\mathfrak{g})_\beta$$

of $U_q(\mathfrak{g})$, and we have $U_q(\mathfrak{g})_\alpha U_q(\mathfrak{g})_\beta \subset U_q(\mathfrak{g})_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbf{Q}$.

If $q \in \mathbb{K}^\times$ is not a root of unity the weight grading is determined by the adjoint action of $U_q(\mathfrak{g})$ on itself in the same way as for $\tilde{U}_q(\mathfrak{g})$.

2.2.5. The restricted integral form. Instead of working over a field \mathbb{K} , it is sometimes necessary to consider more general coefficient rings.

Let us fix a semisimple Lie algebra \mathfrak{g} and put $q = s^L \in \mathbb{Q}(s)$. We consider the ring $\mathcal{A} = \mathbb{Z}[s, s^{-1}] \subset \mathbb{Q}(s)$. With this notation in place, we define the restricted integral form of the quantized universal enveloping algebra, compare [56].

Definition 2.17. The restricted integral form $U_q^{\mathcal{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ is the \mathcal{A} -subalgebra of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(s)$ generated by the elements K_λ for $\lambda \in \mathbf{P}$ and

$$[K_i; 0] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

for $i = 1, \dots, N$, together with the divided powers

$$E_i^{(r)} = \frac{1}{[r]_{q_i}!} E_i^r, \quad F_i^{(r)} = \frac{1}{[r]_{q_i^{-1}}!} F_i^r$$

for $i = 1, \dots, N$.

Let us show that $U_q^{\mathcal{A}}(\mathfrak{g})$ is a Hopf algebra over the commutative ring \mathcal{A} in a natural way.

Lemma 2.18. *The comultiplication, counit and antipode of $U_q(\mathfrak{g})$ induce on $U_q^{\mathcal{A}}(\mathfrak{g})$ the structure of a Hopf algebra over \mathcal{A} .*

Proof. Let us verify that the formulas for the coproducts of all generators of $U_q^{\mathcal{A}}(\mathfrak{g})$ make sense as elements in $U_q^{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{g})$. For the generators K_λ this is obvious. Moreover we observe

$$\hat{\Delta}([K_i; 0]) = [K_i; 0] \otimes K_i + K_i^{-1} \otimes [K_i; 0].$$

According to Lemma 2.6 we have

$$\begin{aligned} \hat{\Delta}(E_j^{(r)}) &= \sum_{i=0}^r q_j^{i(r-i)} E_j^{(i)} \otimes E_j^{(r-i)} K_j^i \\ \hat{\Delta}(F_j^{(r)}) &= \sum_{i=0}^r q_j^{i(r-i)} F_j^{(r-i)} K_j^{-i} \otimes F_j^{(i)} \end{aligned}$$

for all j , so the assertion also holds for the divided powers.

It is clear that the antipode and counit of $U_q(\mathfrak{g})$ induce corresponding maps on the level of $U_q^{\mathcal{A}}(\mathfrak{g})$. The Hopf algebra axioms are verified in the same way as in Lemma 2.4. \square

We next define the bar involution of $U_q^{\mathcal{A}}(\mathfrak{g})$. The field automorphism β of $\mathbb{Q}(s)$ determined by $\beta(s) = s^{-1}$ restricts to a ring automorphism of \mathcal{A} . This extends to $U_q^{\mathcal{A}}(\mathfrak{g})$ as follows.

Lemma 2.19. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(s)$ admits an automorphism $\beta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ of \mathbb{Q} -algebras such that $\beta(s) = s^{-1}$ and*

$$\beta(K_\mu) = K_{-\mu}, \quad \beta(E_i) = E_i, \quad \beta(F_i) = F_i$$

for all $\mu \in \mathbf{P}$ and $1 \leq i \leq N$. This restricts to an automorphism of $U_q^{\mathcal{A}}(\mathfrak{g})$.

Proof. We can view $U_q(\mathfrak{g})$ as a $\mathbb{Q}(s)$ -algebra using the scalar action $c \bullet X = \beta(c)X$ for $c \in \mathbb{Q}(s)$. Let us write $U_q(\mathfrak{g})^\beta$ for the resulting $\mathbb{Q}(s)$ -algebra. Sending K_μ to $K_{-\mu}$ and fixing the generators E_i, F_j determines a homomorphism $U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^\beta$ of $\mathbb{Q}(s)$ -algebras. By slight abuse of notation we can view this as the desired automorphism $\beta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ of \mathbb{Q} -algebras.

The fact that β restricts to $U_q^{\mathcal{A}}(\mathfrak{g})$ follows directly from the definition of the integral form. \square

In the literature, the bar involution β of $U_q(\mathfrak{g})$ is usually denoted by a bar. We shall write $\beta(X)$ instead of \bar{X} for $X \in U_q(\mathfrak{g})$ in order to avoid confusion with $*$ -structures later on. Note that $\beta : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is not a Hopf algebra automorphism.

For $l \in \mathbb{Z}$ and $1 \leq i \leq N$ we will use the notation

$$[K_i; l]_{q_i} = [K_i; l] = \frac{q_i^l K_i - q_i^{-l} K_i^{-1}}{q_i - q_i^{-1}},$$

and for any $m \in \mathbb{N}_0$ we define

$$\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} = \frac{[K_i; l]_{q_i} [K_i; l-1]_{q_i} \cdots [K_i; l-m+1]_{q_i}}{[m]_{q_i}!} = \prod_{j=1}^m \frac{q_i^{l+1-j} K_i - q_i^{-(l+1-j)} K_i^{-1}}{q_i^j - q_i^{-j}},$$

where, as usual, we interpret this as 1 when $m = 0$. Note that

$$\begin{bmatrix} K_i; l \\ 1 \end{bmatrix}_{q_i} = [K_i; l]_{q_i}.$$

These expressions should be considered as elements in $U_q(\mathfrak{g})$. We will see in Proposition 2.24 that they in fact belong to the restricted integral form $U_q^{\mathcal{A}}(\mathfrak{g})$.

Lemma 2.20. *Let $m \in \mathbb{N}_0, l \in \mathbb{Z}$ and $1 \leq i \leq N$. Then the following relations hold.*

a) *We have*

$$q_i^{-(m+1)} \begin{bmatrix} K_i; l \\ m+1 \end{bmatrix}_{q_i} = K_i^{-1} q_i^{-l} \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} + \begin{bmatrix} K_i; l-1 \\ m+1 \end{bmatrix}_{q_i}$$

and

$$q_i^{m+1} \begin{bmatrix} K_i; l \\ m+1 \end{bmatrix}_{q_i} = K_i q_i^l \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} + \begin{bmatrix} K_i; l-1 \\ m+1 \end{bmatrix}_{q_i}.$$

b) *If $l \geq 0$ then*

$$\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} = \sum_{j=0}^m q_i^{l(m-j)} \begin{bmatrix} l \\ j \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i}.$$

c) *If $l < 0$ then*

$$\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} = \sum_{j=0}^m (-1)^j q_i^{l(j-m)} \begin{bmatrix} -l+j-1 \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i}.$$

Proof. a) To verify the first equation we use induction on m . For $m = 0$ one computes

$$\begin{aligned}
q_i^{-1} \begin{bmatrix} K_i; l \\ 1 \end{bmatrix}_{q_i} &= \frac{q_i^{l-1} K_i - q_i^{-l-1} K_i^{-1}}{q_i - q_i^{-1}} \\
&= \frac{q_i^{l-1} K_i - q_i^{-(l-1)} K_i^{-1} + q_i^{-l} (q_i - q_i^{-1}) K_i^{-1}}{q_i - q_i^{-1}} \\
&= q_i^{-l} K_i^{-1} + \frac{q_i^{l-1} K_i - q_i^{-(l-1)} K_i^{-1}}{q_i - q_i^{-1}} \\
&= q_i^{-l} K_i^{-1} \begin{bmatrix} K_i; l-1 \\ 0 \end{bmatrix}_{q_i} + \begin{bmatrix} K_i; l-1 \\ 1 \end{bmatrix}_{q_i}
\end{aligned}$$

as desired. Assume that the assertion is proved for $m-1 \geq 0$ and compute

$$\begin{aligned}
q_i^{-(m+1)} \begin{bmatrix} K_i; l \\ m+1 \end{bmatrix}_{q_i} &= q_i^{-m} \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} \frac{q_i^{l-m-1} K_i - q_i^{-(l-m+1)} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \\
&= \left(K_i^{-1} q_i^{-l} \begin{bmatrix} K_i; l-1 \\ m-1 \end{bmatrix}_{q_i} + \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} \right) \frac{q_i^{l-m-1} K_i - q_i^{-(l-m+1)} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \\
&= K_i^{-1} q_i^{-l} \begin{bmatrix} K_i; l-1 \\ m-1 \end{bmatrix}_{q_i} \frac{q_i^{l-m-1} K_i - q_i^{-(l-m+1)} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \\
&\quad + \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} \frac{q_i^{-(l-m-1)} K_i^{-1} - q_i^{-(l-m+1)} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} + \begin{bmatrix} K_i; l-1 \\ m+1 \end{bmatrix}_{q_i} \\
&= K_i^{-1} q_i^{-l} \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} \left(\frac{q_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} (q_i^m - q_i^{-m}) + \frac{q_i^{m+1} - q_i^{m-1}}{q_i^{m+1} - q_i^{-(m+1)}} \right) \\
&\quad + \begin{bmatrix} K_i; l-1 \\ m+1 \end{bmatrix}_{q_i} \\
&= K_i^{-1} q_i^{-l} \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} + \begin{bmatrix} K_i; l-1 \\ m+1 \end{bmatrix}_{q_i}.
\end{aligned}$$

This yields the first equality. Applying the automorphism β to this equality yields the second claim, where we note that the terms $\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i}$ are fixed under β for all $l \in \mathbb{Z}$ and $m \in \mathbb{N}_0$.

b) We use induction on l . Assume without loss of generality that $m > 0$. For $l = 0$ the assertion clearly holds. To check the inductive step we use a) and the

inductive hypothesis to calculate

$$\begin{aligned}
 \begin{bmatrix} K_i; l+1 \\ m \end{bmatrix}_{q_i} &= K_i^{-1} q_i^{m-l-1} \begin{bmatrix} K_i; l \\ m-1 \end{bmatrix}_{q_i} + q_i^m \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} \\
 &= K_i^{-1} q_i^{m-l-1} \sum_{j=0}^{m-1} q_i^{l(m-1-j)} \begin{bmatrix} l \\ j \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-1-j \end{bmatrix}_{q_i} \\
 &\quad + q_i^m \sum_{j=0}^m q_i^{l(m-j)} \begin{bmatrix} l \\ j \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\
 &= q_i^{m-l-1} \sum_{j=1}^m q_i^{l(m-j)} \begin{bmatrix} l \\ j-1 \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\
 &\quad + q_i^m \sum_{j=0}^m q_i^{l(m-j)} \begin{bmatrix} l \\ j \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\
 &= \sum_{j=0}^m q_i^{(l+1)(m-j)} \begin{bmatrix} l+1 \\ j \end{bmatrix}_{q_i} K_i^{-j} \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i},
 \end{aligned}$$

taking into account Lemma 2.1 in the last step.

c) We first show

$$\prod_{k=1}^m \frac{q_i^{-k} K_i - q_i^k K_i^{-1}}{q_i^k - q_i^{-k}} = \sum_{j=0}^m (-1)^j q_i^{-(j-m)} K_i^j \prod_{k=1}^{m-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}}$$

by induction on m . For $m = 0$ this relation clearly holds. If it holds for $m \geq 0$ then using the inductive hypothesis we get

$$\begin{aligned}
 \prod_{k=1}^{m+1} \frac{q_i^{-k} K_i - q_i^k K_i^{-1}}{q_i^k - q_i^{-k}} &= \prod_{k=1}^{m+1} \frac{q_i^{-k} K_i - q_i^k K_i^{-1}}{q_i^k - q_i^{-k}} + q_i^{m+1} \prod_{k=1}^{m+1} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &\quad - q_i^{m+1} \frac{K_i - K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \prod_{k=1}^m \frac{q_i^{-k} K_i - q_i^k K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &= q_i^{m+1} \prod_{k=1}^{m+1} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} + \left(\frac{q_i^{-(m+1)} K_i - q_i^{m+1} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \right. \\
 &\quad \left. - \frac{q_i^{m+1} K_i - q_i^{m+1} K_i^{-1}}{q_i^{m+1} - q_i^{-(m+1)}} \right) \sum_{j=0}^m (-1)^j q_i^{-(j-m)} K_i^j \prod_{k=1}^{m-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &= q_i^{m+1} \prod_{k=1}^{m+1} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &\quad - K_i \sum_{j=0}^m (-1)^j q_i^{-(j-m)} K_i^j \prod_{k=1}^{m-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &= q_i^{m+1} \prod_{k=1}^{m+1} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &\quad + K_i \sum_{j=1}^{m+1} (-1)^j q_i^{-(j-m-1)} K_i^{j-1} \prod_{k=1}^{m+1-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &= \sum_{j=0}^{m+1} (-1)^j q_i^{-(j-m-1)} K_i^j \prod_{k=1}^{m+1-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}}
 \end{aligned}$$

as desired.

Let us now prove the assertion. For $l = -1$ we obtain

$$\begin{aligned} \begin{bmatrix} K_i; -1 \\ m \end{bmatrix}_{q_i} &= \prod_{k=1}^m \frac{q_i^{-k} K_i - q_i^k K_i^{-1}}{q_i^k - q_i^{-k}} \\ &= \sum_{j=0}^m (-1)^j q_i^{-(j-m)} K_i^j \prod_{k=1}^{m-j} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\ &= \sum_{j=0}^m (-1)^j q_i^{-(j-m)} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \end{aligned}$$

using the above calculation. Assume now that the claim holds for some $l < 0$ and all $m \in \mathbb{N}_0$. For $l-1$ and $m=0$ the assertion clearly holds too. Using induction on m and applying the second formula from a) we get

$$\begin{aligned} \begin{bmatrix} K_i; l-1 \\ m \end{bmatrix}_{q_i} &= q_i^m \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} - K_i q_i^l \begin{bmatrix} K_i; l-1 \\ m-1 \end{bmatrix}_{q_i} \\ &= q_i^m \sum_{j=0}^m (-1)^j q_i^{l(j-m)} \begin{bmatrix} -l+j-1 \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\ &\quad - K_i q_i^l \sum_{j=0}^{m-1} (-1)^j q_i^{(l-1)(j-m+1)} \begin{bmatrix} -l+j \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-1-j \end{bmatrix}_{q_i} \\ &= q_i^m \sum_{j=0}^m (-1)^j q_i^{l(j-m)} \begin{bmatrix} -l+j-1 \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\ &\quad + q_i^l \sum_{j=1}^m (-1)^j q_i^{(l-1)(j-m)} \begin{bmatrix} -l+j-1 \\ j-1 \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \\ &= \sum_{j=0}^m (-1)^j q_i^{(l-1)(j-m)} \begin{bmatrix} -l+j \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m-j \end{bmatrix}_{q_i} \end{aligned}$$

as claimed, again taking into account Lemma 2.1. \square

We collect some more formulas.

Lemma 2.21. *Assume $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$. Then we have*

$$\begin{aligned} \sum_{j=0}^n (-1)^j q_i^{m(n-j)} \begin{bmatrix} m+j-1 \\ j \end{bmatrix}_{q_i} K_i^j \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} \begin{bmatrix} K_i; 0 \\ n-j \end{bmatrix}_{q_i} &= \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} \begin{bmatrix} K_i; -m \\ n \end{bmatrix}_{q_i} \\ &= \begin{bmatrix} m+n \\ n \end{bmatrix}_{q_i} \begin{bmatrix} K_i; 0 \\ m+n \end{bmatrix}_{q_i} \end{aligned}$$

for all $i = 1, \dots, N$.

Proof. The first equality follows from Lemma 2.20 c). For the second equality we compute

$$\begin{aligned}
 & \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} \begin{bmatrix} K_i; -m \\ n \end{bmatrix}_{q_i} \\
 &= \prod_{j=1}^m \frac{q_i^{1-j} K_i - q_i^{-(1-j)} K_i^{-1}}{q_i^j - q_i^{-j}} \prod_{k=1}^n \frac{q_i^{-m+1-k} K_i - q_i^{-(-m+1-k)} K_i^{-1}}{q_i^k - q_i^{-k}} \\
 &= \prod_{j=1}^m \frac{q_i^{1-j} K_i - q_i^{-(1-j)} K_i^{-1}}{q_i^j - q_i^{-j}} \prod_{k=m+1}^{m+n} \frac{q_i^{1-k} K_i - q_i^{-(1-k)} K_i^{-1}}{q_i^{k-m} - q_i^{-(k-m)}} \\
 &= \begin{bmatrix} m+n \\ n \end{bmatrix}_{q_i} \prod_{j=1}^{m+n} \frac{q_i^{1-j} K_i - q_i^{-(1-j)} K_i^{-1}}{q_i^j - q_i^{-j}} \\
 &= \begin{bmatrix} m+n \\ n \end{bmatrix}_{q_i} \begin{bmatrix} K_i; 0 \\ m+n \end{bmatrix}_{q_i}
 \end{aligned}$$

as desired. \square

Lemma 2.22. *Given any $l \in \mathbb{Z}, m \in \mathbb{N}_0$ we have*

$$\begin{aligned}
 \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} E_i^{(r)} &= E_i^{(r)} \begin{bmatrix} K_i; l+2r \\ m \end{bmatrix}_{q_i} \\
 \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} F_i^{(r)} &= F_i^{(r)} \begin{bmatrix} K_i; l-2r \\ m \end{bmatrix}_{q_i}
 \end{aligned}$$

for all $r \in \mathbb{N}_0$.

Proof. We compute

$$\begin{aligned}
 \begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} E_i^{(r)} &= \prod_{j=1}^m \frac{q_i^{l+1-j} K_i - q_i^{-(l+1-j)} K_i^{-1}}{q_i^j - q_i^{-j}} E_i^{(r)} \\
 &= E_i^{(r)} \prod_{j=1}^m \frac{q_i^{l+1-j+2r} K_i - q_i^{-(l+1-j+2r)} K_i^{-1}}{q_i^j - q_i^{-j}} \\
 &= E_i^{(r)} \begin{bmatrix} K_i; l+2r \\ m \end{bmatrix}_{q_i}
 \end{aligned}$$

as desired. The proof of the second formula is analogous. \square

According to Lemma 2.11 We have

$$[E_i, F_i^m] = [m]_{q_i} F_i^{m-1} [K_i; 1-m]_{q_i}$$

for $m \geq 1$. This can also be written in the form

$$[E_i, F_i^{(m)}] = F_i^{(m-1)} [K_i; 1-m]_{q_i}.$$

Let us record a more general commutation relation.

Proposition 2.23. *We have*

$$\begin{aligned}
 E_i^{(r)} F_i^{(s)} &= \sum_{r, s \geq t \geq 0} F_i^{(s-t)} E_i^{(r-t)} \begin{bmatrix} K_i; r-s \\ t \end{bmatrix}_{q_i} \\
 &= \sum_{r, s \geq t \geq 0} F_i^{(s-t)} \begin{bmatrix} K_i; 2t-r-s \\ t \end{bmatrix}_{q_i} E_i^{(r-t)}
 \end{aligned}$$

for all $r, s \in \mathbb{N}_0$.

Proof. Let us verify the first equality. For $r = 0$ and $s \in \mathbb{N}_0$ arbitrary the formula clearly holds. For $r = 1$ and $s \in \mathbb{N}_0$ arbitrary the formula follows from the computation just before the proposition.

Assume that the equality is true for some $r \geq 1$ and all $s \in \mathbb{N}_0$. Then we compute

$$\begin{aligned}
E_i^{(r+1)} F_i^{(s)} &= \frac{1}{[r+1]_{q_i}} E_i^{(r)} E_i F_i^{(s)} \\
&= \frac{1}{[r+1]_{q_i}} (E_i^{(r)} F_i^{(s-1)} [K_i; 1-s] + E_i^{(r)} F_i^{(s)} E_i) \\
&= \frac{1}{[r+1]_{q_i}} \sum_{r, s-1 \geq t \geq 0} F_i^{(s-1-t)} E_i^{(r-t)} \begin{bmatrix} K_i; r-s+1 \\ t \end{bmatrix}_{q_i} [K_i; 1-s] \\
&\quad + \frac{1}{[r+1]_{q_i}} \sum_{r, s \geq t \geq 0} F_i^{(s-t)} E_i^{(r-t)} \begin{bmatrix} K_i; r-s \\ t \end{bmatrix}_{q_i} E_i \\
&= \frac{1}{[r+1]_{q_i}} \sum_{r+1, s \geq t+1 \geq 1} F_i^{(s-1-t)} E_i^{(r-t)} \begin{bmatrix} K_i; r-s+1 \\ t \end{bmatrix}_{q_i} [K_i; 1-s] \\
&\quad + \frac{1}{[r+1]_{q_i}} \sum_{r, s \geq t \geq 0} [r+1-t]_{q_i} F_i^{(s-t)} E_i^{(r+1-t)} \begin{bmatrix} K_i; r-s+2 \\ t \end{bmatrix}_{q_i} \\
&= \frac{1}{[r+1]_{q_i}} \sum_{r+1, s \geq t \geq 1} F_i^{(s-t)} E_i^{(r+1-t)} \begin{bmatrix} K_i; r-s+1 \\ t-1 \end{bmatrix}_{q_i} [K_i; 1-s] \\
&\quad + \frac{1}{[r+1]_{q_i}} \sum_{r+1, s \geq t \geq 0} [r+1-t]_{q_i} F_i^{(s-t)} E_i^{(r+1-t)} \begin{bmatrix} K_i; r-s+2 \\ t \end{bmatrix}_{q_i},
\end{aligned}$$

using Lemma 2.22. Therefore it suffices to show

$$\begin{aligned}
\begin{bmatrix} K_i; r-s+1 \\ t-1 \end{bmatrix}_{q_i} [K_i; 1-s] + [r+1-t]_{q_i} \begin{bmatrix} K_i; r-s+2 \\ t \end{bmatrix}_{q_i} \\
= [r+1]_{q_i} \begin{bmatrix} K_i; r+1-s \\ t \end{bmatrix}_{q_i}
\end{aligned}$$

for $r, s \geq t \geq 1$; note that the contribution corresponding to $t = 0$ is covered by the second term in the previous expression.

We compute the left-hand side,

$$\begin{aligned}
&\begin{bmatrix} K_i; r-s+1 \\ t-1 \end{bmatrix}_{q_i} [K_i; 1-s] + [r+1-t]_{q_i} \begin{bmatrix} K_i; r-s+2 \\ t \end{bmatrix}_{q_i} \\
&= \frac{[K_i; r-s+1] \dots [K_i; r-s-t+3]}{[t]_{q_i}!} \left([t]_{q_i} [K_i; 1-s] + [r+1-t]_{q_i} [K_i; r-s+2] \right)
\end{aligned}$$

The expression in parentheses here is

$$\begin{aligned}
&([t]_{q_i} [K_i; 1-s] + [r+1-t]_{q_i} [K_i; r-s+2]) \\
&= \frac{1}{(q_i - q_i^{-1})^2} \left((q_i^t - q_i^{-t})(q_i^{1-s} K_i - q_i^{-(1-s)} K_i^{-1}) \right. \\
&\quad \left. + (q_i^{r+1-t} - q_i^{-(r+1-t)})(q_i^{r-s+2} K_i - q_i^{-(r-s+2)} K_i^{-1}) \right) \\
&= \frac{1}{(q_i - q_i^{-1})^2} \left((q_i^{2r+3-t-s} - q_i^{-t+1-s}) K_i + (q_i^{-2r-3+t+s} - q_i^{t-1+s}) K_i^{-1} \right) \\
&= \frac{1}{(q_i - q_i^{-1})^2} (q_i^{r+1} - q_i^{-(r+1)})(q_i^{r-s+2-t} K_i - q_i^{-(r-s+2-t)} K_i^{-1}) \\
&= [r+1]_{q_i} [K_i; r-s+2-t],
\end{aligned}$$

which yields the claim.

The second equality follows from Lemma 2.22. \square

We denote by $U_q^{\mathcal{A}}(\mathfrak{n}_+)$ the subalgebra of $U_q^{\mathcal{A}}(\mathfrak{g})$ generated by the divided powers $E_1^{(t_1)}, \dots, E_N^{(t_N)}$ with $t_j \in \mathbb{N}_0$, and let $U_q^{\mathcal{A}}(\mathfrak{n}_-)$ be the subalgebra generated by the corresponding divided powers $F_1^{(t_1)}, \dots, F_N^{(t_N)}$. Moreover we let $U_q^{\mathcal{A}}(\mathfrak{h})$ be the intersection of $U_q^{\mathcal{A}}(\mathfrak{g})$ with $U_q(\mathfrak{h})$.

The algebras $U_q^{\mathcal{A}}(\mathfrak{n}_{\pm})$ will be studied in greater detail later on. Here we shall determine the structure of $U_q^{\mathcal{A}}(\mathfrak{h})$, see Theorem 6.7 in [56].

Proposition 2.24. *The algebra $U_q^{\mathcal{A}}(\mathfrak{h})$ is free as an \mathcal{A} -module, with basis given by all elements of the form*

$$K_{\lambda} \prod_{i=1}^N K_i^{s_i} \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i}$$

where $s_i \in \{0, 1\}$, $m_i \in \mathbb{N}_0$ and λ belongs to a set of representatives of the cosets of \mathbf{Q} in \mathbf{P} .

Proof. Let us first show by induction on m that $U_q^{\mathcal{A}}(\mathfrak{h})$ contains all elements $\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i}$ for $l \in \mathbb{Z}$ and $m \in \mathbb{N}_0$. The assertion clearly holds for $m = 0$. Assume that it holds for some $m \geq 0$. Then Proposition 2.23 applied to $r = s = m + 1$ and induction shows that $\begin{bmatrix} K_i; 0 \\ m + 1 \end{bmatrix}_{q_i}$ is contained in $U_q^{\mathcal{A}}(\mathfrak{h})$. Finally, parts b) and c) of Lemma 2.20 and induction allow us to conclude that $\begin{bmatrix} K_i; l \\ m + 1 \end{bmatrix}_{q_i}$ is in $U_q^{\mathcal{A}}(\mathfrak{h})$ for all positive and negative integers l , respectively.

For $\lambda \in \mathbf{P}$ and $\mathbf{s} = (s_1, \dots, s_N) \in \mathbb{Z}^N$, $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{N}_0^N$ let us abbreviate

$$K(\lambda, \mathbf{s}, \mathbf{m}) = K_{\lambda} \prod_{i=1}^N K_i^{s_i} \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i}.$$

According to the definition of $U_q^{\mathcal{A}}(\mathfrak{h})$ and our above considerations, these elements are all contained in $U_q^{\mathcal{A}}(\mathfrak{h})$.

Let us write H for the \mathcal{A} -linear span of all elements of the form $K_{\lambda} \prod_{i=1}^N \begin{bmatrix} K_i; l_i \\ m_i \end{bmatrix}_{q_i}$ with $\lambda \in \mathbf{P}$, $l_i \in \mathbb{Z}$ and $m_i \in \mathbb{N}_0$. Then $H \subset U_q^{\mathcal{A}}(\mathfrak{h})$. We claim that H is closed under multiplication. For this, it suffices to consider products of the form $\begin{bmatrix} K_i; l \\ m \end{bmatrix}_{q_i} \begin{bmatrix} K_i; l' \\ m' \end{bmatrix}_{q_i}$ with $l, l' \in \mathbb{Z}$, $m, m' \in \mathbb{N}_0$. Using Lemma 2.20, we can reduce to the case $l = l' = 0$, which follows from Lemma 2.21 by induction on m' . Next, note that according to Proposition 2.23 and Lemma 2.22, the \mathcal{A} -module $U_q^{\mathcal{A}}(\mathfrak{n}_-) H U_q^{\mathcal{A}}(\mathfrak{n}_+)$ is closed under multiplication, and therefore agrees with $U_q^{\mathcal{A}}(\mathfrak{g})$. This implies $H = U_q^{\mathcal{A}}(\mathfrak{h})$.

Now fix a set of coset representatives of \mathbf{Q} in \mathbf{P} , which we will denote temporarily by \mathbf{R} . Let us write L for the \mathcal{A} -linear span of all $K(\lambda, \mathbf{s}, \mathbf{m})$ with $\lambda \in \mathbf{R}$ and $\mathbf{s} = (s_1, \dots, s_N) \in \{0, 1\}^N$, $\mathbf{m} = (m_1, \dots, m_N) \in \mathbb{N}_0^N$. We will prove that L contains $K(\lambda, \mathbf{t}, \mathbf{m})$ for all $\lambda \in \mathbf{P}$, $\mathbf{t} \in \mathbb{Z}^N$, $\mathbf{m} \in \mathbb{N}_0^N$ by induction on \mathbf{t} . For this, it is enough to show that L contains $\prod_{i=1}^N K_i^{t_i} \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i}$ for all $\mathbf{t} \in \mathbb{Z}^N$ and $\mathbf{m} \in \mathbb{N}_0^N$.

This is trivially true for $\mathbf{t} \in \{0, 1\}^N$ and any \mathbf{m} . From the definition of $\begin{bmatrix} K_i; -m_i \\ t_i + 1 \end{bmatrix}_{q_i}$

we can write

$$K_i^{t_i+2} = aK_i \begin{bmatrix} K_i; -m_i \\ t_i + 1 \end{bmatrix}_{q_i} + \sum_{k=-t_i}^{t_i} b_k K_i^k$$

for some $a, b_{-t_i}, \dots, b_{t_i} \in \mathcal{A}$, and likewise

$$K_i^{-t_i-1} = a' \begin{bmatrix} K_i; -m_i \\ t_i + 1 \end{bmatrix}_{q_i} + \sum_{k=-t_i+1}^{t_i+1} b'_k K_i^k$$

for some $a', b'_{-t_i+1}, \dots, b'_{t_i+1} \in \mathcal{A}$. Using the second equality from Lemma 2.21 we therefore obtain

$$\begin{aligned} K_i^{t_i+2} \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i} &= a \begin{bmatrix} m_i + t_i + 1 \\ t_i + 1 \end{bmatrix}_{q_i} K_i \begin{bmatrix} K_i; 0 \\ m_i + t_i + 1 \end{bmatrix}_{q_i} + \sum_{k=-t_i}^{t_i} b_k K_i^k \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i}, \\ K_i^{-t_i-1} \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i} &= a' \begin{bmatrix} m_i + t_i + 1 \\ t_i + 1 \end{bmatrix}_{q_i} \begin{bmatrix} K_i; 0 \\ m_i + t_i + 1 \end{bmatrix}_{q_i} + \sum_{k=-t_i+1}^{t_i+1} b'_k K_i^k \begin{bmatrix} K_i; 0 \\ m_i \end{bmatrix}_{q_i}. \end{aligned}$$

Performing an induction on t_i successively for $i = 1, \dots, N$, we see that every $K(\lambda, \mathbf{t}, \mathbf{m})$ is a linear combination of the elements $K(\lambda, \mathbf{s}, \mathbf{m})$ with $\mathbf{s} \in \{0, 1\}^N$. Taking into account Lemma 2.20 b), c) we conclude $L = H = U_q^{\mathcal{A}}(\mathfrak{h})$ as desired.

It remains to show that the elements $K(\lambda, \mathbf{s}, \mathbf{m})$ in L are linearly independent over \mathcal{A} . For this it suffices to observe that the elements $K(0, \mathbf{s}, \mathbf{m})$ with $\mathbf{s} \in \{0, 1\}^N$ and $\mathbf{m} = (m_1, \dots, m_N)$ satisfying $m_j \leq m$ for all $j = 1, \dots, N$ span the $2^N(m+1)^N$ -dimensional subspace of $U_q(\mathfrak{g})$ over $\mathbb{Q}(s)$ generated by all elements of the form $K_1^{r_1} \cdots K_N^{r_N}$ such that $-m \leq r_i \leq m+1$ for all i . \square

It will be shown later on that $U_q^{\mathcal{A}}(\mathfrak{g})$ is indeed an integral form of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{K} = \mathbb{Q}(s)$ in the sense that there is a canonical isomorphism $\mathbb{Q}(s) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{g}) \cong U_q(\mathfrak{g})$.

Using the integral form one can also show that $U_q(\mathfrak{g})$ tends to the classical universal enveloping algebra $U(\mathfrak{g})$ as q tends to 1. More precisely, let \mathbb{K} be a field and let $U_1(\mathfrak{g}) = U_q^{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} \mathbb{K}$ be the algebra obtained by extension of scalars from \mathcal{A} to \mathbb{K} such that s is mapped to $1 \in \mathbb{K}$.

Recall that the classical universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} over \mathbb{K} is the \mathbb{K} -algebra with generators E_i, F_i, H_i for $1 \leq i \leq N$ satisfying

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, E_j] &= a_{ij} E_j \\ [H_i, F_j] &= -a_{ij} F_j \\ [E_i, F_j] &= \delta_{ij} H_j \end{aligned}$$

for all $1 \leq i, j \leq N$ and the Serre relations

$$\begin{aligned} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^k E_j E_i^{1-a_{ij}-k} &= 0 \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^k F_j F_i^{1-a_{ij}-k} &= 0 \end{aligned}$$

for $i \neq j$.

Proposition 2.25. *Let $U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} . Then there exists a canonical surjective homomorphism $U_1(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ of Hopf algebras.*

Proof. Let us write e_i, f_i, h_i for the generators $E_i, F_i, [K_i; 0]$ viewed as elements of $U_1(\mathfrak{g})$, and similarly write k_λ for K_λ . Moreover let $I_q \subset U_q^{\mathcal{A}}(\mathfrak{g})$ be the ideal generated by all elements $K_\lambda - 1$. We claim that $U_1(\mathfrak{g})/I_1$ is isomorphic to $U(\mathfrak{g})$.

For this we need to verify that the generators $e_i, f_i, h_i \in U_1(\mathfrak{g})/I_1$ satisfy the defining relations of $U(\mathfrak{g})$. Clearly the elements h_i commute among themselves. We compute

$$\begin{aligned} [[K_i; 0], E_j] &= \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} E_j - E_j \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\ &= \frac{(1 - q_i^{-a_{ij}})K_i - (1 - q_i^{a_{ij}})K_i^{-1}}{q_i - q_i^{-1}} E_j \\ &= [a_{ij}]_{q_i} E_j \end{aligned}$$

in $U_q(\mathfrak{g})/I_q$, and hence deduce $[h_i, e_j] = a_{ij}e_j$ in $U_1(\mathfrak{g})/I_1$. Similarly one obtains $[h_i, f_j] = -a_{ij}f_j$, and the relation $[e_i, f_j] = \delta_{ij}h_j$ follows from the definitions. Finally, to check the Serre relations for the e_i and f_j it suffices to observe that quantum binomial coefficients become ordinary binomial coefficients at $q = 1$.

Hence we obtain an algebra homomorphism $\iota : U_1(\mathfrak{g}) \rightarrow U(\mathfrak{g})$ by sending e_i, f_i, h_i to E_i, F_i, H_i , respectively. This map is clearly surjective. It induces an isomorphism $U_1(\mathfrak{g})/I_1 \rightarrow U(\mathfrak{g})$ whose inverse is obtained by using the universal property of $U(\mathfrak{g})$.

To check that this map is compatible with coproducts we use Lemma 2.18 to calculate

$$\hat{\Delta}([K_i; 0]) = [K_i; 0] \otimes 1 + 1 \otimes [K_i; 0]$$

in $U_q(\mathfrak{g})/I_q$, and the corresponding formulas for $\hat{\Delta}(E_i), \hat{\Delta}(F_j)$ follow from the definitions. This yields the claim. \square

Let us finally note that the integral form $U_q^{\mathcal{A}}(\mathfrak{g})$ plays a prominent role in the construction of canonical bases. More recently, it has appeared in connection with categorification of quantum groups.

2.3. Verma modules. This section contains some basic definitions and results on weight modules and Verma modules for $U_q(\mathfrak{g})$. Throughout, our basic assumption is that \mathbb{K} is a field and $q = s^L \in \mathbb{K}^\times$ is not a root of unity. At this stage, some constructions and results work for more general parameters q , but our interest is in the case that q is not a root of unity anyway.

2.3.1. The parameter space \mathfrak{h}_q^* . The notion of a (non-integral) weight for a representation of the quantum enveloping algebra $U_q(\mathfrak{g})$ is more subtle than in the classical case. To illustrate this, let us start with the case where the ground field is $\mathbb{K} = \mathbb{C}$ and q is a strictly positive real. This will be the main case of interest in later chapters.

Let $h \in \mathbb{R}^\times$ with $q = e^h$ and put $\hbar = h/2\pi$. One says that a vector v in a representation of $U_q(\mathfrak{g})$ is a weight vector of weight $\lambda \in \mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ if it is a common eigenvector for the action of $U_q(\mathfrak{h})$ with

$$K_\mu \cdot v = q^{(\lambda, \mu)} v, \quad \text{for all } \mu \in \mathbf{P}.$$

Note, however, that the characters

$$U_q(\mathfrak{h}) \rightarrow \mathbb{C}; \quad K_\mu \mapsto q^{(\lambda, \mu)}$$

are not all distinct. Specifically, $q^{(\lambda, \cdot)} \equiv 1$ as a function on \mathbf{P} whenever $\lambda \in \frac{2\pi i}{h} \mathbf{Q}^\vee = i\hbar^{-1} \mathbf{Q}^\vee$. Therefore, the appropriate parameter space for weights is

$$\mathfrak{h}_q^* = \mathfrak{h}^* / i\hbar^{-1} \mathbf{Q}^\vee.$$

For more general ground fields \mathbb{K} , weights should be understood as algebra characters of the Cartan part $U_q(\mathfrak{h})$ of $U_q(\mathfrak{g})$. The set of such characters is an abelian group with product dual to the coproduct on $U_q(\mathfrak{h})$, namely for any characters χ_1, χ_2 of $U_q(\mathfrak{h})$,

$$\chi_1 \chi_2(K) = (\chi_1 \otimes \chi_2)(\hat{\Delta}K), \quad \text{for all } K \in U_q(\mathfrak{h}).$$

This character group is isomorphic to $\text{Hom}(\mathbf{P}, \mathbb{K}^\times)$, upon observing that $U_q(\mathfrak{h})$ is the group algebra $\mathbb{K}[\mathbf{P}]$. In order to be consistent with the notation when $\mathbb{K} = \mathbb{C}$, we will use a formal q -exponential notation for elements of $\text{Hom}(\mathbf{P}, \mathbb{K}^\times)$, as follows.

Definition 2.26. The parameter space \mathfrak{h}_q^* is the character group of $U_q(\mathfrak{h})$, written additively. For an element $\lambda \in \mathfrak{h}_q^*$ we denote the corresponding character of $U_q(\mathfrak{h})$ by χ_λ . The associated elements of $\text{Hom}(\mathbf{P}, \mathbb{K}^\times)$ will be written as formal q -exponentials,

$$\mathbf{P} \rightarrow \mathbb{K}^\times; \quad \mu \mapsto q^{(\lambda, \mu)},$$

so that $\chi_\lambda(K_\mu) = q^{(\lambda, \mu)}$ for all $\mu \in \mathbf{P}$.

To justify this notation, let us make some observations according to the properties of the base field \mathbb{K} .

Firstly, consider the case where \mathbb{K} is a field satisfying no further assumptions. Let $\lambda \in \mathbf{P}$ be an integral weight. Note that $(\lambda, \mu) \in \frac{1}{L}\mathbb{Z}$ for all $\mu \in \mathbf{P}$, so it makes sense to define a character of $U_q(\mathfrak{h}_q)$ by

$$\chi_\lambda(K_\mu) = q^{(\lambda, \mu)} = s^{L(\lambda, \mu)}.$$

The map $\lambda \mapsto \chi_\lambda$ is a homomorphism of \mathbf{P} into the characters of $U_q(\mathfrak{h})$, and we claim that it is injective. Indeed, if λ is in the kernel of this map, then we have $q^{(\lambda, \mu)} = 1$ for all $\mu \in \mathbf{P}$. Since q is not a root of unity, this implies $\lambda = 0$. Therefore, we obtain an embedding

$$\mathbf{P} \subset \mathfrak{h}_q^*$$

which is compatible with the formal q -exponential notation $q^{(\lambda, \mu)}$ above.

Occasionally, we will consider the stronger condition that $q = s^{2L}$, in which case we have an embedding $\frac{1}{2}\mathbf{P} \subset \mathfrak{h}_q^*$.

The other case of interest to us is that the field \mathbb{K} is exponential and the parameter q is obtained by exponentiating an element $h \in \mathbb{K}$. Recall that an exponential field is a field \mathbb{K} of characteristic zero together with a group homomorphism $e^\bullet : \mathbb{K} \rightarrow \mathbb{K}^\times$, which we will denote by $e^\bullet(h) = e^h$. We shall assume that $q = e^h$ for some element $h \in \mathbb{K}$, and write q^\bullet for the group homomorphism $\mathbb{K} \rightarrow \mathbb{K}^\times$ given by $q^\bullet(x) = e^{hx}$.

In this case we set $\mathfrak{h}^* = \mathbb{K} \otimes_{\mathbb{Z}} \mathbf{P}$. We can associate to any $\lambda \in \mathfrak{h}^*$ the character χ_λ given by

$$\chi_\lambda(K_\mu) = q^{(\lambda, \mu)} = e^{h(\lambda, \mu)}.$$

Again, the map $\lambda \mapsto \chi_\lambda$ is a homomorphism from \mathfrak{h}^* to the characters of $U_q(\mathfrak{h})$, and therefore induces a homomorphism

$$\mathfrak{h}^* \rightarrow \mathfrak{h}_q^*; \quad \lambda \mapsto \chi_\lambda,$$

which is compatible with the formal q -exponential notation. Under the continuing assumption that q is not a root of unity, the kernel of this map is $\ker(q^\bullet)\mathbb{Q}^\vee$, so we obtain an embedding

$$\mathfrak{h}^* / \ker(q^\bullet)\mathbb{Q}^\vee \subset \mathfrak{h}_q^*.$$

Of course, the prototypical example of an exponential field is $\mathbb{K} = \mathbb{C}$ with the standard exponential function. In this case, $q^\bullet : \mathbb{C} \rightarrow \mathbb{C}^\times$ is surjective, with kernel $\ker(q^\bullet) = (2\pi i/h)\mathbb{Z} = i\hbar^{-1}\mathbb{Z}$. Using the fact that any character of $U_q(\mathfrak{h})$ is uniquely

determined by its values on the generators K_{ϖ_i} for $i = 1, \dots, n$, one can check that the map $\mathfrak{h}^* \rightarrow \mathfrak{h}_q^*$ is a surjection, and we have an isomorphism

$$\mathfrak{h}_q^* \cong \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee.$$

In other words, Definition 2.26 is consistent with our definition of \mathfrak{h}_q^* at the start of this subsection in the case $\mathbb{K} = \mathbb{C}$.

There is an action of the Weyl group W on the quantum weight space \mathfrak{h}_q^* , induced by the action on $\text{Hom}(\mathbf{P}, \mathbb{K}^\times)$. Specifically, if $w \in W$ and $\lambda \in \mathfrak{h}_q^*$ then $w\lambda$ is defined by

$$q^{(w\lambda, \mu)} = q^{(\lambda, w^{-1}\mu)}$$

for all $\mu \in \mathbf{P}$. This extends the usual action of W on $\mathbf{P} \subset \mathfrak{h}_q^*$.

We can extend the usual ordering on \mathbf{Q} to \mathfrak{h}_q^* as follows. Recall that we write \mathbf{Q}^+ for the set of non-negative integer combinations of the simple roots.

Definition 2.27. We define a relation \geq on \mathfrak{h}_q^* by saying that

$$\lambda \geq \mu \quad \text{if} \quad \lambda - \mu \in \mathbf{Q}^+.$$

Here we are identifying \mathbf{Q}^+ with its image in \mathfrak{h}_q^* .

Since \mathbf{Q}^+ is embedded in \mathfrak{h}_q^* , it is a simple matter to check that this is indeed a partial order.

2.3.2. Weight modules and highest weight modules. Let V be a left module over $U_q(\mathfrak{g})$. For any $\lambda \in \mathfrak{h}_q^*$ we define the weight space

$$V_\lambda = \{v \in V \mid K_\mu \cdot v = q^{(\mu, \lambda)}v \text{ for all } \mu \in \mathbf{P}\}.$$

A vector $v \in V$ is said to have weight λ iff $v \in V_\lambda$.

Definition 2.28. A module V over $U_q(\mathfrak{g})$ is called a weight module if it is the direct sum of its weight spaces V_λ for $\lambda \in \mathfrak{h}_q^*$. We say that λ is a weight of V if $V_\lambda \neq 0$.

Every submodule of a weight module is again a weight module. This is a consequence of Artin's Theorem on linear independence of characters.

Definition 2.29. A $U_q(\mathfrak{g})$ -module V is called integrable if it is a weight module whose weights all belong to \mathbf{P} , and the operators E_i, F_j are locally nilpotent on V for all $1 \leq i, j \leq N$.

Such modules are sometimes referred to as type 1 modules.

Definition 2.30. A vector v in a weight module V is called primitive if

$$E_i \cdot v = 0 \quad \text{for all } i = 1, \dots, N.$$

A module of highest weight $\lambda \in \mathfrak{h}_q^*$ is a weight module V with a primitive cyclic vector $v_\lambda \in V$ of weight λ .

If V is a weight module such that all weight spaces are finite dimensional, we define the restricted dual of V to be the $U_q(\mathfrak{g})$ -module

$$V^\vee = \bigoplus_{\lambda \in \mathfrak{h}_q^*} \text{Hom}(V_\lambda, \mathbb{K}),$$

with the left $U_q(\mathfrak{g})$ -module structure given by

$$(X \cdot f)(v) = f(\tau(X) \cdot v).$$

Here τ is the automorphism from Lemma 2.16. It is clear that V^\vee is again a weight module with $(V^\vee)_\lambda = \text{Hom}(V_\lambda, \mathbb{K})$. Notice that we have a canonical isomorphism $V^{\vee\vee} \cong V$ since τ is involutive. If $0 \rightarrow K \rightarrow M \rightarrow Q \rightarrow 0$ is an exact sequence

of weight modules with finite dimensional weight spaces then the dual sequence $0 \rightarrow Q^\vee \rightarrow M^\vee \rightarrow K^\vee \rightarrow 0$ is again exact.

2.3.3. The definition of Verma modules. Let us now come to the definition of Verma modules. Recall that $U_q(\mathfrak{b}_+)$ denotes the subalgebra of $U_q(\mathfrak{g})$ generated by the elements E_i and K_λ . The projection

$$U_q(\mathfrak{b}_+) \cong U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+) \xrightarrow{\text{id} \otimes \epsilon} U_q(\mathfrak{h})$$

which kills the generators E_i is an algebra morphism, and we can use this to extend any character $\chi : U_q(\mathfrak{h}) \rightarrow \mathbb{K}$ to a character of $U_q(\mathfrak{b}_+)$ which sends the generators E_i to zero. We write again χ for the resulting homomorphism $U_q(\mathfrak{b}_+) \rightarrow \mathbb{K}$.

Conversely, every algebra homomorphism $U_q(\mathfrak{b}_+) \rightarrow \mathbb{K}$ vanishes on the elements E_i , and therefore is determined by a homomorphism $U_q(\mathfrak{h}) \rightarrow \mathbb{K}$. In particular, for each $\lambda \in \mathfrak{h}_q^*$ we have the character χ_λ of $U_q(\mathfrak{b}_+)$ determined, as above, by

$$\chi_\lambda(K_\mu) = q^{(\lambda, \mu)}, \quad \text{for all } \mu \in \mathbf{P}.$$

Definition 2.31. The Verma module $M(\lambda)$ associated to $\lambda \in \mathfrak{h}_q^*$ is the induced $U_q(\mathfrak{g})$ -module

$$M(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{K}_\lambda$$

where \mathbb{K}_λ denotes the one-dimensional $U_q(\mathfrak{b}_+)$ -module \mathbb{K} with the action induced from the character χ_λ . The vector $v_\lambda = 1 \otimes 1 \in U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_+)} \mathbb{K}_\lambda$ is called the highest weight vector of $M(\lambda)$.

By construction, $M(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is a highest weight module of highest weight λ , and every other highest weight module of highest weight λ is a quotient of $M(\lambda)$.

Using Proposition 2.14 one checks that $M(\lambda)$ is free as a $U_q(\mathfrak{n}_-)$ -module. More specifically, the natural map $U_q(\mathfrak{n}_-) \rightarrow M(\lambda)$ given by $Y \mapsto Y \cdot v_\lambda$ sends $U_q(\mathfrak{n}_-)^\mu$ bijectively on to $M(\lambda)_{\lambda - \mu}$.

As in the classical case, the Verma module $M(\lambda)$ contains a unique maximal submodule.

Lemma 2.32. *The Verma module $M(\lambda)$ contains a unique maximal proper submodule $I(\lambda)$, namely the linear span of all submodules not containing the highest weight vector v_λ .*

Proof. Since every submodule U of $M(\lambda)$ is a weight module, it is a proper submodule iff it does not contain the highest weight vector. That is, U is contained in the sum of all weight spaces different from $M(\lambda)_\lambda$. In particular, the sum $I(\lambda)$ of all proper submodules does not contain v_λ , which means that $I(\lambda)$ is the unique maximal proper submodule of $M(\lambda)$. \square

The resulting simple quotient module $M(\lambda)/I(\lambda)$ will be denoted $V(\lambda)$. It is again a weight module, and all its weight spaces are finite dimensional. We may therefore form the restricted dual $V(\lambda)^\vee$ of $V(\lambda)$. Note that $V(\lambda)^\vee$ is simple for any $\lambda \in \mathfrak{h}_q^*$ because $V(\lambda)^{\vee\vee} \cong V(\lambda)$ is simple. Since $V(\lambda)^\vee$ is again a highest weight module of highest weight λ we conclude $V(\lambda)^\vee \cong V(\lambda)$.

2.4. Characters of $U_q(\mathfrak{g})$. Unlike the classical case, the quantized enveloping algebras $U_q(\mathfrak{g})$ admit several one-dimensional representations, with a corresponding multiplicity for all finite dimensional representations of $U_q(\mathfrak{g})$. We will only really be interested in the integrable (type 1) finite dimensional modules, but at certain points we will need to acknowledge the existence of their non-integrable analogues.

Definition 2.33. We write \mathbf{X}_q for the set of weights $\omega \in \mathfrak{h}_q^*$ satisfying

$$q^{(\omega, \alpha)} = \pm 1, \quad \text{for all } \alpha \in \mathbf{Q}.$$

Proposition 2.34. *The algebra characters of $U_q(\mathfrak{g})$ are in one-to-one correspondence with elements of \mathbf{X}_q . Specifically, for every $\omega \in \mathbf{X}_q$, there is an algebra character $\chi_\omega : U_q(\mathfrak{g}) \rightarrow \mathbb{K}$ defined on generators by*

$$\chi_\omega(K_\mu) = q^{(\omega, \mu)}, \quad \chi_\omega(E_i) = \chi_\omega(F_i) = 0,$$

and every algebra character of $U_q(\mathfrak{g})$ is of this form.

Proof. One can directly check that the formulas for χ_ω respect the defining relations of $U_q(\mathfrak{g})$. Conversely, suppose χ is an algebra character of $U_q(\mathfrak{g})$. Considering the restriction of χ to $U_q(\mathfrak{h})$, there is $\omega \in \mathfrak{h}_q^*$ with $\chi(K_\mu) = q^{(\omega, \mu)}$ for all $\mu \in \mathbf{P}$. Since

$$0 = [\chi(E_i), \chi(F_i)] = \frac{q^{(\omega, \alpha_i)} - q^{-(\omega, \alpha_i)}}{q_i - q_i^{-1}},$$

we have $q^{(\omega, \alpha_i)} = \pm 1$ for all i , so $\omega \in \mathbf{X}_q$. Finally, from the relation $K_\mu E_j K_\mu^{-1} = q^{(\alpha_j, \mu)} E_j$ and the fact that q is not a root of unity, we deduce that $\chi(E_j) = 0$, and similarly $\chi(F_j) = 0$. \square

Definition 2.35. We define the extended integral weight lattice by

$$\mathbf{P}_q = \mathbf{P} + \mathbf{X}_q.$$

We also put $\mathbf{P}_q^+ = \mathbf{P}^+ + \mathbf{X}_q$.

Note that $\mathbf{P} \cap \mathbf{X}_q = \{0\}$ by the assumption that q is not a root of unity.

2.5. Finite dimensional representations of $U_q(\mathfrak{sl}(2, \mathbb{K}))$. In this section we discuss the finite dimensional representation theory of $U_q(\mathfrak{sl}(2, \mathbb{K}))$. We work over a field \mathbb{K} containing the non-root of unity $q = s^2 \in \mathbb{K}^\times$, and we write $s = q^{\frac{1}{2}}$ for simplicity.

When discussing $U_q(\mathfrak{sl}(2, \mathbb{K}))$, we will use the following notation for the generators. Let α be the unique simple root and $\varpi = \frac{1}{2}\alpha$ be the fundamental weight. We write

$$E = E_1, \quad F = F_1, \quad K = K_\varpi, \quad K^2 = K_\alpha = K_1.$$

We warn the reader that with this notation one must replace K_i with K^2 when specializing formulas for $U_q(\mathfrak{g})$ in terms of E_i, F_i, K_i to formulas for $U_q(\mathfrak{sl}(2, \mathbb{K}))$.

We will often identify \mathbf{P} with $\frac{1}{2}\mathbb{Z}$ in the sequel, so that ϖ corresponds to $\frac{1}{2}$ and α corresponds to 1. Note that K acts on any vector v of weight $m \in \frac{1}{2}\mathbb{Z}$ by

$$K \cdot v = q^m v.$$

It is not hard to check that the formulas

$$\pi_{\frac{1}{2}}(E) = \begin{pmatrix} 0 & q^{-\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \quad \pi_{\frac{1}{2}}(K) = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad \pi_{\frac{1}{2}}(F) = \begin{pmatrix} 0 & 0 \\ q^{\frac{1}{2}} & 0 \end{pmatrix}$$

define a 2-dimensional irreducible representation $\pi_{\frac{1}{2}} : U_q(\mathfrak{sl}(2, \mathbb{K})) \rightarrow M_2(\mathbb{K})$. We will write $V(\frac{1}{2})$ for this module.

In order to construct further simple modules we shall use the following specific instance of Lemma 2.11. We recall the notation

$$[K^2; m] = \frac{q^m K^2 - q^{-m} K^{-2}}{q - q^{-1}}$$

for $m \in \mathbb{Z}$.

Lemma 2.36. *In $U_q(\mathfrak{sl}(2, \mathbb{K}))$ we have*

$$[E, F^{m+1}] = \frac{1}{q - q^{-1}} [m + 1]_q F^m (q^{-m} K^2 - q^m K^{-2}) = [m + 1]_q F^m [K^2; -m]$$

for all $m \in \mathbb{N}_0$.

Proposition 2.37. *Let $\lambda \in \mathfrak{h}_q^*$.*

- a) *The Verma module $M(\lambda)$ is simple if and only if $\lambda \notin \mathbf{P}_q^+$.*
- b) *If $\lambda \in \mathbf{P}_q^+$, so that $\lambda = n\alpha + \omega$ with $n \in \frac{1}{2}\mathbb{N}_0$, $\omega \in \mathbf{X}_q$ then $M(\lambda)$ has a unique simple quotient $V(\lambda)$ of highest weight λ and dimension $2n + 1$. Up to isomorphism, these are the only finite dimensional simple weight modules of $U_q(\mathfrak{sl}(2, \mathbb{K}))$.*
- c) *If $\lambda \in \mathbf{P}^+$ then $V(\lambda)$ is integrable, and these are the only simple integrable $U_q(\mathfrak{sl}(2, \mathbb{K}))$ -modules up to isomorphism.*

Proof. a) The Verma module $M(\lambda)$ is non-simple if and only if it contains a primitive vector of the form $F^{m+1} \cdot v_\lambda$ for some $m \geq 0$. According to Lemma 2.36, this occurs if and only if $q^{-m} q^{(\lambda, \alpha)} - q^m q^{-(\lambda, \alpha)} = 0$, or equivalently $q^{(\lambda - m\varpi, \alpha)} = \pm 1$. Putting $\omega = \lambda - m\varpi$ we have $\omega \in \mathbf{X}_q$ and so $\lambda = m\varpi + \omega \in \mathbf{P}_q^+$.

b) In the case $\lambda = m\varpi + \omega \in \mathbf{P}_q^+$, the simple quotient is spanned by $\{F^i \cdot v_\lambda \mid i = 0, \dots, m\}$. Putting $m = 2n$ we obtain the dimension formula. Moreover, any finite dimensional simple $U_q(\mathfrak{sl}(2, \mathbb{K}))$ -module is necessarily a highest weight module, and so a quotient of some Verma module $M(\lambda)$.

c) Likewise, any simple integrable module is necessarily a highest weight module with highest weight in \mathbf{P} . Since F must act locally nilpotently, a) implies that $\lambda \in \mathbf{P}^+$. \square

In the sequel we will write $V(n)$ for the integrable module $V(n\alpha)$ with $n \in \frac{1}{2}\mathbb{N}_0$. Let us give an explicit description of $V(n)$.

Lemma 2.38. *Let $n \in \frac{1}{2}\mathbb{N}_0$. The module $V(n)$ over $U_q(\mathfrak{sl}(2, \mathbb{K}))$ has a \mathbb{K} -linear basis $v_n, v_{n-1}, \dots, v_{-n+1}, v_{-n}$ such that*

$$\begin{aligned} K \cdot v_j &= q^j v_j, \\ F \cdot v_j &= v_{j-1}, \\ E \cdot v_j &= [n - j][n + j + 1] v_{j+1}. \end{aligned}$$

Here we interpret $v_k = 0$ if k is not contained in the set $\{n, n - 1, \dots, -n\}$. If we rescale this basis as

$$v_{(j)} = \frac{1}{[n - j]!} v_j$$

we have

$$\begin{aligned} K \cdot v_{(j)} &= q^j v_{(j)}, \\ F \cdot v_{(j)} &= [n - j + 1] v_{(j-1)}, \\ E \cdot v_{(j)} &= [n + j + 1] v_{(j+1)}. \end{aligned}$$

Proof. This is an explicit description of the basis obtained from applying the operator F to the highest weight vector v_n . Since $FK = qKF$ we obtain the action of K on the vectors v_j by induction, for the inductive step use

$$K \cdot v_{j-1} = KF \cdot v_j = q^{-1} FK \cdot v_j = q^{j-1} F \cdot v_j = q^{j-1} v_{j-1}.$$

Moreover, Lemma 2.36 yields

$$EF^{r+1} \cdot v_n = [E, F^{r+1}] \cdot v_n = [r + 1][2n - r] F^r \cdot v_n$$

for all $r \in \mathbb{N}_0$, hence setting $r = n - j - 1$ we get

$$\begin{aligned} E \cdot v_j &= EF^{n-j} \cdot v_n = [n-j][n+j+1]F^{n-j-1} \cdot v_n \\ &= [n-j][n+j+1]v_{j+1} \end{aligned}$$

The second set of equations follows immediately from these formulas, indeed, we have

$$\begin{aligned} E \cdot v_{(j)} &= \frac{1}{[n-j]!} E \cdot v_j = \frac{[n+j+1]}{[n-j-1]!} v_{j+1} = [n+j+1]v_{(j+1)}, \\ F \cdot v_{(j)} &= \frac{1}{[n-j]!} F \cdot v_j = \frac{[n-j+1]}{[n-j+1]!} v_{j-1} = [n-j+1]v_{(j-1)} \end{aligned}$$

as desired. \square

We point out that the labels of the basis vectors in Lemma 2.38 run over half-integers if n is a half-integer, and over integers if $n \in \mathbb{N}_0$.

The non-integrable simple modules are obtained by twisting the integrable simple modules by a character. Specifically, if $n \in \frac{1}{2}\mathbb{N}$ and $\omega \in \mathbf{X}_q$ then

$$V(n\alpha + \omega) \cong V(n) \otimes V(\omega),$$

since both are simple modules of highest weight $n\alpha + \omega$. More generally, we have the following.

Lemma 2.39. *All weights of a finite dimensional weight module V for $U_q(\mathfrak{sl}(2, \mathbb{K}))$ belong to \mathbf{P}_q . Moreover, V admits a direct sum decomposition*

$$V \cong \bigoplus_{\omega \in \mathbf{X}_q} W^\omega \otimes V(\omega)$$

where the W^ω are finite dimensional integrable $U_q(\mathfrak{sl}(2, \mathbb{K}))$ -modules.

Proof. Let $\lambda \in \mathfrak{h}_q^*$ be a weight of V . If we fix $v \in V_\lambda$ nonzero then $E^k \cdot v$ is primitive vector for some $k \in \mathbb{N}_0$. By Proposition 2.37 we must have $\lambda + k\alpha \in \mathbf{P}_q^+$ and so $\lambda \in \mathbf{P}_q$.

Now fix $\omega \in \mathbf{X}_q$. The subspace

$$V_{\mathbf{P}+\omega} = \bigoplus_{\mu \in \mathbf{P}} V_{\mu+\omega}$$

is invariant under the $U_q(\mathfrak{g})$ -action. We obtain a direct sum decomposition $V = \bigoplus_{\omega \in \mathbf{X}_q} V_{\mathbf{P}+\omega}$. Putting $W^\omega = V_{\mathbf{P}+\omega} \otimes V(-\omega)$ yields the result. \square

We can now prove complete reducibility for finite dimensional weight modules of $U_q(\mathfrak{sl}(2, \mathbb{K}))$.

Proposition 2.40. *Every finite dimensional weight module of $U_q(\mathfrak{sl}(2, \mathbb{K}))$ is completely reducible.*

Proof. By Lemma 2.39, it suffices to proof the proposition for finite dimensional integrable modules V . In this case, the weights of V belong to \mathbf{P} .

Let $\mu \in \mathbf{P}$ be maximal among the weights of V . Any vector $v \in V_\mu$ generates a finite dimensional $U_q(\mathfrak{sl}(2, \mathbb{K}))$ -module isomorphic to $V(\mu)$. Since the intersection of this submodule with V_μ is $\mathbb{K}v$, one checks that V contains a direct sum of $\dim(V_\mu)$ copies of $V(\mu)$ as a submodule.

Writing K for this submodule and $Q = V/K$ for the corresponding quotient, we obtain a short exact sequence $0 \rightarrow K \rightarrow V \rightarrow Q \rightarrow 0$ of finite dimensional weight modules. Using induction on dimension we may assume that Q is a direct sum of simple modules. Moreover the dual sequence $0 \rightarrow Q^\vee \rightarrow V^\vee \rightarrow K^\vee \rightarrow 0$, see the constructions after Definition 2.28, is split by construction because the highest

weight subspace of V^\vee maps isomorphically onto the highest weight subspace of K^\vee . Applying duality again shows that the original sequence is split exact, which means that V is a direct sum of simple modules. \square

2.6. Finite dimensional representations of $U_q(\mathfrak{g})$. In this section we begin the study of finite dimensional representations of $U_q(\mathfrak{g})$. Ultimately, in Section 2.9, we will prove complete reducibility of finite dimensional weight modules and obtain a classification the finite dimensional simple modules, although those results require considerably more machinery than in the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ discussed above. For more information we refer to [39], [57].

Throughout this section we assume that \mathfrak{g} is a semisimple Lie algebra and $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

2.6.1. Rank-one quantum subgroups. For each $1 \leq i \leq N$, we write $U_{q_i}(\mathfrak{g}_i) \subset U_q(\mathfrak{g})$ for the subalgebra generated by $E_i, F_i, K_i^{\pm 1}$. It is isomorphic as a Hopf algebra to the subalgebra of $U_{q_i}(\mathfrak{sl}(2, \mathbb{K}))$ generated by $E, F, K^{\pm 2}$.

The classification of finite dimensional simple modules for $U_{q_i}(\mathfrak{g}_i)$ is essentially identical to that of $U_{q_i}(\mathfrak{sl}(2, \mathbb{K}))$: such modules are indexed by highest weights of the form $\mu = n\alpha_i + \omega$ where $n \in \frac{1}{2}\mathbb{N}_0$ and $q_i^{(\omega, \cdot)}$ denotes a character of $\mathbb{Z}\alpha_i$ with values in $\{\pm 1\}$.

Note that if $v \in V_\lambda$ is a vector of weight $\lambda \in \mathfrak{h}_q^*$ for some $U_q(\mathfrak{g})$ -module V , then K_i acts on v by

$$K_i \cdot v = q_i^{(\lambda, \alpha_i^\vee)} v.$$

In particular, if V is an integrable $U_q(\mathfrak{g})$ -module then it becomes an integrable $U_{q_i}(\mathfrak{g}_i)$ -module by restriction, wherein vectors in V_λ have weight

$$\frac{1}{2}(\lambda, \alpha_i^\vee) \in \frac{1}{2}\mathbb{Z}$$

under the usual identification of half-integers with integral weights for $U_{q_i}(\mathfrak{sl}(2, \mathbb{K}))$.

2.6.2. Finite dimensional modules. As a consequence of the above remarks, we obtain some basic structural results for finite dimensional $U_q(\mathfrak{g})$ -modules. We begin with the analogue of Lemma 2.39, which will let us restrict our attention to integrable modules.

Lemma 2.41. *All the weights of a finite dimensional weight module V for $U_q(\mathfrak{g})$ belong to \mathbf{P}_q . Moreover, V admits a direct sum decomposition*

$$V \cong \bigoplus_{\omega \in \mathbf{X}_q} W^\omega \otimes V(\omega)$$

where each W^ω is a finite dimensional integrable $U_q(\mathfrak{g})$ -module and $V(\omega)$ is the one dimensional representation with weight $\omega \in \mathbf{X}_q$.

Proof. Let $\lambda \in \mathfrak{h}_q^*$ and $v \in V_\lambda$ be a nonzero weight vector. For each $i = 1, \dots, N$, we have

$$K_i \cdot v = q^{(\lambda, \alpha_i)} v.$$

Considering V as a $U_{q_i}(\mathfrak{sl}(2, \mathbb{K}))$ -module by restriction, Lemma 2.39 implies that $q^{(\lambda, \alpha_i)} \in \pm q^{\mathbb{Z}}$. It follows that $\lambda \in \mathbf{P}_q$. The direct sum decomposition then follows exactly as in the proof of Lemma 2.39. \square

Proposition 2.42. *The set of weights of any integrable $U_q(\mathfrak{g})$ -module V , and their multiplicities, are invariant under the Weyl group.*

Proof. Let $1 \leq i \leq N$. By Proposition 2.40 V decomposes as a direct sum of irreducible integrable $U_{q_i}(\mathfrak{g}_i)$ -modules. It follows that the set of weights of V and their multiplicities are invariant under the simple reflection s_i . Since these generate W , the result follows. \square

Let us fix a dominant integral weight $\mu \in \mathbf{P}^+$ and $1 \leq i \leq N$. Then $(\alpha_i^\vee, \mu) \in \mathbb{N}_0$, and we claim that $v = F_i^{(\alpha_i^\vee, \mu) + 1} \cdot v_\mu$ is a primitive vector in $M(\mu)$. Indeed, Lemma 2.11 shows that $E_i \cdot v = 0$, and $E_j \cdot v = 0$ for $j \neq i$ follows from the fact that $[E_j, F_i] = 0$ and the primitivity of v_λ . The vector v has weight $s_i \cdot \mu$, where $s_i \in W$ denotes the simple reflection corresponding to α_i and

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

denotes the shifted Weyl group action. Hence v generates a homomorphic image of $M(s_i \cdot \mu)$ inside $M(\mu)$. By slight abuse of notation we write $M(s_i \cdot \mu) \subset M(\mu)$ for this submodule, since it will be shown later that the homomorphism $M(s_i \cdot \mu) \rightarrow M(\mu)$ is indeed injective.

We are now ready to describe the finite dimensional integrable quotients of Verma modules, compare Section 5.9. in [39].

Theorem 2.43. *Let $\lambda \in \mathbf{P}^+$. Then the largest integrable quotient $L(\lambda)$ of $M(\lambda)$ is finite dimensional and given by*

$$L(\lambda) = M(\lambda) \Big/ \sum_{i=1}^N M(s_i \cdot \lambda).$$

If $\lambda \in \mathfrak{h}_q^ \setminus \mathbf{P}^+$ then the Verma module $M(\lambda)$ does not admit any nontrivial integrable quotients.*

Proof. It is clear that any integrable quotient of $M(\lambda)$ must annihilate the sum of the modules $M(s_i \cdot \lambda)$, because otherwise the action of F_i on v_λ fails to be locally finite.

In order to show that the action of F_i on any $v \in L(\lambda)$ is locally nilpotent we use that v can be written as a sum of terms $F \cdot v_\lambda$ where F is a monomial in F_1, \dots, F_N . Let us show by induction on the degree r of $F = F_{i_1} \cdots F_{i_r}$, that $F_i^k \cdot F \cdot v_\lambda = 0$ for some $k \in \mathbb{N}$. For $r = 0$ the claim is obvious from the definition of $L(\lambda)$. Assume now $F = F_j Y \cdot v_\lambda$ where Y is a monomial of degree r and $1 \leq j \leq N$. If $j = i$ then the claim follows from our inductive hypothesis, so let us assume $j \neq i$. Writing $u = Y \cdot v_\lambda$ we have

$$F_i \cdot (F_j \cdot u) = (F_i \rightarrow F_j) \cdot u + K_i^{-1} F_j K_i F_i \cdot u.$$

By applying this relation iteratively we obtain $F_i^k \cdot (F_j \cdot u) = 0$ in $L(\lambda)$ for a suitable $k \in \mathbb{N}$, using the inductive hypothesis and the fact that the Serre relations imply $F_i^{1-a_{ij}} \rightarrow F_j = 0$. It follows that $L(\lambda)$ is integrable, and it is indeed the largest integrable quotient of $M(\lambda)$.

By Proposition 2.42, the set of weights of $L(\lambda)$ is invariant under the Weyl group action. Since each weight space is finite dimensional this implies that $L(\lambda)$ is finite dimensional.

Finally, suppose $M(\lambda)$ admits a nontrivial integrable quotient for some $\lambda \in \mathfrak{h}_q^* \setminus \mathbf{P}^+$. The definition of integrability implies $\lambda \in \mathbf{P}$. But since $\lambda \notin \mathbf{P}^+$, we have $(\alpha_i^\vee, \lambda) \notin \mathbb{N}_0$ for some i . It follows that the $U_{q_i}(\mathfrak{g}_i)$ -module generated by v_λ is infinite dimensional and simple by Proposition 2.37, and hence F_i does not act locally nilpotently on any quotient of $M(\lambda)$. \square

Note that we will ultimately observe, in Theorem 2.97, that when $\mu \in \mathbf{P}$, the finite dimensional quotients $L(\mu)$ are irreducible, and hence $L(\mu) = V(\mu)$.

Corollary 2.44. *The Verma module $M(\lambda)$ admits a finite dimensional quotient if and only if $\lambda \in \mathbf{P}_q^+$.*

Proof. If $M(\lambda)$ admits a nontrivial finite dimensional quotient then Lemma 2.41 implies $\lambda \in \mathbf{P}_q$, so $\lambda = \mu + \omega$ for some $\mu \in \mathbf{P}$, $\omega \in \mathbf{X}_q$. It is easy to check that

$M(\mu) \cong M(\lambda) \otimes V(-\omega)$. It follows that $M(\mu)$ has a nontrivial finite dimensional hence integrable quotient. Theorem 2.43 implies $\mu \in \mathbf{P}^+$, so $\lambda \in \mathbf{P}_q^+$.

Conversely, if $\lambda = \mu + \omega \in \mathbf{P}_q^+$ then $M(\lambda) \cong M(\mu) \otimes V(\omega)$ admits the finite dimensional quotient $L(\lambda) \otimes V(\omega)$. \square

The following technical lemma says roughly that in weight spaces sufficiently close to the highest weight, the finite dimensional module $L(\mu)$ resembles the Verma module $M(\mu)$.

Lemma 2.45. *Let $\mu = \sum_i m_i \varpi_i \in \mathbf{P}^+$. If $\lambda = \sum_i l_i \alpha_i \in \mathbf{Q}^+$ with $l_i \leq m_i$ for all $i = 1, \dots, N$ then $L(\mu)_{\mu-\lambda} \cong M(\mu)_{\mu-\lambda} \cong U_q(\mathfrak{n}_-)_{-\lambda}$ as vector spaces, via the map which sends $F \in U_q(\mathfrak{n}_-)_{-\lambda}$ to $F \cdot v_\mu \in L(\mu)_{\mu-\lambda}$.*

Proof. Note that $s_i \cdot \mu = \mu - m_i \alpha_i$. Therefore, with the given conditions on λ we have $\mu - \lambda \not\leq s_i \cdot \mu$ for every $i = 1, \dots, N$, and the first claim follows from Theorem 2.43. The second claim then follows from the definition of ω . \square

If $\nu = \sum_i n_i \varpi_i \in \mathbf{P}^+$, we will use the notation $L(-\nu)$ to denote the finite dimensional module $L(\nu)$ twisted by the automorphism ω from Lemma 2.15, so that the action of $X \in U_q(\mathfrak{g})$ on $v \in L(-\nu) = L(\nu)$ is given by $\omega(X) \cdot v$. In this case we have $L(-\nu)_{-\nu+\lambda} \cong U_q(\mathfrak{n}_+)_{\lambda}$ provided $l_i \leq n_i$ for all $i = 1, \dots, N$.

An important fact is that finite dimensional integrable representations separate the points of $U_q(\mathfrak{g})$, compare for instance Section 7.1.5 of [48].

Theorem 2.46. *The finite dimensional integrable representations of $U_q(\mathfrak{g})$ separate points. That is, if $X \in U_q(\mathfrak{g})$ satisfies $\pi(X) = 0$ for all finite dimensional integrable representations $\pi : U_q(\mathfrak{g}) \rightarrow \text{End}(V)$, then $X = 0$.*

Proof. Let us write v_μ for the highest weight vector in $L(\mu)$ and $v_{-\nu}$ for the lowest weight vector in $L(-\nu)$. Assume $X \neq 0$ and write this element as a finite sum

$$X = \sum_{i,\eta,j} f_{ij} K_\eta e_{ij},$$

where $f_{ij} \in U_q(\mathfrak{n}_-)_{-\beta_i}$, $e_{ij} \in U_q(\mathfrak{n}_+)_{\gamma_j}$ for pairwise distinct weights β_1, \dots, β_m and $\gamma_1, \dots, \gamma_n$, and $\eta \in \mathbf{P}$. We shall show that X acts in a nonzero fashion on $L(-\nu) \otimes L(\mu)$ for suitable $\mu, \nu \in \mathbf{P}^+$.

Assume this is not the case, so that X acts by zero on all $L(-\nu) \otimes L(\mu)$ for $\mu, \nu \in \mathbf{P}^+$. Let γ_t be maximal among the γ_j such that some $f_{it} K_\eta e_{it}$ is nonzero. For each i, j we have $\hat{\Delta}(e_{ij}) = e_{ij} \otimes K_{\gamma_j} + b$ where b is a sum of terms in $U_q(\mathfrak{n}_+) \otimes (U_q(\mathfrak{b}_+) \cap \ker(\hat{\epsilon}))$ which vanish on $v_{-\nu} \otimes v_\mu$. We thus get

$$e_{ij} \cdot (v_{-\nu} \otimes v_\mu) = q^{(\gamma_j, \mu)} e_{ij} \cdot v_{-\nu} \otimes v_\mu.$$

Moreover, since $\hat{\Delta}(K_\eta) = K_\eta \otimes K_\eta$ we obtain

$$K_\eta e_{ij} \cdot (v_{-\nu} \otimes v_\mu) = q^{(\eta, \mu - \nu + \gamma_j)} q^{(\gamma_j, \mu)} e_{ij} \cdot v_{-\nu} \otimes v_\mu.$$

Finally, using that $\hat{\Delta}(f_{ij}) = K_{-\beta_i} \otimes f_{ij} + c$ where $c \in (U_q(\mathfrak{b}_-) \cap \ker(\hat{\epsilon})) \otimes (U_q(\mathfrak{n}_-))$ we get

$$f_{ij} K_\eta e_{ij} \cdot (v_{-\nu} \otimes v_\mu) = q^{(\beta_i, \nu - \gamma_j)} q^{(\eta, \mu - \nu + \gamma_j)} q^{(\gamma_j, \mu)} e_{ij} \cdot v_{-\nu} \otimes f_{ij} \cdot v_\mu + r,$$

where r is a sum of terms in $L(-\nu)_{-\nu+\delta} \otimes L(\mu)$ with $\delta \neq \gamma_t$. It follows that the component of $X \cdot (v_{-\nu} \otimes v_\mu)$ contained in $L(-\nu)_{-\nu+\gamma_t} \otimes L(\mu)$ is

$$0 = \sum_{i,\eta} q^{(\beta_i, \nu - \gamma_t)} q^{(\eta, \mu - \nu + \gamma_t)} q^{(\gamma_t, \mu)} e_{it} \cdot v_{-\nu} \otimes f_{it} \cdot v_\mu.$$

Assume without loss of generality that all e_{it}, f_{it} for $1 \leq i \leq m$ are nonzero, and using Lemma 2.45, fix ν large enough such that $e_{it} \cdot v_{-\nu}$ is nonzero for all i . Choosing

μ large enough we get that the vectors $e_{it} \cdot v_{-\nu} \otimes f_{it} \cdot v_\mu$ are linearly independent. Hence

$$0 = \sum_{\eta} q^{(\beta_i, \nu - \gamma_t)} q^{(\eta, \mu - \nu + \gamma_t)} q^{(\gamma_t, \mu)}$$

for all i in this case. Cancelling the common nonzero factors $q^{(\beta_i, \nu - \gamma_t)} q^{(\gamma_t, \mu)}$ and writing $c_\eta = q^{(\eta, \gamma_t - \nu)}$ we deduce

$$0 = \sum_{\eta} q^{(\eta, \mu - \nu + \gamma_t)} = \sum_{\eta} c_\eta q^{(\eta, \mu)} = \sum_{\eta} c_\eta \chi_\eta(K_\mu).$$

Consider the subsemigroup $P \subset \mathbf{P}^+$ of all $\mu \in \mathbf{P}^+$ such that the vectors $e_{it} \cdot v_{-\nu} \otimes f_{it} \cdot v_\mu$ are linearly independent. Since the characters χ_η on P are pairwise distinct, Artin's Theorem on the linear independence of characters implies $q^{(\eta, \gamma_t - \nu)} = c_\eta = 0$ for all η , which is clearly impossible.

Hence our initial assumption that X acts by zero on all modules of the form $V(-\nu) \otimes V(\mu)$ was wrong. This finishes the proof. \square

2.7. Braid group action and PBW basis. In this section we explain the construction of the PBW basis for $U_q(\mathfrak{g})$ and its integral form. A detailed exposition is given in part VI of [57] and chapter 8 of [39]; we shall follow the discussion in [39]. In order to define the PBW basis one first constructs an action of the braid group $B_{\mathfrak{g}}$ of \mathfrak{g} on $U_q(\mathfrak{g})$ and its integrable modules. Throughout this section we assume that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

Let \mathfrak{g} be a semisimple Lie algebra of rank N . The braid group $B_{\mathfrak{g}}$ is obtained from the Cartan matrix (a_{ij}) of \mathfrak{g} as follows, see [17] Section 8.1. Let m_{ij} be equal to 2, 3, 4, 6 iff $a_{ij}a_{ji}$ equals 0, 1, 2, 3, respectively. By definition, $B_{\mathfrak{g}}$ is the group with generators T_j for $j = 1, \dots, N$ and relations

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots$$

for all $1 \leq i, j \leq N$ with $i \neq j$, where there are m_{ij} terms on each side of the equation.

For \mathfrak{g} of type A one obtains the classical braid groups in this way. In general, there is a canonical quotient homomorphism $B_{\mathfrak{g}} \rightarrow W$, sending the generators T_j to the simple reflections s_j . Indeed, the Weyl group W is the quotient of $B_{\mathfrak{g}}$ by the relations $T_j^2 = 1$ for $j = 1, \dots, N$.

2.7.1. The case of $\mathfrak{sl}(2, \mathbb{K})$. We use the notation for $U_q(\mathfrak{sl}(2, \mathbb{K}))$ from Section 2.5, in particular we write $K = K_{\varpi}$ and $K^2 = K_{\alpha}$. Let V be an integrable module over $U_q(\mathfrak{sl}(2, \mathbb{K}))$. We label the weights by $\frac{1}{2}\mathbb{Z}$, so that $v \in V$ is of weight m if $K \cdot v = q^m v$. The corresponding weight space is denoted by V_m .

We define operators $\mathcal{T}_{\pm} : V \rightarrow V$ by

$$\begin{aligned} \mathcal{T}_+(v) &= \sum_{\substack{r, s, t \geq 0 \\ r - s + t = 2m}} (-1)^s q^{s - rt} F^{(r)} E^{(s)} F^{(t)} \cdot v \\ \mathcal{T}_-(v) &= \sum_{\substack{r, s, t \geq 0 \\ r - s + t = 2m}} (-1)^s q^{rt - s} F^{(r)} E^{(s)} F^{(t)} \cdot v \end{aligned}$$

for $v \in V_m$, where $E^{(i)} = E^i / [i]!$ and $F^{(j)} = F^j / [j]!$ are the divided powers. Note that these operators are well-defined because the action of E and F on V is locally finite. It is also clear that \mathcal{T}_{\pm} maps V_m to V_{-m} for any weight m .

Recall the basis $v_{(n)}, v_{(n-1)}, \dots, v_{(-n)}$ of $V(n)$ from Lemma 2.38. According to Lemma 2.38 we obtain the formulas

$$\begin{aligned} E^{(r)} \cdot v_{(j)} &= \frac{[n+j+1][n+j+2] \cdots [n+j+r]}{[r]} \cdot v_{(j)} = \begin{bmatrix} n+j+r \\ r \end{bmatrix} v_{(j+r)} \\ F^{(r)} \cdot v_{(j)} &= \frac{[n-j+1][n-j+2] \cdots [n-j+r]}{[r]} \cdot v_{(j)} = \begin{bmatrix} n-j+r \\ r \end{bmatrix} v_{(j-r)} \end{aligned}$$

for the action of divided powers on these basis vectors, where we adopt the convention $v_{(n+1)} = 0 = v_{(-n-1)}$.

Using these formulas shall derive explicit expressions for the action of the operators \mathcal{T}_{\pm} on the basis vectors $v_{(j)}$. As a preparation we state some properties of q -binomial coefficients.

Lemma 2.47. *Let $a \in \mathbb{Z}$ and $b, m \in \mathbb{N}_0$. Then*

$$\sum_{i=0}^m q^{ai-b(m-i)} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} = \begin{bmatrix} a+b \\ m \end{bmatrix}.$$

Proof. We proceed using induction on b . For $b = 0$ the sum reduces to the single term

$$q^{a0-0(m-0)} \begin{bmatrix} a \\ m-0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a+0 \\ m \end{bmatrix}.$$

Suppose the assertion holds for some b . Using Lemma 2.1 with q replaced by q^{-1} and the inductive hypothesis we compute

$$\begin{aligned} &\sum_{i=0}^m q^{ai-(b+1)(m-i)} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b+1 \\ i \end{bmatrix} \\ &= \sum_{i=0}^m q^{ai-b(m-i)-m+i} \begin{bmatrix} a \\ m-i \end{bmatrix} \left(q^{-i} \begin{bmatrix} b \\ i \end{bmatrix} + q^{b-i+1} \begin{bmatrix} b \\ i-1 \end{bmatrix} \right) \\ &= \sum_{i=0}^m q^{ai-b(m-i)-m} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} \\ &\quad + \sum_{i=1}^m q^{ai-b(m-i-1)-m+1} \begin{bmatrix} a \\ m-i \end{bmatrix} \begin{bmatrix} b \\ i-1 \end{bmatrix} \\ &= q^{-m} \begin{bmatrix} a+b \\ m \end{bmatrix} + \sum_{i=0}^{m-1} q^{ai+a-b(m-1-i)+b-m+1} \begin{bmatrix} a \\ m-i-1 \end{bmatrix} \begin{bmatrix} b \\ i \end{bmatrix} \\ &= q^{-m} \begin{bmatrix} a+b \\ m \end{bmatrix} + q^{a+b-m+1} \begin{bmatrix} a+b \\ m-1 \end{bmatrix} \\ &= \begin{bmatrix} a+b+1 \\ m \end{bmatrix} \end{aligned}$$

as desired. \square

Lemma 2.48. *For any $k, l \in \mathbb{N}_0$ we have*

$$\sum_{a=0}^k \sum_{c=0}^k (-1)^{a+c+k} q^{\pm(a+c-ac-k-kl)} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} a+l \\ k-c \end{bmatrix} \begin{bmatrix} c+l \\ c \end{bmatrix} = 1.$$

Proof. Using $\begin{bmatrix} c+l \\ c \end{bmatrix} = (-1)^c \begin{bmatrix} -l-1 \\ c \end{bmatrix}$ and $-(a+l)c - (l+1)(k-c) = c - ac - k - kl$ we obtain

$$\begin{aligned} & \sum_{a=0}^k \sum_{c=0}^k (-1)^{a+c+k} q^{\pm(a+c-ac-k-kl)} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} a+l \\ k-c \end{bmatrix} \begin{bmatrix} c+l \\ c \end{bmatrix} \\ &= \sum_{a=0}^k \sum_{c=0}^k (-1)^{a+k} q^{\pm(a+c-ac-k-kl)} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} a+l \\ k-c \end{bmatrix} \begin{bmatrix} -l-1 \\ c \end{bmatrix} \\ &= \sum_{a=0}^k (-1)^{a+k} q^{\pm a} \begin{bmatrix} k \\ a \end{bmatrix} \sum_{c=0}^k q^{\pm(-(a+l)c - (l+1)(k-c))} \begin{bmatrix} a+l \\ k-c \end{bmatrix} \begin{bmatrix} -l-1 \\ c \end{bmatrix} \\ &= \sum_{a=0}^k (-1)^{a+k} q^{\pm a} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} a-1 \\ k \end{bmatrix}, \end{aligned}$$

using Lemma 2.47 for the sum over c in the final step. The last factor in this expression vanishes for $0 \leq a-1 < k$, so only the summand for $a=0$ survives. We conclude

$$\begin{aligned} \sum_{a=0}^k \sum_{c=0}^k (-1)^{a+c+k} q^{\pm(a+c-ac-k-kl)} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} a+l \\ k-c \end{bmatrix} \begin{bmatrix} c+l \\ c \end{bmatrix} &= (-1)^k \begin{bmatrix} k \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ k \end{bmatrix} \\ &= (-1)^k \cdot 1 \cdot (-1)^k = 1 \end{aligned}$$

as claimed. \square

We are now ready to obtain explicit formulas for the action of \mathcal{T}_{\pm} on the simple module $V(n)$.

Proposition 2.49. *For $V = V(n)$ and any $j \in \{n, n-1, \dots, -n\}$ we have*

$$\mathcal{T}_{\pm}(v_{(j)}) = (-1)^{n-j} q^{\pm(n-j)(n+j+1)} v_{(-j)}.$$

In particular, the operators \mathcal{T}_{\pm} are invertible with inverses given by

$$\mathcal{T}_{\pm}^{-1}(v_{(j)}) = (-1)^{n+j} q^{\mp(n+j)(n-j+1)} v_{(-j)} = (-q)^{\mp 2j} \mathcal{T}_{\mp}(v_{(j)}).$$

Proof. The formulas for the action of divided powers before Lemma 2.47 imply

$$\begin{aligned} \mathcal{T}_{\pm}(v_{(j)}) &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=2j}} (-1)^s q^{\pm(s-rt)} F^{(r)} E^{(s)} F^{(t)} \cdot v_{(j)} \\ &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=2j}} (-1)^s q^{\pm(s-rt)} \begin{bmatrix} n-j+t \\ t \end{bmatrix} F^{(r)} E^{(s)} \cdot v_{(j-t)} \\ &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=2j}} (-1)^s q^{\pm(s-rt)} \begin{bmatrix} n+j-t+s \\ s \end{bmatrix} \begin{bmatrix} n-j+t \\ t \end{bmatrix} F^{(r)} \cdot v_{(j+s-t)} \\ &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=2j}} (-1)^s q^{\pm(s-rt)} \begin{bmatrix} n-j+t-s+r \\ r \end{bmatrix} \begin{bmatrix} n+j-t+s \\ s \end{bmatrix} \begin{bmatrix} n-j+t \\ t \end{bmatrix} v_{(-j)}. \end{aligned}$$

Note that all terms in this sum with $t > n+j$ or $r > n+j$ must vanish, since the vectors $v_{(j-t)}$ or $v_{(j+s-t)} = v_{(r-j)}$ vanish in these cases. Also, when $t \leq n+j$, the middle binomial coefficient can be rewritten as

$$\begin{bmatrix} n+j-t+s \\ s \end{bmatrix} = \begin{bmatrix} n+j-t+s \\ n+j-t \end{bmatrix},$$

and the latter expression vanishes when $s < 0$. Using $s = r + t - 2j$ we therefore have to show

$$\begin{aligned} \sum_{r,t=0}^{n+j} (-1)^{r+t-2j} q^{\pm(r+t-2j-rt)} \begin{bmatrix} n+j \\ r \end{bmatrix} \begin{bmatrix} n-j+r \\ n+j-t \end{bmatrix} \begin{bmatrix} n-j+t \\ t \end{bmatrix} \\ = (-1)^{n-j} q^{\pm(n-j)(n+j+1)}. \end{aligned}$$

Setting $k = n + j, l = n - j, a = r, c = t$ this amounts to

$$\sum_{a,c=0}^k (-1)^{l-k+a+c} q^{\pm(l-k+a+c-ac)} \begin{bmatrix} k \\ a \end{bmatrix} \begin{bmatrix} l+a \\ k-c \end{bmatrix} \begin{bmatrix} l+c \\ c \end{bmatrix} = (-1)^l q^{\pm l(k+1)},$$

hence the first claim follows from Lemma 2.48.

In order to verify the formula for \mathcal{T}_{\pm}^{-1} we compute

$$\begin{aligned} (-1)^{n+j} q^{\mp(n+j)(n-j+1)} \mathcal{T}_{\pm}(v_{(-j)}) \\ = (-1)^{n+j} q^{\mp(n+j)(n-j+1)} (-1)^{n+j} q^{\pm(n+j)(n-j+1)} v_{(j)} = v_{(j)}, \end{aligned}$$

and to check $\mathcal{T}_{\pm}^{-1}(v_{(j)}) = (-q)^{\mp 2j} \mathcal{T}_{\mp}(v_{(j)})$ we observe

$$\begin{aligned} q^{\mp(n+j)(n-j+1)} &= q^{\mp((n+j)(n-j)+n+j)} \\ &= q^{\mp 2j} q^{\mp((n+j)(n-j)+n-j)} = q^{\mp 2j} q^{\mp(n-j)(n+j+1)}. \end{aligned}$$

This finishes the proof. \square

The inverses of \mathcal{T}_{\pm} can be described in a similar fashion to the way we defined the operators \mathcal{T}_{\pm} themselves, as we discuss next.

Corollary 2.50. *Let V be an integrable module over $U_q(\mathfrak{sl}(2, \mathbb{K}))$. Then we have*

$$\begin{aligned} \mathcal{T}_{+}^{-1}(v) &= \sum_{\substack{r,s,t \geq 0 \\ -r+s-t=2m}} (-1)^s q^{rt-s} E^{(r)} F^{(s)} E^{(t)} \cdot v, \\ \mathcal{T}_{-}^{-1}(v) &= \sum_{\substack{r,s,t \geq 0 \\ -r+s-t=2m}} (-1)^s q^{s-rt} E^{(r)} F^{(s)} E^{(t)} \cdot v \end{aligned}$$

for $v \in V_m$.

Proof. It suffices to consider $V = V(n)$. Consider the involutive linear automorphism $\omega_V : V \rightarrow V$ given by $\omega_V(v_{(j)}) = v_{(-j)}$. Using the automorphism ω from Lemma 2.15 and the formulas before Lemma 2.47 we obtain

$$\omega(X) \cdot \omega_V(v) = \omega_V(X \cdot v)$$

for all $X \in U_q(\mathfrak{sl}(2, \mathbb{K}))$ and $v \in V$. Accordingly, the claim becomes $\mathcal{T}_{\pm}^{-1} \omega_V = \omega_V \mathcal{T}_{\mp}$. This in turn follows immediately from Proposition 2.49. \square

Let us also record the following commutation relations.

Lemma 2.51. *Let V be an integrable $U_q(\mathfrak{sl}(2, \mathbb{K}))$ -module and $v \in V$. Then we have*

$$\begin{aligned} \mathcal{T}_{\pm}(E \cdot v) &= -K^{\pm 2} F \cdot \mathcal{T}_{\pm}(v) \\ \mathcal{T}_{\pm}(K \cdot v) &= K^{-1} \cdot \mathcal{T}_{\pm}(v) \\ \mathcal{T}_{\pm}(F \cdot v) &= -EK^{\mp 2} \cdot \mathcal{T}_{\pm}(v) \end{aligned}$$

and

$$\begin{aligned} E \cdot \mathcal{T}_\pm(v) &= -\mathcal{T}_\pm(FK^{\mp 2} \cdot v) \\ K \cdot \mathcal{T}_\pm(v) &= \mathcal{T}_\pm(K^{-1} \cdot v) \\ F \cdot \mathcal{T}_\pm(v) &= -\mathcal{T}_\pm(K^{\pm 2}E \cdot v). \end{aligned}$$

Proof. It suffices to consider $V = V(n)$ for $n \in \frac{1}{2}\mathbb{N}_0$. Since \mathcal{T}_\pm interchanges the weight spaces with weights $\pm j$ for all j the formulas for the action of K are obvious. Using Lemma 2.38 and Lemma 2.49 we compute

$$\begin{aligned} \mathcal{T}_\pm(E \cdot v_{(j)}) &= [n + j + 1]\mathcal{T}_\pm(v_{(j+1)}) \\ &= (-1)^{n-j-1}[n + j + 1]q^{\pm(n-j-1)(n+j+2)}v_{(-j-1)} \\ &= (-1)^{n-j-1}[n + j + 1]q^{\pm(n-j)(n+j)\pm(-n-j-2+2n-2j)}v_{(-j-1)} \\ &= -q^{\pm(-2j-2)}(-1)^{n-j}q^{\pm(n-j)(n+j)\pm(n-j)}[n + j + 1]v_{(-j-1)} \\ &= -q^{\pm(-2j-2)}(-1)^{n-j}q^{\pm(n-j)(n+j+1)}F \cdot v_{(-j)} \\ &= -q^{\pm(-2j-2)}F \cdot \mathcal{T}_\pm(v_{(j)}) \\ &= -K^{\pm 2}F \cdot \mathcal{T}_\pm(v_{(j)}), \end{aligned}$$

and similarly

$$\begin{aligned} \mathcal{T}_\pm(F \cdot v_{(j)}) &= [n - j + 1]\mathcal{T}_\pm(v_{(j-1)}) \\ &= (-1)^{n-j+1}[n - j + 1]q^{\pm(n-j+1)(n+j)}v_{(-j+1)} \\ &= (-1)^{n-j+1}[n - j + 1]q^{\pm(n-j)(n+j)\pm(n+j)}v_{(-j+1)} \\ &= -q^{\pm 2j}(-1)^{n-j}q^{\pm(n-j)(n+j)\pm(n-j)}[n - j + 1]v_{(-j+1)} \\ &= -q^{\pm 2j}(-1)^{n-j}q^{\pm(n-j)(n+j+1)}E \cdot v_{(-j)} \\ &= -q^{\pm 2j}E \cdot \mathcal{T}_\pm(v_{(j)}) \\ &= -EK^{\mp 2} \cdot \mathcal{T}_\pm(v_{(j)}). \end{aligned}$$

The remaining formulas can be easily deduced from these relations. \square

2.7.2. The case of general \mathfrak{g} . Now let \mathfrak{g} be an arbitrary semisimple Lie algebra. Based on the constructions in the previous subsection we will define operators \mathcal{T}_i for $i = 1, \dots, N$ acting on every integrable $U_q(\mathfrak{g})$ -module.

Specifically, let V be an integrable $U_q(\mathfrak{g})$ -module and recall that $U_{q_i}(\mathfrak{g}_i)$ for $1 \leq i \leq N$ denotes the subalgebra of $U_q(\mathfrak{g})$ generated by $E_i, F_i, K_i^{\pm 1}$. Using the canonical embedding $U_{q_i}(\mathfrak{g}_i) \subset U_q(\mathfrak{g})$ we can view V as an integrable $U_{q_i}(\mathfrak{g}_i)$ -module, and we define $\mathcal{T}_i : V \rightarrow V$ to be the operator corresponding to $\mathcal{T} = \mathcal{T}_+$ in the notation of the previous subsection. Explicitly,

$$\mathcal{T}_i(v) = \sum_{\substack{r,s,t \geq 0 \\ r-s+t=m}} (-1)^s q_i^{s-rt} F_i^{(r)} E_i^{(s)} F_i^{(t)} \cdot v$$

for $v \in V_\lambda$ and $m = (\alpha_i^\vee, \lambda)$. According to Proposition 2.49, the operators \mathcal{T}_i are bijective, and \mathcal{T}_i maps V_λ onto $V_{s_i \lambda}$. From this observation it follows that

$$\mathcal{T}_i(K_\mu \cdot v) = K_{s_i \mu} \cdot \mathcal{T}_i(v)$$

for all $\mu \in \mathbf{P}$. Let us also record the following formulas.

Lemma 2.52. *Let V be an integrable $U_q(\mathfrak{g})$ -module and $v \in V$. Then we have*

$$\begin{aligned}\mathcal{T}_i(E_i \cdot v) &= -K_i F_i \cdot \mathcal{T}_i(v) \\ \mathcal{T}_i(F_i \cdot v) &= -E_i K_i^{-1} \cdot \mathcal{T}_i(v) \\ E_i \cdot \mathcal{T}_i(v) &= -\mathcal{T}_i(F_i K_i^{-1} \cdot v) \\ F_i \cdot \mathcal{T}_i(v) &= -\mathcal{T}_i(K_i E_i \cdot v)\end{aligned}$$

for all $i = 1, \dots, N$.

Proof. This is an immediate consequence of Lemma 2.51. \square

Our first aim is to obtain explicit formulas for $\mathcal{T}_i(E_j \cdot v)$ and $E_j \cdot \mathcal{T}_i(v)$ also in the case $i \neq j$, and similarly for F_j instead of E_j .

Rewriting the results from Lemma 2.9 in terms of divided powers we obtain

$$\begin{aligned}E_i^{(m)} \rightarrow Y &= \sum_{k=0}^m (-1)^{m-k} q_i^{-(m-k)(m-1)} E_i^{(k)} Y E_i^{(m-k)} K_i^{-m} \\ F_i^{(m)} \rightarrow Y &= \sum_{k=0}^m (-1)^k q_i^{k(m-1)} F_i^{(m-k)} K_i^{-k} Y K_i^k F_i^{(k)}.\end{aligned}$$

In particular, we have

$$F_i^{(m)} \rightarrow F_j = \sum_{k=0}^m (-1)^k q_i^{k(m-1+a_{ij})} F_i^{(m-k)} F_j F_i^{(k)},$$

using $K_i^{-k} F_j K_i^k = q_i^{ka_{ij}} F_j$.

Lemma 2.53. *If $i \neq j$ then for all $m, l \in \mathbb{N}_0$ we have*

$$\begin{aligned}(F_i^{(m)} \rightarrow F_j) E_i^{(l)} &= \sum_{k=0}^l (-1)^k \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{k(l-1)} E_i^{(l-k)} (F_i^{(m-k)} \rightarrow F_j) K_i^k \\ (F_i^{(m)} \rightarrow F_j) F_i^{(l)} &= \sum_{k=0}^l (-1)^k \begin{bmatrix} m + k \\ k \end{bmatrix}_{q_i} q_i^{-l(a_{ij}+2m)-k(l-1)} F_i^{(l-k)} (F_i^{(m+k)} \rightarrow F_j).\end{aligned}$$

Proof. We use induction on l for both formulas. For $l = 0$ there is nothing to show.

Assume the first formula holds for some l . Note that since E_i and F_j commute we have $E_i \rightarrow F_j = 0$, and hence

$$(E_i F_i^{(m)}) \rightarrow F_j = [-a_{ij} + 1 - m]_{q_i} F_i^{(m-1)} \rightarrow F_j$$

due to Lemma 2.36. Writing $a(m) = F_i^{(m)} \rightarrow F_j$ we deduce

$$\begin{aligned}a(m) E_i K_i^{-1} &= E_i a(m) K_i^{-1} - E_i \rightarrow a(m) \\ &= E_i a(m) K_i^{-1} - [-a_{ij} + 1 - m]_{q_i} a(m-1),\end{aligned}$$

which implies

$$a(m-k) E_i = E_i a(m-k) - [-a_{ij} + 1 + k - m]_{q_i} a(m-k-1) K_i$$

for all $0 \leq k \leq m$. Now from the induction hypothesis we obtain

$$\begin{aligned}a(m) E_i^{(l+1)} &= a(m) \frac{E_i^{(l)} E_i}{[l+1]_{q_i}} \\ &= \sum_{k=0}^l \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{k(l-1)} E_i^{(l-k)} (F_i^{(m-k)} \rightarrow F_j) K_i^k E_i.\end{aligned}$$

Since $K_i^k E_i = q_i^{2k} E_i K_i^k$ our above computation yields

$$a(m)E_i^{(l+1)} = \sum_{k=0}^{l+1} b_k E_i^{(l+1-k)} a(m-k)K_i^k,$$

where

$$\begin{aligned} b_k &= \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{k(l-1)+2k} [l+1-k]_{q_i} \\ &\quad - \frac{(-1)^{k-1}}{[l+1]_{q_i}} \begin{bmatrix} -a_{ij} - m + k - 1 \\ k-1 \end{bmatrix}_{q_i} q_i^{(k-1)(l-1)+2k-2} [-a_{ij} - m + k]_{q_i} \\ &= \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{kl+k} [l+1-k]_{q_i} \\ &\quad + \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{kl+k-l-1} [k]_{q_i}, \end{aligned}$$

note here that the first summand vanishes for $k = l + 1$, and the second summand vanishes for $k = 0$. Using

$$\frac{q_i^k [l+1-k]_{q_i} + q_i^{k-l-1} [k]_{q_i}}{[l+1]_{q_i}} = \frac{q_i^k (q_i^{l+1-k} - q^{-l-1+k}) + q_i^{k-l-1} (q_i^k - q_i^{-k})}{q_i^{l+1} - q_i^{-l-1}} = 1$$

we conclude

$$b_k = (-1)^k \begin{bmatrix} -a_{ij} - m + k \\ k \end{bmatrix}_{q_i} q_i^{kl}$$

as desired.

For the second formula we proceed in the same way. Assume that the formula holds for some l . Writing again $a(m) = F_i^{(m)} \rightarrow F_j$ we compute

$$\begin{aligned} a(m)F_i^{(l+1)} &= a(m) \frac{F_i^{(l)} F_i}{[l+1]_{q_i}} \\ &= \sum_{k=0}^l \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} q_i^{-l(a_{ij}+2m)-k(l-1)} F_i^{(l-k)} a(m+k)F_i \end{aligned}$$

according to the induction hypothesis. Note also that

$$\begin{aligned} a(m)F_i &= q_i^{-2m-a_{ij}} K_i^{-1} a(m) K_i F_i \\ &= q_i^{-2m-a_{ij}} (F_i a(m) - F_i \rightarrow a(m)) \\ &= q_i^{-2m-a_{ij}} (F_i a(m) - [m+1]_{q_i} a(m+1)), \end{aligned}$$

using $[m+1]_{q_i} F_i^{(m+1)} = F_i F_i^{(m)}$ in the last step. We thus have

$$a(m+k)F_i = q_i^{-2m-2k-a_{ij}} (F_i a(m+k) - [m+k+1]_{q_i} a(m+k+1)),$$

and combining these formulas we obtain

$$a(m)F_i^{(l+1)} = \sum_{k=0}^{l+1} c_k F_i^{(l+1-k)} a(m+k)$$

where

$$\begin{aligned}
c_k &= \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} q_i^{-(l+1)(a_{ij}+2m)-2k-k(l-1)} [l+1-k]_{q_i} \\
&\quad - \frac{(-1)^{k-1}}{[l+1]_{q_i}} \begin{bmatrix} m+k-1 \\ k-1 \end{bmatrix}_{q_i} q_i^{-(l+1)(a_{ij}+2m)-2k+2-(k-1)(l-1)} [m+k]_{q_i} \\
&= \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} q_i^{-(l+1)(a_{ij}+2m)-kl} q^{-k} [l+1-k]_{q_i} \\
&\quad + \frac{(-1)^k}{[l+1]_{q_i}} \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} q_i^{-(l+1)(a_{ij}+2m)-kl-k+l+1} [k]_{q_i},
\end{aligned}$$

note in particular that the first summand vanishes for $k = l + 1$, and the second summand vanishes for $k = 0$. Using

$$\frac{q_i^{-k} [l+1-k]_{q_i} + q_i^{l+1-k} [k]_{q_i}}{[l+1]_{q_i}} = \frac{q_i^{-k} (q_i^{l+1-k} - q^{-l-1+k}) + q_i^{l+1-k} (q_i^k - q_i^{-k})}{q_i^{l+1} - q_i^{-l-1}} = 1$$

we conclude

$$c_k = (-1)^k \begin{bmatrix} m+k \\ k \end{bmatrix}_{q_i} q_i^{-(l+1)(a_{ij}+2m)-kl}.$$

This finishes the proof. \square

Proposition 2.54. *Let V be an integrable $U_q(\mathfrak{g})$ -module. If $i \neq j$ we have*

$$\begin{aligned}
\mathcal{T}_i(E_j \cdot v) &= (-q_i)^{-a_{ij}} (E_j \leftarrow \hat{S}^{-1}(E_i^{(-a_{ij})})) \cdot \mathcal{T}_i(v) \\
\mathcal{T}_i(F_j \cdot v) &= (F_i^{(-a_{ij})} \rightarrow F_j) \cdot \mathcal{T}_i(v)
\end{aligned}$$

for all $v \in V$.

Proof. Let us first consider the second formula. In the same way as in the proof of Lemma 2.53 we shall abbreviate $a(m) = F_i^{(m)} \rightarrow F_j$. Since $F_i^{-a_{ij}+1} \rightarrow F_j = u_{ij}^- = 0$, the second part of Lemma 2.53 implies

$$a(-a_{ij})F_i^{(l)} = (F_i^{(-a_{ij})} \rightarrow F_j)F_i^{(l)} = q_i^{la_{ij}} F_i^{(l)} (F_i^{(-a_{ij})} \rightarrow F_j) = q_i^{la_{ij}} F_i^{(l)} a(-a_{ij})$$

for any l . Moreover we obtain

$$\begin{aligned}
a(-a_{ij})E_i^{(l)} &= (F_i^{(-a_{ij})} \rightarrow F_j)E_i^{(l)} \\
&= \sum_{k=0}^l (-1)^k q_i^{k(l-1)} E_i^{(l-k)} (F_i^{(-a_{ij}-k)} \rightarrow F_j) K_i^k \\
&= \sum_{k=0}^l (-1)^k q_i^{k(l-1)} E_i^{(l-k)} a(-a_{ij}-k) K_i^k
\end{aligned}$$

from the first part of Lemma 2.53.

Assume $v \in V_\lambda$ for some $\lambda \in \mathbf{P}$. Writing $p = (\alpha_i^\vee, \lambda)$, the above formulas imply

$$\begin{aligned}
 a(-a_{ij})\mathcal{T}_i(v) &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} (-1)^s q_i^{s-rt} a(-a_{ij}) F_i^{(r)} E_i^{(s)} F_i^{(t)} \cdot v \\
 &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} (-1)^s q_i^{s-rt+ra_{ij}} F_i^{(r)} a(-a_{ij}) E_i^{(s)} F_i^{(t)} \cdot v \\
 &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} \sum_{k=0}^s (-1)^{k+s} q_i^{s-rt+ra_{ij}+k(s-1)} F_i^{(r)} E_i^{(s-k)} a(-a_{ij}-k) K_i^k F_i^{(t)} \cdot v \\
 &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} \sum_{k=0}^s (-1)^{k+s} q_i^{s-r(t-a_{ij})+k(s-1+p-2t)} F_i^{(r)} E_i^{(s-k)} a(-a_{ij}-k) F_i^{(t)} \cdot v \\
 &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} \sum_{k=0}^s \sum_{l=0}^t (-1)^{k+s} q_i^{s-r(t-a_{ij})+k(s-1+p-2t)} F_i^{(r)} E_i^{(s-k)} \times \\
 &\quad (-1)^l \begin{bmatrix} -a_{ij}-k+l \\ l \end{bmatrix}_{q_i} q_i^{-t(-a_{ij}-2k)-l(t-1)} F_i^{(t-l)} a(-a_{ij}-k+l) \cdot v \\
 &= \sum_{\substack{r,s,t \geq 0 \\ r-s+t=p}} \sum_{k=0}^s \sum_{l=0}^t (-1)^{k+s+l} \begin{bmatrix} -a_{ij}-k+l \\ l \end{bmatrix}_{q_i} \times \\
 &\quad q_i^{s-r(t-a_{ij})+k(s-1+p)+ta_{ij}-l(t-1)} F_i^{(r)} E_i^{(s-k)} F_i^{(t-l)} a(-a_{ij}-k+l) \cdot v,
 \end{aligned}$$

using

$$K_i^k F_i^{(t)} \cdot v = q_i^{-2kt} F_i^{(t)} K_i^k \cdot v = q_i^{kp-2kt} F_i^{(t)} \cdot v$$

for $k, t \in \mathbb{N}_0$ and the second part of Lemma 2.53. Setting $b = s - k, c = t - l$ we have $-r + b - c = -p - k + l$, and since $a(-a_{ij} - k + l) = 0$ for $l > k$ we see that $a(-a_{ij})\mathcal{T}_i(v)$ is a linear combination of terms $F_i^{(r)} E_i^{(b)} F_i^{(c)} a(-a_{ij} - h) \cdot v$ with $-r + b - c = -p - h$ and $h \geq 0$. Explicitly, the coefficient of $F_i^{(r)} E_i^{(b)} F_i^{(c)} a(-a_{ij} - h) \cdot v$ is

$$\sum_{l=0}^{-a_{ij}-h} (-1)^{b+l} \begin{bmatrix} -a_{ij}-h \\ l \end{bmatrix}_{q_i} q_i^{b+k-r(c+l-a_{ij})+k(b+k-1+p)+(c+l)a_{ij}-l(c+l-1)},$$

where $k = l + h$. If we insert $k = l + h$ in this expression we obtain

$$\begin{aligned}
 &(-1)^b q_i^{b+h-r(c-a_{ij})+h(b+h-1+p)+ca_{ij}} \sum_{l=0}^{-a_{ij}-h} (-1)^l \begin{bmatrix} -a_{ij}-h \\ l \end{bmatrix}_{q_i} \times \\
 &\quad q_i^{l(1-r+b+l+h-1+p+h+a_{ij}-c-l+1)} \\
 &= (-1)^b q_i^{b-r(c-a_{ij})+h(b+h+p)+ca_{ij}} \sum_{l=0}^{-a_{ij}-h} (-1)^l \begin{bmatrix} -a_{ij}-h \\ l \end{bmatrix}_{q_i} \times \\
 &\quad q_i^{l(1-r+b+2h+p+a_{ij}-c)}.
 \end{aligned}$$

Using $-r + b - c = -p - h$, the sum over l in the previous formula becomes

$$\sum_{l=0}^{-a_{ij}-h} (-1)^l \begin{bmatrix} -a_{ij}-h \\ l \end{bmatrix}_{q_i} q_i^{l(1+h+a_{ij})} = \delta_{-a_{ij}-h,0}$$

according to Lemma 2.2. Hence only the term with $h = -a_{ij}$ survives, and since

$$q_i^{b-r(c-a_{ij})+h(b+h+p)+ca_{ij}} = q_i^{b-rc+h(-r+b-c+h+p)} = q_i^{b-rc}$$

for $h = -a_{ij}$, we conclude

$$a(-a_{ij})\mathcal{T}_i(v) = \sum_{\substack{r,b,c \geq 0 \\ r-b+c=p-a_{ij}}} (-1)^b q_i^{b-rc} F_i^{(r)} E_i^{(b)} F_i^{(c)} a(0)v = \mathcal{T}_i(F_j \cdot v)$$

as desired.

Let us now consider the first relation. Fix $v \in V_\lambda$ and let ω_V denote the automorphism of V defined as in the proof of Corollary 2.50 by viewing V as a $U_{q_i}(\mathfrak{g}_i)$ -module. Then combining Proposition 2.49 and the proof Corollary 2.50 yields

$$\omega_V \mathcal{T}_i \omega_V(v) = (-q_i)^{(\alpha_i^\vee, \lambda)} \mathcal{T}_i(v).$$

Moreover we recall that

$$\begin{aligned} \omega(X \rightarrow Y) &= \omega(X_{(1)} Y \hat{S}(X_{(2)})) \\ &= \omega(X_{(1)}) \omega(Y) \hat{S}^{-1}(\omega(X_{(2)})) \\ &= \omega(X)_{(2)} \omega(Y) \hat{S}^{-1}(\omega(X)_{(1)}) = \omega(Y) \leftarrow \hat{S}^{-1}(\omega(X)) \end{aligned}$$

for $X, Y \in U_q(\mathfrak{g})$, where $Y \leftarrow X = \hat{S}(X_{(1)}) Y X_{(2)}$. Using that the vector $F_j \cdot \omega_V(v)$ has weight $-\lambda - \alpha_j$ we therefore obtain

$$\begin{aligned} \mathcal{T}_i(E_j \cdot v) &= \mathcal{T}_i(\omega(F_j) \cdot v) \\ &= \mathcal{T}_i(\omega_V(F_j \cdot \omega_V(v))) \\ &= (-q_i)^{(\alpha_i^\vee, -\lambda - \alpha_j)} \omega_V(\mathcal{T}_i(F_j \cdot \omega_V(v))) \\ &= (-q_i)^{-(\alpha_i^\vee, \lambda + \alpha_j)} \omega_V((F_i^{(-a_{ij})} \rightarrow F_j) \cdot \mathcal{T}_i(\omega_V(v))) \\ &= (-q_i)^{-(\alpha_i^\vee, \lambda) - a_{ij}} \omega(F_i^{(-a_{ij})} \rightarrow F_j) \cdot \omega_V(\mathcal{T}_i(\omega_V(v))) \\ &= (-q_i)^{-(\alpha_i^\vee, \lambda) - a_{ij}} (-q_i)^{(\alpha_i^\vee, \lambda)} \omega(F_i^{(-a_{ij})} \rightarrow F_j) \cdot \mathcal{T}_i(v) \\ &= (-q_i)^{-a_{ij}} (\omega(F_j) \leftarrow \hat{S}^{-1}(\omega(F_i^{(-a_{ij})}))) \cdot \mathcal{T}_i(v) \\ &= (-q_i)^{-a_{ij}} (E_j \leftarrow \hat{S}^{-1}(E_i^{(-a_{ij})})) \cdot \mathcal{T}_i(v). \end{aligned}$$

This finishes the proof. \square

We obtain an algebra antiautomorphism $\gamma : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ by defining

$$\gamma(E_i) = E_i, \quad \gamma(F_i) = F_i, \quad \gamma(K_\mu) = K_{-\mu}$$

on generators. Notice that γ is involutive, that is, $\gamma^2 = \text{id}$.

Let us define the γ -twisted adjoint action by

$$X \xrightarrow{\gamma} Y = \gamma(X \rightarrow \gamma(Y))$$

for $X, Y \in U_q(\mathfrak{g})$. Note that

$$\begin{aligned} E_i \xrightarrow{\gamma} Y &= \gamma([E_i, \gamma(Y)] K_i^{-1}) = K_i[Y, E_i] \\ F_i \xrightarrow{\gamma} Y &= \gamma(F_i \gamma(Y) - K_i^{-1} \gamma(Y) K_i F_i) = Y F_i - F_i K_i^{-1} Y K_i \\ K_\mu \xrightarrow{\gamma} Y &= \gamma(K_\mu \gamma(Y) K_\mu^{-1}) = K_\mu Y K_\mu^{-1} \end{aligned}$$

for all $Y \in U_q(\mathfrak{g})$, where $1 \leq i \leq N$ and $\mu \in \mathbf{P}$.

Lemma 2.55. *Suppose $X, Y \in U_q(\mathfrak{g})$ satisfy $X \cdot \mathcal{T}_i(v) = \mathcal{T}_i(Y \cdot v)$ for any vector v in an integrable $U_q(\mathfrak{g})$ -module. Then*

$$\begin{aligned} (E_i \rightarrow X) \cdot \mathcal{T}_i(v) &= \mathcal{T}_i((F_i \xrightarrow{\gamma} Y) \cdot v) \\ (F_i \rightarrow X) \cdot \mathcal{T}_i(v) &= \mathcal{T}_i((E_i \xrightarrow{\gamma} Y) \cdot v). \end{aligned}$$

Proof. Using the formula $E_i \cdot \mathcal{T}_i(v) = -\mathcal{T}_i(F_i K_i^{-1} \cdot v)$ from Lemma 2.52 we compute

$$\begin{aligned} (E_i \rightarrow X) \cdot \mathcal{T}_i(v) &= [E_i, X] K_i^{-1} \cdot \mathcal{T}_i(v) \\ &= \mathcal{T}_i((-F_i K_i^{-1} Y + Y F_i K_i^{-1}) K_i \cdot v) \\ &= \mathcal{T}_i((Y F_i - F_i K_i^{-1} Y K_i) \cdot v) \\ &= \mathcal{T}_i((F_i \xrightarrow{\gamma} Y) \cdot v) \end{aligned}$$

for $v \in V$. Similarly, using $F_i \cdot \mathcal{T}_i(v) = -\mathcal{T}_i(K_i E_i \cdot v)$ we obtain

$$\begin{aligned} (F_i \rightarrow X) \cdot \mathcal{T}_i(v) &= (F_i X - K_i^{-1} X K_i F_i) \cdot \mathcal{T}_i(v) \\ &= \mathcal{T}_i((-K_i E_i Y + K_i Y K_i^{-1} K_i E_i) \cdot v) \\ &= \mathcal{T}_i(K_i [Y, E_i] \cdot v) \\ &= \mathcal{T}_i((E_i \xrightarrow{\gamma} Y) \cdot v). \end{aligned}$$

This yields the claim. \square

Lemma 2.56. *Let V be an integrable $U_q(\mathfrak{g})$ -module and $v \in V$. If $i \neq j$ we have*

$$(F_i^{(-a_{ij}-l)} \rightarrow F_j) \cdot \mathcal{T}_i(v) = \mathcal{T}_i((F_i^{(l)} \xrightarrow{\gamma} F_j) \cdot v)$$

for any $0 \leq l \leq -a_{ij}$.

Proof. Using induction on l , Lemma 2.55 and Proposition 2.54 imply

$$\mathcal{T}_i((F_i^{(l)} \xrightarrow{\gamma} F_j) \cdot v) = (E_i^{(l)} F_i^{(-a_{ij})} \rightarrow F_j) \cdot \mathcal{T}_i(v).$$

As already observed in the proof of Lemma 2.53, we have

$$(E_i F_i^{(m)}) \rightarrow F_j = [-a_{ij} + 1 - m]_{q_i} F_i^{(m-1)} \rightarrow F_j.$$

Applying this formula iteratively we obtain

$$(E_i^{(l)} F_i^{(-a_{ij})}) \rightarrow F_j = F_i^{(-a_{ij}-l)} \rightarrow F_j,$$

which yields the desired formula. \square

Proposition 2.57. *Let V be an integrable $U_q(\mathfrak{g})$ -module and $v \in V$. If $i \neq j$ we have*

$$\begin{aligned} &\sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{k(l+1)} E_i^{(k)} E_j E_i^{(-a_{ij}-l-k)} \cdot \mathcal{T}_i(v) \\ &= \sum_{k=0}^l (-1)^k q_i^{-k(l-1+a_{ij})} \mathcal{T}_i(E_i^{(l-k)} E_j E_i^{(k)} \cdot v) \end{aligned}$$

and

$$\begin{aligned} &\sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{-k(l+1)} F_i^{(-a_{ij}-l-k)} F_j F_i^{(k)} \cdot \mathcal{T}_i(v) \\ &= \sum_{k=0}^l (-1)^k q_i^{k(l-1+a_{ij})} \mathcal{T}_i(F_i^{(k)} F_j F_i^{(l-k)} \cdot v) \end{aligned}$$

for any $0 \leq l \leq -a_{ij}$.

Proof. Let us first consider the second formula. Recall from the calculations before Lemma 2.53 that

$$F_i^{(m)} \rightarrow F_j = \sum_{k=0}^m (-1)^k q_i^{k(m-1+a_{ij})} F_i^{(m-k)} F_j F_i^{(k)}.$$

Since γ is an anti-automorphism fixing the generators F_k for all k we therefore obtain

$$F_i^{(m)} \xrightarrow{\gamma} F_j = \gamma(F_i^{(m)} \rightarrow F_j) = \sum_{k=0}^m (-1)^k q_i^{k(m-1+a_{ij})} F_i^{(k)} F_j F_i^{(m-k)}.$$

Combining these formulas with Lemma 2.56 yields

$$\begin{aligned} & \sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{k(-l-1)} F_i^{(-a_{ij}-l-k)} F_j F_i^{(k)} \cdot \mathcal{T}_i(v) \\ &= \sum_{k=0}^l (-1)^k q_i^{k(l-1+a_{ij})} \mathcal{T}_i(F_i^{(k)} F_j F_i^{(l-k)} \cdot v) \end{aligned}$$

as desired.

Next observe that $\omega(X) \cdot \mathcal{T}_i(v) = (-q_i)^{(\alpha_i^\vee, \mu)} \mathcal{T}_i(\omega(Y) \cdot v)$ if X has weight μ with respect to the adjoint action and $X \cdot \mathcal{T}_i(v) = \mathcal{T}_i(Y \cdot v)$ for all vectors v in integrable $U_q(\mathfrak{g})$ -modules. Hence the formula proved above implies

$$\begin{aligned} & \sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{-k(-l-1)} E_i^{(k)} E_j E_i^{(-a_{ij}-l-k)} \cdot \mathcal{T}_i(v) \\ &= q_i^{(-a_{ij}-l)(l+1)} \sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{(-a_{ij}-l-k)(-l-1)} E_i^{(k)} E_j E_i^{(-a_{ij}-l-k)} \cdot \mathcal{T}_i(v) \\ &= (-1)^{a_{ij}+l} q_i^{(-a_{ij}-l)(l+1)} \sum_{k=0}^{-a_{ij}-l} (-1)^k q_i^{k(-l-1)} E_i^{(-a_{ij}-l-k)} E_j E_i^{(k)} \cdot \mathcal{T}_i(v) \\ &= (-1)^{a_{ij}+l} q_i^{(-a_{ij}-l)(l+1)} (-q_i)^{2l+a_{ij}} \sum_{k=0}^l (-1)^k q_i^{k(l-1+a_{ij})} \mathcal{T}_i(E_i^{(k)} E_j E_i^{(l-k)} \cdot v) \\ &= q_i^{(-a_{ij}-l)(l+1)} q_i^{2l+a_{ij}} q_i^{l(l-1+a_{ij})} \sum_{k=0}^l (-1)^k q_i^{-k(l-1+a_{ij})} \mathcal{T}_i(E_i^{(l-k)} E_j E_i^{(k)} \cdot v) \\ &= \sum_{k=0}^l (-1)^k q_i^{-k(l-1+a_{ij})} \mathcal{T}_i(E_i^{(l-k)} E_j E_i^{(k)} \cdot v), \end{aligned}$$

upon reindexing k to $-a_{ij}-l-k$, and further below reindexing k to $l-k$, respectively. Note here that $F_i^{(-a_{ij}-l)} \rightarrow F_j$ has weight $(a_{ij}+l)\alpha_i - \alpha_j$, so that

$$(\alpha_i^\vee, (a_{ij}+l)\alpha_i - \alpha_j) = 2l + a_{ij}.$$

This finishes the proof. \square

Note that putting $l = 0$ in Proposition 2.57 yields in particular

$$\begin{aligned} \mathcal{T}_i(E_j \cdot v) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^k E_i^{(k)} E_j E_i^{(-a_{ij}-k)} \cdot \mathcal{T}_i(v) \\ \mathcal{T}_i(F_j \cdot v) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} F_i^{(-a_{ij}-k)} F_j F_i^{(k)} \cdot \mathcal{T}_i(v). \end{aligned}$$

2.7.3. The braid group action on $U_q(\mathfrak{g})$. We are now ready to construct automorphisms \mathcal{T}_i of $U_q(\mathfrak{g})$ for $i = 1, \dots, N$.

Theorem 2.58. *For $i = 1, \dots, N$ there exist algebra automorphisms $\mathcal{T}_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ satisfying*

$$\begin{aligned}\mathcal{T}_i(K_\mu) &= K_{s_i\mu}, & \mathcal{T}_i(E_i) &= -K_i F_i, & \mathcal{T}_i(F_i) &= -E_i K_i^{-1} \\ \mathcal{T}_i(E_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^k E_i^{(k)} E_j E_i^{(-a_{ij}-k)}, & i &\neq j \\ \mathcal{T}_i(F_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} F_i^{(-a_{ij}-k)} F_j F_i^{(k)}, & i &\neq j.\end{aligned}$$

Proof. We shall define $\mathcal{T}_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ using conjugation with the operators \mathcal{T}_i on integrable $U_q(\mathfrak{g})$ -modules. More precisely, we consider the canonical map $U_q(\mathfrak{g}) \rightarrow \prod \text{End}(V) \subset \text{End}(\bigoplus V)$ where V runs over all integrable $U_q(\mathfrak{g})$ -modules. According to Theorem 2.46 this map is an embedding, and we shall identify $X \in U_q(\mathfrak{g})$ with its image in $\prod \text{End}(V)$. We then define $\mathcal{T}_i(X) \in \prod \text{End}(V)$ by setting

$$\mathcal{T}_i(X) = \mathcal{T}_i X \mathcal{T}_i^{-1},$$

where the operators $\mathcal{T}_i \in \prod \text{End}(V)$ are defined as in the previous subsection. In order to show that this prescription yields a well-defined automorphism of $U_q(\mathfrak{g})$ it is enough to check that conjugation with $\mathcal{T}_i \in \prod \text{End}(V)$ preserves the action of $U_q(\mathfrak{g})$. In fact, it is sufficient to verify this for the standard generators of $U_q(\mathfrak{g})$.

For the generators K_μ for $\mu \in \mathbf{P}$ the claim follows from elementary weight considerations, indeed the corresponding relation $\mathcal{T}_i K_\mu \mathcal{T}_i^{-1} = K_{s_i\mu}$ was already pointed out in the discussion before Lemma 2.52. For the generators E_i, F_i the assertion follows from Proposition 2.57. \square

Next we shall verify the braid group relations.

Theorem 2.59. *The automorphisms \mathcal{T}_i of $U_q(\mathfrak{g})$ satisfy the braid relations in $B_{\mathfrak{g}}$, that is,*

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \cdots = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \cdots$$

for all $1 \leq i, j \leq N$ such that $i \neq j$, with m_{ij} operators on each side of the equation. In other words, the maps $\mathcal{T}_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ induce an action of the braid group $B_{\mathfrak{g}}$ on $U_q(\mathfrak{g})$ by algebra automorphisms.

Proof. Recall that $m_{ij} = 2, 3, 4, 6$ iff $a_{ij}a_{ji} = 0, 1, 2, 3$, respectively. We have $a_{ij} = 0 = a_{ji}$ if $m_{ij} = 2$, and $a_{ij} = -1 = a_{ji}$ if $m_{ij} = 3$. Note that we may assume that $a_{ij} = -2, a_{ji} = -1$ and $a_{ij} = -3, a_{ji} = -1$ in the cases $m_{ij} = 4, 6$, respectively. The cases $m = m_{ij} = 2, 3, 4, 6$ are verified separately by explicit calculations, compare [57] and [39].

$m = 2$: This corresponds to $(\alpha_i, \alpha_j) = 0$. In this case we clearly have $\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i$ on modules because the operators E_i, F_i commute pairwise with E_j, F_j . Since the action of \mathcal{T}_i on $U_q(\mathfrak{g})$ is obtained by conjugating with the operator \mathcal{T}_i on modules, it follows that $\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i$ as automorphisms of $U_q(\mathfrak{g})$.

$m = 3$: Since $a_{ij} = -1 = a_{ji}$ we have $q_i = q_j$, and the formulas in Proposition 2.57 give

$$\begin{aligned}\mathcal{T}_i(E_j) &= E_j E_i - q_i E_i E_j = \mathcal{T}_j^{-1}(E_i) \\ \mathcal{T}_i(F_j) &= F_i F_j - q_i^{-1} F_j F_i = \mathcal{T}_j^{-1}(F_i).\end{aligned}$$

Symmetrically, we obtain

$$\begin{aligned}\mathcal{T}_j(E_i) &= E_i E_j - q_i E_j E_i = \mathcal{T}_i^{-1}(E_j) \\ \mathcal{T}_j(F_i) &= F_j F_i - q_i^{-1} F_i F_j = \mathcal{T}_i^{-1}(F_j).\end{aligned}$$

It suffices to verify $\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(X) = \mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(X)$ for $X = E_k, F_k$ and $k = 1, \dots, N$ as well as $X = K_\lambda$ for $\lambda \in \mathbf{P}$. For the generators K_λ the claim follows from the fact that the action of the Weyl group on \mathfrak{h}^* satisfies the braid relations.

Let us consider $X = E_j$. The above relations imply $\mathcal{T}_j\mathcal{T}_i(E_j) = E_i$, and hence we obtain

$$\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(E_j) = \mathcal{T}_i(E_i) = -K_i F_i.$$

On the other hand we get

$$\mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(E_j) = \mathcal{T}_j\mathcal{T}_i(-K_j F_j) = -\mathcal{T}_j\mathcal{T}_i(K_j)\mathcal{T}_j\mathcal{T}_i(F_j) = -K_i F_i$$

since the above relations imply $\mathcal{T}_j\mathcal{T}_i(F_j) = F_i$, and moreover we have $\mathcal{T}_j\mathcal{T}_i(K_j) = K_i$. For the latter use

$$\begin{aligned} s_j s_i(\alpha_j) &= s_j(\alpha_j - (\alpha_i^\vee, \alpha_j)\alpha_i) \\ &= s_j(\alpha_j + \alpha_i) = -\alpha_j + \alpha_j + \alpha_i = \alpha_i. \end{aligned}$$

For $X = F_j$ analogous considerations give

$$\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(F_j) = \mathcal{T}_i(F_i) = -E_i K_i^{-1}$$

and

$$\mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(F_j) = \mathcal{T}_j\mathcal{T}_i(-E_j K_j^{-1}) = -\mathcal{T}_j\mathcal{T}_i(E_j)\mathcal{T}_j\mathcal{T}_i(K_j^{-1}) = -E_i K_i^{-1}.$$

Since the situation is symmetric in i and j we also get the claim for $X = E_i, F_i$.

Therefore it remains to consider the case $X = E_k, F_k$ for $k \neq i, j$. For $k \neq i, j$ we have $a_{ik} = 0$ or $a_{jk} = 0$. Again by symmetry we may assume $a_{jk} = 0$. Then $[E_k, E_j] = 0 = [E_k, F_j]$. From the construction of the automorphism \mathcal{T}_j using conjugation by the operator \mathcal{T}_j we therefore get $\mathcal{T}_j(E_k) = E_k$. Moreover recall that $\mathcal{T}_j\mathcal{T}_i(E_j) = E_i$ and $\mathcal{T}_j\mathcal{T}_i(F_j) = F_i$. Hence we also get

$$\begin{aligned} [\mathcal{T}_j\mathcal{T}_i(E_k), E_i] &= \mathcal{T}_j\mathcal{T}_i([E_k, E_j]) = 0, \\ [\mathcal{T}_j\mathcal{T}_i(E_k), F_i] &= \mathcal{T}_j\mathcal{T}_i([E_k, F_j]) = 0. \end{aligned}$$

Again by the construction of \mathcal{T}_i we therefore get $\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(E_k) = \mathcal{T}_j\mathcal{T}_i(E_k)$. Combining these considerations yields

$$\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(E_k) = \mathcal{T}_j\mathcal{T}_i(E_k) = \mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(E_k)$$

as desired. Upon replacing E_k by F_k , the above argument shows

$$\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(F_k) = \mathcal{T}_j\mathcal{T}_i(F_k) = \mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(F_k)$$

as well.

$m = 4$: Let us assume $a_{ij} = -2, a_{ji} = -1$. In this case we have $q_i = q$ and $q_j = q^2$. As in the case $m = 3$, Proposition 2.57 implies

$$\begin{aligned} \mathcal{T}_j(E_i) &= E_i E_j - q^2 E_j E_i, \\ \mathcal{T}_j^{-1}(E_i) &= E_j E_i - q^2 E_i E_j. \end{aligned}$$

Moreover, taking the first formula of Proposition 2.57 for $l = 1$ gives

$$\mathcal{T}_i(E_i E_j - q^2 E_j E_i) = E_j E_i - q^2 E_i E_j.$$

Hence $\mathcal{T}_i\mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i)$, or equivalently $\mathcal{T}_j\mathcal{T}_i\mathcal{T}_j(E_i) = E_i$.

Swapping the roles of i and j , Proposition 2.57 gives

$$\begin{aligned} \mathcal{T}_i(E_j) &= E_j E_i^{(2)} - q E_i E_j E_i + q^2 E_i^{(2)} E_j, \\ \mathcal{T}_i^{-1}(E_j) &= E_i^{(2)} E_j - q E_i E_j E_i + q^2 E_j E_i^{(2)}. \end{aligned}$$

Multiplying these equations with $[2]_{q_i} = q_i + q_i^{-1} = q + q^{-1}$ we obtain

$$\begin{aligned} [2]_{q_i} \mathcal{T}_i(E_j) &= E_j E_i^2 - (1 + q^2) E_i E_j E_i + q^2 E_i^2 E_j \\ &= \mathcal{T}_j^{-1}(E_i) E_i - E_i \mathcal{T}_j^{-1}(E_i) \end{aligned}$$

and

$$\begin{aligned} [2]_{q_i} \mathcal{T}_i^{-1}(E_j) &= E_i^2 E_j - (1 + q^2) E_i E_j E_i + q^2 E_j E_i^2 \\ &= E_i \mathcal{T}_j(E_i) - \mathcal{T}_j(E_i) E_i. \end{aligned}$$

Combining these equations yields $\mathcal{T}_j \mathcal{T}_i(E_j) = \mathcal{T}_i^{-1}(E_j)$, or equivalently $\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_j) = E_j$.

In a completely analogous fashion we obtain

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(F_j) = F_j, \quad \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(F_i) = F_i.$$

Let us now show $\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(X) = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(X)$ for $X = E_k, F_k$ where $k = 1, \dots, N$. Since

$$\begin{aligned} s_j s_i s_j(\alpha_i) &= s_j s_i(\alpha_i - a_{ji} \alpha_j) \\ &= s_j(-\alpha_i + \alpha_j - a_{ji} \alpha_i) \\ &= s_j(\alpha_i + \alpha_j) = \alpha_i + \alpha_j - \alpha_j = \alpha_i \end{aligned}$$

we have $\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(K_i) = K_i$, and hence our above computations imply

$$\begin{aligned} \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) &= \mathcal{T}_i(E_i) = -K_i F_i \\ &= -\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(K_i) \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(F_i) \\ &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(-K_i F_i) \\ &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_i). \end{aligned}$$

In a similar way one obtains the assertion for $X = E_j$ and $X = F_i, F_j$.

Consider now $k \neq i, j$. Then we have $(\alpha_i^\vee, \alpha_k) = 0$ or $(\alpha_j^\vee, \alpha_k) = 0$. If $(\alpha_i^\vee, \alpha_k) = 0$ we obtain $[E_k, E_j] = 0 = [E_k, F_j]$ and hence $\mathcal{T}_j(E_k) = E_k$. Moreover, our above computations yield

$$\begin{aligned} [\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k), E_j] &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i([E_k, E_j]) = 0, \\ [\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k), F_j] &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i([E_k, F_j]) = 0. \end{aligned}$$

This implies $\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k)$, and thus

$$\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_k).$$

Replacing E_k by F_k in this argument yields the claim for $X = F_k$. Finally, if $(\alpha_j^\vee, \alpha_k) = 0$ we swap the roles of i and j in the above argument, and obtain the assertion in the same way.

$m = 6$: Let us assume $a_{ij} = -3, a_{ji} = -1$. In this case we have $q_i = q$ and $q_j = q^3$. Given $0 \leq l \leq 3$, let us abbreviate

$$\begin{aligned} E_{ij}^-(l) &= \sum_{k=0}^l (-1)^k q_i^{k(4-l)} E_i^{(k)} E_j E_i^{(l-k)}, \\ E_{ij}^+(l) &= \sum_{k=0}^l (-1)^k q_i^{k(4-l)} E_i^{(l-k)} E_j E_i^{(k)}. \end{aligned}$$

As in the case $m = 3$, Proposition 2.57 implies

$$\begin{aligned} \mathcal{T}_j(E_i) &= E_i E_j - q^3 E_j E_i = E_{ij}^+(1) \\ \mathcal{T}_j^{-1}(E_i) &= E_j E_i - q^3 E_i E_j = E_{ij}^-(1). \end{aligned}$$

Moreover the relations from Proposition 2.57 give

$$\begin{aligned} E_j &= \mathcal{T}_i(E_{ij}^+(3)), \\ E_{ij}^-(1) &= \mathcal{T}_i(E_{ij}^+(2)), \\ E_{ij}^-(2) &= \mathcal{T}_i(E_{ij}^+(1)), \\ E_{ij}^-(3) &= \mathcal{T}_i(E_j). \end{aligned}$$

Explicitly, we have

$$\begin{aligned} E_{ij}^+(3) &= E_i^{(3)} E_j - q E_i^{(2)} E_j E_i + q^2 E_i E_j E_i^{(2)} - q^3 E_j E_i^{(3)}, \\ E_{ij}^-(3) &= E_j E_i^{(3)} - q E_i E_j E_i^{(2)} + q^2 E_i^{(2)} E_j E_i - q^3 E_i^{(3)} E_j, \\ E_{ij}^+(2) &= E_i^{(2)} E_j - q^2 E_i E_j E_i + q^4 E_j E_i^{(2)}, \\ E_{ij}^-(2) &= E_j E_i^{(2)} - q^2 E_i E_j E_i + q^4 E_i^{(2)} E_j. \end{aligned}$$

Hence we obtain

$$\begin{aligned} [3]_q E_{ij}^+(3) &= [2]_q^{-1} E_i^3 E_j - q [3]_q E_i^{(2)} E_j E_i \\ &\quad + q^2 [3]_q E_i E_j E_i^{(2)} - q^3 [2]_q^{-1} E_j E_i^3 \\ &= E_i (E_i^{(2)} E_j - q^2 E_i E_j E_i + q^4 E_j E_i^{(2)}) \\ &\quad - q^{-1} (E_i^{(2)} E_j - q^2 E_i E_j E_i + q^4 E_j E_i^{(2)}) E_i \\ &= E_i E_{ij}^+(2) - q^{-1} E_{ij}^+(2) E_i \end{aligned}$$

using $q(q^2+1+q^{-2}) = q^2(q+q^{-1})+q^{-1}$ and $q^2(q^2+1+q^{-2}) = q^4+q(q+q^{-1})$. Similarly,

$$\begin{aligned} [3]_q E_{ij}^-(3) &= [2]_q^{-1} E_j E_i^3 - q [3]_q E_i E_j E_i^{(2)} \\ &\quad + q^2 [3]_q E_i^{(2)} E_j E_i - q^3 [2]_q^{-1} E_i^3 E_j \\ &= (E_j E_i^{(2)} - q^2 E_i E_j E_i + q^4 E_i^{(2)} E_j) E_i \\ &\quad - q^{-1} E_i (E_j E_i^{(2)} - q^2 E_i E_j E_i + q^4 E_i^{(2)} E_j) \\ &= E_{ij}^-(2) E_i - q^{-1} E_i E_{ij}^-(2). \end{aligned}$$

Let us also record

$$\begin{aligned} [2]_q E_{ij}^+(2) &= E_i^2 E_j - q^2 (q + q^{-1}) E_i E_j E_i + q^4 E_j E_i^2 \\ &= E_i E_{ij}^+(1) - q E_{ij}^+(1) E_i \end{aligned}$$

and

$$\begin{aligned} [2]_q E_{ij}^-(2) &= E_j E_i^2 - q^2 (q + q^{-1}) E_i E_j E_i + q^4 E_i^2 E_j \\ &= E_{ij}^-(1) E_i - q E_i E_{ij}^-(1). \end{aligned}$$

Using $E_{ij}^\pm(1) = \mathcal{T}_j^{\pm 1}(E_i)$ we obtain

$$\begin{aligned} E_{ij}^+(2) &= [2]_q^{-1} (E_i \mathcal{T}_j(E_i) - q \mathcal{T}_j(E_i) E_i) \\ &= [2]_q^{-1} \mathcal{T}_j(E_{ij}^-(1) E_i - q E_i E_{ij}^-(1)) \\ &= \mathcal{T}_j(E_{ij}^-(2)). \end{aligned}$$

Now we compute

$$\begin{aligned}
 \mathcal{T}_j \mathcal{T}_i(E_j) &= \mathcal{T}_j(E_{ij}^-(3)) \\
 &= [3]_q^{-1} \mathcal{T}_j(E_{ij}^-(2)E_i - q^{-1}E_i E_{ij}^-(2)) \\
 &= [3]_q^{-1} (\mathcal{T}_j(E_{ij}^-(2))E_{ij}^+(1) - q^{-1}E_{ij}^+(1)\mathcal{T}_j^{-1}(E_{ij}^-(2))) \\
 &= [3]_q^{-1} (E_{ij}^+(2)E_{ij}^+(1) - q^{-1}E_{ij}^+(1)E_{ij}^+(2))
 \end{aligned}$$

and

$$\begin{aligned}
 \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) &= \mathcal{T}_j^{-1}(E_{ij}^+(3)) \\
 &= [3]_q^{-1} \mathcal{T}_j^{-1}(E_i E_{ij}^+(2) - q^{-1}E_{ij}^+(2)E_i) \\
 &= [3]_q^{-1} (E_{ij}^-(1)\mathcal{T}_j^{-1}(E_{ij}^+(2)) - q^{-1}\mathcal{T}_j^{-1}(E_{ij}^+(2))E_{ij}^-(1)) \\
 &= [3]_q^{-1} (E_{ij}^-(1)E_{ij}^-(2) - q^{-1}E_{ij}^-(2)E_{ij}^-(1)).
 \end{aligned}$$

We thus obtain

$$\begin{aligned}
 \mathcal{T}_i^{-1} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) &= [3]_q^{-1} \mathcal{T}_i^{-1}(E_{ij}^-(1)E_{ij}^-(2) - q^{-1}E_{ij}^-(2)E_{ij}^-(1)) \\
 &= [3]_q^{-1} (E_{ij}^+(2)E_{ij}^+(1) - q^{-1}E_{ij}^+(1)E_{ij}^+(2)) \\
 &= \mathcal{T}_j \mathcal{T}_i(E_j),
 \end{aligned}$$

or equivalently, $\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_j) = E_j$.

We also have

$$\mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_i(E_{ij}^+(1)) = E_{ij}^-(2)$$

and

$$\mathcal{T}_i^{-1} \mathcal{T}_j^{-1}(E_i) = \mathcal{T}_i^{-1}(E_{ij}^-(1)) = E_{ij}^+(2).$$

Therefore

$$\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j(E_{ij}^-(2)) = E_{ij}^+(2) = \mathcal{T}_i^{-1} \mathcal{T}_j^{-1}(E_i),$$

and hence $\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i$.

In a completely analogous fashion one calculates

$$\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(F_j) = F_j, \quad \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(F_i) = F_i.$$

Let us now verify $\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(X) = \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(X)$ for $X = E_k, F_k$ where $k = 1, \dots, N$. Since

$$\begin{aligned}
 s_j s_i s_j s_i s_j(\alpha_i) &= s_j s_i s_j s_i(\alpha_i - a_{ji}\alpha_j) \\
 &= s_j s_i s_j(-\alpha_i + \alpha_j - a_{ij}\alpha_i) \\
 &= s_j s_i s_j(2\alpha_i + \alpha_j) \\
 &= s_j(4\alpha_i + 2\alpha_j - \alpha_j + a_{ij}\alpha_i) \\
 &= s_j(\alpha_i + \alpha_j) = \alpha_i + \alpha_j - \alpha_j = \alpha_i
 \end{aligned}$$

we have $\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(K_i) = K_i$, and hence our above computations imply

$$\begin{aligned}
 \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) &= \mathcal{T}_i(E_i) = -K_i F_i \\
 &= -\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(K_i) \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(F_i) \\
 &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(-K_i F_i) \\
 &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_i).
 \end{aligned}$$

In a similar way one obtains the assertion for $X = E_j$ and $X = F_i, F_j$.

Consider now $k \neq i, j$. Then we have $(\alpha_i^\vee, \alpha_k) = 0$ or $(\alpha_j^\vee, \alpha_k) = 0$. If $(\alpha_j^\vee, \alpha_k) = 0$ we obtain $[E_k, E_j] = 0 = [E_k, F_j]$ and hence $\mathcal{T}_j(E_k) = E_k$. Moreover, our above computations yield

$$\begin{aligned} [\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k), E_j] &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i([E_k, E_j]) = 0, \\ [\mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k), F_j] &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i([E_k, F_j]) = 0. \end{aligned}$$

This implies $\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k)$, and thus

$$\mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_k) = \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_k).$$

Replacing E_k by F_k in this argument yields the claim for $X = F_k$. Finally, if $(\alpha_i^\vee, \alpha_k) = 0$ we swap i and j in the above computations and obtain the assertion in the same way.

This finishes the proof. \square

We remark that the algebra automorphisms \mathcal{T}_i in Theorem 2.59 are not coalgebra automorphisms.

The action of the braid group $B_{\mathfrak{g}}$ on $U_q(\mathfrak{g})$ does not descend to an action of the Weyl group W , since $\mathcal{T}_i^2 \neq \text{id}$. Nevertheless, there is a standard way to associate to each $w \in W$ an automorphism \mathcal{T}_w , as we now explain.

If $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of $w \in W$, then any other reduced expression is obtained by applying a sequence of *elementary moves* to $s_{i_1} \cdots s_{i_k}$. Here an elementary move is the replacement of a substring $s_i s_j s_i \cdots$ of length m_{ij} in the given expression by the string $s_j s_i s_j \cdots$ of the same length. We refer to Section 8.1 in [36] for the details. As a consequence, according to Theorem 2.59 the automorphism

$$\mathcal{T}_w = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_k}$$

of $U_q(\mathfrak{g})$ depends only on w and not on the reduced expression for w .

Note that we have $\mathcal{T}_w(K_\mu) = K_{w\mu}$ for all $\mu \in \mathbf{P}$ by Theorem 2.58.

2.7.4. The Poincaré-Birkhoff-Witt Theorem. Let $w \in W$ and let us fix a reduced expression $w = s_{i_1} \cdots s_{i_t}$. If $w \neq e$ then the roots $\beta_r = s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}$ for $1 \leq r \leq t$ are pairwise distinct and positive, see 5.6 in [36]. We define the root vectors of $U_q(\mathfrak{g})$ associated to the reduced expression $w = s_{i_1} \cdots s_{i_t}$ by

$$E_{\beta_r} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{r-1}}(E_{i_r}), \quad F_{\beta_r} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{r-1}}(F_{i_r})$$

for $1 \leq r \leq t$; in the special case $w = e$ we declare 1 to be the unique associated root vector. Let us point out that these vectors do depend on the choice of the reduced expression for w in general, see for instance the discussion in Section 8.15 of [39].

Our first aim is to show that the root vectors E_{β_r} associated to a reduced expression for w are contained in $U_q(\mathfrak{n}_+)$. For this we need some preparations.

Lemma 2.60. *Let $i \neq j$ and assume w is contained in the subgroup of W generated by s_i and s_j . If $w\alpha_i \in \Delta^+$ then $\mathcal{T}_w(E_i)$ is contained in the subalgebra of $U_q(\mathfrak{n}_+)$ generated by E_i and E_j , and if $w\alpha_i = \alpha_k$ for some $1 \leq k \leq N$ then $\mathcal{T}_w(E_i) = E_k$.*

Proof. Let us remark first that $w\alpha_i = \alpha_k$ can only happen for $k = i, j$.

Without loss of generality we may assume $w \neq e$. The proof will proceed case-by-case depending on the order m of $s_i s_j$ in W . Let us write $\langle s_i, s_j \rangle$ for the subgroup of W generated by s_i, s_j .

$m = 2$: In this case s_i and s_j commute, and the only nontrivial element $w \in \langle s_i, s_j \rangle$ satisfying $w\alpha_i \in \Delta^+$ is $w = s_j$; note that $s_j\alpha_i = \alpha_i$. The formula in Theorem 2.58 gives $\mathcal{T}_{s_j}(E_i) = \mathcal{T}_j(E_i) = E_i$.

- $m = 3$: In this case we have $s_i s_j s_i = s_j s_i s_j$, and the nontrivial elements $w \in \langle s_i, s_j \rangle$ satisfying $w\alpha_i \in \Delta^+$ are $w = s_j$ and $w = s_i s_j$; note that in this case $\langle s_i, s_j \rangle \cong S_3$. Using the computations in the proof of Theorem 2.59 we obtain $\mathcal{T}_{s_j}(E_i) = \mathcal{T}_j(E_i) = E_i E_j - q_i E_j E_i$ and $\mathcal{T}_{s_i s_j}(E_i) = \mathcal{T}_i \mathcal{T}_j(E_i) = E_j$.
- $m = 4$: In this case we have $s_i s_j s_i s_j = s_j s_i s_j s_i$, and the nontrivial elements $w \in \langle s_i, s_j \rangle$ satisfying $w\alpha_i \in \Delta^+$ are $w = s_j, w = s_i s_j, w = s_j s_i s_j$; in fact applying these elements to α_i yields two distinct positive roots. Note that we have here $\langle s_i, s_j \rangle \cong D_4 = S_2 \wr S_2$, the dihedral group of order 8. This is the Weyl group of type B_2 .

If $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$, then as in the proof of Theorem 2.59 we obtain

$$\begin{aligned}\mathcal{T}_{s_j}(E_i) &= \mathcal{T}_j(E_i) = E_i E_j - q^2 E_j E_i, \\ \mathcal{T}_{s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i) = E_j E_i - q^2 E_i E_j, \\ \mathcal{T}_{s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i\end{aligned}$$

If $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$, we have to swap the roles of i and j in the proof of Theorem 2.59 and obtain

$$\begin{aligned}\mathcal{T}_{s_j}(E_i) &= \mathcal{T}_j(E_i) = E_i E_j^{(2)} - q E_j E_i E_j + q^2 E_j^{(2)} E_i, \\ \mathcal{T}_{s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i) = E_j^{(2)} E_i - q E_j E_i E_j + q^2 E_i E_j^{(2)}, \\ \mathcal{T}_{s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i\end{aligned}$$

In both cases the required properties hold.

- $m = 6$: In this case we have $s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$, and for the nontrivial elements $w \in \langle s_i, s_j \rangle$ satisfying $w\alpha_i \in \Delta^+$ one finds $w = s_j, w = s_i s_j, w = s_j s_i s_j, w = s_i s_j s_i s_j, w = s_j s_i s_j s_i s_j$; in fact applying these elements to α_i yields three distinct positive roots. Note that we have $\langle s_i, s_j \rangle \cong D_6$, the dihedral group of order 12. This is the Weyl group of type G_2 .

If $(\alpha_i, \alpha_i) < (\alpha_j, \alpha_j)$, then using the notation in the proof of Theorem 2.59 we obtain

$$\begin{aligned}\mathcal{T}_{s_j}(E_i) &= \mathcal{T}_j(E_i) = E_{ij}^+(1), \\ \mathcal{T}_{s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j(E_i) = E_{ij}^-(2), \\ \mathcal{T}_{s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_{ij}^+(2), \\ \mathcal{T}_{s_i s_j s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i) = E_{ij}^-(1), \\ \mathcal{T}_{s_j s_i s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i.\end{aligned}$$

If $(\alpha_i, \alpha_i) > (\alpha_j, \alpha_j)$, we have to swap the roles of i and j in the proof of Theorem 2.59 and obtain

$$\begin{aligned}\mathcal{T}_{s_j}(E_i) &= \mathcal{T}_j(E_i) = E_{ji}^-(3), \\ \mathcal{T}_{s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j(E_i) = [3]_q^{-1} (E_{ji}^+(2) E_{ji}^+(1) - q^{-1} E_{ji}^+(1) E_{ji}^+(2)), \\ \mathcal{T}_{s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = [3]_q^{-1} (E_{ji}^-(1) E_{ji}^-(2) - q^{-1} E_{ji}^-(2) E_{ji}^-(1)), \\ \mathcal{T}_{s_i s_j s_i s_j}(E_i) &= \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i) = E_{ji}^+(3) \\ \mathcal{T}_{s_j s_i s_j s_i s_j}(E_i) &= \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i.\end{aligned}$$

In both cases the required properties hold.

This finishes the proof. \square

Lemma 2.61. *Let $w \in W$. If $w\alpha_i \in \Delta^+$ then $\mathcal{T}_w(E_i) \in U_q(\mathfrak{n}_+)$, and if $w\alpha_i = \alpha_k$ for some $1 \leq k \leq N$ then $\mathcal{T}_w(E_i) = E_k$.*

Proof. We use induction on the length $l(w)$ of w , the case $l(w) = 0$ being trivial. Hence suppose $l(w) > 0$. Then according to Section 10.2 in [35] there exists $1 \leq j \leq N$ such that $w\alpha_j \in \Delta^-$, note in particular that we must have $i \neq j$. Let us write again $\langle s_i, s_j \rangle \subset W$ for the subgroup generated by s_i and s_j . According to Section 1.10 in [36] we find $u \in W$ and $v \in \langle s_i, s_j \rangle$ such that $w = uv$ and $u\alpha_i, u\alpha_j \in \Delta^+$, with lengths satisfying $l(w) = l(u) + l(v)$. In particular, since u maps both α_i and α_j into Δ^+ we have $u \neq w$, and we conclude $l(u) < l(w)$. Applying the inductive hypothesis to u we get $\mathcal{T}_u(E_i) \in U_q(\mathfrak{n}_+)$ and $\mathcal{T}_u(E_j) \in U_q(\mathfrak{n}_+)$. A similar reasoning as above shows $v\alpha_i \in \Delta^+$ since $w\alpha_i \in \Delta^+$. Hence Lemma 2.60 implies that $\mathcal{T}_v(E_i)$ is contained in the subalgebra of $U_q(\mathfrak{n}_+)$ generated by E_i and E_j . We conclude $\mathcal{T}_w(E_i) = \mathcal{T}_u\mathcal{T}_v(E_i) \in U_q(\mathfrak{n}_+)$ as desired.

For the second claim we proceed again by induction on $l(w)$. Using the above considerations, it suffices to show in the inductive step that $v\alpha_i$ is a simple root. Indeed, the second part of Lemma 2.60 will then yield the claim. However, if $v\alpha_i$ is not simple we can write $v\alpha_i = m\alpha_i + n\alpha_j$ with $m, n \in \mathbb{N}$ because $v \in \langle s_i, s_j \rangle$ and $v\alpha_i \in \Delta^+$. Then $w\alpha_i = mv\alpha_i + nv\alpha_j$ is a sum of positive roots because $u\alpha_i, u\alpha_j \in \Delta^+$. This contradicts the assumption that $w\alpha_i = \alpha_k$ is simple. Hence $v\alpha_i$ is simple, and this finishes the proof. \square

If $w \in W$ and $w = s_{i_1} \cdots s_{i_t}$ is a reduced expression of w then we shall call the vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_t}^{a_t}$ with $a_j \in \mathbb{N}_0$ for all j the associated PBW-vectors. We will show next that these vectors are always contained in $U_q(\mathfrak{n}_+)$.

Proposition 2.62. *Let $w \in W$ and $w = s_{i_1} \cdots s_{i_t}$ be a reduced expression of w . Then the associated PBW-vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_t}^{a_t}$ are contained in $U_q(\mathfrak{n}_+)$.*

Proof. Since $U_q(\mathfrak{n}_+)$ is a subalgebra of $U_q(\mathfrak{g})$ it suffices to show that E_{β_r} is contained in $U_q(\mathfrak{n}_+)$ for all $1 \leq r \leq t$. As indicated earlier on, $s_{i_1} \cdots s_{i_{r-1}}\alpha_{i_r}$ is a positive root. Considering the element $s_{i_1} \cdots s_{i_{r-1}} \in W$ and $i = i_r$ in Lemma 2.61 we see that $\mathcal{T}_{s_{i_1} \cdots s_{i_{r-1}}}(E_{i_r})$ is contained in $U_q(\mathfrak{n}_+)$. Moreover, observe that $s_{i_1} \cdots s_{i_{r-1}}$ is a reduced expression since it is part of a reduced expression. It follows that $\mathcal{T}_{s_{i_1} \cdots s_{i_{r-1}}} = \mathcal{T}_{s_{i_1}} \cdots \mathcal{T}_{s_{i_{r-1}}}$, and hence $\mathcal{T}_{s_{i_1} \cdots s_{i_{r-1}}}(E_{i_r}) = E_{\beta_r}$. This finishes the proof. \square

Next we discuss linear independence.

Proposition 2.63. *Let $w \in W$ and $w = s_{i_1} \cdots s_{i_t}$ be a reduced expression of w . Then the associated PBW-vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_t}^{a_t}$ are linearly independent.*

Proof. As a preparation, assume first that $X_1, \dots, X_m \in U_q(\mathfrak{n}_+)$ satisfy $\mathcal{T}_i^{-1}(X_j) \in U_q(\mathfrak{n}_+)$ for all j and $\sum_j X_j E_i^j = 0$ or $\sum_j E_i^j X_j = 0$. We claim that $X_j = 0$ for $j = 1, \dots, m$.

In order to verify this consider the case $\sum_j E_i^j X_j = 0$. Applying \mathcal{T}_i^{-1} yields

$$0 = \mathcal{T}_i^{-1} \left(\sum_{j=1}^m E_i^j X_j \right) = \sum_{j=1}^m \mathcal{T}_i^{-1}(E_i)^j \mathcal{T}_i^{-1}(X_j) = \sum_{j=1}^m (-F_i K_i^{-1})^j \mathcal{T}_i^{-1}(X_j).$$

The vectors $(-F_i K_i^{-1})^j \in U_q(\mathfrak{b}_-)$ have pairwise distinct weights and $\mathcal{T}_i^{-1}(X_j) \in U_q(\mathfrak{n}_+)$ by assumption. Hence the triangular decomposition of Proposition 2.14 yields the claim. The case $\sum_j X_j E_i^j = 0$ is analogous.

Let us now use induction on t to prove the Proposition. For $t = 0$, there is only one PBW-vector, namely the element 1, corresponding to the empty word. The assertion clearly holds in this case. Assume now that $t > 0$. Then the PBW-vectors have the form $E_{i_1}^k \mathcal{T}_{i_1}(X_{kl})$ where $k \in \mathbb{N}_0$ and X_{kl} is a PBW-vector for the reduced expression $s_{i_2} \cdots s_{i_t}$ of $s_{i_1} w$. If c_{kl} are scalars with $\sum_{k,l} c_{kl} E_{i_1}^k \mathcal{T}_{i_1}(X_{kl}) = 0$, then

the first part of our proof combined with Proposition 2.62 implies $\sum_l c_{kl} X_{kl} = 0$ for all k . By the inductive hypothesis we get $c_{kl} = 0$ for all k, l , and this finishes the proof. \square

Let $w \in W$ and $w = s_{i_1} \cdots s_{i_t}$ be a reduced expression of w . We show next that the subspace of $U_q(\mathfrak{n}_+)$ spanned by the associated PBW-vectors is independent of the reduced expression of w .

Lemma 2.64. *If $i \neq j$ and w is the longest element in the subgroup of W generated by s_i and s_j , then the PBW-vectors associated to a reduced expression of w span the subalgebra of $U_q(\mathfrak{n}_+)$ generated by E_i and E_j .*

Proof. Note that there are precisely two reduced expressions for w , namely $w = s_i s_j s_i \cdots$ and $w = s_j s_i s_j \cdots$. Both consist of m factors, where m is the order of $s_i s_j$. We shall proceed case-by-case depending on m .

$m = 2$: In this case we have $\mathcal{T}_j(E_i) = E_i, \mathcal{T}_i(E_j) = E_j$, the two reduced expressions for w have associated PBW-vectors $E_i^{r_1} E_j^{r_2}$ and $E_j^{t_1} E_i^{t_2}$, respectively, with $r_k, t_k \in \mathbb{N}_0$. Since E_i and E_j commute, the span is in both cases equal to the subalgebra of $U_q(\mathfrak{n}_+)$ generated by E_i and E_j .

$m = 3$: In this case the situation is symmetric in i and j . Hence it suffices to show that the PBW-vectors associated with $s_j s_i s_j$ span the subalgebra generated by E_i and E_j . From the proof of Theorem 2.59 we have $\mathcal{T}_{s_j}(E_i) = \mathcal{T}_j(E_i) = E_i E_j - q_i E_j E_i$ and $\mathcal{T}_{s_j s_i}(E_j) = \mathcal{T}_j \mathcal{T}_i(E_j) = E_i$. Then the claim is that the vectors $E_j^{r_1} \mathcal{T}_j(E_i)^{r_2} E_i^{r_3}$ with $r_k \in \mathbb{N}_0$ span the subalgebra generated by E_i and E_j .

It suffices to show that the linear span of these vectors is closed under left multiplication by E_i and E_j . For E_j this is trivial, so let us consider E_i . Using the quantum Serre relations we get

$$\begin{aligned} E_i \mathcal{T}_j(E_i) &= E_i^2 E_j - q_i E_i E_j E_i \\ &= q_i^{-1} E_i E_j E_i - E_j E_i^2 = q_i^{-1} \mathcal{T}_j(E_i) E_i, \end{aligned}$$

and we have

$$E_i E_j = q_i E_j E_i + \mathcal{T}_j(E_i)$$

by definition. Applying these relations iteratively we see that any term of the form $E_i E_j^{r_1} \mathcal{T}_j(E_i)^{r_2} E_i^{r_3}$ can again be written as a linear combination of such terms.

$m = 4$: As in the proof of Theorem 2.59 we shall use the convention $a_{ij} = -2, a_{ji} = -1$. We have $q_i = q$ and $q_j = q^2$ in this case.

For the reduced expression $w = s_j s_i s_j s_i$ we obtain the associated root vectors $E_j, \mathcal{T}_j(E_i), \mathcal{T}_j \mathcal{T}_i(E_j) = \mathcal{T}_i^{-1}(E_j), \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_i$. As before, it suffices to check that left multiplication by E_i preserves the linear span of all terms of the form

$$E_j^{r_1} \mathcal{T}_j(E_i)^{r_2} \mathcal{T}_i^{-1}(E_j)^{r_3} E_i^{r_4}$$

with $r_k \in \mathbb{N}_0$.

To this end we observe

$$E_i E_j = q^2 E_j E_i + \mathcal{T}_j(E_i),$$

and

$$E_i \mathcal{T}_j(E_i) = \mathcal{T}_j(E_i) E_i + [2]_q \mathcal{T}_i^{-1}(E_j).$$

The quantum Serre relations give

$$\begin{aligned} [2]_q E_i \mathcal{T}_i^{-1}(E_j) &= E_i^3 E_j - (q + q^{-1}) q E_i^2 E_j E_i + q^2 E_i E_j E_i^2 \\ &= q^{-2} E_i^2 E_j E_i - (q + q^{-1}) q^{-1} E_i E_j E_i^2 + E_j E_i^3 \\ &= q^{-2} [2]_q \mathcal{T}_i^{-1}(E_j) E_i. \end{aligned}$$

Moreover we have

$$E_i \mathcal{T}_j^{-1}(E_j) = -E_i F_j K_j^{-1} = -q_j^{a_{ji}} F_j K_j^{-1} E_i = q^{-2} \mathcal{T}_j^{-1}(E_j) E_i$$

and hence

$$\mathcal{T}_j(E_i) E_j = q^{-2} E_j \mathcal{T}_j(E_i).$$

Finally, the quantum Serre relations give

$$\begin{aligned} [2]_q \mathcal{T}_i(E_j) E_i &= E_j E_i^3 - (q + q^{-1}) q E_i E_j E_i^2 + q^2 E_i^2 E_j E_i \\ &= q^{-2} E_i E_j E_i^2 - (q + q^{-1}) q^{-1} E_i^2 E_j E_i + E_i^3 E_j \\ &= q^{-2} [2]_q E_i \mathcal{T}_i(E_j) \end{aligned}$$

and hence using $\mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) = \mathcal{T}_i(E_j)$ we get

$$\mathcal{T}_i^{-1}(E_j) \mathcal{T}_j(E_i) = q^{-2} \mathcal{T}_j(E_i) \mathcal{T}_i^{-1}(E_j).$$

Combining these commutation relations yields the assertion for $w = s_j s_i s_j s_i$.

For the reduced expression $w = s_i s_j s_i s_j$ we obtain the associated root vectors $E_i, \mathcal{T}_i(E_j), \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i), \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_j) = E_j$. We check that right multiplication by E_i preserves the linear span of all terms of the form

$$E_i^{r_1} \mathcal{T}_i(E_j^{r_2}) \mathcal{T}_j^{-1}(E_i^{r_3}) E_j^{r_4}$$

with $r_k \in \mathbb{N}_0$.

To this end we observe

$$E_j E_i = q^2 E_i E_j + \mathcal{T}_j^{-1}(E_i),$$

and

$$\mathcal{T}_j^{-1}(E_i) E_i = E_i \mathcal{T}_j^{-1}(E_i) + [2]_q \mathcal{T}_i(E_j).$$

As above, the quantum Serre relations give

$$[2]_q \mathcal{T}_i(E_j) E_i = q^{-2} [2]_q E_i \mathcal{T}_i(E_j)$$

and

$$[2]_q E_i \mathcal{T}_i^{-1}(E_j) = q^{-2} [2]_q \mathcal{T}_i^{-1}(E_j) E_i,$$

the second of which implies

$$\mathcal{T}_j^{-1}(E_i) \mathcal{T}_i(E_j) = q^{-2} \mathcal{T}_i(E_j) \mathcal{T}_j^{-1}(E_i).$$

Finally, we have the relation

$$\mathcal{T}_j(E_j) E_i = -F_j K_j E_i = -q_j^{a_{ji}} E_i F_j K_j = q^{-2} E_i \mathcal{T}_j(E_j)$$

which implies

$$E_j \mathcal{T}_j^{-1}(E_i) = q^{-2} \mathcal{T}_j^{-1}(E_i) E_j.$$

Again, combining these relations yields the assertion.

$m = 6$: As in the proof of Theorem 2.59 we shall use the convention $a_{ij} = -3, a_{ji} = -1$. We have $q_i = q$ and $q_j = q^3$ in this case.

Consider first the reduced expression $w = s_i s_j s_i s_j s_i s_j$. We obtain the associated root vectors $E_i, \mathcal{T}_i(E_j) = E_{ij}^-(3), \mathcal{T}_i \mathcal{T}_j(E_i) = E_{ij}^-(2), \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_j) = \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j), \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = \mathcal{T}_j^{-1}(E_i) = E_{ij}^-(1), E_j$. We check that right multiplication by E_i preserves the linear span of the vectors

$$E_i^{r_1} E_{ij}^-(3)^{r_2} E_{ij}^-(2)^{r_4} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) E_{ij}^-(1)^{r_5} E_j^{r_6}$$

with $r_j \in \mathbb{N}_0$.

We first note

$$E_j E_i = q^3 E_i E_j + E_{ij}^-(1)$$

and

$$E_{ij}^-(1) E_i = q E_i E_{ij}^-(1) + [2]_q E_{ij}^-(2).$$

Using

$$\begin{aligned} [2]_q E_j E_{ij}^-(2) &= E_j E_{ij}^-(1) E_i - q E_j E_i E_{ij}^-(1) \\ &= q^{-3} E_{ij}^-(1) E_j E_i - q E_{ij}^-(1)^2 - q^4 E_i E_j E_{ij}^-(1) \\ &= q^{-3} E_{ij}^-(1)^2 + E_{ij}^-(1) E_i E_j - q E_{ij}^-(1)^2 - q E_i E_{ij}^-(1) E_j \\ &= (q^{-3} - q) E_{ij}^-(1)^2 + E_{ij}^-(1) E_i E_j - q E_i E_{ij}^-(1) E_j \\ &= (q^{-3} - q) E_{ij}^-(1)^2 + [2]_q E_{ij}^-(2) E_j \end{aligned}$$

and $\mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_{ij}^-(1)) = \mathcal{T}_j^{-1}(E_{ij}^+(2)) = E_{ij}^-(2)$ we get

$$[2]_q \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) E_i = (q^{-3} - q) E_{ij}^-(2)^2 + [2]_q E_i \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j).$$

In addition,

$$\begin{aligned} E_{ij}^-(2) E_i - q^{-1} E_i E_{ij}^-(2) &= [3]_q E_j E_i^{(3)} - q^2 [2]_q E_i E_j E_i^{(2)} + q^4 E_i^{(2)} E_j E_i \\ &\quad - q^{-1} E_i E_j E_i^{(2)} + q [2]_q E_i^{(2)} E_j E_i - q^3 [3]_q E_i^{(3)} E_j \\ &= [3]_q E_{ij}^-(3), \end{aligned}$$

using $q[3]_q = q^{-1} + q + q^3 = q^2 [2]_q + q^{-1}$ and $q^2 [3]_q = 1 + q^2 + q^4 = q^4 + q [2]_q$. Finally, we recall $E_{ij}^-(3) E_i = -q^{-3} E_i E_{ij}^-(3)$.

Moreover $E_j E_{ij}^-(1) = q^{-3} E_{ij}^-(1) E_j$ and

$$E_{ij}^-(1) E_{ij}^-(2) = q E_{ij}^-(2) E_{ij}^-(1) + [3]_q \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j).$$

We have $E_i \mathcal{T}_i^{-1}(E_j) = E_i E_{ij}^+(3) = -q^{-3} E_{ij}^+(3) E_i$, and since $\mathcal{T}_j^{-1}(E_i) = E_{ij}^-(1)$ we get

$$E_{ij}^-(1) \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) = -q^{-3} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) E_{ij}^-(1).$$

In addition, $E_{ij}^+(3) E_{ij}^+(2) = q^{-3} E_{ij}^+(2) E_{ij}^+(3)$ implies

$$\mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) E_{ij}^-(2) = q^{-3} E_{ij}^-(2) \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j).$$

Finally, we recall $E_j E_{ij}^+(1) = q^3 E_{ij}^+(1) E_j$, which gives

$$E_{ij}^-(3) E_{ij}^-(2) = q^3 E_{ij}^-(2) E_{ij}^-(3).$$

Now consider the reduced expression $w = s_j s_i s_j s_i s_j s_i$. The associated root vectors are $E_j, \mathcal{T}_j(E_i) = E_{ij}^+(1), \mathcal{T}_j \mathcal{T}_i(E_j) = \mathcal{T}_i^{-1} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j), \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j(E_i) = E_{ij}^+(2), \mathcal{T}_j \mathcal{T}_i \mathcal{T}_j \mathcal{T}_i(E_j) = \mathcal{T}_i^{-1}(E_j) = E_{ij}^+(3), E_i$. We have to check that left multiplication by E_i preserves the linear span of all terms of the form

$$E_j^{r_1} E_{ij}^+(1)^{r_2} \mathcal{T}_i^{-1} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j^{r_3}) E_{ij}^+(2)^{r_4} E_{ij}^+(3)^{r_5} E_i^{r_6}$$

with $r_j \in \mathbb{N}_0$.

To this end we observe

$$E_i E_j = q^3 E_j E_i + E_{ij}^+(1),$$

and

$$E_i E_{ij}^+(1) = q E_{ij}^+(1) E_i + [2]_q E_{ij}^+(2),$$

Next observe that $\mathcal{T}_i^{-1}\mathcal{T}_j^{-1}(E_i) = \mathcal{T}_i^{-1}(E_{ij}^-(1)) = E_{ij}^+(2)$ and $\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i\mathcal{T}_j\mathcal{T}_i(E_j) = E_j$. We obtain

$$\begin{aligned} [2]_q E_{ij}^+(2)E_j &= E_i E_{ij}^+(1)E_j - q E_{ij}^+(1)E_i E_j \\ &= q^{-3} E_i E_j E_{ij}^+(1) - q E_{ij}^+(1)^2 + q^4 E_{ij}^+(1)E_j E_i \\ &= q^{-3} E_{ij}^+(1)^2 - E_j E_i E_{ij}^+(1) - q E_{ij}^+(1)^2 + q E_j E_{ij}^+(1)E_i \\ &= (q^{-3} - q) E_{ij}^+(1)^2 - E_j E_i E_{ij}^+(1) + q E_j E_{ij}^+(1)E_i \\ &= (q^{-3} - q) E_{ij}^+(1)^2 - [2]_q E_j E_{ij}^+(2) \end{aligned}$$

Using $\mathcal{T}_j\mathcal{T}_i(E_{ij}^+(1)) = E_{ij}^+(2)$ we get

$$[2]_q E_i \mathcal{T}_i^{-1}\mathcal{T}_j^{-1}\mathcal{T}_i^{-1}(E_j) = (q + q^{-3}) E_{ij}^+(2)^2 - [2]_q \mathcal{T}_i^{-1}\mathcal{T}_j^{-1}\mathcal{T}_i^{-1}(E_j)E_i.$$

In addition,

$$\begin{aligned} E_i E_{ij}^+(2) - q^{-1} E_{ij}^+(2)E_i &= [3]_q E_i^{(3)} E_j - q^2 [2]_q E_i^{(2)} E_j E_i + q^4 E_i E_j E_i^{(2)} \\ &\quad - q^{-1} E_i^{(2)} E_j E_i + q [2]_q E_i E_j E_i^{(2)} - q^3 [3]_q E_j E_i^{(3)} \\ &= [3]_q E_{ij}^+(3), \end{aligned}$$

using $q[3]_q = q^{-1} + q + q^3 = q^2 [2]_q + q^{-1}$ and $q^2 [3]_q = 1 + q^2 + q^4 = q^4 + q [2]_q$. Using the quantum Serre relations we obtain

$$\begin{aligned} [3]_q! E_i E_{ij}^+(3) &= E_i^4 E_j - q [3]_q E_i^3 E_j E_i + q^2 [3]_q E_i^2 E_j E_i^2 - q^3 E_i E_j E_i^3 \\ &= -q^{-3} E_i^3 E_j E_i + q^{-2} [3]_q E_i^2 E_j E_i^2 - q^{-1} [3]_q E_i E_j E_i^3 + E_j E_i^4 \\ &= -q^{-3} [3]_q! E_{ij}^+(3) E_i, \end{aligned}$$

taking into account $[4]_q = q^{-3} + q [3]_q = q^3 + q^{-1} [3]_q$ and $[4]_q = (q^2 + q^{-2}) [2]_q$. Moreover,

$$\begin{aligned} E_{ij}^+(1)E_j &= E_i E_j^2 - q^3 E_j E_i E_j \\ &= q^{-3} E_j E_i E_j - E_j^2 E_i \\ &= q^{-3} E_j E_{ij}^+(1) \end{aligned}$$

using the quantum Serre relation

$$E_i E_j^2 - (q^3 + q^{-3}) E_j E_i E_j + E_j^2 E_i = 0.$$

We have

$$E_{ij}^+(2)E_{ij}^+(1) = [3]_q \mathcal{T}_i^{-1}\mathcal{T}_j^{-1}\mathcal{T}_i^{-1}(E_j) + q^{-1} E_{ij}^+(1)E_{ij}^+(2),$$

and from $E_{ij}^+(1)E_j = q^{-3} E_j E_{ij}^+(1)$ we get

$$\begin{aligned} E_{ij}^+(2)\mathcal{T}_i^{-1}\mathcal{T}_j^{-1}\mathcal{T}_i^{-1}(E_j) &= \mathcal{T}_j\mathcal{T}_i(E_{ij}^+(1)E_j) \\ &= q^{-3} \mathcal{T}_j\mathcal{T}_i(E_j E_{ij}^+(1)) = q^{-3} \mathcal{T}_i^{-1}\mathcal{T}_j^{-1}\mathcal{T}_i^{-1}(E_j)E_{ij}^+(2). \end{aligned}$$

Using again the quantum Serre relations we compute

$$\begin{aligned} [3]_q! E_{ij}^-(3)E_i &= E_j E_i^4 - q [3]_q E_i E_j E_i^3 + q^2 [3]_q E_i^2 E_j E_i^2 - q^3 E_i^3 E_j E_i \\ &= -q^{-3} E_i E_j E_i^3 + q^{-2} [3]_q E_i^2 E_j E_i^2 - q^{-1} [3]_q E_i^3 E_j E_i + E_i^4 E_j \\ &= -q^{-3} [3]_q! E_i E_{ij}^-(3), \end{aligned}$$

taking into account $[4]_q = q^{-3} + q[3]_q = q^3 + q^{-1}[3]_q$ and $[4]_q = (q^2 + q^{-2})[2]_q$. This yields

$$\begin{aligned} \mathcal{T}_i^{-1} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j) E_{ij}^+(1) &= \mathcal{T}_j(E_{ij}^-(3) E_i) \\ &= -q^{-3} \mathcal{T}_j(E_i E_{ij}^-(3)) = -q^{-3} E_{ij}^+(1) \mathcal{T}_i^{-1} \mathcal{T}_j^{-1} \mathcal{T}_i^{-1}(E_j). \end{aligned}$$

We have

$$\begin{aligned} E_j E_{ij}^-(1) &= E_j^2 E_i - q^3 E_j E_i E_j \\ &= q^{-3} E_j E_i E_j - E_i E_j^2 \\ &= q^{-3} E_{ij}^-(1) E_j, \end{aligned}$$

whence

$$\begin{aligned} E_{ij}^+(3) E_{ij}^+(2) &= \mathcal{T}_i^{-1}(E_j E_{ij}^-(1)) \\ &= q^{-3} \mathcal{T}_i^{-1}(E_{ij}^-(1) E_j) = q^{-3} E_{ij}^+(2) E_{ij}^+(3). \end{aligned}$$

Combining these relations yields the assertion.

This finishes the proof. \square

Proposition 2.65. *Let $w \in W$ and $w = s_{i_1} \cdots s_{i_t}$ be a reduced expression of w . The linear subspace of $U_q(\mathfrak{n}_+)$ spanned by the associated PBW-vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_t}^{a_t}$ depends on w , but not on the choice of the reduced expression for w .*

Proof. We use induction on $l(w)$, the cases $l(w) = 0$ and $l(w) = 1$ being trivial. Now assume $l(w) > 1$ and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression. Any other reduced expression $s_{j_1} \cdots s_{j_r}$ is obtained from a sequence of elementary moves applied to $s_{i_1} \cdots s_{i_r}$, so it suffices to show that the subspace spanned by the associated PBW-vectors does not change if $s_{j_1} \cdots s_{j_r}$ is obtained by applying a single elementary move to $s_{i_1} \cdots s_{i_r}$. Set $\alpha = \alpha_{i_1}$ and $\beta = \alpha_{j_1}$.

Assume first $\alpha = \beta$ write $u = s_{i_1} w = s_{j_1} w$. The the inductive hypothesis applied to u shows that the subspace U associated to u is the same for the reduced expressions $s_{i_2} \cdots s_{i_r}$ and $s_{j_2} \cdots s_{j_r}$ of u . It follows from the definitions that the space spanned by the PBW-vectors associated to both $s_{i_1} \cdots s_{i_r}$ and $s_{j_1} \cdots s_{j_r}$ is the sum of the subspaces $E_{i_1}^k \mathcal{T}_{\alpha_i}(U)$ for $k \in \mathbb{N}_0$. In particular, these spaces agree for both reduced expressions of w .

Assume now $\alpha \neq \beta$. Then the elementary move changes the first letter of the expression $s_{i_1} \cdots s_{i_r}$, and therefore the corresponding elementary move has to take place at the beginning of the expression. Let u be the longest word in the subgroup of W generated by s_{i_1} and s_{j_1} . Then $s_{i_1} \cdots s_{i_r}$ and $s_{j_1} \cdots s_{j_r}$ start with the two distinct reduced expressions for u and agree after that. Let us write $w = uv$ with $l(w) = l(u) + l(v)$. According to Lemma 2.64 the subspace U spanned by the PBW-vectors associated to u is independent of the two expressions. Let us write V for the span of PBW-vectors associated to the reduced expression of v induced by $s_{i_1} \cdots s_{i_r}$, or equivalently $s_{j_1} \cdots s_{j_r}$. Then the span of the PBW-vectors associated to the given expressions for w is equal to the $U\mathcal{T}_u(V)$, in particular, it does not depend on the reduced expressions. \square

In the sequel we will write $U_q(\mathfrak{n}_+)[w] \subset U_q(\mathfrak{n}_+)$ for the subspace spanned by the PBW-vectors associated with any reduced expression of w .

Let us fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_n}$ for the longest element w_0 of W . Then the positive roots of \mathfrak{g} can be uniquely written in the form $\beta_r = s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}$ for $1 \leq r \leq n$.

Definition 2.66. Fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_n}$ for the longest element of W . We define the *quantum root vectors* of $U_q(\mathfrak{g})$ to be the associated PBW-vectors

$$E_{\beta_r} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{r-1}}(E_{i_r}), \quad F_{\beta_r} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{r-1}}(F_{i_r}).$$

Let us now formulate and prove the Poincaré-Birkhoff-Witt Theorem for $U_q(\mathfrak{g})$.

Theorem 2.67 (PBW-basis - Non-root of unity case). *Assume $q \in \mathbb{K}^\times$ is not a root of unity. Then the elements*

$$F_{\beta_1}^{b_1} \cdots F_{\beta_n}^{b_n} K_\lambda E_{\beta_1}^{a_1} \cdots E_{\beta_n}^{a_n}$$

where $b_j, a_j \in \mathbb{N}_0$ for all j , $\lambda \in \mathbf{P}$ and β_1, \dots, β_n are the positive roots of \mathfrak{g} , form a vector space basis of $U_q(\mathfrak{g})$ over \mathbb{K} .

Proof. According to the triangular decomposition of $U_q(\mathfrak{g})$, see Proposition 2.14, it suffices to show that the vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_n}^{a_n}$ with $a_j \in \mathbb{N}_0$ form a basis of $U_q(\mathfrak{n}_+)$. The corresponding claim for $U_q(\mathfrak{n}_-)$ can then be obtained by applying the automorphism ω . Indeed, one checks that $\mathcal{T}_i(\omega(E_j))$ and $\omega(\mathcal{T}_i(E_j))$ agree up to invertible elements in \mathbb{K} , for every $1 \leq i, j \leq N$.

Firstly, according to Proposition 2.63 the PBW-vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_n}^{a_n}$ with $a_j \in \mathbb{N}_0$ are linearly independent. To show that these vectors span $U_q(\mathfrak{n}_+)$ we need an auxiliary consideration. If $w \in W$ and $\alpha_i \in \Sigma$ satisfy $w^{-1}\alpha_i \in \Delta^-$ then we have $E_i U_q(\mathfrak{n}_+)[w] \subset U_q(\mathfrak{n}_+)[w]$. Indeed, in this case we find a reduced decomposition $w = s_{i_1} \cdots s_{i_t}$ with $i_1 = i$, hence the PBW-vectors start with powers of E_i . Clearly, the resulting space is invariant under left multiplication by E_i .

Let us now show that the PBW-vectors span $U_q(\mathfrak{n}_+)$. By definition, these vectors span the subspace $U_q(\mathfrak{n}_+)[w_0]$. Since w_0 is the longest word of W we have $w_0^{-1}(\alpha_i) \in \Delta^-$ and thus $E_i U_q(\mathfrak{n}_+)[w_0] \subset U_q(\mathfrak{n}_+)[w_0]$ for all i . Hence $U_q(\mathfrak{n}_+)[w_0]$ is closed under left multiplication by the generators E_1, \dots, E_N . By construction, the space $U_q(\mathfrak{n}_+)[w_0]$ also contains 1. It follows that $U_q(\mathfrak{n}_+)[w_0] = U_q(\mathfrak{n}_+)$ as desired. \square

We point out that one can reverse the ordering in the basis obtained in Theorem 2.67. That is, the vectors

$$F_{\beta_n}^{b_n} \cdots F_{\beta_1}^{b_1} K_\lambda E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}$$

where $b_j, a_j \in \mathbb{N}_0$ for all j , $\lambda \in \mathbf{P}$ and β_1, \dots, β_n are the positive roots of \mathfrak{g} , form again a vector space basis of $U_q(\mathfrak{g})$. This can be seen using the anti-automorphism δ of $U_q(\mathfrak{g})$ which preserves all generators but exchanges q with q^{-1} . One checks that the automorphisms $\mathcal{T}_i^{-1} \delta \mathcal{T}_i \delta$ act by certain scalars on the generators, and hence diagonally on weight spaces of $U_q(\mathfrak{g})$. Hence we have for all $1 \leq j \leq n$ that $\delta(E_{\beta_j})$ is equal to E_{β_j} up to a scalar, and since δ is anti-multiplicative it follows from Theorem 2.67 that the vectors $E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}$ form indeed a basis. We will frequently make use of this fact in the sequel, and refer to the vectors $E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}$ again as PBW-vectors. In a similar way we proceed for $F_{\beta_n}^{b_n} \cdots F_{\beta_1}^{b_1}$ of course.

Let us also remark that the rank 1 case of Theorem 2.67 follows immediately from Proposition 2.14.

As a consequence of the PBW-basis Theorem, we see that the dimension of the weight spaces $U_q(\mathfrak{n}_\pm)_\nu$ for $\nu \in \mathbf{Q}^\pm$ coincide with their classical counterparts. Here we are writing $\mathbf{Q}^- = -\mathbf{Q}^+$. These dimensions can be described in terms of the Kostant partition function $P : \mathbf{Q} \rightarrow \mathbb{N}_0$, defined by

$$P(\nu) = |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1 \beta_1 + \cdots + r_n \beta_n = \nu\}|.$$

Proposition 2.68. *For all $\nu \in \mathbf{Q}^+$ we have $\dim U_q(\mathfrak{n}_\pm)_{\pm\nu} = P(\nu)$, where P denotes Kostant's partition function.*

Proof. Let us only consider the case of $U_q(\mathfrak{n}_+)$. According to Theorem 2.67, a linear basis for $U_q(\mathfrak{n}_+)_\nu$ for $\nu \in \mathbf{Q}^+$ is given by all PBW-vectors $E_{\beta_1}^{a_1} \cdots E_{\beta_n}^{a_n}$ such that $a_1\beta_1 + \cdots + a_n\beta_n = \nu$. The number of these vectors is precisely $P(\nu)$. \square

Let us finish this subsection by the following result, originally due to Levenskii, which will help us to analyze the structure of $U_q(\mathfrak{n}_\pm)$ further below.

Proposition 2.69. *If $0 \leq r < s \leq n$ then we have*

$$E_{\beta_r} E_{\beta_s} - q^{(\beta_r, \beta_s)} E_{\beta_s} E_{\beta_r} = \sum_{t_{r+1}, \dots, t_{s-1}=0}^{\infty} c_{t_{r+1}, \dots, t_{s-1}} E_{\beta_{s-1}}^{t_{s-1}} \cdots E_{\beta_{r+1}}^{t_{r+1}},$$

with only finitely many coefficients $c_{t_{r+1}, \dots, t_{s-1}} \in \mathbb{K}$ in the sum on the right hand side being nonzero. Likewise,

$$F_{\beta_r} F_{\beta_s} - q^{(\beta_r, \beta_s)} F_{\beta_s} F_{\beta_r} = \sum_{t_{r+1}, \dots, t_{s-1}=0}^{\infty} d_{t_{r+1}, \dots, t_{s-1}} F_{\beta_{s-1}}^{t_{s-1}} \cdots F_{\beta_{r+1}}^{t_{r+1}},$$

with only finitely many $d_{t_{r+1}, \dots, t_{s-1}} \in \mathbb{K}$ nonzero.

Proof. According to the PBW-Theorem 2.67 we can write

$$E_{\beta_r} E_{\beta_s} = \sum_{t_1, \dots, t_n \in \mathbb{N}_0} c_{t_1, \dots, t_n} E_{\beta_n}^{t_n} \cdots E_{\beta_1}^{t_1},$$

with only finitely many nonzero terms on the right hand side. Assume that there exists $a < r$ such that some $c_{0, \dots, 0, t_a, \dots, t_n} \neq 0$ with $t_a > 0$, and without loss of generality pick the smallest such a . If we apply $\mathcal{T}_{i_a}^{-1} \cdots \mathcal{T}_{i_1}^{-1}$ to both sides of the above equation then the left hand side, and all terms on the right hand side with $t_a = 0$ are contained in $U_q(\mathfrak{n}_+)$. Indeed, all these terms are products of root vectors of the form $\mathcal{T}_{i_{a+1}} \cdots \mathcal{T}_{i_{l-1}}(E_{i_l})$ such that $a+1 \leq l$, recall that $E_{\beta_l} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{l-1}}(E_{i_l})$. The word $s_{i_{a+1}} \cdots s_{i_l}$ is a reduced expression and hence $s_{i_{a+1}} \cdots s_{i_{l-1}} \alpha_{i_l}$ is a positive root. According to Lemma 2.61 this implies $E_{\beta_l} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{l-1}}(E_{i_l}) \in U_q(\mathfrak{n}_+)$.

The terms with $t_a > 0$ are of the form $X \mathcal{T}_{i_a}^{-1}(E_{i_a})^{t_a}$ for $X \in U_q(\mathfrak{n}_+)$. Since $\mathcal{T}_{i_a}^{-1}(E_{i_a}) = -F_{i_a} K_{i_a}^{-1}$ this is a contradiction to the PBW-Theorem 2.67.

In a similar way one checks that $c_{t_1, \dots, t_b, 0, \dots, 0} \neq 0$ for $b > s$ is impossible. Indeed, assume that a nonzero coefficient of this form exists and pick b maximal with this property. Applying $\mathcal{T}_{i_{b-1}}^{-1} \cdots \mathcal{T}_{i_1}^{-1}$ to both sides of the equation leads to expressions in $U_q(\mathfrak{n}_-)$, except for the terms with $t_b \neq 0$.

Next, consider the nonzero coefficients $c_{0, \dots, 0, t_r, \dots, t_s, 0, \dots}$ with t_r or t_s nonzero. In a similar way as above we apply $\mathcal{T}_{i_r}^{-1} \cdots \mathcal{T}_{i_1}^{-1}$ and $\mathcal{T}_{i_{s-1}}^{-1} \cdots \mathcal{T}_{i_1}^{-1}$, respectively, and compare both sides of the equation. The highest powers of F_{i_r} or E_{i_s} appearing in the resulting expressions must have the same degree on both sides, and this forces $t_r = 1$ and $t_s = 1$, respectively. As a consequence, for any nonzero coefficient of this form we obtain $t_j = 0$ for all $r < j < s$ by comparing weights.

It follows that we can write

$$E_{\beta_r} E_{\beta_s} - c E_{\beta_s} E_{\beta_r} = \sum_{t_{r+1}, \dots, t_{s-1}=0}^{\infty} c_{t_{r+1}, \dots, t_{s-1}} E_{\beta_{s-1}}^{t_{s-1}} \cdots E_{\beta_{r+1}}^{t_{r+1}},$$

with an as yet unspecified constant c . In order to compute c we apply $\mathcal{T}_{i_r}^{-1} \cdots \mathcal{T}_{i_1}^{-1}$ to both sides of the previous equation. Then the left hand side becomes

$$-F_{i_r} K_{i_r}^{-1} E + c E F_{i_r} K_{i_r}^{-1}$$

where $E = \mathcal{T}_{i_{r+1}} \cdots \mathcal{T}_{i_{s-1}}(E_{i_s})$. Since E has weight $s_{i_r} \cdots s_{i_1}(\beta_s)$ we get $K_{i_r}^{-1} E = q^{-(\alpha_{i_r}, s_{i_r} \cdots s_{i_1}(\beta_s))} E K_{i_r}^{-1}$. Moreover

$$-(\alpha_{i_r}, s_{i_r} \cdots s_{i_1}(\beta_s)) = (s_{i_1} \cdots s_{i_{r-1}}(\alpha_{i_r}), \beta_s) = (\beta_r, \beta_s),$$

using that $s_{i_r}(\alpha_{i_r}) = -\alpha_{i_r}$. Accordingly, we obtain

$$-F_{i_r}K_{i_r}^{-1}E + cEF_{i_r}K_{i_r}^{-1} = (-q^{(\beta_r, \beta_s)}F_{i_r}E + cEF_{i_r})K_{i_r}^{-1}.$$

Writing E as a sum of monomials in the generators E_1, \dots, E_N we get $F_{i_r}E - EF_{i_r} \in U_q(\mathfrak{b}_+)$. Comparing with the right hand side of the formula and using once more the PBW-Theorem 2.67 yields $c = q^{(\beta_r, \beta_s)}$ as desired.

The claim for the PBW-vectors of $U_q(\mathfrak{n}_-)$ follows by applying the automorphism ω to the formula in $U_q(\mathfrak{n}_+)$ and using the fact that $\mathcal{T}_i(\omega(E_j))$ and $\omega(\mathcal{T}_i(E_j))$ agree up to an invertible element in \mathbb{K} , for every i, j . \square

2.7.5. The integral PBW-basis. Recall that the integral version $U_q^{\mathcal{A}}(\mathfrak{g})$ of the quantized universal enveloping algebra is defined over $\mathcal{A} = \mathbb{Z}[s, s^{-1}]$ where $q = s^L$. Let us fix a reduced expression $w_0 = w_{i_1} \cdots w_{i_n}$ for the longest element w_0 of W . We define the *restricted PBW-elements* for $U_q(\mathfrak{n}_+)$ by

$$E_{\beta_1}^{(a_1)} \cdots E_{\beta_n}^{(a_n)},$$

where

$$E_{\beta_r}^{(a_r)} = \frac{1}{[a_r]_{q_{\beta_r}}!} E_{\beta}^{a_r}$$

and $q_{\beta_r} = q_{i_r}$ if $E_{\beta_r} = \mathcal{T}_{i_1} \cdots \mathcal{T}_{i_{r-1}}(E_{i_r})$. That is, the difference to the ordinary PBW-elements is that we multiply by the inverse of $[a_1]_{q_{\beta_1}}! \cdots [a_n]_{q_{\beta_n}}!$. In a similar way one obtains restricted PBW-elements for $U_q(\mathfrak{n}_-)$.

Before we can state the integral version of the PBW-Theorem we need a refinement of Proposition 2.54.

Lemma 2.70. *Let V be an integrable $U_q(\mathfrak{g})$ -module. For any $n \in \mathbb{N}$ we have*

$$\begin{aligned} \mathcal{T}_i(E_j^{(n)} \cdot v) &= (-q_i)^{-na_{ij}} (E_j^{(n)} \leftarrow \hat{S}^{-1}(E_i^{(-na_{ij})})) \cdot \mathcal{T}_i(v) \\ \mathcal{T}_i(F_j^{(n)} \cdot v) &= (F_i^{(-na_{ij})} \rightarrow F_j^{(n)}) \cdot \mathcal{T}_i(v) \end{aligned}$$

for all $v \in V$.

Proof. Let us first consider the second formula. According to Proposition 2.54 we have the claim for $n = 1$. In order to prove the assertion for general n it suffices to show $(F_i^{(-a_{ij})} \rightarrow F_j)^n = F_i^{(-na_{ij})} \rightarrow (F_j^n)$ by induction. Assuming that the claim holds for $n - 1$, we compute, using Lemma 2.6,

$$\begin{aligned} F_i^{(-na_{ij})} \rightarrow (F_j^n) &= \left((F_i^{(-na_{ij})})_{(1)} \rightarrow F_j^{n-1} \right) \left((F_i^{(-na_{ij})})_{(2)} \rightarrow F_j \right) \\ &= \sum_{k=0}^{-na_{ij}} q_i^{k(-na_{ij}-k)} (F_i^{(-na_{ij}-k)} K_i^{-k} \rightarrow F_j^{n-1}) (F_i^{(k)} \rightarrow F_j). \end{aligned}$$

Using that $F_i^{(-a_{ij}+1)} \rightarrow F_j = 0$ and the inductive hypothesis, all of the terms in this sum vanish except when $k = -a_{ij}$, leaving

$$\begin{aligned} F_i^{(-na_{ij})} \rightarrow (F_j^n) &= q_i^{-a_{ij}(-(n-1)a_{ij})} (F_i^{-(n-1)a_{ij}} K_i^{a_{ij}} \rightarrow F_j^{n-1}) (F_j^{(-a_{ij})} \rightarrow F_j) \\ &= (F_i^{-(n-1)a_{ij}} \rightarrow F_j^{n-1}) (F_j^{(-a_{ij})} \rightarrow F_j) \\ &= (F_j^{(-a_{ij})} \rightarrow F_j)^{n-1} (F_j^{(-a_{ij})} \rightarrow F_j) = (F_j^{(-a_{ij})} \rightarrow F_j)^n \end{aligned}$$

as desired.

Using this result, one obtains the first formula in the same way as in the proof of Proposition 2.54. \square

Let us now present an integral version of the PBW Theorem 2.67. Recall that $U_q^{\mathcal{A}}(\mathfrak{n}_{\pm}) \subset U_q^{\mathcal{A}}(\mathfrak{g})$ is the subalgebra generated by all divided powers $E_i^{(a_i)}$ and $F_j^{(b_j)}$, respectively.

Theorem 2.71 (PBW-basis - Integral Case). *The elements*

$$E_{\beta_1}^{(a_1)} \cdots E_{\beta_n}^{(a_n)}$$

where $a_j \in \mathbb{N}_0$ and β_1, \dots, β_n are the positive roots of \mathfrak{g} , form a free basis of $U_q^{\mathcal{A}}(\mathfrak{n}_+)$. Similarly, the elements

$$F_{\beta_1}^{(b_1)} \cdots F_{\beta_n}^{(b_n)}$$

where $b_j \in \mathbb{N}_0$ and β_1, \dots, β_n are the positive roots of \mathfrak{g} , form a free basis of $U_q^{\mathcal{A}}(\mathfrak{n}_-)$.

Proof. Recall that we can view $U_q^{\mathcal{A}}(\mathfrak{g})$ as a subring of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ defined over $\mathbb{K} = \mathbb{Q}(s)$. Using the formulas in Theorem 2.58 and Lemma 2.70 we see that the operators \mathcal{T}_i map $U_q^{\mathcal{A}}(\mathfrak{g})$ into itself. It follows in particular that the PBW-vectors are indeed contained in $U_q^{\mathcal{A}}(\mathfrak{g})$.

In the same way as in the proof of Theorem 2.67 one checks that left multiplication with $E_i^{(r)}$ preserves the \mathcal{A} -span of the PBW-basis vectors $E_{\beta_1}^{(a_1)} \cdots E_{\beta_n}^{(a_n)}$ for all $1 \leq i \leq N$ and $r \in \mathbb{N}_0$. We point out in particular that the commutation relations obtained in Lemma 2.64 only involve coefficients from \mathcal{A} , so that the integral version of Proposition 2.65 holds true. That is, the \mathcal{A} -linear span of the restricted PBW-vectors $E_{\beta_1}^{(a_1)} \cdots E_{\beta_n}^{(a_n)}$ inside $U_q(\mathfrak{n}_+)$ is independent of the reduced expression of w_0 . Linear independence of the restricted PBW-elements over \mathcal{A} follows from linear independence over $\mathbb{Q}(s)$.

The assertion for $U_q(\mathfrak{n}_-)$ is obtained in an analogous way. \square

Using Theorem 2.71 we see that the triangular decomposition of $U_q(\mathfrak{g})$ carries over to the integral form as well. Recall that $U_q^{\mathcal{A}}(\mathfrak{h})$ denotes the intersection of $U_q^{\mathcal{A}}(\mathfrak{g})$ with $U_q(\mathfrak{h})$.

Proposition 2.72. *Multiplication in $U_q^{\mathcal{A}}(\mathfrak{g})$ induces an isomorphism*

$$U_q^{\mathcal{A}}(\mathfrak{n}_-) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{h}) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{n}_+) \cong U_q^{\mathcal{A}}(\mathfrak{g})$$

of \mathcal{A} -modules, and we have

$$\mathbb{Q}(s) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{g}) \cong U_q(\mathfrak{g}),$$

where $U_q(\mathfrak{g})$ is the quantized universal enveloping algebra over $\mathbb{Q}(s)$.

Proof. It is easy to see that the image in $U_q^{\mathcal{A}}(\mathfrak{g})$ under the multiplication map is a subalgebra containing all generators, and therefore yields all of $U_q^{\mathcal{A}}(\mathfrak{g})$. According to Theorem 2.71 and Theorem 2.67 we have $\mathbb{Q}(s) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{n}_{\pm}) \cong U_q(\mathfrak{n}_{\pm})$. From Proposition 2.24 we know $\mathbb{Q}(s) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{h}) \cong U_q(\mathfrak{h})$. Hence the multiplication map $U_q^{\mathcal{A}}(\mathfrak{n}_-) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{h}) \otimes_{\mathcal{A}} U_q^{\mathcal{A}}(\mathfrak{n}_+) \rightarrow U_q^{\mathcal{A}}(\mathfrak{g})$ is injective. The remaining claim then follows taking into account Proposition 2.14. \square

Proposition 2.72 shows in particular that $U_q^{\mathcal{A}}(\mathfrak{g})$ is indeed an integral version of $U_q(\mathfrak{g})$ in a natural sense.

2.8. The Drinfeld pairing and the quantum Killing form. In this section we discuss the Drinfeld pairing and the quantum Killing form, compare [71].

2.8.1. Drinfeld pairing. Let us start by defining the Drinfeld pairing. We assume that \mathbb{K} is a field and that $q = s^L \in \mathbb{K}^{\times}$ is not a root of unity.

We recall that a skew-pairing of Hopf algebras L and K means a bilinear map $\rho : L \times K \rightarrow \mathbb{K}$ such that

$$\begin{aligned} \rho(xy, f) &= \rho(x, f_{(1)})\rho(y, f_{(2)}), & \rho(x, fg) &= \rho(x_{(1)}, g)\rho(x_{(2)}, f), \\ \rho(1, f) &= \epsilon_K(f), & \rho(x, 1) &= \epsilon_L(x). \end{aligned}$$

Proposition 2.73. *There exists a unique skew-pairing $\rho : U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{b}_+) \rightarrow \mathbb{K}$ determined by*

$$\rho(K_\alpha, K_\beta) = q^{(\alpha, \beta)}, \quad \rho(K_\alpha, E_i) = 0 = \rho(F_j, K_\beta), \quad \rho(F_j, E_i) = \frac{\delta_{ij}}{q_i - q_i^{-1}}$$

for all $\alpha, \beta \in \mathbf{P}$ and $i, j = 1, \dots, N$.

Proof. Uniqueness of the form is clear since the elements K_α, E_i and K_β, F_j generate $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$, respectively.

We define linear functionals $\kappa_\lambda, \phi_i \in U_q(\mathfrak{b}_+)^*$ for $\lambda \in \mathbf{P}$ and $i = 1, \dots, N$ by

$$\kappa_\lambda(K_\beta X) = q^{(\lambda, \beta)} \hat{\epsilon}(X)$$

for $X \in U_q(\mathfrak{n}_+)$ and

$$\phi_i(K_\beta E_i) = \frac{1}{q_i - q_i^{-1}}, \quad \phi_i(K_\beta X) = 0 \quad \text{if } X \in U_q(\mathfrak{n}_+)_\gamma, \gamma \neq \alpha_i.$$

Consider the convolution product on $U_q(\mathfrak{b}_+)^*$ dual to comultiplication on $U_q(\mathfrak{b}_+)$, namely

$$\varphi\varphi' = (\varphi \otimes \varphi') \circ \hat{\Delta}, \quad \text{for } \varphi, \varphi' \in U_q(\mathfrak{b}_+)^*.$$

With this product, κ_λ is convolution invertible with inverse $\kappa_\lambda^{-1} = \kappa_{-\lambda}$. Moreover, we claim that

$$\kappa_\lambda \phi_j = q^{-(\alpha_j, \lambda)} \phi_j \kappa_\lambda.$$

Indeed, for $X \in U_q(\mathfrak{n}_+)$ we have $(\kappa_\lambda \phi_j \kappa_\lambda^{-1})(K_\mu X) = 0 = \phi_j(K_\mu X)$ if $X \in U_q(\mathfrak{n}_+)_\beta$ for $\beta \neq \alpha_j$ and

$$\begin{aligned} (\kappa_\lambda \phi_j)(K_\mu E_j) &= \kappa_\lambda(K_\mu) \phi_j(K_\mu E_j) + \kappa_\lambda(K_\mu E_j) \phi_j(K_\mu K_j) \\ &= q^{(\lambda, \mu)} \phi_j(K_\mu E_j) \\ &= q^{-(\alpha_j, \lambda)} (\phi_j(K_\mu) \kappa_\lambda(K_\mu E_j) + \phi_j(K_\mu E_j) \kappa_\lambda(K_\mu K_j)) \\ &= q^{-(\alpha_j, \lambda)} (\phi_j \kappa_\lambda)(K_\mu E_j). \end{aligned}$$

It follows that we obtain an algebra homomorphism $\tilde{\gamma} : \tilde{U}_q(\mathfrak{b}_-) \rightarrow U_q(\mathfrak{b}_+)^*$ by setting $\tilde{\gamma}(K_\lambda) = \kappa_\lambda$ and $\tilde{\gamma}(F_j) = \phi_j$. Here $\tilde{U}_q(\mathfrak{b}_-)$ denotes the algebra without Serre relations discussed in Section 2.2.

We define $\tilde{\rho}(Y, X) = \tilde{\gamma}(Y)(X)$. Then we have $\tilde{\rho}(YZ, X) = \tilde{\rho}(Y, X_{(1)})\tilde{\rho}(Z, X_{(2)})$ by construction. In order to show $\tilde{\rho}(Y, WX) = \tilde{\rho}(Y_{(2)}, W)\tilde{\rho}(Y_{(1)}, X)$ it suffices to consider generators in the first variable, since the relation holds for $\tilde{\rho}(Y_1 Y_2, WX)$ for arbitrary W, X iff it holds for all $\tilde{\rho}(Y_1, WX)$ and $\tilde{\rho}(Y_2, WX)$, for arbitrary W, X .

Let $E_i = E_{i_1} \cdots E_{i_k}, E_j = E_{j_1} \cdots E_{j_l}$ and $\mu, \nu \in \mathbf{P}$ be given, and consider $\tilde{\rho}(K_\lambda, K_\mu E_i K_\nu E_j)$. If either one of E_i or E_j is different from 1 we get

$$\tilde{\rho}(K_\lambda, K_\mu E_i K_\nu E_j) = 0 = \tilde{\rho}(K_\lambda, K_\nu E_j) \tilde{\rho}(K_\lambda, K_\mu E_i).$$

Otherwise we get

$$\tilde{\rho}(K_\lambda, K_\mu K_\nu) = \kappa_\lambda(K_\mu K_\nu) = q^{(\lambda, \mu + \nu)} = \kappa_\lambda(K_\nu) \kappa_\lambda(K_\mu) = \tilde{\rho}(K_\lambda, K_\nu) \tilde{\rho}(K_\lambda, K_\mu).$$

Moreover we have

$$\tilde{\rho}(F_k, K_\mu E_i K_\nu E_j) = \phi_k(K_\mu E_i K_\nu E_j) = 0 = (\phi_k \otimes 1 + \kappa_k^{-1} \otimes \phi_k)(K_\nu E_j \otimes K_\mu E_i)$$

unless precisely one of the terms E_i or E_j has length one. In these cases we obtain

$$\begin{aligned}\tilde{\rho}(F_k, K_\mu E_i K_\nu) &= \phi_k(K_\mu E_i K_\nu) = q^{-(\alpha_i, \nu)} \phi_k(K_\mu K_\nu E_i) \\ &= q^{-(\alpha_i, \nu)} \delta_{ik} \frac{1}{q_i - q_i^{-1}} \\ &= (\phi_k \otimes 1 + \kappa_k^{-1} \otimes \phi_k)(K_\nu \otimes K_\mu E_i) \\ &= \tilde{\rho}((F_k)_{(1)}, K_\nu) \tilde{\rho}((F_k)_{(2)}, K_\mu E_i)\end{aligned}$$

and

$$\begin{aligned}\tilde{\rho}(F_k, K_\mu K_\nu E_j) &= \phi_k(K_\mu K_\nu E_j) \\ &= \delta_{jk} \frac{1}{q_j - q_j^{-1}} \\ &= (\phi_k \otimes 1 + \kappa_k^{-1} \otimes \phi_k)(K_\nu E_j \otimes K_\mu) \\ &= \tilde{\rho}((F_k)_{(1)}, K_\nu E_j) \tilde{\rho}((F_k)_{(2)}, K_\mu),\end{aligned}$$

respectively.

Next we claim that

$$\tilde{\gamma}(u_{ij}^-) = \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} \phi_i^{1-a_{ij}-k} \phi_j \phi_i^k = 0,$$

where we recall that the Serre elements u_{ij}^- were introduced in Subsection 2.2.3. For this, notice that the individual summands vanish on all monomials in the E_k except on terms of the form $K_\nu E_i^r E_j E_i^{1-a_{ij}-r}$. According to Proposition 2.10 and the relation $\tilde{\rho}(Y, WX) = \tilde{\rho}(Y_{(2)}, W) \tilde{\rho}(Y_{(1)}, X)$ for all $W, X \in U_q(\mathfrak{b}_+)$ established above we have

$$\tilde{\rho}(u_{ij}^-, WX) = \tilde{\rho}(u_{ij}^-, W) \tilde{\rho}(\kappa_i^{-(1-a_{ij})} \kappa_j^{-1}, X) + \tilde{\rho}(1, W) \tilde{\rho}(u_{ij}^-, X)$$

for all $W, X \in U_q(\mathfrak{b}_+)$, therefore the claim follows from degree considerations. We conclude that the map $\tilde{\gamma}$ factorises to an algebra homomorphism $\gamma : U_q(\mathfrak{b}_-) \rightarrow U_q(\mathfrak{b}_+)^*$ such that $\gamma(K_\lambda) = \kappa_\lambda$ and $\gamma(F_j) = \phi_j$. We define $\rho(Y, X) = \gamma(Y)(X)$.

By construction, we have $\rho(YZ, X) = \rho(Y, X_{(1)}) \rho(Z, X_{(2)})$ for all $Y, Z \in U_q(\mathfrak{b}_-)$ and $X \in U_q(\mathfrak{b}_+)$, and the relation $\rho(Y, WX) = \rho(Y_{(2)}, W) \rho(Y_{(1)}, X)$ for $Y \in U_q(\mathfrak{b}_-)$ and $W, X \in U_q(\mathfrak{b}_+)$ is inherited from $\tilde{\rho}$. Moreover, we notice that $\rho(1, X) = \hat{\epsilon}(X)$ because γ is an algebra homomorphism. Finally, $\rho(K_\lambda, 1) = \kappa_\lambda(1) = 1 = \hat{\epsilon}(K_\lambda)$ and $\rho(F_i, 1) = \phi_i(1) = 0 = \hat{\epsilon}(F_i)$, which implies $\rho(Y, 1) = \hat{\epsilon}(Y)$ for all $Y \in U_q(\mathfrak{b}_-)$. This finishes the proof. \square

Note that the form ρ satisfies

$$\rho(Y, X) = 0 \quad \text{for } X \in U_q(\mathfrak{n}_+)_\alpha, Y \in U_q(\mathfrak{n}_-)_\beta, \alpha \neq -\beta.$$

In the sequel we will work both with ρ and the closely related form τ defined as follows.

Definition 2.74. We shall refer to the skew-pairing $\tau : U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{K}$ given by

$$\tau(X, Y) = \rho(\hat{S}(Y), X).$$

as the Drinfeld pairing.

Lemma 2.75. *The Drinfeld pairing τ satisfies*

$$\tau(K_\alpha, K_\beta) = q^{-(\alpha, \beta)}, \quad \tau(E_i, K_\alpha) = 0 = \tau(K_\beta, F_j), \quad \tau(E_i, F_j) = -\frac{\delta_{ij}}{q_i - q_i^{-1}}$$

for all $\alpha, \beta \in \mathbf{P}$ and $i, j = 1, \dots, N$. Moreover

$$\tau(XK_\mu, YK_\nu) = q^{-(\mu, \nu)} \tau(X, Y) \quad \text{for } X \in U_q(\mathfrak{n}_+), Y \in U_q(\mathfrak{n}_-).$$

Proof. The formulas in the first part of the Lemma are verified by direct computation from the definition of ρ .

For $Y = 1 \in U_q(\mathfrak{n}_-)$ the second claim follows immediately from $\tau(XK_\mu, K_\nu) = \rho(K_\nu^{-1}, XK_\mu)$. For $Y = F_j$ we see that $\tau(XK_\mu, F_j)$ and $\tau(X, F_j)$ vanish unless X is a multiple of E_j , and we compute

$$\begin{aligned} \tau(E_j K_\mu, F_j) &= \tau(E_j, F_j) \tau(K_\mu, 1) + \tau(E_j, K_j^{-1}) \tau(K_\mu, F_j) \\ &= -\frac{1}{q_j - q_j^{-1}} = \tau(E_j, F_j). \end{aligned}$$

We get the formula for general Y by induction, using

$$\begin{aligned} \tau(XK_\mu, F_j Y K_\nu) &= \tau(X_{(1)} K_\mu, Y K_\nu) \tau(X_{(2)} K_\mu, F_j) \\ &= q^{-(\mu, \nu)} \tau(X_{(1)}, Y) \tau(X_{(2)}, F_j) \\ &= q^{-(\mu, \nu)} \tau(X, F_j Y), \end{aligned}$$

in the inductive step. Notice here that $\hat{\Delta}(X)$ is contained in $U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{b}_+)$. \square

A crucial property of the pairings ρ and τ is that they can be used to exhibit $U_q(\mathfrak{g})$ as a quotient of the Drinfeld double $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$.

Lemma 2.76. *For all $X \in U_q(\mathfrak{b}_+)$ and $Y \in U_q(\mathfrak{b}_-)$ we have*

$$\begin{aligned} XY &= \rho(Y_{(1)}, X_{(1)}) Y_{(2)} X_{(2)} \rho(\hat{S}(Y_{(3)}), X_{(3)}) \\ &= \tau(\hat{S}(X_{(1)}), Y_{(1)}) Y_{(2)} X_{(2)} \tau(X_{(3)}, Y_{(3)}) \end{aligned}$$

in $U_q(\mathfrak{g})$.

Proof. It suffices to check this on generators. We compute

$$\begin{aligned} K_\mu K_\nu &= K_\nu K_\mu = q^{(\mu, \nu)} K_\nu K_\mu q^{-(\mu, \nu)}, \\ E_i K_\mu &= q^{-(\mu, \alpha_i)} K_\mu E_i \\ &= \rho(K_\mu, 1) K_\mu E_i \rho(\hat{S}(K_\mu), K_i) \\ &= \rho(K_\mu, 1) K_\mu E_i \rho(\hat{S}(K_\mu), K_i) + \rho(K_\mu, E_i) K_\mu K_i \rho(\hat{S}(K_\mu), K_i) \\ &\quad + \rho(K_\mu, 1) K_\mu \rho(\hat{S}(K_\mu), E_i), \\ E_i F_j &= F_j E_i - \delta_{ij} \frac{K_i^{-1}}{q_i - q_i^{-1}} + \delta_{ij} \frac{K_i}{q_i - q_i^{-1}} \\ &= \rho(K_j^{-1}, 1) F_j E_i \rho(\hat{S}(1), K_i) + \rho(K_j^{-1}, 1) K_j^{-1} \rho(\hat{S}(F_j), E_i) \\ &\quad + \rho(F_j, E_i) K_i \rho(\hat{S}(1), K_i). \end{aligned}$$

This yields the claim. \square

We note that we can rephrase the commutation relation of Lemma 2.76 as

$$X_{(1)} Y_{(1)} \rho(Y_{(2)}, X_{(2)}) = \rho(Y_{(1)}, X_{(1)}) Y_{(2)} X_{(2)}$$

for $X \in U_q(\mathfrak{b}_+)$ and $Y \in U_q(\mathfrak{b}_-)$.

In the double $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$ we shall denote the Cartan generators of $U_q(\mathfrak{b}_+)$ by K_λ^+ and the Cartan generators of $U_q(\mathfrak{b}_-)$ by K_λ^- .

Corollary 2.77. *The two-sided ideal I of the Drinfeld double $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$ generated by the elements $K_\lambda^- - K_\lambda^+$ for $\lambda \in \mathbf{P}$ is a Hopf ideal, and there is a canonical isomorphism*

$$(U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+))/I \cong U_q(\mathfrak{g})$$

of Hopf algebras.

Proof. The canonical embedding maps $U_q(\mathfrak{b}_+) \rightarrow U_q(\mathfrak{g})$ and $U_q(\mathfrak{b}_-) \rightarrow U_q(\mathfrak{g})$ define a Hopf algebra homomorphism $f : (U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+))/I \rightarrow U_q(\mathfrak{g})$ according to Lemma 2.76. Conversely, we obtain an algebra homomorphism $g : U_q(\mathfrak{g}) \rightarrow (U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+))/I$ by setting

$$g(K_\lambda) = K_\lambda^- = K_\lambda^+, \quad g(E_i) = 1 \bowtie E_i, \quad g(F_j) = F_j \bowtie 1,$$

since the defining relations of $U_q(\mathfrak{g})$ are satisfied by these elements. It is evident that f and g define inverse isomorphisms. \square

2.8.2. The quantum Killing form. Using the above constructions we can define the quantum Killing form on $U_q(\mathfrak{g})$ in terms of the Rosso form for the Drinfeld double. In this subsection, we will need a slightly stronger assumption on q , namely $q = s^{2L} \in \mathbb{K}^\times$, again not a root of unity.

According to Definition 1.18, the Rosso form on the double $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$ is given by

$$\kappa(Y_1 \bowtie X_1, Y_2 \bowtie X_2) = \tau(\hat{S}(X_1), Y_2) \tau(X_2, \hat{S}(Y_1)),$$

for $X_1, X_2 \in U_q(\mathfrak{b}_+)$, $Y_1, Y_2 \in U_q(\mathfrak{b}_-)$, or equivalently

$$\kappa(Y_1 \bowtie \hat{S}^{-1}(X_1), Y_2 \bowtie \hat{S}(X_2)) = \tau(X_1, Y_2) \tau(X_2, Y_1).$$

These formulas inspire the following definition.

Definition 2.78. Assume that $q = s^{2L} \in \mathbb{K}^\times$. The quantum Killing form for $U_q(\mathfrak{g})$ is the bilinear form $\kappa : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \rightarrow \mathbb{K}$ given by

$$\kappa(Y_1 K_\mu \hat{S}^{-1}(X_1), Y_2 K_\nu \hat{S}(X_2)) = q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(X_2, Y_1).$$

for $X_1, X_2 \in U_q(\mathfrak{n}_+)$, $Y_1, Y_2 \in U_q(\mathfrak{n}_-)$, $\mu, \nu \in \mathbf{P}$.

We remark that we need here a slightly stronger requirement on q than usual in order for the terms $q^{(\lambda, \mu)/2}$ to be well-defined. The factor of $\frac{1}{2}$ is needed to compensate for the doubling of the Cartan part in $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$, see also the remarks further below.

Heuristically, the formula in Definition 2.78 can be explained by formally writing

$$\begin{aligned} \kappa(Y_1 K_{\mu/2} \bowtie K_{\mu/2} \hat{S}^{-1}(X_1), Y_2 K_{\nu/2} \bowtie K_{\nu/2} \hat{S}(X_2)) \\ = \kappa(Y_1 K_{\mu/2} \bowtie \hat{S}^{-1}(X_1 K_{-\mu/2}), Y_2 K_{\nu/2} \bowtie \hat{S}(X_2 K_{-\nu/2})) \\ = q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(X_2, Y_1), \end{aligned}$$

in the Rosso form. This amounts to “splitting up” the Cartan part of $U_q(\mathfrak{g})$ evenly among $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$.

Proposition 2.79. *The quantum Killing form $\kappa : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \rightarrow \mathbb{K}$ is ad-invariant, that is,*

$$\kappa(Z \rightarrow X, Y) = \kappa(X, \hat{S}(Z) \rightarrow Y)$$

for all $X, Y, Z \in U_q(\mathfrak{g})$.

Proof. Let us consider $X = Y_1 K_\mu \hat{S}^{-1}(X_1)$ and $Y = Y_2 K_\nu \hat{S}(X_2)$. For $Z = K_\lambda$ and $X_j \in U_q(\mathfrak{n}_+)_{\beta_j}$ and $Y_j \in U_q(\mathfrak{n}_-)_{\gamma_j}$ we get

$$\begin{aligned} \kappa(K_\lambda \rightarrow (Y_1 K_\mu \hat{S}^{-1}(X_1)), K_\lambda \rightarrow (Y_2 K_\nu \hat{S}(X_2))) \\ = q^{(\lambda, \beta_1 + \gamma_1)} q^{(\lambda, \beta_2 + \gamma_2)} \kappa(Y_1 K_\mu \hat{S}^{-1}(X_1), Y_2 K_\nu \hat{S}(X_2)) \\ = \delta_{\beta_1, -\gamma_2} \delta_{\beta_2, -\gamma_1} \kappa(Y_1 K_\mu \hat{S}^{-1}(X_1), Y_2 K_\nu \hat{S}(X_2)) \\ = \kappa(Y_1 K_\alpha \hat{S}^{-1}(X_1), Y_2 K_\beta \hat{S}(X_2)). \end{aligned}$$

Now consider $Z = \hat{S}^{-1}(E_i)$. We compute

$$\begin{aligned} E_i \rightarrow (Y_2 K_\nu \hat{S}(X_2)) &= E_i Y_2 K_\nu \hat{S}(X_2) \hat{S}(K_i) + Y_2 K_\nu \hat{S}(X_2) \hat{S}(E_i) \\ &= q^{(\alpha_i, \beta_2)} E_i Y_2 K_{\nu - \alpha_i} \hat{S}(X_2) + Y_2 K_\nu \hat{S}(E_i X_2) \\ &= q^{(\alpha_i, \beta_2)} \left(\tau(\hat{S}(E_i), (Y_2)_{(1)})(Y_2)_{(2)} K_i \tau(K_i, (Y_2)_{(3)}) K_{\nu - \alpha_i} \hat{S}(X_2) \right. \\ &\quad + \tau(1, (Y_2)_{(1)})(Y_2)_{(2)} E_i \tau(K_i, (Y_2)_{(3)}) K_{\nu - \alpha_i} \hat{S}(X_2) \\ &\quad \left. + \tau(1, (Y_2)_{(1)})(Y_2)_{(2)} \tau(E_i, (Y_2)_{(3)}) K_{\nu - \alpha_i} \hat{S}(X_2) \right) + Y_2 K_\nu \hat{S}(E_i X_2) \\ &= q^{(\alpha_i, \beta_2)} \left(\tau(\hat{S}(E_i), (Y_2)_{(1)})(Y_2)_{(2)} K_\nu \hat{S}(X_2) \right. \\ &\quad \left. + Y_2 E_i K_{\nu - \alpha_i} \hat{S}(X_2) + \tau(E_i, (Y_2)_{(2)})(Y_2)_{(1)} K_{\nu - \alpha_i} \hat{S}(X_2) \right) + Y_2 K_\nu \hat{S}(E_i X_2) \\ &= q^{(\alpha_i, \beta_2)} \tau(\hat{S}(E_i), (Y_2)_{(1)})(Y_2)_{(2)} K_\nu \hat{S}(X_2) - q^{(\alpha_i, \beta_2)} Y_2 \hat{S}(E_i) K_\nu \hat{S}(X_2) \\ &\quad + q^{(\alpha_i, \beta_2)} \tau(E_i, (Y_2)_{(2)})(Y_2)_{(1)} K_{\nu - \alpha_i} \hat{S}(X_2) + Y_2 K_\nu \hat{S}(E_i X_2) \\ &= -q^{(\alpha_i, \beta_2)} \tau(E_i K_i^{-1}, (Y_2)_{(1)})(Y_2)_{(2)} K_\nu \hat{S}(X_2) \\ &\quad - q^{(\alpha_i, \beta_2)} q^{-(\alpha_i, \nu)} Y_2 K_\nu \hat{S}(X_2 E_i) \\ &\quad + q^{(\alpha_i, \beta_2)} \tau(E_i, (Y_2)_{(2)})(Y_2)_{(1)} K_{\nu - \alpha_i} \hat{S}(X_2) \\ &\quad + Y_2 K_\nu \hat{S}(E_i X_2). \end{aligned}$$

In the third term, note that $\tau(E_i, \cdot)$ annihilates all PBW vectors in $U_q(\mathfrak{b}_-)$ except those of the form $F_i K_\lambda$ for some $\lambda \in \mathbf{P}$. We therefore obtain

$$\begin{aligned} \kappa(Y_1 K_\mu \hat{S}^{-1}(X_1), E_i \rightarrow (Y_2 K_\nu \hat{S}(X_2))) \\ = -q^{(\alpha_i, \beta_2)} q^{(\mu, \nu)/2} \tau(E_i K_i^{-1}, (Y_2)_{(1)}) \tau(X_1, (Y_2)_{(2)}) \tau(X_2, Y_1) \\ \quad - q^{(\alpha_i, \beta_2)} q^{-(\alpha_i, \nu)} q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(X_2 E_i, Y_1) \\ \quad + q^{(\alpha_i, \beta_2)} q^{(\mu, -\alpha_i)/2} q^{(\mu, \nu - \alpha_i)/2} \tau(E_i, (Y_2)_{(2)}) \tau(X_1, (Y_2)_{(1)}) \tau(X_2, Y_1) \\ \quad + q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(E_i X_2, Y_1) \\ = -q^{(\alpha_i, \beta_2 - \beta_1)} q^{(\mu, \nu)/2} \tau(E_i X_1, Y_2) \tau(X_2, Y_1) \\ \quad - q^{(\alpha_i, \beta_2)} q^{-(\alpha_i, \nu)} q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(X_2 E_i, Y_1) \\ \quad + q^{(\alpha_i, \beta_2)} q^{-(\mu, \alpha_i)} q^{(\mu, \nu)/2} \tau(X_1 E_i, Y_2) \tau(X_2, Y_1) \\ \quad + q^{(\mu, \nu)/2} \tau(X_1, Y_2) \tau(E_i X_2, Y_1). \end{aligned}$$

On the other hand, using

$$\begin{aligned}
 E_i \rightarrow (Y_1 K_\mu \hat{S}^{-1}(X_1)) &= E_i Y_1 K_\mu \hat{S}^{-1}(X_1) \hat{S}(K_i) + Y_1 K_\mu \hat{S}^{-1}(X_1) \hat{S}(E_i) \\
 &= q^{(\alpha_i, \beta_1)} E_i Y_1 K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) + q^{(\alpha_i, \alpha_i)} Y_1 K_\mu \hat{S}^{-1}(X_i) \hat{S}^{-1}(E_i) \\
 &= q^{(\alpha_i, \beta_1)} \left(\tau(\hat{S}(E_i), (Y_1)_{(1)})(Y_1)_{(2)} K_i \tau(K_i, (Y_1)_{(3)}) K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) \right. \\
 &\quad + \tau(1, (Y_1)_{(1)})(Y_1)_{(2)} E_i \tau(K_i, (Y_1)_{(3)}) K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) \\
 &\quad \left. + \tau(1, (Y_1)_{(1)})(Y_1)_{(2)} \tau(E_i, (Y_1)_{(3)}) K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) \right) + q^{(\alpha_i, \alpha_i)} Y_1 K_\mu \hat{S}^{-1}(E_i X_1) \\
 &= q^{(\alpha_i, \beta_1)} \left(\tau(\hat{S}(E_i), (Y_1)_{(1)})(Y_1)_{(2)} K_\mu \hat{S}^{-1}(X_1) \right. \\
 &\quad \left. + Y_1 E_i K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) + \tau(E_i, (Y_1)_{(2)})(Y_1)_{(1)} K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) \right) + q^{(\alpha_i, \alpha_i)} Y_1 K_\mu \hat{S}^{-1}(E_i X_1) \\
 &= q^{(\alpha_i, \beta_1)} \tau(\hat{S}(E_i), (Y_1)_{(1)})(Y_1)_{(2)} K_\mu \hat{S}^{-1}(X_1) - q^{(\alpha_i, \beta_1 + \alpha_i)} Y_1 \hat{S}^{-1}(E_i) K_\mu \hat{S}^{-1}(X_1) \\
 &\quad + q^{(\alpha_i, \beta_1)} \tau(E_i, (Y_1)_{(2)})(Y_1)_{(1)} K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) + q^{(\alpha_i, \alpha_i)} Y_1 K_\mu \hat{S}^{-1}(E_i X_1) \\
 &= -q^{(\alpha_i, \beta_1)} \tau(E_i K_i^{-1}, (Y_1)_{(1)})(Y_1)_{(2)} K_\mu \hat{S}^{-1}(X_1) \\
 &\quad - q^{(\alpha_i, \beta_1 + \alpha_i)} q^{-(\alpha_i, \mu)} Y_1 K_\mu \hat{S}^{-1}(X_1 E_i) \\
 &\quad + q^{(\alpha_i, \beta_1)} \tau(E_i, (Y_1)_{(2)})(Y_1)_{(1)} K_{\mu - \alpha_i} \hat{S}^{-1}(X_1) \\
 &\quad + q^{(\alpha_i, \alpha_i)} Y_1 K_\mu \hat{S}^{-1}(E_i X_1).
 \end{aligned}$$

we calculate

$$\begin{aligned}
 \kappa(\hat{S}^{-1}(E_i) \rightarrow (Y_1 K_\mu \hat{S}^{-1}(X_1)), Y_2 K_\nu \hat{S}(X_2)) & \\
 &= -\kappa(E_i \rightarrow (Y_1 K_\mu \hat{S}^{-1}(X_1)), K_i \rightarrow (Y_2 K_\nu \hat{S}(X_2))) \\
 &= -q^{(\alpha_i, \gamma_2 + \beta_2)} \kappa(E_i \rightarrow (Y_1 K_\mu \hat{S}^{-1}(X_1)), Y_2 K_\nu \hat{S}(X_2)) \\
 &= -q^{(\alpha_i, \gamma_2 + \beta_2)} \left(-q^{(\alpha_i, \beta_1)} q^{(\mu, \nu)/2} \tau(E_i K_i^{-1}, (Y_1)_{(1)}) \tau(X_2, (Y_1)_{(2)}) \tau(X_1, Y_2) \right. \\
 &\quad - q^{(\alpha_i, \beta_1 + \alpha_i)} q^{-(\alpha_i, \mu)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(X_1 E_i, Y_2) \\
 &\quad + q^{(\alpha_i, \beta_1)} q^{(\nu, -\alpha_i)/2} q^{(\mu - \alpha_i, \nu)/2} \tau(E_i, (Y_1)_{(2)}) \tau(X_2, (Y_1)_{(1)}) \tau(X_1, Y_2) \\
 &\quad \left. + q^{(\alpha_i, \alpha_i)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(E_i X_1, Y_2) \right) \\
 &= -q^{(\alpha_i, \gamma_2 + \beta_2)} \left(-q^{(\alpha_i, \beta_1)} q^{(\mu, \nu)/2} \tau(E_i K_i^{-1} X_2, Y_1) \tau(X_1, Y_2) \right. \\
 &\quad - q^{(\alpha_i, \beta_1 + \alpha_i)} q^{-(\alpha_i, \mu)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(X_1 E_i, Y_2) \\
 &\quad + q^{(\alpha_i, \beta_1)} q^{-(\nu, \alpha_i)} q^{(\mu, \nu)/2} \tau(X_2 E_i, Y_1) \tau(X_1, Y_2) \\
 &\quad \left. + q^{(\alpha_i, \alpha_i)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(E_i X_1, Y_2) \right) \\
 &= q^{(\mu, \nu)/2} \tau(E_i X_2, Y_1) \tau(X_1, Y_2) \\
 &\quad + q^{(\alpha_i, \beta_2)} q^{-(\alpha_i, \mu)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(X_1 E_i, Y_2) \\
 &\quad - q^{(\alpha_i, \beta_2)} q^{(\nu, -\alpha_i)} q^{(\mu, \nu)/2} \tau(X_2 E_i, Y_1) \tau(X_1, Y_2) \\
 &\quad - q^{(\alpha_i, \beta_2 - \beta_1)} q^{(\mu, \nu)/2} \tau(X_2, Y_1) \tau(E_i X_1, Y_2),
 \end{aligned}$$

where in the first term of the last equality we are using the fact that $\tau(E_i X_2, Y_1) \tau(X_1, Y_2)$ is zero unless $-\gamma_1 = \alpha_i + \beta_2$ and $-\gamma_2 = \beta_1$, and similarly for each of the other terms.

In a similar way one proceeds for $Z = \hat{S}^{-1}(F_i)$. This yields the claim. \square

Consider also the bilinear form

$$\begin{aligned} (X_1 K_\mu \hat{S}(Y_1), Y_2 K_\nu \hat{S}(X_2)) &= q^{-(\mu, \nu)/2} \tau(X_1, Y_2) \tau(X_2, Y_1) \\ &= \kappa(\hat{S}^{-1}(X_1 K_\mu \hat{S}(Y_1)), Y_2 K_\nu \hat{S}(X_2)). \end{aligned}$$

on $U_q(\mathfrak{g})$. Using

$$\hat{S}(X) \leftarrow \hat{S}(Z) = \hat{S}^2(Z_{(2)}) \hat{S}(X) \hat{S}(Z_{(1)}) = \hat{S}(Z_{(1)} X \hat{S}(Z_{(2)})) = \hat{S}(Z \rightarrow X)$$

and Proposition 2.79 we get

$$(X \leftarrow Z, Y) = (X, Z \rightarrow Y)$$

for all $X, Y, Z \in U_q(\mathfrak{g})$. This will be crucial when we use the quantum Killing form to express the locally finite part of $U_q(\mathfrak{g})$ in terms of $\mathcal{O}(G_q)$ in Theorem 2.113.

2.8.3. Computation of the Drinfeld pairing. In this subsection we use the PBW-basis to compute the Drinfeld pairing. For the main part of the arguments we follow the approach of Tanisaki [72].

Let us start with some preliminaries. The q -exponential function \exp_q is defined by

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} x^n.$$

Here x is a formal variable, and the expression on the right hand side can be viewed as an element of $\mathbb{Q}(q)[[x]]$.

Lemma 2.80. *The formal series $\exp_q(x)$ is invertible, with inverse*

$$\exp_q(x)^{-1} = \sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{[n]_q!} (-x)^n = \exp_{q^{-1}}(-x).$$

Proof. We formally multiply and collect terms of common powers in x , explicitly,

$$\begin{aligned} \exp_q(x) \exp_{q^{-1}}(-x) &= \left(\sum_{m=0}^{\infty} \frac{q^{m(m-1)/2}}{[m]_q!} x^m \right) \left(\sum_{n=0}^{\infty} \frac{q^{-n(n-1)/2}}{[n]_q!} (-x)^n \right) \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^k (-1)^{k-l} \frac{q^{l(l-1)/2} q^{-(k-l)(k-l-1)/2}}{[l]_q!} \frac{1}{[k-l]_q!} x^k \\ &= \sum_{k=0}^{\infty} (-1)^k q^{(-k^2+k)/2} \left(\sum_{l=0}^k (-1)^l \frac{q^{(l^2-l+k-l+lk-l^2-l)/2}}{[k]_q!} \begin{bmatrix} k \\ l \end{bmatrix}_q \right) x^k \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{q^{(-k^2+k)/2}}{[k]_q!} \left(\sum_{l=0}^k (-1)^l q^{kl-l} \begin{bmatrix} k \\ l \end{bmatrix}_q \right) x^k. \end{aligned}$$

Hence the claim follows from Lemma 2.2. \square

For $1 \leq i \leq N$ and integrable $U_q(\mathfrak{g})$ -modules V, W , define a linear operator Z_i on $V \otimes W$ by

$$Z_i = \exp_{q_i}((q_i - q_i^{-1})(E_i \otimes F_i)).$$

According to Lemma 2.80 the operator Z_i is invertible with inverse

$$Z_i^{-1} = \exp_{q_i^{-1}}(-(q_i - q_i^{-1})(E_i \otimes F_i)).$$

Recall moreover the definition of the operators \mathcal{T}_i acting on integrable $U_q(\mathfrak{g})$ -modules from Section 2.7.

Proposition 2.81. *Let V, W be finite dimensional integrable $U_q(\mathfrak{g})$ -modules and $1 \leq i \leq N$. Then we have*

$$\mathcal{T}_i(v \otimes w) = (\mathcal{T}_i \otimes \mathcal{T}_i)Z_i(v \otimes w)$$

for all $v \otimes w \in V \otimes W$.

Proof. Since the operators \mathcal{T}_i and Z_i are defined in terms of $U_{q_i}(\mathfrak{g}_i) \subset U_q(\mathfrak{g})$ it suffices to consider the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$. We shall therefore restrict to this case in the sequel, writing $Z = \exp_q(-(q - q^{-1})(E \otimes F))$.

Let us first show that both \mathcal{T} and $(\mathcal{T} \otimes \mathcal{T})Z$ commute in the same way with the diagonal action of E and F . By Lemma 2.51 we have

$$\begin{aligned} \mathcal{T}(E \cdot (v \otimes w)) &= -K^2 F \cdot \mathcal{T}(v \otimes w) \\ &= (-K^2 F \otimes K^2 - 1 \otimes K^2 F) \cdot \mathcal{T}(v \otimes w). \end{aligned}$$

On the other hand,

$$\begin{aligned} (\mathcal{T} \otimes \mathcal{T})Z(E \cdot (v \otimes w)) &= (\mathcal{T} \otimes \mathcal{T})Z(E \cdot v \otimes K^2 \cdot w + v \otimes E \cdot w) \\ &= (\mathcal{T} \otimes \mathcal{T})((E \otimes K^2) \cdot \exp_q((q - q^{-1})q^2(E \otimes F))(v \otimes w)) \\ &\quad + (\mathcal{T} \otimes \mathcal{T})Z((1 \otimes E) \cdot (v \otimes w)) \\ &= -(K^2 F \otimes K^{-2})(\mathcal{T} \otimes \mathcal{T})(\exp_q((q - q^{-1})q^2(E \otimes F))(v \otimes w)) \\ &\quad + (\mathcal{T} \otimes \mathcal{T})Z((1 \otimes E) \cdot (v \otimes w)). \end{aligned}$$

In order to compute $(\mathcal{T} \otimes \mathcal{T})Z((1 \otimes E) \cdot (v \otimes w))$ consider $x = (q - q^{-1})(E \otimes F)$. Let us prove by induction that

$$\begin{aligned} [x^n, 1 \otimes E] &= \sum_{r=0}^{n-1} (E \otimes (K^{-2}q^{-2r} - K^2q^{2r}))x^{n-1} \\ &= (q^{-n+1}[n]_q(E \otimes K^{-2}) - q^{n-1}[n]_q(E \otimes K^2))x^{n-1} \end{aligned}$$

for all $n \in \mathbb{N}$. Indeed, for $n = 1$ one checks

$$[x, 1 \otimes E] = (q - q^{-1})(E \otimes [F, E]) = E \otimes (K^{-2} - K^2),$$

and for the inductive step we compute

$$\begin{aligned} [x^n, 1 \otimes E] &= x[x^{n-1}, 1 \otimes E] + [x, 1 \otimes E]x^{n-1} \\ &= x \sum_{r=0}^{n-2} (E \otimes (K^{-2}q^{-2r} - K^2q^{2r}))x^{n-2} + (E \otimes (K^{-2} - K^2))x^{n-1} \\ &= \sum_{r=0}^{n-2} (E \otimes K^{-2}q^{-2r-2} - E \otimes K^2q^{2r+2})x^{n-1} + (E \otimes (K^{-2} - K^2))x^{n-1} \\ &= \sum_{r=0}^{n-1} (E \otimes (K^{-2}q^{-2r} - K^2q^{2r}))x^{n-1}. \end{aligned}$$

Therefore we formally obtain

$$\begin{aligned}
Z(1 \otimes E) &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} x^n (1 \otimes E) \\
&= (1 \otimes E) \exp_q(x) + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} q^{-n+1} [n]_q (E \otimes K^{-2}) x^{n-1} \\
&\quad - \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} q^{n-1} [n]_q (E \otimes K^2) x^{n-1} \\
&= (1 \otimes E) \exp_q(x) + (E \otimes K^{-2}) \sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}}{[n-1]_q!} x^{n-1} \\
&\quad - (E \otimes K^2) \sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}}{[n-1]_q!} q^{2(n-1)} x^{n-1} \\
&= (1 \otimes E) \exp_q(x) + (E \otimes K^{-2}) \exp_q(x) - (E \otimes K^2) \exp_q(q^2 x).
\end{aligned}$$

Summarizing the above calculations we get

$$\begin{aligned}
(\mathcal{T} \otimes \mathcal{T})Z(1 \otimes E) &= -(1 \otimes K^2 F)(\mathcal{T} \otimes \mathcal{T})Z - (K^2 F \otimes K^2)(\mathcal{T} \otimes \mathcal{T})Z \\
&\quad + (K^2 F \otimes K^{-2})(\mathcal{T} \otimes \mathcal{T}) \exp_q(q^2 x),
\end{aligned}$$

and hence

$$\begin{aligned}
(\mathcal{T} \otimes \mathcal{T})Z(E \cdot (v \otimes w)) &= -(K^2 F \otimes K^{-2})(\mathcal{T} \otimes \mathcal{T})(\exp_q((q - q^{-1})q^2(E \otimes F)))(v \otimes w) \\
&\quad - (1 \otimes K^2 F)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) - (K^2 F \otimes K^2)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) \\
&\quad + (K^2 F \otimes K^{-2})(\mathcal{T} \otimes \mathcal{T}) \exp_q((q - q^{-1})q^2(E \otimes F))(v \otimes w) \\
&= -(1 \otimes K^2 F)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) - (K^2 F \otimes K^2)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w).
\end{aligned}$$

This shows that both operators commute in the same way with the diagonal action of E . In a similar way we compute

$$\begin{aligned}
\mathcal{T}(F \cdot (v \otimes w)) &= -EK^{-2} \cdot \mathcal{T}(v \otimes w) \\
&= (-EK^{-2} \otimes 1 - K^{-2} \otimes EK^{-2}) \cdot \mathcal{T}(v \otimes w).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathcal{T} \otimes \mathcal{T})Z(F \cdot (v \otimes w)) &= (\mathcal{T} \otimes \mathcal{T})Z(F \cdot v \otimes w + K^{-2} \cdot v \otimes F \cdot w) \\
&= (\mathcal{T} \otimes \mathcal{T})Z((F \otimes 1) \cdot (v \otimes w)) \\
&\quad + (\mathcal{T} \otimes \mathcal{T})((K^{-2} \otimes F) \cdot \exp_q((q - q^{-1})q^2(E \otimes F)))(v \otimes w) \\
&= (\mathcal{T} \otimes \mathcal{T})Z((F \otimes 1) \cdot (v \otimes w)) \\
&\quad - (K^2 \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T})(\exp_q((q - q^{-1})q^2(E \otimes F)))(v \otimes w).
\end{aligned}$$

If $x = (q - q^{-1})(E \otimes F)$, then we obtain

$$\begin{aligned}
[x^n, F \otimes 1] &= \sum_{r=0}^{n-1} ((K^2 q^{-2r} - K^{-2} q^{2r}) \otimes F) x^{n-1} \\
&= (q^{-n+1} [n]_q K^2 \otimes F - q^{n-1} [n]_q K^{-2} \otimes F) x^{n-1}
\end{aligned}$$

for all $n \in \mathbb{N}$ by induction. Indeed, for $n = 1$ one checks

$$[x, F \otimes 1] = (q - q^{-1})([E, F] \otimes F) = (K^2 - K^{-2}) \otimes F,$$

and for the inductive step we compute

$$\begin{aligned}
 [x^n, F \otimes 1] &= x[x^{n-1}, F \otimes 1] + [x, F \otimes 1]x^{n-1} \\
 &= x \sum_{r=0}^{n-2} ((K^2 q^{-2r} - K^{-2} q^{2r}) \otimes F) x^{n-2} + ((K^2 - K^{-2}) \otimes F) x^{n-1} \\
 &= \sum_{r=0}^{n-2} ((K^2 q^{-2r-2} - K^{-2} q^{2r+2}) \otimes F) x^{n-2} + ((K^2 - K^{-2}) \otimes F) x^{n-1} \\
 &= \sum_{r=0}^{n-1} ((K^2 q^{-2r} - K^{-2} q^{2r}) \otimes F) x^{n-1}.
 \end{aligned}$$

We formally obtain

$$\begin{aligned}
 Z(F \otimes 1) &= \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} x^n (F \otimes 1) \\
 &= (F \otimes 1) \exp_q(x) + \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} q^{-n+1} [n]_q (K^2 \otimes F) x^{n-1} \\
 &\quad - \sum_{n=1}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} q^{n-1} [n]_q (K^{-2} \otimes F) x^{n-1} \\
 &= (F \otimes 1) \exp_q(x) + (K^2 \otimes F) \sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}}{[n-1]_q!} x^{n-1} \\
 &\quad - (K^{-2} \otimes F) \sum_{n=1}^{\infty} \frac{q^{(n-1)(n-2)/2}}{[n]_q!} q^{2(n-1)} x^{n-1} \\
 &= (F \otimes 1) \exp_q(x) + (K^2 \otimes F) \exp_q(x) - (K^{-2} \otimes F) \exp_q(q^2 x).
 \end{aligned}$$

Summarising the above calculations we get

$$\begin{aligned}
 (\mathcal{T} \otimes \mathcal{T})Z(F \otimes 1) &= -(EK^{-2} \otimes 1)(\mathcal{T} \otimes \mathcal{T})Z - (K^{-2} \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T})Z \\
 &\quad + (K^2 \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T}) \exp_q(q^2 x),
 \end{aligned}$$

and hence

$$\begin{aligned}
 (\mathcal{T} \otimes \mathcal{T})Z(F \cdot (v \otimes w)) &= -(EK^{-2} \otimes 1)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) - (K^{-2} \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) \\
 &\quad + (K^2 \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T}) \exp_q(q^2(q - q^{-1})(E \otimes F)(v \otimes w)) \\
 &\quad - (K^2 \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T})(\exp_q((q - q^{-1})q^2(E \otimes F))(v \otimes w)) \\
 &= -(EK^{-2} \otimes 1)(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w) - (K^{-2} \otimes EK^{-2})(\mathcal{T} \otimes \mathcal{T})Z(v \otimes w).
 \end{aligned}$$

That is, both operators commute in the same way with the diagonal action of F . Let us point out that \mathcal{T} and $(\mathcal{T} \otimes \mathcal{T})Z$ commute also in the same way with the diagonal action of K_μ for $\mu \in \mathbf{P}$, as one checks easily by weight considerations.

We show next that \mathcal{T} and $(\mathcal{T} \otimes \mathcal{T})Z$ agree on $V(m) \otimes V(1/2)$ for all $m \in \frac{1}{2}\mathbb{N}_0$. For $m = 0$ there is nothing to prove. Hence assume that $m \geq \frac{1}{2}$ and consider the tensor product decomposition

$$V(m) \otimes V(1/2) \cong V(m + 1/2) \oplus V(m - 1/2).$$

Since both operators commute in the same way with the diagonal action of $U_q(\mathfrak{g})$ it suffices to show that \mathcal{T} and $(\mathcal{T} \otimes \mathcal{T})Z$ agree on the lowest weight vectors in the

irreducible components $V(m + 1/2)$ and $V(m - 1/2)$ of the tensor product. Up to a scalar, these lowest weight vectors are given by

$$\begin{aligned} v_{(-m-1/2)} &= v_{(-m)} \otimes v_{(-1/2)}, \\ v_{(-m+1/2)} &= [2m]_q v_{(-m)} \otimes v_{(1/2)} - q^{2m} v_{(-m+1)} \otimes v_{(-1/2)}, \end{aligned}$$

as can be checked by computing the diagonal action of F on each.

For the case of $v_{(-m-1/2)}$, note that Z fixes $v_{(-m)} \otimes v_{(-1/2)}$, and so Proposition 2.49 gives

$$(\mathcal{T} \otimes \mathcal{T})Z(v_{(-m)} \otimes v_{(-1/2)}) = (-1)^{2m} q^{2m} v_{(m)} \otimes -q v_{(1/2)},$$

while

$$\mathcal{T}(v_{(-m-1/2)}) = (-1)^{2m+1} q^{2m+1} v_{(m+1/2)}.$$

For $v_{(-m+1/2)}$, we have

$$\begin{aligned} Z(v_{(-m+1/2)}) &= v_{(-m+1/2)} + (q - q^{-1})(E \otimes F) \cdot v_{(-m+1/2)} \\ &= [2m]_q v_{(-m)} \otimes v_{(1/2)} - q^{2m} v_{(-m+1)} \otimes v_{(-1/2)} \\ &\quad + (q - q^{-1})[2m]_q v_{(-m+1)} \otimes v_{(-1/2)} \\ &= [2m]_q v_{(-m)} \otimes v_{(1/2)} - q^{-2m} v_{(-m+1)} \otimes v_{(-1/2)}. \end{aligned}$$

Using the formulas from Proposition 2.49 we get

$$\begin{aligned} &(\mathcal{T} \otimes \mathcal{T})Z(v_{(-m+1/2)}) \\ &= (\mathcal{T} \otimes \mathcal{T})([2m]_q v_{(-m)} \otimes v_{(1/2)} - q^{-2m} v_{(-m+1)} \otimes v_{(-1/2)}) \\ &= (-1)^{2m} q^{2m} [2m]_q v_{(m)} \otimes v_{(-1/2)} - (-1)^{2m-1} q^{-2m} q^{(2m-1)^2} v_{(m-1)} \otimes (-q v_{(1/2)}) \\ &= (-1)^{2m} q^{2m} [2m]_q v_{(m)} \otimes v_{(-1/2)} - (-1)^{2m} q^{2m-1} v_{(m-1)} \otimes v_{(1/2)}. \end{aligned}$$

On the other hand, from the definition of \mathcal{T} and Lemma 2.6 we obtain

$$\begin{aligned} &\mathcal{T}(v_{(-m+1/2)}) = (-1)^{2m-1} q^{2m-1} E^{(2m-1)} \cdot v_{(-m+1/2)} \\ &= (-1)^{2m-1} q^{2m-1} E^{(2m-1)} \cdot ([2m]_q v_{(-m)} \otimes v_{(1/2)} - q^{2m} v_{(-m+1)} \otimes v_{(-1/2)}) \\ &= (-1)^{2m-1} q^{2m-1} ([2m]_q E^{(2m-1)} \cdot v_{(-m)} \otimes K^{2(2m-1)} \cdot v_{(1/2)} \\ &\quad - q^{2m} E^{(2m-1)} \cdot v_{(-m+1)} \otimes K^{2(2m-1)} \cdot v_{(-1/2)} \\ &\quad - q^{2m} q^{2m-2} E^{(2m-2)} \cdot v_{(-m+1)} \otimes EK^{2(2m-2)} \cdot v_{(-1/2)}) \\ &= (-1)^{2m-1} q^{2m-1} [2m]_q q^{2m-1} v_{(m-1)} \otimes v_{(1/2)} \\ &\quad + (-1)^{2m} q^{2m-1} q^{2m} [2m]_q v_{(m)} \otimes q^{-(2m-1)} v_{(-1/2)} \\ &\quad + (-1)^{2m} q^{2m-1} q^{2m} q^{2m-2} q^{-(2m-2)} [2m-1]_q v_{(m-1)} \otimes v_{(1/2)} \\ &= (-1)^{2m-1} q^{4m-2} [2m]_q v_{(m-1)} \otimes v_{(1/2)} \\ &\quad + (-1)^{2m} q^{2m} [2m]_q v_{(m)} \otimes v_{(-1/2)} \\ &\quad + (-1)^{2m} q^{4m-1} [2m-1]_q v_{(m-1)} \otimes v_{(1/2)} \\ &= (-1)^{2m-1} q^{2m-1} v_{(m-1)} \otimes v_{(1/2)} + (-1)^{2m} q^{2m} [2m]_q v_{(m)} \otimes v_{(-1/2)}, \end{aligned}$$

using

$$q^{4m-2} [2m]_q - q^{4m-1} [2m-1]_q = q^{2m-1}$$

in the last step.

Now we want to show that \mathcal{T} and $(\mathcal{T} \otimes \mathcal{T})Z$ agree on $V(m) \otimes V(n)$ for all $n \in \frac{1}{2}\mathbb{N}_0$. Since both operators commute in the same way with the diagonal action of $U_q(\mathfrak{g})$ it suffices to show that they agree on $V(m) \otimes v_{(-n)}$, where $v_{(-n)} \in V(n)$ is a lowest weight vector. The case $n = 0$ is trivial, and the case $n = 1/2$ follows from our above calculations. Let $n > 1/2$ and assume that the assertion is proved

for all m and all $k < n$. Since $Z(v \otimes v_{(-n)}) = v \otimes v_{(-n)}$ for all $v \in V(m)$ we have to show $\mathcal{T}(v \otimes v_{(-n)}) = \mathcal{T}(v) \otimes \mathcal{T}(v_{(-n)})$. To this end consider the inclusion $V(n) \subset V(n-1/2) \otimes V(1/2)$. Then, up to a scalar, the vector $v_{(-n)}$ identifies with $v_{(-n+1/2)} \otimes v_{(-1/2)}$, and we obtain

$$\begin{aligned} \mathcal{T}(v \otimes v_{(-n+1/2)} \otimes v_{(-1/2)}) &= \mathcal{T}(v \otimes v_{(-n+1/2)}) \otimes \mathcal{T}(v_{(-1/2)}) \\ &= \mathcal{T}(v) \otimes \mathcal{T}(v_{(-n+1/2)}) \otimes \mathcal{T}(v_{(-1/2)}) \\ &= \mathcal{T}(v) \otimes \mathcal{T}(v_{(-n+1/2)} \otimes v_{(-1/2)}) \end{aligned}$$

by our induction hypothesis. This finishes the proof. \square

Recall that the automorphism $\mathcal{T}_i : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$ is induced by conjugation with the operators \mathcal{T}_i on integrable $U_q(\mathfrak{g})$ -modules. As a consequence of Proposition 2.81 we obtain

$$\hat{\Delta}(\mathcal{T}_i^{-1}(X)) = Z_i^{-1}(\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})\hat{\Delta}(X)(\mathcal{T}_i \otimes \mathcal{T}_i)Z_i = Z_i^{-1}(\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})(\hat{\Delta}(X))Z_i$$

for all $X \in U_q(\mathfrak{g})$.

Lemma 2.82. *We have*

$$\begin{aligned} \hat{\Delta}(\mathcal{T}_i(U_q(\mathfrak{b}_+))) &\subset \mathcal{T}_i(U_q(\mathfrak{b}_+)) \otimes U_q(\mathfrak{g}), \\ \hat{\Delta}(\mathcal{T}_i(U_q(\mathfrak{b}_-))) &\subset U_q(\mathfrak{g}) \otimes \mathcal{T}_i(U_q(\mathfrak{b}_-)) \end{aligned}$$

and

$$\begin{aligned} U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{b}_+)) &= U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)), \\ U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{b}_-)) &= U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)). \end{aligned}$$

Proof. For the first formula we compute

$$\begin{aligned} \hat{\Delta}(\mathcal{T}_i(X)) &= (\mathcal{T}_i \otimes \mathcal{T}_i)(Z_i \hat{\Delta}(X) Z_i^{-1})(\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1}) \\ &= (\mathcal{T}_i \otimes \mathcal{T}_i)(Z_i \hat{\Delta}(X) Z_i^{-1}), \end{aligned}$$

and if $X \in U_q(\mathfrak{b}_+)$ the term on the right hand side is contained in $\mathcal{T}_i(U_q(\mathfrak{b}_+)) \otimes U_q(\mathfrak{g})$ by the explicit formula for Z_i . The second formula is obtained in exactly the same way.

For the third formula it clearly suffices to show that the space on the left hand side is contained in the space on the right hand side. Let $X \in U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{b}_+))$. Moreover let V be an integrable $U_q(\mathfrak{g})$ -module and $v \in V$. If $\lambda \in \mathbf{P}^+$ and $v_\lambda \in V(\lambda)$ is a highest weight vector we have $Z_i(v_\lambda \otimes v) = v_\lambda \otimes v$, hence the formula after Proposition 2.81 yields

$$\mathcal{T}_i^{-1}(X) \cdot (v_\lambda \otimes v) = Z_i^{-1}(\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})(\hat{\Delta}(X)) \cdot (v_\lambda \otimes v).$$

Since $X \in U_q(\mathfrak{n}_+)$ we have

$$\hat{\Delta}(X) - 1 \otimes X \in \bigoplus_{\mu \in \mathbf{Q}^+ \setminus \{0\}} U_q(\mathfrak{n}_+)_{\mu} \otimes U_q(\mathfrak{b}_+),$$

and combining this with $X \in \mathcal{T}_i(U_q(\mathfrak{b}_+))$, the first formula of the present Lemma yields

$$\hat{\Delta}(X) - 1 \otimes X \in \bigoplus_{\mu \in \mathbf{Q}^+ \setminus \{0\}} (U_q(\mathfrak{n}_+)_{\mu} \cap \mathcal{T}_i(U_q(\mathfrak{b}_+))) \otimes U_q(\mathfrak{b}_+).$$

Hence

$$(\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})(\hat{\Delta}(X)) - 1 \otimes \mathcal{T}_i^{-1}(X) \in U_q(\mathfrak{b}) \left(\bigoplus_{\mu \in \mathbf{Q}^+ \setminus \{0\}} U_q(\mathfrak{n}_+)_{\mu} \right) \otimes U_q(\mathfrak{g}),$$

observing that the automorphism \mathcal{T}_i^{-1} preserves the sum of all nonzero weight spaces in $U_q(\mathfrak{g})$. We conclude

$$\mathcal{T}_i^{-1}(X) \cdot (v_\lambda \otimes v) = Z_i^{-1}(v_\lambda \otimes \mathcal{T}_i^{-1}(X) \cdot v) = v_\lambda \otimes \mathcal{T}_i^{-1}(X) \cdot v.$$

Since $\mathcal{T}_i^{-1}(X) \in U_q(\mathfrak{b}_+)$ we can write

$$\mathcal{T}_i^{-1}(X) = \sum_{\nu \in \mathbf{Q}^+} X_\nu K_\nu$$

for elements $X_\nu \in U_q(\mathfrak{n}_+)$. Then

$$\mathcal{T}_i^{-1}(X) \cdot (v_\lambda \otimes v) = \sum_{\nu \in \mathbf{Q}^+} X_\nu \cdot (K_\nu \cdot v_\lambda \otimes K_\nu \cdot v) = \sum_{\nu \in \mathbf{Q}^+} q^{(\nu, \lambda)} v_\lambda \otimes X_\nu K_\nu \cdot v.$$

On the other hand,

$$v_\lambda \otimes (\mathcal{T}_i^{-1}(X) \cdot v) = v_\lambda \otimes \sum_{\nu \in \mathbf{Q}^+} X_\nu K_\nu \cdot v,$$

and thus

$$\sum_{\nu \in \mathbf{Q}^+} q^{(\nu, \lambda)} v_\lambda \otimes X_\nu K_\nu \cdot v = v_\lambda \otimes \sum_{\nu \in \mathbf{Q}^+} X_\nu K_\nu \cdot v.$$

This holds for all $v \in V$, where V is an arbitrary integrable $U_q(\mathfrak{g})$ -module. Therefore we get

$$\sum_{\nu \in \mathbf{Q}^+} (1 - q^{(\nu, \lambda)}) X_\nu K_\nu = 0$$

for all $\lambda \in \mathbf{P}^+$. This implies $X_\nu = 0$ for $\nu \neq 0$, and thus $\mathcal{T}_i^{-1}(X) \in U_q(\mathfrak{n}_+)$. In other words, we obtain $X \in \mathcal{T}_i(U_q(\mathfrak{n}_+))$ as desired.

The final formula is obtained in a similar way. \square

Lemma 2.83. *We have*

$$\begin{aligned} \hat{\Delta}(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) &\subset (U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) \otimes U_q(\mathfrak{b}_+), \\ \hat{\Delta}(U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))) &\subset U_q(\mathfrak{b}_-) \otimes (U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))) \end{aligned}$$

for $1 \leq i \leq N$.

Proof. According to the first formula of Lemma 2.82 we obtain

$$\hat{\Delta}(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) \subset (U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{b}_+))) \otimes U_q(\mathfrak{b}_+).$$

Hence the third formula of Lemma 2.82 yields the first claim. The second assertion is proved in the same way using the second and fourth formulas of Lemma 2.82. \square

Lemma 2.84. *We have*

$$\begin{aligned} U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)) &= \{X \in U_q(\mathfrak{n}_+) \mid \tau(X, U_q(\mathfrak{n}_-) F_i) = 0\}, \\ U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)) &= \{Y \in U_q(\mathfrak{n}_-) \mid \tau(U_q(\mathfrak{n}_+) E_i, Y) = 0\} \end{aligned}$$

for $1 \leq i \leq N$.

Proof. Assume first $X \in U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$. Using the skew-pairing property we have

$$\tau(X, Y F_i) = \tau(X_{(1)}, F_i) \tau(X_{(2)}, Y)$$

for any $Y \in U_q(\mathfrak{n}_-)$. According to the first part of Lemma 2.83, the first tensor factors of $\hat{\Delta}(X)$ consist of sums of terms $Z_\gamma \in U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$ of weight γ for some $\gamma \in (\mathbf{Q}^+ \cap s_i \mathbf{Q}^+) \setminus \{0\}$. Since $\gamma \neq \alpha_i$, we must have $\rho(Z_\gamma, F_i) = 0$ for any such Z_γ , and thus $\tau(X, Y F_i) = 0$.

Assume conversely that $X \in U_q(\mathfrak{n}_+)$ satisfies $\tau(X, U_q(\mathfrak{n}_-)F_i) = 0$, and let us show $X \in \mathcal{T}_i(U_q(\mathfrak{n}_+))$. According to Lemma 2.82 it suffices to prove $\mathcal{T}_i^{-1}(X) \in U_q(\mathfrak{b}_+)$, and for this it is enough to verify

$$\mathcal{T}_i^{-1}(X)(v_\lambda \otimes V) \subset v_\lambda \otimes V$$

for all integrable modules V and $v_\lambda \in V(\lambda)$ for $\lambda \in \mathbf{P}^+$. Let us define $X_r \in U_q(\mathfrak{b}_+)$ for $r \in \mathbb{N}$ by

$$\hat{\Delta}(X) - \sum_{r \in \mathbb{N}} E_i^r \otimes X_r K_i^r \in \left(\bigoplus_{\mu \in \mathbf{Q}^+ \setminus \mathbb{N}\alpha_i} U_q(\mathfrak{n}_+)_{\mu} \right) \otimes U_q(\mathfrak{b}_+).$$

For $Y \in U_q(\mathfrak{n}_-)$ and $m \in \mathbb{N}$ we get

$$\begin{aligned} 0 &= \tau(X, Y F_i^m) = \tau(X_{(1)}, F_i^m) \tau(X_{(2)}, Y) \\ &= \sum_{r \in \mathbb{N}} \tau(E_i^r, F_i^m) \tau(X_r K_i^r, Y) = \sum_{r \in \mathbb{N}} \tau(E_i^r, F_i^m) \tau(X_r, Y). \end{aligned}$$

Since $\tau(E_i^r, F_i^m) = 0$ for $m \neq r$ we deduce $X_m = 0$ for all $m \in \mathbb{N}$, using the nondegeneracy of τ obtained in Proposition 2.92. Hence

$$\hat{\Delta}(X) \in \left(\bigoplus_{\mu \in \mathbf{Q}^+ \setminus \mathbb{N}\alpha_i} U_q(\mathfrak{n}_+)_{\mu} \right) \otimes U_q(\mathfrak{b}_+).$$

On the other hand, we have $U_q(\mathfrak{n}_+)_{\gamma} F_i^m \subset \sum_{r=0}^m U_q(\mathfrak{h}) F_i^r U_q(\mathfrak{n}_+)_{\gamma-r\alpha_i}$. If $\gamma \in \mathbf{Q}^+ \setminus \mathbb{N}_0\alpha_i$ we thus get $U_q(\mathfrak{n}_+)_{\gamma} F_i^m \cdot v_\lambda = 0$.

We now compute

$$\begin{aligned} \mathcal{T}_i^{-1}(X) \cdot (v_\lambda \otimes V) &= Z_i^{-1} \cdot (\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})(\hat{\Delta}(X) \cdot (\mathcal{T}_i(v_\lambda) \otimes V)) \\ &= Z_i^{-1} \cdot (\mathcal{T}_i^{-1} \otimes \mathcal{T}_i^{-1})(\hat{\Delta}(X) \cdot (F_i^{(\alpha_i^\vee, \lambda)} \cdot v_\lambda \otimes V)) \end{aligned}$$

using the formula for the coproduct of $\mathcal{T}_i^{-1}(X)$ obtained after Proposition 2.81 and the definition of the action of \mathcal{T}_i on a highest weight vector. From our above considerations we get

$$\hat{\Delta}(X) \cdot (F_i^{(\alpha_i^\vee, \lambda)} \cdot v_\lambda \otimes V) = (1 \otimes X) \cdot (F_i^{(\alpha_i^\vee, \lambda)} \cdot v_\lambda \otimes V) \subset F_i^{(\alpha_i^\vee, \lambda)} \cdot v_\lambda \otimes V.$$

Hence

$$\mathcal{T}_i^{-1}(X) \cdot (v_\lambda \otimes V) \subset Z_i^{-1} \cdot (\mathcal{T}_i^{-1}(F_i^{(\alpha_i^\vee, \lambda)} \cdot v_\lambda) \otimes V) = Z_i^{-1} \cdot (v_\lambda \otimes V) = v_\lambda \otimes V$$

as desired.

The second assertion is proved in a similar way. \square

Lemma 2.85. *The multiplication of $U_q(\mathfrak{g})$ induces linear isomorphisms*

$$\begin{aligned} U_q(\mathfrak{n}_+) &\cong \bigoplus_{r=0}^{\infty} \mathbb{K} E_i^r \otimes (U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) \\ U_q(\mathfrak{n}_-) &\cong \bigoplus_{r=0}^{\infty} \mathbb{K} F_i^r \otimes (U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))) \end{aligned}$$

for each $1 \leq i \leq N$.

Proof. Let $w_0 = s_{i_1} \cdots s_{i_n}$ be a reduced expression of the longest word of W beginning with $i_1 = i$. Then according to Theorem 2.67 the vectors $E_i^{a_1} E_{\beta_2}^{a_2} \cdots E_{\beta_n}^{a_n}$ form a basis of $U_q(\mathfrak{n}_+)$. By construction, the linear span of $E_{\beta_2}^{a_2} \cdots E_{\beta_n}^{a_n}$ is contained in $U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$. Moreover, the multiplication map $\bigoplus_{r=0}^{\infty} \mathbb{K} E_i^r \otimes (U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) \rightarrow U_q(\mathfrak{n}_+)$ is injective because $\mathcal{T}_i^{-1}(E_i) \in U_q(\mathfrak{b}_-)$ by the definition of \mathcal{T}_i and $\mathcal{T}_i^{-1}(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))) \subset U_q(\mathfrak{n}_+)$. This yields the first claim.

The second isomorphism is obtained in a similar way. \square

Using the notation $U_q(\mathfrak{n}_\pm)[w]$ which we introduced before Definition 2.66, we may rephrase the assertion of Lemma 2.85 as

$$\begin{aligned} U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)) &= \mathcal{T}_i(U_q(\mathfrak{n}_+)[s_i w_0]), \\ U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)) &= \mathcal{T}_i(U_q(\mathfrak{n}_-)[s_i w_0]) \end{aligned}$$

for all $1 \leq i \leq N$, where $w_0 \in W$ is the longest element of W .

Definition 2.86. The Harish-Chandra map is the linear map $\mathcal{P} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ given by $\hat{\epsilon} \otimes \text{id} \otimes \hat{\epsilon}$ under the triangular isomorphism $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+)$.

Proposition 2.87. *Let $\gamma \in \mathbf{Q}^+$ and assume $X \in U_q(\mathfrak{n}_+)_\gamma, Y \in U_q(\mathfrak{n}_-)_{-\gamma}$. Then*

$$\mathcal{P}(XY) - \tau(X, Y)K_{-\gamma} \in \sum_{\nu \in \mathbf{Q}^+ \setminus \{0\}} \mathbb{K}K_{2\nu - \gamma}.$$

Proof. Let us write

$$\hat{\Delta}(X) = \sum_r X_r^{[1]} \otimes X_r^{[2]} K_{\gamma_r}, \quad \hat{\Delta}(Y) = \sum_s K_{-\delta_s} Y_s^{[1]} \otimes Y_s^{[2]}$$

where $\gamma_r, \delta_s \in \mathbf{Q}^+$ and $X_r^{[1]} \in U_q(\mathfrak{n}_+)_{\gamma_r}, X_r^{[2]} \in U_q(\mathfrak{n}_+)_{\gamma - \gamma_r}$, and similarly $Y_s^{[1]} \in U_q(\mathfrak{n}_-)_{-\gamma + \delta_s}, Y_s^{[2]} \in U_q(\mathfrak{n}_-)_{-\delta_s}$. From the structure of the comultiplication we see that

$$(\text{id} \otimes \hat{\Delta})\hat{\Delta}(X) - \sum_r X_r^{[1]} \otimes K_{\gamma_r} \otimes X_r^{[2]} K_{\gamma_r}$$

is contained in $U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{h})(U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon})) \otimes U_q(\mathfrak{b}_+)$. Similarly,

$$(\hat{\Delta} \otimes \text{id})\hat{\Delta}(Y) - \sum_s K_{-\delta_s} Y_s^{[1]} \otimes K_{-\delta_s} \otimes Y_s^{[2]}$$

is contained in $U_q(\mathfrak{b}_-) \otimes (U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_-)$. Hence according to the commutation relations in $U_q(\mathfrak{g})$ from Lemma 2.76 and the definition of \mathcal{P} we obtain

$$\begin{aligned} \mathcal{P}(XY) &= \tau(X_{(1)}, \hat{S}^{-1}(Y_{(1)}))\mathcal{P}(Y_{(2)}X_{(2)})\tau(X_{(3)}, Y_{(3)}) \\ &= \sum_{r,s} \tau(X_r^{[1]}, \hat{S}^{-1}(K_{-\delta_s} Y_s^{[1]}))K_{\gamma_r - \delta_s} \tau(X_r^{[2]} K_{\gamma_r}, Y_s^{[2]}) \\ &= \sum_{r,s} \tau(X_r^{[1]}, \hat{S}^{-1}(K_{-\delta_s} Y_s^{[1]}))\tau(X_r^{[2]} K_{\gamma_r}, Y_s^{[2]})K_{2\gamma_r - \gamma}; \end{aligned}$$

here we note that only summands r, s such that $\gamma_r + \delta_s = \gamma$ contribute. The term for $\gamma_r = 0$ is

$$\sum_s \tau(1, K_{-\delta_s} Y_s^{[1]})\tau(X, Y_s^{[2]}) = \tau(X, Y).$$

This yields the claim. \square

We prove now that the Drinfeld pairing is invariant in a suitable sense.

Theorem 2.88. *Let $X \in U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$ and $Y \in U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$. Then*

$$\tau(\mathcal{T}_i^{-1}(X), \mathcal{T}_i^{-1}(Y)) = \tau(X, Y).$$

Proof. We may assume $X \in U_q(\mathfrak{n}_+)_\gamma \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$, $Y \in U_q(\mathfrak{n}_-)_{-\gamma} \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$ for some $\gamma \in \mathbf{Q}^+$. Applying Proposition 2.87 to $\mathcal{T}_i^{-1}(X), \mathcal{T}_i^{-1}(Y)$, we see that it suffices to show

$$\mathcal{P}(\mathcal{T}_i^{-1}(XY)) - \tau(X, Y)K_{-s_i\gamma} \in \sum_{\nu \in \mathbf{Q}^+ \setminus \{0\}} \mathbb{K}K_{2\nu - s_i\gamma},$$

since we then obtain the claim by comparing the coefficients of $K_{-s_i\gamma}$.

As in the proof of Proposition 2.87 we may write

$$\hat{\Delta}(X) = \sum_r X_r^{[1]} \otimes X_r^{[2]} K_{\gamma_r}, \quad \hat{\Delta}(Y) = \sum_s K_{-\delta_s} Y_s^{[1]} \otimes Y_s^{[2]}$$

where $\gamma_r, \delta_s \in \mathbf{Q}^+ \cap s_i \mathbf{Q}^+$ and $X_r^{[1]} \in U_q(\mathfrak{n}_+)_{\gamma_r}, Y_s^{[2]} \in U_q(\mathfrak{n}_-)_{-\delta_s}$. According to Lemma 2.83 we have in fact $X_r^{[1]} \in U_q(\mathfrak{n}_+)_{\gamma_r} \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)), X_r^{[2]} \in U_q(\mathfrak{n}_+)_{\gamma-\gamma_r}$, and similarly $Y_s^{[1]} \in U_q(\mathfrak{n}_-)_{-\gamma+\delta_s}, Y_s^{[2]} \in U_q(\mathfrak{n}_-)_{-\delta_s} \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$. Using again Lemma 2.83 and Lemma 2.85 we find elements $X_{rm}^{[1]} \in U_q(\mathfrak{n}_+)_{\gamma_r-m\alpha_i} \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$ such that

$$\hat{\Delta}(X_r^{[1]}) - \sum_{m \in \mathbb{N}_0} X_{rm}^{[1]} \otimes E_i^{(m)} K_{\gamma_r-m\alpha_i}$$

is contained in $U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{b}_+)(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)) \cap \ker(\hat{\epsilon}))$. Then

$$(\hat{\Delta} \otimes \text{id})\hat{\Delta}(X) - \sum_r \sum_{m \in \mathbb{N}_0} X_{rm}^{[1]} \otimes E_i^{(m)} K_{\gamma_r-m\alpha_i} \otimes X_r^{[2]} K_{\gamma_r}$$

is contained in $U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{b}_+)(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)) \cap \ker(\hat{\epsilon})) \otimes U_q(\mathfrak{b}_+)$.

Similarly, we find $Y_{sm}^{[2]} \in U_q(\mathfrak{n}_-)_{-\delta_s+m\alpha_i} \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$ such that

$$\hat{\Delta}(Y_s^{[2]}) - \sum_{m \in \mathbb{N}_0} K_{-\delta_s+m\alpha_i} F_i^{(m)} \otimes Y_{sm}^{[2]}$$

is contained in $(U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{n}_-)$, and then

$$(\text{id} \otimes \hat{\Delta})\hat{\Delta}(Y) - \sum_s \sum_{m \in \mathbb{N}_0} K_{-\delta_s} Y_s^{[1]} \otimes K_{-\delta_s+m\alpha_i} F_i^{(m)} \otimes Y_{sm}^{[2]}$$

is contained in $U_q(\mathfrak{b}_-) \otimes (U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{b}_-) \otimes U_q(\mathfrak{n}_-)$.

Due to the commutation relations from Lemma 2.76 and the invariance properties of τ we get

$$\begin{aligned} XY - \sum_{m \in \mathbb{N}_0} \sum_{\gamma_r+\delta_s=\gamma+m\alpha_i} \tau(X_{rm}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) K_{-\delta_s+m\alpha_i} F_i^{(m)} E_i^{(m)} K_{\gamma_r-m\alpha_i} \tau(X_r^{[2]}, Y_{sm}^{[2]}) \\ \in (U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-)) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{g}) + U_q(\mathfrak{g})(U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+)) \cap \ker(\hat{\epsilon})). \end{aligned}$$

In the sequel, this relation will be evaluated in two ways.

Firstly, by the definition of \mathcal{P} we obtain

$$\mathcal{P}(XY) = \sum_{\gamma_r+\delta_s=\gamma} \tau(X_{r0}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) K_{-\gamma+2\gamma_r} \tau(X_r^{[2]}, Y_{s0}^{[2]}).$$

Therefore

$$\tau(X, Y) = \sum_{\gamma_r=0, \delta_s=\gamma} \tau(X_{r0}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) \tau(X_r^{[2]}, Y_{s0}^{[2]})$$

according to Proposition 2.87.

Secondly, applying \mathcal{T}_i^{-1} to the above formula gives

$$\begin{aligned} \mathcal{T}_i^{-1}(XY) - \sum_{m \in \mathbb{N}_0} \sum_{\gamma_r+\delta_s=\gamma+m\alpha_i} \tau(X_{rm}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) \tau(X_r^{[2]}, Y_{sm}^{[2]}) \times \\ K_{s_i(-\delta_s+m\alpha_i)} E_i^{(m)} F_i^{(m)} K_{s_i(\gamma_r-m\alpha_i)} \\ \in (U_q(\mathfrak{n}_-) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{g}) + U_q(\mathfrak{g})(U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon})), \end{aligned}$$

taking into account

$$\mathcal{T}_i^{-1}(F_i^{(m)} E_i^{(m)}) = \frac{1}{[m]_q!^2} (-K_i E_i)^m (-F_i K_i^{-1})^m = \frac{1}{[m]_q!^2} E_i^m F_i^m = E_i^{(m)} F_i^{(m)}$$

by the definition of \mathcal{T}_i . According to the commutation relations in Proposition 2.23 we obtain

$$E_i^{(m)} F_i^{(m)} - \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} \in U_q(\mathfrak{g})(U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon})) + (U_q(\mathfrak{n}_-) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{g})$$

where we recall that

$$\begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} = \prod_{j=1}^m \frac{q_i^{1-j} K_i - q_i^{-(1-j)} K_i^{-1}}{q_i^j - q_i^{-j}}.$$

Hence,

$$\begin{aligned} \mathcal{T}_i^{-1}(XY) - \sum_{m \in \mathbb{N}_0} \sum_{\gamma_r + \delta_s = \gamma + m\alpha_i} \tau(X_{rm}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) \tau(X_r^{[2]}, Y_{sm}^{[2]}) \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} K_{s_i(\gamma_r - \delta_s)} \\ \in (U_q(\mathfrak{n}_-) \cap \ker(\hat{\epsilon}))U_q(\mathfrak{g}) + U_q(\mathfrak{g})(U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon})), \end{aligned}$$

and therefore

$$\mathcal{P}(\mathcal{T}_i^{-1}(XY)) = \sum_{m \in \mathbb{N}_0} \sum_{\gamma_r + \delta_s = \gamma + m\alpha_i} \tau(X_{rm}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) \tau(X_r^{[2]}, Y_{sm}^{[2]}) \begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} K_{s_i(\gamma_r - \delta_s)}.$$

Observe that

$$\begin{bmatrix} K_i; 0 \\ m \end{bmatrix}_{q_i} \in K_{-m\alpha_i} \left(\mathbb{K}^\times + \sum_{l \in \mathbb{N}} \mathbb{K} K_{2l\alpha_i} \right).$$

We may therefore write $\mathcal{P}(\mathcal{T}_i^{-1}(XY))$ as a linear combination of elements of the form $K_{s_i(\gamma_r - \delta_s) - m\alpha_i} K_{2l\alpha_i}$ where $l \in \mathbb{N}_0$.

Let us recall that $\gamma_r \in \mathbf{Q}^+ \cap s_i \mathbf{Q}^+$ by assumption, which implies $s_i \gamma_r \in \mathbf{Q}^+$. Now $\gamma_r + \delta_s = \gamma + m\alpha_i$ yields

$$\gamma_r - \delta_s + m\alpha_i = -\gamma + 2\gamma_r,$$

and thus

$$s_i(\gamma_r - \delta_s) - m\alpha_i = -s_i \gamma + 2s_i \gamma_r.$$

Recall also that $X_{rm}^{[1]} \in U_q(\mathfrak{n}_+)_{\gamma_r - m\alpha_i}$. Hence if $\gamma_r = 0$ then $X_{rm}^{[1]} = 0$ unless $m = 0$. Combining these considerations we obtain

$$\begin{aligned} \mathcal{P}(\mathcal{T}_i^{-1}(XY)) &\in K_{-s_i \gamma} \left(\sum_{\gamma_r=0, \delta_s=\gamma} \tau(X_{r0}^{[1]}, \hat{S}^{-1}(Y_s^{[1]})) \tau(X_r^{[2]}, Y_{s0}^{[2]}) + \sum_{\nu \in \mathbf{Q}^+ \setminus \{0\}} \mathbb{K} K_{2\nu} \right) \\ &= K_{-s_i \gamma} \left(\tau(X, Y) + \sum_{\nu \in \mathbf{Q}^+ \setminus \{0\}} \mathbb{K} K_{2\nu} \right), \end{aligned}$$

using our previous formula for $\tau(X, Y)$. This finishes the proof. \square

Lemma 2.89. *If $X \in U_q(\mathfrak{n}_+)[s_i w_0]$, $Y \in U_q(\mathfrak{n}_-)[s_i w_0]$ and $r, s \in \mathbb{N}_0$, then*

$$\tau(\mathcal{T}_i(X) E_i^r, \mathcal{T}_i(Y) F_i^s) = \delta_{rs} \tau(\mathcal{T}_i(X), \mathcal{T}_i(Y)) \tau(E_i^r, F_i^r)$$

for any $1 \leq i \leq N$.

Proof. Let us first show

$$\tau(E_i^r, F_i^r) = q_i^{r(r-1)/2} [r]_{q_i}! \tau(E_i, F_i)^r = \left(\prod_{k=1}^r \frac{q_i^{2k} - 1}{q_i^2 - 1} \right) \tau(E_i, F_i)^r$$

for $r \in \mathbb{N}$. For $r = 1$ this claim is trivially true. Assume that it holds for r , then using Lemma 2.6 we obtain

$$\begin{aligned}\tau(E_i^{r+1}, F_i^{r+1}) &= \tau(E_i^r, (F_i^{r+1})_{(1)})\tau(E_i, (F_i^{r+1})_{(2)}) \\ &= q_i^r [r+1]_{q_i} \tau(E_i^r, F_i^r K_i^{-1})\tau(E_i, F_i) \\ &= q_i^{(r+1)r/2} [r+1]_{q_i}! \tau(E_i, F_i)^{r+1}.\end{aligned}$$

This proves the first equality, and the second is elementary.

In order to prove the Lemma we use induction on $r+s$. For $r+s=0$, that is, $r=s=0$, the claim trivially holds.

Assume now $r+s=k>0$, and that the claim holds for $k-1$. Recall from the remark after Lemma 2.85 that $\mathcal{T}_i(U_q(\mathfrak{n}_+)[s_i w_0]) = U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$ and $\mathcal{T}_i(U_q(\mathfrak{n}_-)[s_i w_0]) = U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$, so that $\mathcal{T}_i(X) \in U_q(\mathfrak{n}_+) \cap \mathcal{T}_i(U_q(\mathfrak{n}_+))$ and $\mathcal{T}_i(Y) \in U_q(\mathfrak{n}_-) \cap \mathcal{T}_i(U_q(\mathfrak{n}_-))$. If $r=0$ then Lemma 2.84 yields $\tau(\mathcal{T}_i(X), \mathcal{T}_i(Y)F_i^s) = 0$, and similarly we get $\tau(\mathcal{T}_i(X)E_i^r, \mathcal{T}_i(Y)) = 0$ if $s=0$.

Let us therefore assume that both r, s are positive, and let us consider the case $s \geq r$. Lemma 2.6 yields

$$\begin{aligned}\tau(\mathcal{T}_i(X)E_i^r, \mathcal{T}_i(Y)F_i^s) &= \tau(\mathcal{T}_i(X)_{(2)}(E_i^r)_{(2)}, \mathcal{T}_i(Y)F_i^{s-1})\tau(\mathcal{T}_i(X)_{(1)}(E_i^r)_{(1)}, F_i) \\ &= \sum_{l=0}^r q_i^{l(r-l)} \begin{bmatrix} r \\ l \end{bmatrix}_{q_i} \tau(\mathcal{T}_i(X)_{(2)}E_i^{r-l}K_i^l, \mathcal{T}_i(Y)F_i^{s-1})\tau(\mathcal{T}_i(X)_{(1)}E_i^l, F_i).\end{aligned}$$

According to Lemma 2.83 we know $\hat{\Delta}(\mathcal{T}_i(X)) \in (\mathcal{T}_i(U_q(\mathfrak{n}_+)) \cap U_q(\mathfrak{n}_+)) \otimes U_q(\mathfrak{b}_+)$, and so the above terms are all zero except when $l=1$. Therefore,

$$\begin{aligned}\tau(\mathcal{T}_i(X)E_i^r, \mathcal{T}_i(Y)F_i^s) &= q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)_{(2)}E_i^{r-1}K_i, \mathcal{T}_i(Y)F_i^{s-1})\tau(\mathcal{T}_i(X)_{(1)}E_i, F_i) \\ &= q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)_{(2)}E_i^{r-1}K_i, \mathcal{T}_i(Y)F_i^{s-1}) \times \\ &\quad (\tau(\mathcal{T}_i(X)_{(1)}, F_i)\tau(E_i, 1) + \tau(\mathcal{T}_i(X)_{(1)}, K_i^{-1})\tau(E_i, F_i)) \\ &= q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)_{(2)}E_i^{r-1}K_i, \mathcal{T}_i(Y)F_i^{s-1})\tau(\mathcal{T}_i(X)_{(1)}, 1)\tau(E_i, F_i) \\ &= q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)E_i^{r-1}K_i, \mathcal{T}_i(Y)F_i^{s-1})\tau(E_i, F_i) \\ &= q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)E_i^{r-1}, \mathcal{T}_i(Y)F_i^{s-1})\tau(E_i, F_i).\end{aligned}$$

Now the inductive hypothesis and our considerations at the beginning imply

$$\begin{aligned}q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X)E_i^{r-1}, \mathcal{T}_i(Y)F_i^{s-1})\tau(E_i, F_i) &= \delta_{r,s} q_i^{r-1} [r]_{q_i} \tau(\mathcal{T}_i(X), \mathcal{T}_i(Y))\tau(E_i^{r-1}, F_i^{s-1})\tau(E_i, F_i) \\ &= \delta_{r,s} q_i^{r-1} [r]_{q_i} q_i^{(r-1)(r-2)/2} [r-1]_{q_i}! \tau(\mathcal{T}_i(X), \mathcal{T}_i(Y))\tau(E_i, F_i)^{r-1} \tau(E_i, F_i) \\ &= \delta_{r,s} \tau(\mathcal{T}_i(X), \mathcal{T}_i(Y))\tau(E_i^r, F_i^r).\end{aligned}$$

This yields the claim. \square

Theorem 2.90. *The PBW-basis vectors are orthogonal with respect to the Drinfeld pairing. More precisely, we have*

$$\tau(E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}, F_{\beta_n}^{b_n} \cdots F_{\beta_1}^{b_1}) = \prod_{k=1}^n \delta_{a_k, b_k} \tau(E_{\beta_k}^{a_k}, F_{\beta_k}^{b_k}).$$

Proof. We shall prove more generally that the PBW-vectors associated to any reduced expression $w = s_{i_1} \cdots s_{i_t}$ of $w \in W$ satisfy the above orthogonality relations. This clearly implies the desired statement.

We use induction on the length of w . For $l(w) = 0$ the claim is trivial, and for $l(w) = 1$ it follows immediately from the construction of the pairing. Assume that $l(w) > 1$ and write

$$\begin{aligned} X &= (\mathcal{T}_{i_2} \cdots \mathcal{T}_{i_{t-1}})(E_{i_t}^{a_t}) \cdots \mathcal{T}_{i_2}(E_{i_3}^{a_3})E_{i_2}^{a_2} \\ Y &= (\mathcal{T}_{i_2} \cdots \mathcal{T}_{i_{n-1}})(F_{i_t}^{s_t}) \cdots \mathcal{T}_{i_2}(F_{i_3}^{s_3})F_{i_2}^{s_2}. \end{aligned}$$

Since $\mathcal{T}_{i_1}(X) \in U_q(\mathfrak{n}_+)$ the remark after Lemma 2.85 shows that $X \in U_q(\mathfrak{n}_+)[s_{i_1}w_0]$ where w_0 is the longest element of W . Similarly we obtain $Y \in U_q(\mathfrak{n}_-)[s_{i_1}w_0]$.

Hence by Lemma 2.89 and Theorem 2.88 we calculate

$$\begin{aligned} \tau(\mathcal{T}_{i_1}(X)E_{i_1}^{a_1}, \mathcal{T}_{i_1}(Y)F_{i_1}^{b_1}) &= \tau(\mathcal{T}_{i_1}(X), \mathcal{T}_{i_1}(Y))\tau(E_{i_1}^{a_1}, F_{i_1}^{b_1}) \\ &= \tau(X, Y)\tau(E_{i_1}^{a_1}, F_{i_1}^{b_1}) = \delta_{a_1, b_1}\tau(E_{i_1}^{a_1}, F_{i_1}^{b_1})\tau(X, Y). \end{aligned}$$

Now we can apply the induction hypothesis to $s_{i_1}w = s_{i_2} \cdots s_{i_t}$. This yields the claim. \square

To conclude this subsection we complete the description obtained in Theorem 2.90 with the following formula. We use the notation

$$q_\beta = q^{(\beta, \beta)/2}$$

for any root $\beta \in \Delta$.

Lemma 2.91. *For any $1 \leq k \leq n$ and $r \geq 0$ we have*

$$\tau(E_{\beta_k}^r, F_{\beta_k}^r) = (-1)^r q_{\beta_k}^{r(r-1)/2} \frac{[r]_{q_{\beta_k}}!}{(q_{\beta_k} - q_{\beta_k}^{-1})^r},$$

and therefore

$$\tau(E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}, F_{\beta_n}^{b_n} \cdots F_{\beta_1}^{b_1}) = \prod_{k=1}^n \delta_{a_k, b_k} (-1)^{a_k} q_{\beta_k}^{a_k(a_k-1)/2} \frac{[a_k]_{q_{\beta_k}}!}{(q_{\beta_k} - q_{\beta_k}^{-1})^{a_k}}.$$

Proof. According to Theorem 2.88 it suffices to prove the first assertion in the case that $\beta_k = \alpha_i$ is a simple root. In this case, the computation in the proof of Lemma 2.89 combined with Lemma 2.75 yields

$$\tau(E_i^r, F_i^r) = q_i^{r(r-1)/2} [r]_{q_i}! \tau(E_i, F_i)^r = (-1)^r q_i^{r(r-1)/2} [r]_{q_i}! \frac{1}{(q_i - q_i^{-1})^r}$$

as desired. \square

2.8.4. Nondegeneracy of the Drinfeld pairing. The computations of the previous subsection are key to proving nondegeneracy of the Drinfeld pairing and the quantum Killing form. We keep the assumptions and notation from above.

Theorem 2.92. *The Drinfeld pairing $\tau : U_q(\mathfrak{b}_+) \times U_q(\mathfrak{b}_-) \rightarrow \mathbb{K}$ and the quantum Killing form $\kappa : U_q(\mathfrak{g}) \times U_q(\mathfrak{g}) \rightarrow \mathbb{K}$ are nondegenerate.*

Proof. For the first claim it suffices to show that the restriction of τ to $U_q(\mathfrak{b}_+)_{\beta} \times U_q(\mathfrak{b}_-)_{-\beta}$ is nondegenerate. From Lemma 2.75 we know that τ is diagonal with respect to the tensor product decomposition $U_q(\mathfrak{b}_{\pm})_{\pm\beta} \cong U_q(\mathfrak{n}_{\pm})_{\pm\beta} \otimes U_q(\mathfrak{h})$.

Since q is not a root of unity the characters χ_{μ} of $U_q(\mathfrak{h})$ given by $\chi_{\mu}(K_{\lambda}) = q^{(\mu, \lambda)}$ for $\mu \in \mathbf{P}$ are pairwise distinct, and hence linearly independent by Artin's Theorem. It follows that the restriction of τ to $U_q(\mathfrak{h})$ is nondegenerate. Hence the claim follows from Theorem 2.90 combined with the PBW-Theorem 2.67.

The second assertion is verified in a similar way using the triangular decomposition of $U_q(\mathfrak{g})$ and nondegeneracy of τ . \square

We remark that Theorem 2.92 shows also that the Rosso form on $U_q(\mathfrak{b}_-) \bowtie U_q(\mathfrak{b}_+)$ is nondegenerate. In particular, this form does not pass to the quotient $U_q(\mathfrak{g})$. Therefore the quantum Killing cannot be obtained directly from the Rosso form on the double.

2.9. The quantum Casimir element and simple modules. In this section we resume our study of finite dimensional weight modules for $U_q(\mathfrak{g})$. We define a Casimir element Ω in a certain algebraic completion of $U_q(\mathfrak{g})$ in order to prove irreducibility of the modules $L(\mu)$ constructed in Theorem 2.43. Throughout we assume that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

According to Theorem 2.92 the restriction of the Drinfeld pairing τ to $U_q(\mathfrak{n}_+)_\gamma \times U_q(\mathfrak{n}_-)_{-\gamma}$ is nondegenerate for every $\gamma \in \mathbf{Q}^+$. Therefore, we can choose bases consisting of $b_j^\gamma \in U_q(\mathfrak{n}_+)_\gamma$ and $a_k^\gamma \in U_q(\mathfrak{n}_-)_{-\gamma}$ such that $\tau(b_i^\gamma, a_j^\gamma) = \delta_{ij}$. For $\gamma \in \mathbf{Q}^+$ we define

$$C_\gamma = \sum_j a_j^\gamma \otimes b_j^\gamma, \quad \Omega_\gamma = \sum_j \hat{S}^{-1}(a_j^\gamma) b_j^\gamma,$$

in particular we obtain $C_\gamma = 1 \otimes 1$ and $\Omega_\gamma = 1$ for $\gamma = 0$. By definition, the quantum Casimir element is the formal sum

$$\Omega = \sum_{\gamma \in \mathbf{Q}^+} \Omega_\gamma,$$

which is not an element of $U_q(\mathfrak{g})$, but rather of a suitable completion of this algebra. We shall not discuss any completions here, instead we will only consider the “image” of Ω in $\text{End}(V)$ for representations V of $U_q(\mathfrak{g})$ in which for any $v \in V$ all but finitely many terms Ω_α for $\alpha \in \mathbf{Q}^+$ act by zero. Note that this applies in particular to all Verma modules, and hence also all quotients of Verma modules. Indeed, if $v \in M(\lambda)$ for some $\lambda \in \mathfrak{h}_q^*$ then we have $U_q(\mathfrak{n}_+)_{\gamma} \cdot v = 0$ for all $\gamma \in \mathbf{Q}^+$ sufficiently large. Hence the Casimir operator Ω yields a well-defined endomorphism of $M(\lambda)$.

Lemma 2.93. *Let $\gamma \in \mathbf{Q}^+$. Then we have*

$$(1 \otimes F_i)C_\gamma - C_\gamma(1 \otimes F_i) = C_{\gamma - \alpha_i}(F_i \otimes K_i) - (F_i \otimes K_i^{-1})C_{\gamma - \alpha_i}$$

for all $1 \leq i \leq N$, where we interpret $C_{\gamma - \alpha_i} = 0$ if $\gamma - \alpha_i \notin \mathbf{Q}^+$.

Proof. For $\gamma - \alpha_i \notin \mathbf{Q}^+$ the element F_i commutes with all b_j^γ since the latter must be sums of monomials in E_1, \dots, E_N not containing any powers of E_i . Hence the equality holds in this case.

Consider now the case that $\gamma = \beta + \alpha_i$ for some $\beta \in \mathbf{Q}^+$. Since the pairing τ is nondegenerate it is enough to compare the pairing of both sides against $X \in U_q(\mathfrak{n}_+)_{\alpha_i + \beta}$ with respect to τ in the first tensor factor. For the left hand side this yields

$$\sum_j \tau(X, a_j^{\beta + \alpha_i}) [F_i, b_j^{\beta + \alpha_i}] = [F_i, X] = F_i X - X F_i.$$

To compute the right hand side let us write $\hat{\Delta}(X) = \sum_\gamma c_\gamma X_\gamma^{[1]} \otimes X_{\beta + \alpha_i - \gamma}^{[2]} K_\gamma$ with monomials $X_\gamma^{[1]}, X_{\beta + \alpha_i - \gamma}^{[2]}$ in the generators F_k of weight γ and $\beta + \alpha_i - \gamma$,

respectively, and coefficients c_γ . We obtain

$$\begin{aligned}
& \sum_j \tau(X, a_j^\beta F_i) b_j^\beta K_i - \tau(X, F_i a_j^\beta) K_i^{-1} b_j^\beta \\
&= \sum_{j, \gamma} c_\gamma \tau(X_\gamma^{[1]}, F_i) \tau(X_{\beta+\alpha_i-\gamma}^{[2]} K_\gamma, a_j^\beta) b_j^\beta K_i - c_\gamma \tau(X_\gamma^{[1]}, a_j^\beta) \tau(X_{\beta+\alpha_i-\gamma}^{[2]} K_\gamma, F_i) K_i^{-1} b_j^\beta \\
&= \sum_j -c_{\alpha_i} \frac{1}{q_i - q_i^{-1}} \tau(X_\beta^{[2]}, a_j^\beta) b_j^\beta K_i + c_\beta \frac{1}{q_i - q_i^{-1}} \tau(X_\beta^{[1]}, a_j^\beta) K_i^{-1} b_j^\beta \\
&= -\frac{1}{q_i - q_i^{-1}} (c_{\alpha_i} X_\beta^{[2]} K_i - c_\beta K_i^{-1} X_\beta^{[1]}),
\end{aligned}$$

using Lemma 2.75 in the second step. Moreover, combining

$$(\hat{\Delta} \otimes \text{id}) \hat{\Delta}(F_i) = F_i \otimes 1 \otimes 1 + K_i^{-1} \otimes F_i \otimes 1 + K_i^{-1} \otimes K_i^{-1} \otimes F_i,$$

with the commutation relations from Lemma 2.76 gives

$$\begin{aligned}
X F_i &= c_{\alpha_i} \tau(\hat{S}(X_{\alpha_i}^{[1]}), F_i) X_\beta^{[2]} K_i + F_i X + c_\beta \tau(X_{\alpha_i}^{[2]} K_\beta, F_i) K_i^{-1} X_\beta^{[1]} \\
&= -c_{\alpha_i} \tau(X_{\alpha_i}^{[1]}, F_i) X_\beta^{[2]} K_i + F_i X + c_\beta \tau(X_{\alpha_i}^{[2]}, F_i) K_i^{-1} X_\beta^{[1]} \\
&= c_{\alpha_i} \frac{1}{q_i - q_i^{-1}} X_\beta^{[2]} K_i + F_i X - c_\beta \frac{1}{q_i - q_i^{-1}} K_i^{-1} X_\beta^{[1]},
\end{aligned}$$

taking into account that $X_{\alpha_i}^{[1]} = E_i = X_{\alpha_i}^{[2]}$ in the last step. Comparing the above expressions yields the claim. \square

Lemma 2.94. *Let $\lambda \in \mathfrak{h}_q^*$. Then for all $\beta \in \mathbf{Q}^+$ and $Y \in U_q(\mathfrak{n}_-)_{-\beta}$ we have*

$$\Omega Y = q^{2(\rho, \beta) - (\beta, \beta)} Y \Omega K_{2\beta}$$

in $\text{End}(M(\lambda))$.

Proof. Assume that $Y_1 \in U_q(\mathfrak{n}_-)_{-\beta_1}, Y_2 \in U_q(\mathfrak{n}_-)_{-\beta_2}$ for $\beta_1, \beta_2 \in \mathbf{Q}^+$ satisfy the above relation. Then

$$\begin{aligned}
\Omega Y_1 Y_2 &= q^{2(\rho, \beta_1) - (\beta_1, \beta_1)} Y_1 \Omega K_{2\beta_1} Y_2 \\
&= q^{2(\rho, \beta_1) - (\beta_1, \beta_1)} q^{-(2\beta_1, \beta_2)} Y_1 \Omega Y_2 K_{2\beta_1} \\
&= q^{2(\rho, \beta_1) - (\beta_1, \beta_1)} q^{-(2\beta_1, \beta_2)} q^{2(\rho, \beta_2) - (\beta_2, \beta_2)} Y_1 Y_2 \Omega K_{2(\beta_1 + \beta_2)} \\
&= q^{2(\rho, \beta_1 + \beta_2) - (\beta_1 + \beta_2, \beta_1 + \beta_2)} Y \Omega K_{2(\beta_1 + \beta_2)}.
\end{aligned}$$

Hence using induction on the length of β it suffices to consider the case $Y = F_i$ for some $1 \leq i \leq N$. Notice that in this case $\beta = \alpha_i$ and $2(\rho, \alpha_i) = (\alpha_i, \alpha_i)$. Hence the desired relation reduces to

$$\Omega F_i = F_i \Omega K_i^2.$$

This in turn is a consequence of Lemma 2.93. Indeed, if we apply $\hat{S}^{-1} \otimes \text{id}$ to the relation from Lemma 2.93 and multiply the tensor factors, we obtain

$$\begin{aligned}
& \sum_j \hat{S}^{-1}(a_j^\gamma) F_i b_j^\gamma - \hat{S}^{-1}(a_j^\gamma) b_j^\gamma F_i \\
&= \sum_j \hat{S}^{-1}(F_i) \hat{S}^{-1}(a_j^{\gamma - \alpha_i}) b_j^{\gamma - \alpha_i} K_i - \hat{S}^{-1}(a_j^{\gamma - \alpha_i}) \hat{S}^{-1}(F_i) K_i^{-1} b_j^{\gamma - \alpha_i}
\end{aligned}$$

for all $\gamma \in \mathbf{Q}^+$. Summing over γ this yields

$$\sum_{\gamma \in \mathbf{Q}^+} \sum_j \hat{S}^{-1}(a_j^\gamma) b_j^\gamma F_i = \sum_{\gamma \in \mathbf{Q}^+} \sum_j F_i K_i \hat{S}^{-1}(a_j^{\gamma - \alpha_i}) b_j^{\gamma - \alpha_i} K_i,$$

which implies $\Omega F_i = F_i \Omega K_i^2$ as desired. \square

Let us say that $\gamma \in \mathfrak{h}_q^*$ is a primitive weight of a weight module M if there exists a primitive vector $v \in M$ of weight γ .

Lemma 2.95. *Let $\lambda \in \mathfrak{h}_q^*$. If γ is a primitive weight of $M(\lambda)$ then $\gamma = \lambda - \beta$ for some $\beta \in \mathbf{Q}^+$ satisfying*

$$q^{2(\rho+\lambda, \beta) - (\beta, \beta)} = 1.$$

Proof. If $v \in M(\lambda)$ is a primitive vector then $\Omega \cdot v = v$ since all the terms Ω_α for $\alpha \neq 0$ act on v by zero in this case. At the same time we have $v = Y \cdot v_\lambda$ for some $Y \in U_q(\mathfrak{n}_-)_-\beta$. Hence

$$v = \Omega \cdot v = \Omega Y \cdot v_\lambda = q^{2(\rho, \beta) - (\beta, \beta)} Y \Omega K_{2\beta} \cdot v_\lambda = q^{2(\rho+\lambda, \beta) - (\beta, \beta)} v$$

by Lemma 2.94. This yields the claim. \square

Lemma 2.96. *Let $\lambda \in \mathbf{P}^+$ and assume $\gamma = \lambda - \beta$ for some $\beta \in \mathbf{Q}^+$ satisfies*

$$(\gamma + \rho, \gamma + \rho) = (\lambda + \rho, \lambda + \rho).$$

If $\gamma + \rho \in \mathbf{P}^+$ then $\gamma = \lambda$.

Proof. The given relation can be rewritten as

$$(\lambda - \gamma, \lambda + \rho) + (\gamma + \rho, \lambda - \gamma) = (\lambda + \rho, \lambda + \rho) + (\gamma + \rho, \lambda - \gamma - \lambda - \rho) = 0.$$

Note that we have $\lambda + \rho \in \mathbf{P}^+$. If $\gamma + \rho \in \mathbf{P}^+$ then since $\lambda - \gamma \in \mathbf{Q}^+$ it follows that both terms on the left hand side are positive, and therefore vanish separately. The equality $(\lambda - \gamma, \lambda + \rho) = 0$ implies $\lambda - \gamma = 0$, since $\lambda + \rho$ is a linear combination of fundamental weights with all coefficients being strictly positive. \square

Theorem 2.97. *Let $\mu \in \mathbf{P}^+$. Then the largest integrable quotient module $L(\mu)$ of $M(\mu)$ is irreducible, and hence $L(\mu) \cong V(\mu)$.*

Proof. If $L(\mu)$ is not simple then it must contain a proper submodule V . Inside V we can find a primitive vector v of highest weight $\gamma = \mu - \beta$ for some $\beta \in \mathbf{Q}^+$. Note that $\gamma \in \mathbf{P}^+$ by Theorem 2.43.

Then Lemma 2.95 implies $2(\beta, \mu + \rho) - (\beta, \beta) = 0$ and thus

$$\begin{aligned} (\gamma + \rho, \gamma + \rho) &= (\mu + \rho - \beta, \mu + \rho - \beta) \\ &= (\mu + \rho, \mu + \rho) - 2(\beta, \mu + \rho) + (\beta, \beta) = (\mu + \rho, \mu + \rho). \end{aligned}$$

According to Lemma 2.96 we conclude $\gamma = \mu$. This means $V = L(\mu)$, contradicting our assumption that V is a proper submodule. Hence $L(\mu)$ is indeed simple, and therefore the canonical quotient map $L(\mu) \rightarrow V(\mu)$ is an isomorphism. \square

We write $\pi_\mu : U_q(\mathfrak{g}) \rightarrow \text{End}(V(\mu))$ for the algebra homomorphism corresponding to the module structure on $V(\mu)$.

Lemma 2.98. *For every $\mu \in \mathbf{P}^+$ we have $\pi_\mu(U_q(\mathfrak{g})) = \text{End}(V(\mu))$.*

Proof. Assume that $C \in \text{End}(V(\mu))$ is contained in the commutant of $\pi_\mu(U_q(\mathfrak{g}))$, or equivalently, C is an intertwiner of $V(\mu)$ with itself. Then C must map the highest weight vector v_μ to a multiple of itself. It follows that C acts as a scalar on all of $V(\mu)$ because v_μ is cyclic. Now the Jacobson density theorem yields the claim. \square

2.9.1. Complete reducibility. Having determined the irreducible finite dimensional weight modules over $U_q(\mathfrak{g})$, we obtain the following complete reducibility result in analogy to the situation for classical universal enveloping algebras, compare also the proof of Proposition 2.40.

Theorem 2.99. *Every finite dimensional integrable module over $U_q(\mathfrak{g})$ decomposes into a direct sum of irreducible highest weight modules $V(\mu)$ for weights $\mu \in \mathbf{P}^+$.*

Proof. We show that every extension $0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$ of finite dimensional integrable modules splits. Notice that the extension automatically admits a $U_q(\mathfrak{h})$ -linear splitting since all the modules involved are weight modules.

In the first step consider the case that $K = V(\mu)$ and $Q = V(\lambda)$ are simple. If $\mu = \lambda$ then we can lift the highest weight vector of $V(\lambda)$ to a primitive vector in E , which induces a splitting of the sequence by the characterisation of $V(\lambda)$ as the largest integrable quotient of $M(\lambda)$ from Theorem 2.97.

Assume now $\mu \neq \lambda$. If $\lambda \not\prec \mu$, that is, we do not have $\lambda = \mu - \nu$ for a nonzero element $\nu \in \mathbf{Q}^+$ then λ does not appear as a weight in $V(\mu)$. Again we can lift the highest weight vector of $V(\lambda)$ to a primitive vector in E and obtain a splitting. Finally, assume $\mu \not\prec \lambda$, that is we do not have $\lambda = \mu + \nu$ for some nonzero element $\nu \in \mathbf{Q}^+$. We consider the dual sequence $0 \rightarrow V(\lambda)^\vee \rightarrow E \rightarrow V(\mu)^\vee \rightarrow 0$. Since $V(\mu)^\vee \cong V(\mu)$ and $V(\lambda)^\vee \cong V(\lambda)$ we obtain a splitting by our previous argument. Applying duality again shows that the original sequence is split as well.

In the second step we consider arbitrary K and Q and show the existence of a splitting by induction on the dimension of E . For $\dim(E) = 0$ the assertion is clear. Assume that every extension of dimension less than n splits and let $\dim(E) = n$. If $K = 0$ or $Q = 0$ there is nothing to prove. Otherwise both K and Q have dimension $< n$ and hence are direct sums of simple modules. Considering each simple block in Q and its preimage in E independently we see that we can restrict to the case that Q is simple. If K is simple then the claim follows from our above considerations. Otherwise there exists a proper simple submodule $L \subset K$. Then $0 \rightarrow K/L \rightarrow E/L \rightarrow Q \rightarrow 0$ is an extension, and since $\dim(E/L) < n$ it is split. That is, Q is a direct summand in E/L . Considering $0 \rightarrow L \rightarrow E \rightarrow E/L \rightarrow 0$, we thus obtain a splitting $Q \rightarrow E$ according to the first part of the proof. This map splits the sequence $0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$, which finishes the proof. \square

As a consequence of Lemma 2.41, we immediately obtain the analogous result for finite dimensional weight modules.

Corollary 2.100. *Every finite dimensional weight module over $U_q(\mathfrak{g})$ decomposes into a direct sum of irreducible highest weight modules $V(\lambda)$ for weights $\lambda \in \mathbf{P}_q^+$.*

Using Theorem 2.99 we can also strengthen Theorem 2.46.

Theorem 2.101. *The representations $V(\mu)$ of $U_q(\mathfrak{g})$ for $\mu \in \mathbf{P}^+$ separate points. More precisely, if $X \in U_q(\mathfrak{g})$ satisfies $\pi_\mu(X) = 0$ in $\text{End}(V(\mu))$ for all $\mu \in \mathbf{P}^+$ then $X = 0$.*

Proof. According to Theorem 2.46 the finite dimensional integrable representations separate the points of $U_q(\mathfrak{g})$. However, every such representation is a finite direct sum of representations $V(\mu)$ by Theorem 2.99. This yields the claim. \square

2.10. Quantized algebras of functions. We define quantized algebras of functions and discuss their duality with quantized universal enveloping algebras. It is assumed throughout that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

Definition 2.102. Let \mathfrak{g} be a semisimple Lie algebra. The quantized algebra of functions $\mathcal{O}(G_q)$ consists of all matrix coefficients of finite dimensional integrable $U_q(\mathfrak{g})$ -modules, that is,

$$\mathcal{O}(G_q) = \bigoplus_{\mu \in \mathbf{P}^+} \text{End}(V(\mu))^*,$$

with the direct sum on the right hand side being an algebraic direct sum.

The direct sum decomposition of $\mathcal{O}(G_q)$ in Definition 2.102 is called the Peter-Weyl decomposition.

By construction, there exists a bilinear pairing between $U_q(\mathfrak{g})$ and $\mathcal{O}(G_q)$ given by evaluation of linear functionals. We will write (X, f) for the value of this pairing on $X \in U_q(\mathfrak{g})$ and $f \in \mathcal{O}(G_q)$, and we point out that this pairing is nondegenerate. Indeed, $(X, f) = 0$ for all $X \in U_q(\mathfrak{g})$ implies $f = 0$ by Lemma 2.98. On the other hand, $(X, f) = 0$ for all $f \in \mathcal{O}(G_q)$ implies $X = 0$ by Theorem 2.101.

We define the product and coproduct of $\mathcal{O}(G_q)$ such that

$$(XY, f) = (X, f_{(1)})(Y, f_{(2)}), \quad (X, fg) = (X_{(2)}, f)(X_{(1)}, g),$$

and the antipode S , unit 1 and counit ϵ of $\mathcal{O}(G_q)$ are determined by

$$(\hat{S}(X), f) = (X, S^{-1}(f)), \quad (\hat{S}^{-1}(X), f) = (X, S(f))$$

and

$$\hat{\epsilon}(X) = (X, 1), \quad \epsilon(f) = (1, f)$$

for $X, Y \in U_q(\mathfrak{g})$ and $f, g \in \mathcal{O}(G_q)$. We shall write Δ for the coproduct of $\mathcal{O}(G_q)$, and in the above formulas we have used the Sweedler notation $\Delta(f) = f_{(1)} \otimes f_{(2)}$ for $f \in \mathcal{O}(G_q)$. With the above structure maps, the quantized algebra of functions becomes a Hopf algebra.

Using the transpose of the canonical isomorphism $\text{End}(V(\mu)) \cong V(\mu) \otimes V(\mu)^*$ we obtain an isomorphism

$$\mathcal{O}(G_q) \cong \bigoplus_{\mu \in \mathbf{P}^+} V(\mu)^* \otimes V(\mu).$$

More precisely, if $v \in V(\mu)$ and $v^* \in V(\mu)^*$ we shall write $\langle v^* | \bullet | v \rangle \in V(\mu)^* \otimes V(\mu) \subset \mathcal{O}(G_q)$ for the matrix coefficient determined by

$$(X, \langle v^* | \bullet | v \rangle) = v^*(\pi_\mu(X)(v)).$$

The right regular action of $U_q(\mathfrak{g})$ on $\mathcal{O}(G_q)$ given by $X \rightarrow f = f_{(1)}(X, f_{(2)})$ corresponds to $X \rightarrow \langle v^* | \bullet | v \rangle = \langle v^* | \bullet | X \cdot v \rangle$ in this picture. The left regular action given by $X \cdot f = f \leftarrow \hat{S}(X) = (\hat{S}(X), f_{(1)})f_{(2)}$ corresponds to $X \cdot \langle v^* | \bullet | v \rangle = \langle X \cdot v^* | \bullet | v \rangle$. Here we work with the action $(X \cdot v^*)(v) = v^*(\hat{S}(X) \cdot v)$ on the dual space $V(\mu)^*$ as usual.

If e_1^μ, \dots, e_n^μ is a basis of $V(\mu)$ with dual basis $e_\mu^1, \dots, e_\mu^n \in V(\mu)^*$, so that $e_\mu^i(e_j^\mu) = \delta_{ij}$, then the matrix coefficients $u_{ij}^\mu = \langle e_\mu^i | \bullet | e_j^\mu \rangle$ form a basis of $\text{End}(V(\mu))^*$. Notice that we have

$$\Delta(u_{ij}^\mu) = \sum_{k=1}^n u_{ik}^\mu \otimes u_{kj}^\mu$$

for all $\mu \in \mathbf{P}^+$ and $1 \leq i, j \leq n = \dim(V(\mu))$. By the remarks preceding Definition 1.1, this shows that $\mathcal{O}(G_q)$ is a cosemisimple Hopf algebra. By Proposition 1.10, it admits a left and right invariant integral $\phi : \mathcal{O}(G_q) \rightarrow \mathbb{K}$, specifically

$$\phi(u_{ij}^\mu) = \begin{cases} 1 & \text{if } \mu = 0 \\ 0 & \text{else.} \end{cases}$$

Since $\mathcal{O}(G_q)$ is a cosemisimple Hopf algebra, and hence in particular a regular multiplier Hopf algebra with integrals, it admits a dual multiplier Hopf algebra $\mathcal{D}(G_q)$, compare Subsection 1.3.2. Explicitly, we have

$$\mathcal{D}(G_q) = \bigoplus_{\mu \in \mathbf{P}^+} \text{End}(V(\mu)),$$

such that the canonical skew-pairing (x, f) for $x \in \mathcal{D}(G_q)$ and $f \in \mathcal{O}(G_q)$ corresponds to the obvious pairing between $\text{End}(V(\mu))$ and $\text{End}(V(\mu))^*$ in each block.

The product and coproduct of $\mathcal{D}(G_q)$ are fixed such that

$$(xy, f) = (x, f_{(1)})(y, f_{(2)}), \quad (x, fg) = (x_{(2)}, f)(x_{(1)}, g),$$

and the antipode \hat{S} , counit $\hat{\epsilon}$ of $\mathcal{D}(G_q)$ and unit $1 \in \mathcal{M}(\mathcal{D}(G_q))$ are determined by

$$(\hat{S}(x), f) = (x, S^{-1}(f)), \quad (\hat{S}^{-1}(x), f) = (x, S(f))$$

and

$$\hat{\epsilon}(x) = (x, 1), \quad \epsilon(f) = (1, f)$$

for $x, y \in \mathcal{D}(G_q)$ and $f, g \in \mathcal{O}(G_q)$. We shall write $\hat{\Delta}$ for the coproduct of $\mathcal{D}(G_q)$, and use the Sweedler notation $\hat{\Delta}(x) = x_{(1)} \otimes x_{(2)}$ for $x \in \mathcal{D}(G_q)$.

It follows immediately from the construction that there is a canonical homomorphism $U_q(\mathfrak{g}) \rightarrow \mathcal{M}(\mathcal{D}(G_q))$ of multiplier Hopf algebras, namely

$$X \mapsto \prod_{\mu \in \mathbf{P}^+} \pi_\mu(X).$$

This homomorphism is injective according to Theorem 2.101. In the sequel it will often be useful to view elements of $U_q(\mathfrak{g})$ as multipliers of $\mathcal{D}(G_q)$.

Given a basis of matrix coefficients u_{ij}^μ as above, the algebra $\mathcal{D}(G_q)$ admits a linear basis ω_{ij}^μ such that

$$(\omega_{ij}^\mu, u_{kl}^\nu) = \delta_{\mu\nu} \delta_{ik} \delta_{jl}.$$

Note that with respect to the algebra structure of $\mathcal{D}(G_q)$ we have

$$\omega_{ij}^\mu \omega_{kl}^\nu = \delta_{\mu\nu} \delta_{jk} \omega_{il}^\mu$$

that is, the elements ω_{ij}^μ are matrix units for $\mathcal{D}(G_q)$.

We note that the algebra $\mathcal{O}(G_q)$ can be viewed as a deformation of the coordinate ring of the affine algebraic variety associated with the group G corresponding to \mathfrak{g} .

2.11. The universal R -matrix. In this subsection we discuss universal R -matrices and the dual concept of universal r -forms. We derive an explicit formula for the universal r -form of $\mathcal{O}(G_q)$ in terms of the PBW-basis of $U_q(\mathfrak{g})$.

2.11.1. Universal R -matrices. Let us first define what it means for a regular multiplier Hopf algebra to be quasitriangular.

Definition 2.103. Let H be a regular multiplier Hopf algebra. Then H is called quasitriangular if there exists an invertible element $\mathcal{R} \in \mathcal{M}(H \otimes H)$, called a universal R -matrix, such that

$$\Delta^{\text{cop}}(x) = \mathcal{R} \Delta(x) \mathcal{R}^{-1}$$

for all $x \in H$ and

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}.$$

Here Δ^{cop} denotes the opposite comultiplication on H , given by $\Delta^{\text{cop}} = \sigma \Delta$, where $\sigma : H \otimes H \rightarrow H \otimes H$, $\sigma(x \otimes y) = y \otimes x$ is the flip map. Also, we are using the standard leg-numbering notation, whereby $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$ and $\mathcal{R}_{13} = (\sigma \otimes \text{id}) \mathcal{R}_{23}$ in $\mathcal{M}(H \otimes H \otimes H)$.

Notice that the final two relations in Definition 2.103 imply

$$(\epsilon \otimes \text{id})(\mathcal{R}) = 1 = (\text{id} \otimes \epsilon)(\mathcal{R}).$$

Moreover we record the formulas

$$(S \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{-1} = (\text{id} \otimes S^{-1})(\mathcal{R}),$$

obtained using the antipode property applied to the final two relations in Definition 2.103 and the previous formulas.

From a technical point of view it is sometimes easier to view the R -matrix as a linear functional on the dual side. If H is a Hopf algebra and $l, k : H \rightarrow \mathbb{K}$ are linear functionals on H we define the convolution $l * k$ to be the linear functional on H defined by

$$(l * k)(f) = l(f_{(1)})k(f_{(2)})$$

for $f \in H$. It is straightforward to check that the counit ϵ is a neutral element with respect to convolution. Accordingly, we say that $l : H \rightarrow \mathbb{K}$ is convolution invertible if there exists a linear functional $k : H \rightarrow \mathbb{K}$ such that

$$l(f_{(1)})k(f_{(2)}) = \epsilon(f) = k(f_{(1)})l(f_{(2)})$$

for all $f \in H$. In this case the functional k is uniquely determined and we write $k = l^{-1}$.

Definition 2.104. Let H be a Hopf algebra. Then H is called coquasitriangular if there exists a linear functional $r : H \otimes H \rightarrow \mathbb{K}$, called a universal r -form, which is invertible with respect to the convolution multiplication and satisfies

- a) $r(f \otimes gh) = r(f_{(1)} \otimes g)r(f_{(2)} \otimes h)$
- b) $r(fg \otimes h) = r(f \otimes h_{(2)})r(g \otimes h_{(1)})$
- c) $r(f_{(1)} \otimes g_{(1)})f_{(2)}g_{(2)} = g_{(1)}f_{(1)}r(f_{(2)} \otimes g_{(2)})$

for all $f, g, h \in H$.

If H is a cosemisimple Hopf algebra then it is in particular a regular multiplier Hopf algebra with integrals. Moreover, the linear dual space H^* identifies with $\mathcal{M}(\hat{H})$ in a natural way, and similarly $(H \otimes H)^* \cong \mathcal{M}(\hat{H} \otimes \hat{H})$. In particular, a linear map $r : H \otimes H \rightarrow \mathbb{K}$ corresponds to a unique element $\mathcal{R} \in \mathcal{M}(\hat{H} \otimes \hat{H})$ such that $(\mathcal{R}, f \otimes g) = r(f \otimes g)$. With these observations in mind, the relation between universal r -forms and universal R -matrices is as follows.

Lemma 2.105. Let H be a cosemisimple Hopf algebra and let $r : H \otimes H \rightarrow \mathbb{K}$ be a linear form, and let \mathcal{R} be the corresponding element of $\mathcal{M}(\hat{H} \otimes \hat{H})$ as explained above. Then

- a) r is convolution invertible iff \mathcal{R} is invertible.
- b) $r(f \otimes gh) = r(f_{(1)} \otimes g)r(f_{(2)} \otimes h)$ for all $f, g, h \in H$ iff $(\text{id} \otimes \hat{\Delta})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$.
- c) $r(fg \otimes h) = r(f \otimes h_{(2)})r(g \otimes h_{(1)})$ for all $f, g, h \in H$ iff $(\hat{\Delta} \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$.
- d) $r(f_{(1)} \otimes g_{(1)})g_{(2)}f_{(2)} = f_{(1)}g_{(1)}r(f_{(2)} \otimes g_{(2)})$ for all $f, g, h \in H$ iff $\hat{\Delta}^{\text{cop}}(x) = \mathcal{R}\hat{\Delta}(x)\mathcal{R}^{-1}$ for all $x \in \hat{H}$.

In particular, r is a universal r -form for H iff \mathcal{R} is a universal R -matrix for \hat{H} .

Proof. a) This follows immediately from the definition of convolution. The convolution inverse r^{-1} of r corresponds to the inverse \mathcal{R}^{-1} of \mathcal{R} .

b) Using nondegeneracy of the pairing between $H \otimes H$ and $\hat{H} \otimes \hat{H}$, this follows from

$$r(f \otimes gh) = (\mathcal{R}, f \otimes gh) = ((\text{id} \otimes \hat{\Delta})(\mathcal{R}), f \otimes h \otimes g)$$

and

$$r(f_{(1)} \otimes g)r(f_{(2)} \otimes h) = (\mathcal{R}, f_{(1)} \otimes g)(\mathcal{R}, f_{(2)} \otimes h) = (\mathcal{R}_{13}\mathcal{R}_{12}, f \otimes h \otimes g)$$

for $f, g, h \in H$.

c) This follows from

$$r(fg \otimes h) = (\mathcal{R}, fg \otimes h) = ((\hat{\Delta} \otimes \text{id})(\mathcal{R}), g \otimes f \otimes h)$$

and

$$r(f \otimes h_{(2)})r(g \otimes h_{(1)}) = (\mathcal{R}, f \otimes h_{(2)})(\mathcal{R}, g \otimes h_{(1)}) = (\mathcal{R}_{13}\mathcal{R}_{23}, g \otimes f \otimes h)$$

for all $f, g, h \in H$.

d) For any $y \in \hat{H}$ and f, g, h we compute

$$r(f_{(1)} \otimes g_{(1)})(y, g_{(2)}f_{(2)}) = (\mathcal{R}, f_{(1)} \otimes g_{(1)})(\hat{\Delta}(y), f_{(2)} \otimes g_{(2)}) = (\mathcal{R}\hat{\Delta}(y), f \otimes g)$$

and

$$(y, f_{(1)}g_{(1)})r(f_{(2)} \otimes g_{(2)}) = (\hat{\Delta}(y), g_{(1)} \otimes f_{(1)})(\mathcal{R}, f_{(2)} \otimes g_{(2)}) = (\hat{\Delta}^{\text{cop}}(y)\mathcal{R}, f \otimes g).$$

Since the pairing between \hat{H} and H is nondegenerate, the relation $\mathcal{R}\hat{\Delta}(x) = \hat{\Delta}^{\text{cop}}(x)\mathcal{R}$ for all $x \in \hat{H}$ is equivalent to

$$r(f_{(1)} \otimes g_{(1)})g_{(2)}f_{(2)} = f_{(1)}g_{(1)}r(f_{(2)} \otimes g_{(2)})$$

for all $f, g, h \in H$. □

2.11.2. *The R-matrix for $U_q(\mathfrak{g})$.* The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is not quite quasitriangular in the sense of Definition 2.103, but the multiplier Hopf algebra $\mathcal{D}(G_q)$ is. We assume that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

Define $\mathcal{O}(B_q^\pm)$ to be the image of $\mathcal{O}(G_q)$ under the natural projection $U_q(\mathfrak{g})^* \rightarrow U_q(\mathfrak{b}_\pm)^*$. Then there is a unique Hopf algebra structure on $\mathcal{O}(B_q^\pm)$ such that the projection maps $\mathcal{O}(G_q) \rightarrow \mathcal{O}(B_q^\pm)$ are Hopf algebra homomorphisms.

Recall the definition of the Drinfeld pairing from Definition 2.74.

Proposition 2.106. *The Drinfeld pairing τ induces isomorphisms $\iota_\pm : U_q(\mathfrak{b}_\pm) \rightarrow \mathcal{O}(B_q^\mp)$ of Hopf algebras, such that*

$$\iota_+(X)(Y) = \tau(\hat{S}(X), Y), \quad \iota_-(Y)(X) = \tau(X, Y),$$

for all $X \in U_q(\mathfrak{b}_+)$, $Y \in U_q(\mathfrak{b}_-)$.

Proof. By Theorem 2.92 τ is a nondegenerate skew-pairing, so the maps $\iota_\pm : U_q(\mathfrak{b}_\pm) \rightarrow U_q(\mathfrak{b}_\mp)^*$ are injective algebra homomorphisms.

To show that ι_+ is an isomorphism, we consider the direct sum decompositions

$$\begin{aligned} U_q(\mathfrak{b}_+) &= \bigoplus_{\beta \in \mathbf{P}^+, \mu \in \mathbf{P}} A_{\beta, \mu}, \\ \mathcal{O}(B_q^-) &= \bigoplus_{\beta \in \mathbf{P}^+, \mu \in \mathbf{P}} B_{\beta, \mu}, \end{aligned}$$

where

$$A_{\beta, \mu} = \hat{S}^{-1}(U_q(\mathfrak{n}_+)_{\beta} K_{\mu}) = \{\hat{S}^{-1}(XK_{\mu}) \mid X \in U_q(\mathfrak{n}_+)_{\beta}\}$$

and

$$B_{\beta, \mu} = \{\langle v^+ \mid \bullet \mid w^+ \rangle \mid w^+ \in V(\lambda)_{-\mu}, v^+ \in V(\lambda)_{\mu+\beta}^* \text{ for some } \lambda \in \mathbf{P}^+\}.$$

We claim that ι_+ restricts to an isomorphism between the finite dimensional subspaces $A_{\beta, \mu}$ and $B_{\beta, \mu}$ for each $\beta \in \mathbf{P}^+$, $\mu \in \mathbf{P}$.

Let $X \in U_q(\mathfrak{n}_+)_{\beta}$. Using Lemma 2.75 we see that for every $Y \in U_q(\mathfrak{n}_-)$ and $\nu \in \mathbf{P}$,

$$\begin{aligned} \iota_+(\hat{S}^{-1}(XK_{\mu}))(YK_{\nu}) &= \tau(XK_{\mu}, YK_{\nu}) = q^{-(\mu, \nu)}\tau(X, Y) \\ &= q^{-(\mu, \nu)}\tau(XK_{\mu}, Y) \\ &= q^{-(\mu, \nu)}\iota_+(\hat{S}^{-1}(XK_{\mu}))(Y) \end{aligned}$$

Therefore, by the orthogonality of the PBW basis (Theorem 2.90), the functional $\varphi = \iota_+(\hat{S}^{-1}(XK_{\mu})) \in U_q(\mathfrak{b}_-)^*$ satisfies

$$\varphi(YK_{\nu}) = \delta_{\beta\gamma}q^{-(\mu, \nu)}\varphi(Y) \quad \text{for every } Y \in U_q(\mathfrak{n}_-)_{-\gamma}, \gamma \in \mathbf{P}^+, \nu \in \mathbf{P}. \quad (2.1)$$

Conversely, any linear functional $\varphi \in U_q(\mathfrak{b}_-)^*$ satisfying the property (2.1) is completely determined by its restriction to $U_q(\mathfrak{n}_-)_{-\beta}$, so the orthogonality of the PBW basis implies that it is the image of a unique element $\hat{S}^{-1}(XK_\mu) \in A_{\beta,\mu}$.

Therefore, it suffices to show that the set of φ satisfying (2.1) agrees with the subspace of matrix coefficients $\langle v^+ | \bullet | w^+ \rangle$ with $w^+ \in V(\lambda)_{-\mu}$, $v^+ \in V(\lambda)_{\mu+\beta}^*$, as claimed. For such a matrix coefficient, we have

$$\begin{aligned} (YK_\nu, \langle v^+ | \bullet | w^+ \rangle) &= v^+(YK_\nu \cdot w^+) \\ &= q^{-(\nu,\mu)} v^+(Y \cdot w^+) = q^{-(\nu,\mu)} (Y, \langle v^+ | \bullet | w^+ \rangle), \end{aligned}$$

so that it satisfies (2.1). Conversely, let $\varphi \in U_q(\mathfrak{b}_-)^*$ satisfy (2.1). By Lemma 2.45 we can choose $\lambda \in \mathbf{P}^+$ sufficiently large such that, after fixing an arbitrary nonzero $w^+ \in V(\lambda)_{-\mu}$, the map

$$U_q(\mathfrak{n}_-)_{-\beta} \rightarrow V(\lambda)_{-\mu-\beta}, \quad Y \mapsto Y \cdot w^+$$

is injective. Therefore we can find $v^+ \in V(\lambda)_{\mu+\beta}^*$ such that

$$(Y, \langle v^+ | \bullet | w^+ \rangle) = \varphi(Y)$$

for every $Y \in U_q(\mathfrak{n}_-)_{-\beta}$. Since $\langle v^+ | \bullet | w^+ \rangle$ vanishes on $U_q(\mathfrak{n}_-)_{-\gamma}$ for all $\gamma \neq \beta$ by weight considerations, it follows that $\langle v^+ | \bullet | w^+ \rangle = \varphi$ as an element of $U_q(\mathfrak{b}_-)^*$. This completes the proof for ι_+ .

A similar argument shows that ι_- is an isomorphism.

Finally, using again the skew-pairing property of τ it is straightforward to check that the maps ι_\pm are compatible with the comultiplications. \square

Since $U_q(\mathfrak{b}_\pm) \cong \mathcal{O}(B_q^\mp)$, the Drinfeld pairing $\tau : U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-) \rightarrow \mathbb{K}$ induces a pairing $\mathcal{O}(G_q) \otimes \mathcal{O}(G_q) \rightarrow \mathbb{K}$ as follows, compare [29].

Proposition 2.107. *The linear map $r : \mathcal{O}(G_q) \otimes \mathcal{O}(G_q) \rightarrow \mathbb{K}$ given by*

$$\mathcal{O}(G_q) \otimes \mathcal{O}(G_q) \longrightarrow \mathcal{O}(B_q^-) \otimes \mathcal{O}(B_q^+) \cong U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_-) \xrightarrow{\tau} \mathbb{K}$$

is a universal r -form for $\mathcal{O}(G_q)$. Here the first arrow is the flip map composed with the canonical projection, and τ is the Drinfeld pairing. That is

$$r(f \otimes g) = \tau(\iota_+^{-1}(g), \iota_-^{-1}(f))$$

where, by slight abuse of notation, we identify f and g with their images in $\mathcal{O}(B_q^+)$ and $\mathcal{O}(B_q^-)$, respectively.

Proof. Since τ is a skew-pairing we see that r is a convolution invertible linear map satisfying conditions *a*) and *b*) of Definition 2.104.

It remains to verify condition *c*) of Definition 2.104. That is, we have to check

$$r(f_{(1)} \otimes g_{(1)})g_{(2)}f_{(2)} = f_{(1)}g_{(1)}r(f_{(2)} \otimes g_{(2)})$$

for all $f, g \in \mathcal{O}(G_q)$. Since the canonical pairing between $U_q(\mathfrak{g})$ and $\mathcal{O}(G_q)$ is nondegenerate, it suffices to show

$$r(f_{(1)} \otimes g_{(1)})(X, g_{(2)}f_{(2)}) = (X, f_{(1)}g_{(1)})r(f_{(2)} \otimes g_{(2)})$$

for all $X \in U_q(\mathfrak{g})$. Moreover, since $U_q(\mathfrak{g})$ is generated as a Hopf algebra by $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$, it suffices to consider $X \in U_q(\mathfrak{b}_\pm)$. Abbreviating $Z = \iota_-^{-1}(f) \in U_q(\mathfrak{b}_-)$

and taking $X \in U_q(\mathfrak{b}_+)$, we indeed compute

$$\begin{aligned}
r(f_{(1)} \otimes g_{(1)})(X, g_{(2)}f_{(2)}) &= \tau(\iota_+^{-1}(g_{(1)}), \iota_-^{-1}(f_{(1)}))\tau(X, \iota_-^{-1}(g_{(2)}f_{(2)})) \\
&= \tau(\iota_+^{-1}(g_{(1)}), \iota_-^{-1}(f_{(1)}))\tau(X_{(1)}, \iota_-^{-1}(f_{(2)}))\tau(X_{(2)}, \iota_-^{-1}(g_{(2)})) \\
&= \tau(\iota_+^{-1}(g_{(1)}), Z_{(1)})\tau(X_{(1)}, Z_{(2)})\tau(X_{(2)}, \iota_-^{-1}(g_{(2)})) \\
&= (\hat{S}(Z_{(1)}), g_{(1)})\tau(X_{(1)}, Z_{(2)})(X_{(2)}, g_{(2)}) \\
&= \rho(\hat{S}(Z_{(2)}), X_{(1)})(\hat{S}(Z_{(1)})X_{(2)}, g) \\
&= (X_{(1)}\hat{S}(Z_{(2)}), g)\rho(\hat{S}(Z_{(1)}), X_{(2)}) \\
&= \tau(X_{(2)}, Z_{(1)})(X_{(1)}, g_{(1)})(\hat{S}(Z_{(2)}), g_{(2)}) \\
&= \tau(X_{(2)}, Z_{(1)})\tau(X_{(1)}, \iota_-^{-1}(g_{(1)}))\tau(\iota_+^{-1}(g_{(2)}), Z_{(2)}) \\
&= \tau(X_{(2)}, \iota_-^{-1}(f_{(1)}))\tau(X_{(1)}, \iota_-^{-1}(g_{(1)}))\tau(\iota_+^{-1}(g_{(2)}), \iota_-^{-1}(f_{(2)})) \\
&= \tau(X, \iota_-^{-1}(f_{(1)}g_{(1)}))\tau(\iota_+^{-1}(g_{(2)}), \iota_-^{-1}(f_{(2)})) \\
&= (X, f_{(1)}g_{(1)})r(f_{(2)} \otimes g_{(2)}),
\end{aligned}$$

using Lemma 2.76. A similar argument works for $X \in U_q(\mathfrak{b}_-)$. This finishes the proof. \square

Let us now give an explicit formula for the universal r -form of $\mathcal{O}(G_q)$. We will use the notation $H_i \in \mathfrak{h}$ ($1 \leq i \leq N$) for the simple coroots of \mathfrak{g} , identified with $\alpha_i^\vee \in \mathbf{Q}^\vee$ via the pairing:

$$\lambda(H_i) = (\lambda, \alpha_i^\vee)$$

for all $\lambda \in \mathbf{P}$. We also fix quantum root vectors $E_{\beta_1}, \dots, E_{\beta_n}, F_{\beta_1}, \dots, F_{\beta_n}$ associated to a reduced expression of $w_0 \in W$, as in Definition 2.66.

Theorem 2.108. *The multiplier Hopf algebra $\mathcal{D}(G_q)$ has a universal R -matrix given by*

$$\mathcal{R} = q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)} \prod_{\alpha \in \Delta^+} \exp_{q_\alpha}((q_\alpha - q_\alpha^{-1})(E_\alpha \otimes F_\alpha))$$

where H_1, \dots, H_N are the simple coroots of \mathfrak{g} , the matrix (B_{ij}) is the inverse of $(C_{ij}) = (d_j^{-1}a_{ij})$, and $q_\alpha = q_i$ if α and α_i lie on the same W -orbit. Moreover the factors on the right hand side appear in the order β_1, \dots, β_n .

Equivalently, the above formula determines a universal r -form for $\mathcal{O}(G_q)$.

Proof. The formula requires a little explanation. Note that $C_{ij} = (\alpha_i^\vee, \alpha_j^\vee) = (H_i, H_j)$, so that

$$\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j) = \sum_{i=1}^N H_i \otimes \varpi_i = \sum_{i=1}^N \varpi_i \otimes H_i,$$

where we recall that the fundamental weights ϖ_i form the dual basis to the simple coroots H_i . The term $q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)}$ should thus be interpreted as the element in $\mathcal{M}(\mathcal{D}(G_q) \otimes \mathcal{D}(G_q))$ such that

$$q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)} \cdot (v \otimes w) = q^{(\mu, \nu)} v \otimes w,$$

whenever V and W are integrable $U_q(\mathfrak{g})$ -modules and $v \in V$, $w \in W$ have weight μ and ν respectively,

We shall show that the universal r -form obtained in Proposition 2.107 is given by the stated formula. It is technically more convenient however, to compute the inverse r^{-1} , given by

$$r^{-1}(f \otimes g) = \tau(\hat{S}(\iota_+^{-1}(g)), \iota_-^{-1}(f)) = \rho(\iota_-^{-1}(f), \iota_+^{-1}(g)).$$

Note that according to Lemma 2.80 the inverse of the element \mathcal{R} above is given by

$$\mathcal{R}^{-1} = \left(\prod_{\alpha \in \Delta^+} \exp_{q_\alpha^{-1}}(-(q_\alpha - q_\alpha^{-1})(E_\alpha \otimes F_\alpha)) \right) q^{-\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)},$$

where now the ordering of the positive roots in the product is reversed to β_n, \dots, β_1 .

To compute r^{-1} it is sufficient to consider $r^{-1}(f \otimes g)$ for elements of the form

$$f = \iota_-(F_{\beta_n}^{t_n} \cdots F_{\beta_1}^{t_1} K_\nu), \quad g = \iota_+(\hat{S}^{-1}(E_{\beta_n}^{s_n} \cdots E_{\beta_1}^{s_1} K_\mu)).$$

Using the orthogonality relations for the Drinfeld pairing in Lemma 2.91, we calculate

$$\begin{aligned} r^{-1}(f \otimes g) &= \tau(E_{\beta_n}^{s_n} \cdots E_{\beta_1}^{s_1}, F_{\beta_n}^{t_n} \cdots F_{\beta_1}^{t_1}) q^{-(\mu, \nu)} \\ &= \sum_{a_1, \dots, a_n=0}^{\infty} \tau(E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}, F_{\beta_n}^{a_n} \cdots F_{\beta_1}^{a_1})^{-1} \\ &\quad \times \tau(E_{\beta_n}^{s_n} \cdots E_{\beta_1}^{s_1}, F_{\beta_n}^{a_n} \cdots F_{\beta_1}^{a_1}) \tau(E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}, F_{\beta_n}^{t_n} \cdots F_{\beta_1}^{t_1}) q^{-(\mu, \nu)} \\ &= \sum_{a_1, \dots, a_n=0}^{\infty} \prod_{k=1}^n (-1)^{a_k} \frac{q^{-a_k(a_k-1)/2}}{[a_k]_{q_{\beta_k}}!} (q_{\beta_k} - q_{\beta_k}^{-1})^{a_k} \\ &\quad \times (F_{\beta_n}^{a_n} \cdots F_{\beta_1}^{a_1}, g) (E_{\beta_n}^{a_n} \cdots E_{\beta_1}^{a_1}, f) q^{-(\mu, \nu)} \\ &= \left(\prod_{\alpha \in \Delta^+} \exp_{q_\alpha^{-1}}(-(q_\alpha - q_\alpha^{-1})(E_\alpha \otimes F_\alpha)) \right) f \otimes g q^{-(\mu, \nu)}. \end{aligned}$$

Now, according to the proof of Proposition 2.106, we have

$$f = \langle v^- | \bullet | w^- \rangle, \quad g = \langle v^+ | \bullet | w^+ \rangle$$

where w^- has weight $-\nu$ and w^+ has weight $-\mu$. Therefore

$$f_{(1)} \otimes g_{(1)} (q^{-\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)}, f_{(2)} \otimes g_{(2)}) = q^{-(\mu, \nu)} f \otimes g,$$

and we conclude that

$$r^{-1}(f \otimes g) = (\mathcal{R}^{-1}, f \otimes g).$$

This completes the proof. \square

According to Theorem 2.108, we have in particular $\hat{\Delta}^{\text{cop}}(X) = \mathcal{R} \hat{\Delta}(X) \mathcal{R}^{-1}$ for all $X \in U_q(\mathfrak{g}) \subset \mathcal{M}(\mathcal{D}(G_q))$. Since $\mathcal{M}(\mathcal{D}(G_q))$ can be viewed as an algebraic completion of $U_q(\mathfrak{g})$ this makes the latter share most properties of Hopf algebras which are quasitriangular in the precise sense of definition 2.103. In the literature this is often phrased by saying that $U_q(\mathfrak{g})$ is quasitriangular, however note that the universal R -matrix \mathcal{R} from Theorem 2.108 is not contained in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$.

Let us next introduce l -functionals.

Definition 2.109. For $f \in \mathcal{O}(G_q)$, we define the l -functionals $l^\pm(f)$ on $\mathcal{O}(G_q)$ by

$$(l^+(f), h) = (\mathcal{R}, h \otimes f), \quad (l^-(f), h) = (\mathcal{R}^{-1}, f \otimes h).$$

Since \mathcal{R} is contained in $\mathcal{M}(\mathcal{D}(G_q) \otimes \mathcal{D}(G_q))$ these functionals can be naturally viewed as elements of $\mathcal{M}(\mathcal{D}(G_q))$. We can strengthen this observation as follows.

Lemma 2.110. For any $f \in \mathcal{O}(G_q)$ the l -functionals $l^\pm(f)$ are contained in $U_q(\mathfrak{g}) \subset \mathcal{M}(\mathcal{D}(G_q))$.

Proof. Consider the case where $g = \langle v | \bullet | w \rangle$, $h = \langle v' | \bullet | w' \rangle$ are matrix coefficients with $v \in V(\lambda)^*$, $w \in V(\lambda)$, $v' \in V(\lambda')^*$, $w' \in V(\lambda')$ for some $\lambda, \lambda' \in \mathbf{P}^+$. Suppose moreover that w and w' have weight μ and ν , respectively. As in the proof of Theorem 2.108, we calculate

$$(q^{\sum_{i,j} B_{ij}(H_i \otimes H_j)}, h \otimes g) = q^{(\mu, \nu)} \epsilon(h) \epsilon(g) = (\epsilon(g) K_\mu, h).$$

From this we easily deduce $l^+(f) \in U_q(\mathfrak{g})$ for any $f \in \mathcal{O}(G_q)$.

The proof for $l^-(f)$ is analogous. \square

The conclusion of Lemma 2.110 is one of the reasons to work with the *simply connected* version of $U_q(\mathfrak{g})$ with generators K_μ for all $\mu \in \mathbf{P}$. The corresponding assertion fails if one considers the version of the quantized universal enveloping algebra with Cartan part containing only elements K_μ for $\mu \in \mathbf{Q}$.

We record the following basic properties of l -functionals.

Lemma 2.111. *The maps $l^\pm : \mathcal{O}(G_q) \rightarrow U_q(\mathfrak{g})$ are Hopf algebra homomorphisms. Explicitly, we have*

$$\begin{aligned} l^\pm(fg) &= l^\pm(f)l^\pm(g) \\ \hat{\Delta}(l^\pm(f)) &= l^\pm(f_{(1)}) \otimes l^\pm(f_{(2)}) \\ \hat{S}(l^\pm(f)) &= l^\pm(S(f)) \end{aligned}$$

for all $f, g \in \mathcal{O}(G_q)$.

Proof. For $h \in \mathcal{O}(G_q)$ we compute

$$\begin{aligned} (l^+(fg), h) &= (\mathcal{R}, h \otimes fg) \\ &= ((\text{id} \otimes \hat{\Delta})(\mathcal{R}), h \otimes g \otimes f) \\ &= (\mathcal{R}_{13}\mathcal{R}_{12}, h \otimes g \otimes f) \\ &= (\mathcal{R}, h_{(1)} \otimes f)(\mathcal{R}, h_{(2)} \otimes g) \\ &= (l^+(f)l^+(g), h), \end{aligned}$$

and similarly

$$\begin{aligned} (\hat{\Delta}(l^+(f)), g \otimes h) &= ((\hat{\Delta} \otimes \text{id})(\mathcal{R}), g \otimes h \otimes f) \\ &= (\mathcal{R}_{13}\mathcal{R}_{23}, g \otimes h \otimes f) \\ &= (\mathcal{R}, g \otimes f_{(1)})(\mathcal{R}, h \otimes f_{(2)}) \\ &= (l^+(f_{(1)}) \otimes l^+(f_{(2)}), g \otimes h). \end{aligned}$$

The relation concerning the antipodes follows from these two formulas; explicitly we calculate

$$(\hat{S}(l^+(f)), h) = ((\hat{S} \otimes \text{id})(\mathcal{R}), h \otimes f) = ((\text{id} \otimes \hat{S}^{-1})(\mathcal{R}), h \otimes f) = (l^+(S(f)), h).$$

The assertions for l^- are obtained in a similar way. \square

2.12. The locally finite part of $U_q(\mathfrak{g})$. In this subsection we discuss some results on the locally finite part of $U_q(\mathfrak{g})$. For more details we refer to Section 5.3 in [41] and [8], [16]. Throughout we assume that $q = s^{2L} \in \mathbb{K}^\times$ is not a root of unity.

Recall that $U_q(\mathfrak{g})$ is a left module over itself with respect to the adjoint action given by

$$\text{ad}(X)(Y) = X \rightarrow Y = X_{(1)}Y\hat{S}(X_{(2)})$$

for $X, Y \in U_q(\mathfrak{g})$. The space of invariant elements of $U_q(\mathfrak{g})$ with respect to the adjoint action is the centre $ZU_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$.

The locally finite part $FU_q(\mathfrak{g})$ is the sum of all finite dimensional $U_q(\mathfrak{g})$ -submodules of $U_q(\mathfrak{g})$ with respect to the adjoint action. In contrast to the classical case it turns out that $FU_q(\mathfrak{g})$ is a proper subspace of $U_q(\mathfrak{g})$.

Lemma 2.112. *The locally finite part $FU_q(\mathfrak{g})$ is a subalgebra and a left coideal of $U_q(\mathfrak{g})$, that is, $\Delta(FU_q(\mathfrak{g})) \subset U_q(\mathfrak{g}) \otimes FU_q(\mathfrak{g})$.*

Proof. The fact that $FU_q(\mathfrak{g})$ is a subalgebra follows from the fact that the adjoint action is compatible with multiplication. Now let $Y \in FU_q(\mathfrak{g})$ and $X \in U_q(\mathfrak{g})$. Note that

$$\begin{aligned} Y_{(1)} \otimes (X \rightarrow Y_{(2)}) &= S(X_{(1)})X_{(2)}Y_{(1)}S(X_{(5)})X_{(6)} \otimes X_{(3)}Y_{(2)}S(X_{(4)}) \\ &= (S(X_{(1)}) \otimes 1)\Delta(X_{(2)} \rightarrow Y)(X_{(3)} \otimes 1). \end{aligned}$$

Since $U_q(\mathfrak{g}) \rightarrow Y$ is finite dimensional it follows that $\Delta(Y) \in U_q(\mathfrak{g}) \otimes FU_q(\mathfrak{g})$. \square

Recall that the anti-automorphism τ of $U_q(\mathfrak{g})$ defined in Lemma 2.16 satisfies

$$\begin{aligned} \tau(X \rightarrow Y) &= \tau(X_{(1)}Y\hat{S}(X_{(2)})) \\ &= \tau(\hat{S}(X_{(2)}))\tau(Y)\tau(X_{(1)}) \\ &= \tau(\hat{S}(X_{(2)}))\tau(Y)\hat{S}(\tau(\hat{S}(X_{(1)}))) \\ &= \tau(\hat{S}(X)) \rightarrow \tau(Y) \end{aligned}$$

for all $X, Y \in U_q(\mathfrak{g})$. In particular, the involution τ preserves the locally finite part $FU_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$.

More generally, for an arbitrary $U_q(\mathfrak{g})$ -module M we shall denote by $FM \subset M$ its locally finite part, that is, the subspace consisting of all elements m such that $U_q(\mathfrak{g}) \cdot m$ is finite dimensional. We shall say that the action of $U_q(\mathfrak{g})$ on M is locally finite if $FM = M$. Of course, this is the case in particular if $M = V(\mu)$ for $\mu \in \mathbf{P}^+$.

Let us also define the coadjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})^*$ by

$$(X \rightarrow f)(Y) = f(\hat{S}(X_{(1)})YX_{(2)})$$

for $X, Y \in U_q(\mathfrak{g})$ and $f \in U_q(\mathfrak{g})^*$. If $f = \langle v | \bullet | w \rangle \in \mathcal{O}(G_q)$ is a matrix coefficient, then

$$X \rightarrow \langle v | \bullet | w \rangle = \langle X_{(1)} \cdot v | \bullet | X_{(2)} \cdot w \rangle.$$

Therefore the coadjoint action preserves $\mathcal{O}(G_q)$ and is locally finite on $\mathcal{O}(G_q)$. Note that for $f \in \mathcal{O}(G_q)$,

$$(X \rightarrow f)(Y) = (Y, X \rightarrow f)$$

for all $X, Y \in U_q(\mathfrak{g})$.

We define a linear map $J : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^*$ by

$$J(X)(Y) = \kappa(\hat{S}^{-1}(Y), X)$$

where κ is the quantum Killing form, see Definition 2.78. Using J we can determine the structure of $FU_q(\mathfrak{g})$, compare Section 7.1 in [41] and [8].

Theorem 2.113. *The linear map J defines an isomorphism from $FU_q(\mathfrak{g})$ onto $\mathcal{O}(G_q)$ compatible with the adjoint and coadjoint actions, respectively. Moreover*

$$FU_q(\mathfrak{g}) = \bigoplus_{\mu \in \mathbf{P}^+} U_q(\mathfrak{g}) \rightarrow K_{2\mu}$$

as a subspace of $U_q(\mathfrak{g})$.

Proof. We follow the discussion in [16]. According to Theorem 2.92 the map $J : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})^*$ is injective. Moreover J intertwines the adjoint and coadjoint actions since by the \mathbf{ad} -invariance of κ from Proposition 2.79 we obtain

$$\begin{aligned} (X \rightarrow J(Y))(Z) &= J(Y)(\hat{S}(X_{(1)})ZX_{(2)}) \\ &= \kappa(\hat{S}^{-1}(X_{(2)})\hat{S}^{-1}(Z)X_{(1)}, Y) \\ &= \kappa(\hat{S}^{-1}(X) \rightarrow \hat{S}^{-1}(Z), Y) \\ &= \kappa(\hat{S}^{-1}(Z), X \rightarrow Y) \\ &= J(X \rightarrow Y)(Z) \end{aligned}$$

for all $X, Y, Z \in U_q(\mathfrak{g})$.

Let $\mu \in \mathbf{P}^+$ and let $w_0 \in W$ be the longest element of the Weyl group. Using the definition of the quantum Killing form we check that $J(K_{-2w_0\mu})$ vanishes on monomials $XK_\nu\hat{S}(Y)$ for which $X \in U_q(\mathfrak{n}_+)$ or $Y \in U_q(\mathfrak{n}_-)$ is contained in the kernel of $\hat{\varepsilon}$. With this in mind, if v^μ is a lowest weight vector of $V(\mu)$ and v_*^μ is the highest weight vector in $V(\mu)^*$ such that $v_*^\mu(v^\mu) = 1$, one can confirm that

$$J(K_{-2w_0\mu}) = \langle v_*^\mu | \bullet | v^\mu \rangle$$

by comparing both sides on $U_q(\mathfrak{h})$.

Since $V(\mu)$ is irreducible, the vector $\langle v_*^\mu | \bullet | v^\mu \rangle$ is cyclic for the coadjoint action. We conclude that J induces an isomorphism $U_q(\mathfrak{g}) \rightarrow K_{-2w_0\mu} \cong \text{End}(V(\mu))^*$. Since the spaces $\text{End}(V(\mu))^*$ form a direct sum in $\mathcal{O}(G_q) \subset U_q(\mathfrak{g})^*$ it follows that the sum of the spaces $U_q(\mathfrak{g}) \rightarrow K_{-2w_0\mu}$ is a direct sum, and the resulting subspace of $FU_q(\mathfrak{g})$ is isomorphic to $\mathcal{O}(G_q)$ via J .

To finish the proof it suffices to show that $J(FU_q(\mathfrak{g}))$ is contained in $\mathcal{O}(G_q)$. For this let $f \in J(FU_q(\mathfrak{g}))$ be arbitrary. Consider the left and right regular actions of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})^*$ given by

$$(X \cdot f)(Y) = f(YX), \quad (f \cdot X)(Y) = f(XY)$$

for $X, Y \in U_q(\mathfrak{g})$. Inspecting the definition of the quantum Killing form shows that $U_q(\mathfrak{b}_-) \cdot f$ and $f \cdot U_q(\mathfrak{b}_+)$ are finite dimensional subspaces of $U_q(\mathfrak{g})^*$. Moreover f is contained in the locally finite part of $U_q(\mathfrak{g})^*$ with respect to the coadjoint action, so the relation $X \cdot f = X_{(2)} \rightarrow (f \cdot X_{(1)})$ shows that $U_q(\mathfrak{b}_+) \cdot f$ is finite dimensional as well. Hence $U_q(\mathfrak{g}) \cdot f$ is finite dimensional.

From the definition of the quantum Killing form κ it follows that the linear functional $J(YK_\lambda\hat{S}(X)) \in U_q(\mathfrak{g})^*$ for $X \in U_q(\mathfrak{n}_+)_\nu, Y \in U_q(\mathfrak{n}_-)$ has weight $\nu - \frac{1}{2}\lambda \in \frac{1}{2}\mathbf{P} \subset \mathfrak{h}_q^*$ with respect to the left regular action. In particular, $J(U_q(\mathfrak{g}))$ is a weight module with respect to the left regular action, and therefore the same holds for $U_q(\mathfrak{g}) \cdot f$. According to Corollary 2.100, we conclude that $U_q(\mathfrak{g}) \cdot f$ is isomorphic to a finite direct sum of modules $V(\mu_j)$ for $\mu_1, \dots, \mu_r \in \mathbf{P}_q^+$, and since $\mathbf{P}_q^+ \cap \frac{1}{2}\mathbf{P} = \mathbf{P}^+$, we have in fact $\mu_1, \dots, \mu_r \in \mathbf{P}^+$. In particular, we get $X \cdot f = 0$ for any $X \in I(\mu_1) \cap \dots \cap I(\mu_r)$, the intersection of the kernels of the irreducible representations corresponding to μ_1, \dots, μ_r . Therefore $0 = (X \cdot f)(1) = f(X)$ for all such X , which means that f can be viewed as a linear functional on the quotient $U_q(\mathfrak{g})/(I(\mu_1) \cap \dots \cap I(\mu_r)) \cong \text{End}(V(\mu_1)) \oplus \dots \oplus \text{End}(V(\mu_r))$. In other words, we have $f \in \mathcal{O}(G_q)$ as desired. This finishes the proof. \square

As a consequence of Theorem 2.113 we obtain the following properties of $FU_q(\mathfrak{g})$.

Proposition 2.114. *The intersection of $U_q(\mathfrak{h})$ and $FU_q(\mathfrak{g})$ is linearly spanned by the elements K_λ for $\lambda \in 2\mathbf{P}^+$.*

Proof. Let us first show

$$E_j^r \rightarrow K_\lambda = \prod_{k=1}^r (1 - q^{(\lambda - (k-1)\alpha_j, \alpha_j)}) E_j^r K_{\lambda - r\alpha_j},$$

using induction on $r \in \mathbb{N}$. We have

$$E_j \rightarrow K_\lambda = [E_j, K_\lambda] K_j^{-1} = (E_j - K_\lambda E_j K_{-\lambda}) K_{\lambda - \alpha_j} = (1 - q^{(\lambda, \alpha_j)}) E_j K_{\lambda - \alpha_j},$$

and in the inductive step we compute

$$\begin{aligned}
 E_j^r \rightarrow K_\lambda &= [E_j, E_j^{r-1} \rightarrow K_\lambda] K_j^{-1} \\
 &= \prod_{k=1}^{r-1} (1 - q^{(\lambda - (k-1)\alpha_j, \alpha_j)}) (E_j E_j^{r-1} K_{\lambda - (r-1)\alpha_j} K_{-\alpha_j} - E_j^{r-1} K_{\lambda - (r-1)\alpha_j} E_j K_{-\alpha_j}) \\
 &= \prod_{k=1}^{r-1} (1 - q^{(\lambda - (k-1)\alpha_j, \alpha_j)}) (E_j^r K_{\lambda - r\alpha_j} - q^{(\lambda - (r-1)\alpha_j, \alpha_j)} E_j^r K_{\lambda - r\alpha_j}) \\
 &= \prod_{k=1}^r (1 - q^{(\lambda - (k-1)\alpha_j, \alpha_j)}) E_j^r K_{\lambda - r\alpha_j},
 \end{aligned}$$

so that the formula indeed holds.

Since q is not a root of unity it follows that $E_j^r \rightarrow K_\lambda = 0$ for some $r \in \mathbb{N}$ iff $(\alpha_j, \lambda) = n_j(\alpha_j, \alpha_j)$ for some integer $n_j \geq 0$. Writing $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_N \varpi_N$ this amounts to

$$\lambda_j = (\alpha_j^\vee, \lambda) = \frac{2}{(\alpha_j, \alpha_j)} (\alpha_j, \lambda) = 2n_j.$$

We conclude that $K_\lambda \in FU_q(\mathfrak{g})$ implies $\lambda \in 2\mathbf{P}^+$. In fact, the argument shows more generally that an element of $U_q(\mathfrak{h})$ which is contained in $FU_q(\mathfrak{g})$ is necessarily a linear combination of elements K_λ with $\lambda \in 2\mathbf{P}$.

Conversely, according to Theorem 2.113 the elements K_λ for $\lambda \in 2\mathbf{P}^+$ are indeed contained in $FU_q(\mathfrak{g})$. This finishes the proof. \square

Observe that Proposition 2.114 shows in particular that $FU_q(\mathfrak{g})$ is not a Hopf subalgebra of $U_q(\mathfrak{g})$.

The following result illustrates that the subalgebra $FU_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ is reasonably large. For more precise information in this direction see Section 7.1 in [41].

Lemma 2.115. *We have $U_q(\mathfrak{g}) = FU_q(\mathfrak{g})U_q(\mathfrak{h}) = U_q(\mathfrak{h})FU_q(\mathfrak{g})$.*

Proof. We first note that $K_\mu FU_q(\mathfrak{g}) K_{-\mu} = \text{ad}(K_\mu)(FU_q(\mathfrak{g})) = FU_q(\mathfrak{g})$ for all $\mu \in \mathbf{P}$, and hence $FU_q(\mathfrak{g})U_q(\mathfrak{h}) = U_q(\mathfrak{h})FU_q(\mathfrak{g})$. It follows that the latter is a subalgebra.

In order to finish the proof it therefore suffices to show that the generators E_i, F_i are contained in $FU_q(\mathfrak{g})U_q(\mathfrak{h}) = U_q(\mathfrak{h})FU_q(\mathfrak{g})$. From the computation in the proof of Proposition 2.114 we conclude that

$$E_j \rightarrow K_{2\rho} = (1 - q^{(2\rho, \alpha_j)}) E_j K_{2\rho - \alpha_j} = (1 - q_j^2) E_j K_{2\rho - \alpha_j}$$

is contained in $FU_q(\mathfrak{g})$, and hence $E_j \in FU_q(\mathfrak{g})U_q(\mathfrak{h})$. In the same way we see that

$$F_j \rightarrow K_{2\rho} = -K_{2\rho} F_j + F_j K_{2\rho} = (1 - q^{-(2\rho, \alpha_j)}) E_j K_{2\rho} = (1 - q_j^{-2}) E_j K_{2\rho}$$

is contained in $FU_q(\mathfrak{g})$, and hence $F_j \in FU_q(\mathfrak{g})U_q(\mathfrak{h})$. \square

Recall next the definition of the l -functionals $l^\pm(f)$ for $f \in \mathcal{O}(G_q)$ from Definition 2.109. We define a linear map $I : \mathcal{O}(G_q) \rightarrow U_q(\mathfrak{g})$ by

$$I(f) = l^-(f_{(1)}) l^+(S(f_{(2)})).$$

For the following result compare [8].

Proposition 2.116. *The map I induces an isomorphism $\mathcal{O}(G_q) \rightarrow FU_q(\mathfrak{g})$, and this isomorphism is the inverse of the isomorphism $J : FU_q(\mathfrak{g}) \rightarrow \mathcal{O}(G_q)$ constructed above.*

Proof. By the definition of the l -functionals we have

$$(I(f), g) = (\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}, f \otimes g)$$

for $f, g \in \mathcal{O}(G_q)$. Here we write $\mathcal{R}_{12} = \mathcal{R}$ and $\mathcal{R}_{21} = \sigma(\mathcal{R})$ where σ is the flip map. Using

$$\hat{\Delta}(X)\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1} = \mathcal{R}_{12}^{-1}\hat{\Delta}^{\text{cop}}(X)\mathcal{R}_{21}^{-1} = \mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}\hat{\Delta}(X)$$

for all $X \in U_q(\mathfrak{g})$ we compute

$$\begin{aligned} (I(X \rightarrow f), g) &= (\hat{S}(X_{(1)}), f_{(1)})(I(f_{(2)}), g)(X_{(2)}, f_{(3)}) \\ &= (\hat{S}(X_{(1)}), f_{(1)})(\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}, f_{(2)} \otimes g)(X_{(2)}, f_{(3)}) \\ &= (\hat{S}(X_{(1)}), f_{(1)})(\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}\hat{\Delta}(X_{(2)}), f_{(2)} \otimes g_{(1)})(\hat{S}(X_{(3)}), g_{(2)}) \\ &= (\hat{S}(X_{(1)}), f_{(1)})(\hat{\Delta}(X_{(2)})\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}, f_{(2)} \otimes g_{(1)})(\hat{S}(X_{(3)}), g_{(2)}) \\ &= (X_{(1)}, g_{(1)})(\mathcal{R}_{12}^{-1}\mathcal{R}_{21}^{-1}, f \otimes g_{(2)})(\hat{S}(X_{(2)}), g_{(3)}) \\ &= (X_{(1)}, g_{(1)})(I(f), g_{(2)})(\hat{S}(X_{(2)}), g_{(3)}) = (X \rightarrow I(f), g) \end{aligned}$$

for $f, g \in \mathcal{O}(G_q)$. Hence I is $U_q(\mathfrak{g})$ -linear with respect to the coadjoint and adjoint actions, respectively. This means in particular that the image of I is contained in $FU_q(\mathfrak{g})$ since the coadjoint action on $\mathcal{O}(G_q)$ is locally finite.

Let $v^\mu \in V(\mu)$ be a lowest weight vector and $v_*^\mu \in V(\mu)^*$ a highest weight vector such that $v_*^\mu(v^\mu) = 1$. From the explicit description of the R -matrix in Theorem 2.108 we see that $l^-(\langle v_*^\mu | \bullet | v \rangle) = (v_*^\mu, v)K_{-w_0\mu}$ for all $v \in V(\mu)$. Similarly, we obtain $l^+(\langle v^* | \bullet | v^\mu \rangle) = (v^*, v^\mu)K_{w_0\mu}$ for all $v^* \in V(\mu)^*$. Taking a basis v_1, \dots, v_n of $V(\mu)$ with dual basis $v_1^*, \dots, v_n^* \in V(\mu)^*$ this shows

$$\begin{aligned} I(\langle v_*^\mu | \bullet | v^\mu \rangle) &= \sum_{j=1}^n l^-(\langle v_*^\mu | \bullet | v_j \rangle) l^+(S(\langle v_j^* | \bullet | v^\mu \rangle)) \\ &= \sum_{j=1}^n (v_*^\mu, v_j)(v_j^*, v^\mu) K_{-2w_0\mu} = K_{-2w_0\mu}. \end{aligned}$$

Combining this with the results from Theorem 2.113 we conclude that I is indeed the inverse of J . In particular I is an isomorphism. \square

Let $g, f \in \mathcal{O}(G_q)$. Using that I is compatible with the adjoint and coadjoint actions and properties of l -functionals we compute

$$\begin{aligned} I(g)I(f) &= I(g)l^-(f_{(1)})l^+(S(f_{(2)})) \\ &= l^-(f_{(2)})(\hat{S}^{-1}(l^-(f_{(1)})) \rightarrow I(g))l^+(S(f_{(3)})) \\ &= l^-(f_{(2)})I(\hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g)l^+(S(f_{(3)})) \\ &= l^-(f_{(2)})l^-(\langle \hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g \rangle_{(1)})l^+(S(\langle \hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g \rangle_{(2)}))l^+(S(f_{(3)})) \\ &= l^-(f_{(2)})(\hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g)_{(1)}l^+(S(f_{(3)})(\hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g)_{(2)}) \\ &= I(f_{(2)}\hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g). \end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
 I(f)I(g) &= l^-(f_{(1)})l^+(S(f_{(2)}))I(g) \\
 &= l^-(f_{(1)})(l^+(S(f_{(3)})) \rightarrow I(g))l^+(S(f_{(2)})) \\
 &= l^-(f_{(1)})I(l^+(S(f_{(3)})) \rightarrow g)l^+(S(f_{(2)})) \\
 &= l^-(f_{(1)})l^-((l^+(S(f_{(3)})) \rightarrow g)_{(1)})l^+(S(l^+(S(f_{(3)})) \rightarrow g)_{(2)})l^+(S(f_{(2)})) \\
 &= l^-(f_{(1)})(l^+(S(f_{(3)})) \rightarrow g)_{(1)}l^+(S(f_{(2)})(l^+(S(f_{(3)})) \rightarrow g)_{(2)}) \\
 &= I(f_{(1)})\hat{S}^{-1}(l^+(f_{(2)})) \rightarrow g).
 \end{aligned}$$

for all $f, g \in \mathcal{O}(G_q)$.

In particular, the map $I : \mathcal{O}(G_q) \rightarrow FU_q(\mathfrak{g})$ is not an algebra homomorphism. However, assume that g is an element of the invariant part $\mathcal{O}(G_q)^{G_q}$ of $\mathcal{O}(G_q)$ with respect to the coadjoint action. Then the computation above shows that I satisfies the multiplicativity property

$$I(fg) = I(f)I(g) = I(g)I(f)$$

for all $f \in \mathcal{O}(G_q)$. In particular, we obtain the following.

Proposition 2.117. *The map I induces an algebra isomorphism between $\mathcal{O}(G_q)^{G_q}$ and $ZU_q(\mathfrak{g})$.*

2.13. The centre of $U_q(\mathfrak{g})$ and the Harish-Chandra homomorphism. In this section we describe the structure of the centre $ZU_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$. For more details we refer to [9]. Throughout we assume that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

Recall from Proposition 2.117 that we have an algebra isomorphism $\mathcal{O}(G_q)^{G_q} \cong ZU_q(\mathfrak{g})$. Therefore, we start by studying the algebra $\mathcal{O}(G_q)^{G_q}$.

Definition 2.118. Let V be a finite dimensional integrable $U_q(\mathfrak{g})$ -module V . The quantum trace $t_V \in \mathcal{O}(G_q)$ is defined by

$$t_V(X) = \text{tr}_V(XK_{-2\rho})$$

for all $X \in U_q(\mathfrak{g})$, where tr_V denotes the unnormalized trace on $\text{End}(V)$. If $V = V(\mu)$ for some $\mu \in \mathbf{P}^+$, we will write t_μ for $t_{V(\mu)}$.

Lemma 2.119. *The set of quantum traces t_μ for $\mu \in \mathbf{P}^+$ is a linear basis of $\mathcal{O}(G_q)^{G_q}$.*

Proof. First, one checks that an element $f \in \mathcal{O}(G_q)$ is invariant for the coadjoint action if and only if

$$(\hat{S}^2(X)Y, f) = (YX, f)$$

for all $X, Y \in U_q(\mathfrak{g})$. Recall from Lemma 2.7, that $\hat{S}^2(X) = K_{2\rho}XK_{-2\rho}$. Using this, and $f \cdot K_{2\rho} = (K_{2\rho}, f_{(1)})f_{(2)}$, we see that the above condition is equivalent to

$$(XY, f \cdot K_{2\rho}) = (K_{2\rho}XY, f) = (K_{2\rho}YX, f) = (YX, f \cdot K_{2\rho})$$

for all $X, Y \in U_q(\mathfrak{g})$. This holds if and only if $f \cdot K_{2\rho}$ is a linear combination of traces $\text{tr}_{V(\mu)}$. \square

Note that the quantum traces satisfy

$$t_{V \oplus W} = t_V + t_W, \quad t_{V \otimes W} = t_V t_W,$$

for all finite dimensional integrable modules V, W . It follows that $\mathcal{O}(G_q)^{G_q}$ is isomorphic to the polynomial algebra $\mathbb{K}[t_{\varpi_1}, \dots, t_{\varpi_N}]$.

For $\mu \in \mathbf{P}^+$ let us define the Casimir element $z_\mu \in U_q(\mathfrak{g})$ by $z_\mu = I(t_\mu)$. As a direct consequence of Proposition 2.117 and Lemma 2.119 we obtain the following result.

Theorem 2.120. *The centre $ZU_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$ is canonically isomorphic to the polynomial ring $\mathbb{K}[z_{\varpi_1}, \dots, z_{\varpi_N}]$. A linear basis of $ZU_q(\mathfrak{g})$ is given by the Casimir elements z_μ for $\mu \in \mathbf{P}^+$.*

Recall that the Harish-Chandra map is the linear map $\mathcal{P} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ defined by $\mathcal{P} = \hat{e} \otimes \text{id} \otimes \hat{e}$ with respect to the triangular decomposition $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+)$. Our next goal is the quantum analogue of Harish-Chandra's Theorem, which says that \mathcal{P} restricts to an algebra isomorphism of $ZU_q(\mathfrak{g})$ onto a subalgebra of $U_q(\mathfrak{h})$.

In order to specify this subalgebra, we will need to introduce a shifted Weyl group action on $U_q(\mathfrak{h})$. Note that there is a standard action of W on $U_q(\mathfrak{h})$ by algebra automorphisms, defined by

$$w(K_\mu) = K_{w\mu},$$

for $w \in W$ and $\mu \in \mathbf{P}$. The shifted version of this action is defined as follows.

Definition 2.121. Let γ be the algebra automorphism of $U_q(\mathfrak{h})$ defined by

$$\gamma(K_\mu) = q^{(\rho, \mu)} K_\mu \quad (\mu \in \mathbf{P}).$$

The shifted action of W on $U_q(\mathfrak{h})$ is the conjugate of the standard action above by γ . Specifically, the shifted action of $w \in W$ is given by

$$w.K_\mu = q^{(\rho, w\mu - \mu)} K_\mu.$$

for $w \in W$.

Note that the automorphism γ of $U_q(\mathfrak{h})$ induces a map γ^* on the group of characters of $U_q(\mathfrak{h})$ and hence on the parameter space \mathfrak{h}_q^* . Specifically, for the character χ_λ associated to $\lambda \in \mathfrak{h}_q^*$ we have

$$\gamma^*(\chi_\lambda)(K_\mu) = \chi_\lambda(\gamma(K_\mu)) = q^{(\lambda + \rho, \mu)}$$

for all $\mu \in \mathbf{P}$, so that γ^* acts on \mathfrak{h}_q^* by translation by ρ .

We also introduce the algebra homomorphism $\tau : \mathbb{K}[\mathbf{P}] \rightarrow U_q(\mathfrak{h})$ defined by

$$\tau(\mu) = K_{2\mu}$$

for $\mu \in \mathbf{P}$. Note that the subalgebra $\text{im}(\tau) \subset U_q(\mathfrak{h})$ is preserved by the shifted Weyl group action.

The following result appears in various frameworks in [70, 19, 42], see also [67].

Theorem 2.122. *The Harish-Chandra map $\mathcal{P} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ restricts to an algebra isomorphism $ZU_q(\mathfrak{g}) \cong \text{im}(\tau)^W$, the fixed point subalgebra of $\text{im}(\tau)$ under the shifted Weyl group action.*

Proof. It is equivalent to prove the the map $\gamma^{-1} \circ \mathcal{P}$ defines an algebra isomorphism of $ZU_q(\mathfrak{g})$ onto the fixed point subalgebra of $\text{im}(\tau)$ under the standard action of W .

Let $\mu \in \mathbf{P}^+$ and consider the central element z_μ . Fix a basis e_1^μ, \dots, e_n^μ of weight vectors for $V(\mu)$ with dual basis e_μ^1, \dots, e_μ^n and put $u_{ij}^\mu = \langle e_\mu^i | \bullet | e_j^\mu \rangle$ so that the quantum trace is

$$t_\mu = \sum_{j=1}^n (K_{-2\rho}, u_{jj}^\mu) u_{jj}^\mu.$$

Then

$$z_\mu = I(t_\mu) = \sum_{j,k} (K_{-2\rho}, u_{jj}^\mu) l^-(u_{jk}^\mu) l^+(S(u_{kj}^\mu)).$$

Note that $l^-(u_{rs}^\mu)$ is contained in $U_q(\mathfrak{b}_-)$, and similarly $l^+(S(u_{rs}^\mu))$ is contained in $U_q(\mathfrak{b}_+)$. Inspecting the explicit form of the universal R -matrix in Theorem 2.108 we get

$$\mathcal{P}(z_\mu) = \sum_{\nu \in \mathbf{P}(V(\mu))} q^{(-2\rho, \nu)} K_{-\nu} K_{-\nu} = \sum_{\nu \in \mathbf{P}(V(\mu))} q^{(\rho, -2\nu)} K_{-2\nu},$$

where ν runs over the set of weights $\mathbf{P}(V(\mu))$ of $V(\mu)$, counted with multiplicities. We obtain

$$\gamma^{-1} \circ \mathcal{P}(z_\mu) = \sum_{\nu \in \mathbf{P}(V(\mu))} K_{-2\nu},$$

which is obviously in $\text{im}(\tau)$ and invariant under the standard action of W . Moreover, for any $\lambda, \mu \in \mathbf{P}^+$ we have

$$\begin{aligned} (\gamma^{-1} \circ \mathcal{P}(z_\mu))(\gamma^{-1} \circ \mathcal{P}(z_\lambda)) &= \sum_{\substack{\nu' \in \mathbf{P}(V(\mu)), \\ \nu'' \in \mathbf{P}(V(\lambda))}} K_{-2\nu'} K_{-2\nu''} \\ &= \sum_{\nu \in \mathbf{P}(V(\mu) \otimes V(\lambda))} K_{-2\nu} \\ &= \gamma^{-1} \circ \mathcal{P}(z_\mu z_\lambda), \end{aligned}$$

so $\gamma \otimes \mathcal{P}$ is an algebra homomorphism.

Using an induction argument with respect to the partial ordering on \mathbf{P}^+ , one checks that the elements $\gamma^{-1} \circ \mathcal{P}(z_\mu)$ for $\mu \in \mathbf{P}^+$ form a basis of the fixed point subalgebra of $\text{im}(\tau)$ with respect to the standard action. Combining this with Theorem 2.120 finishes the proof. \square

Let us illustrate these considerations in the case of $U_q(\mathfrak{sl}(2, \mathbb{K}))$. In this case one can check directly that the Casimir element $z_{1/2}$ associated with the fundamental representation $V(1/2)$ is, up to a scalar multiple, given by

$$C = FE + \frac{qK^2 + q^{-1}K^{-2}}{(q - q^{-1})^2}.$$

Let us verify that C is in the centre of $U_q(\mathfrak{sl}(2, \mathbb{K}))$. We compute

$$\begin{aligned} EC &= [E, F]E + FEE + \frac{q^{-1}K^2E + qK^{-2}E}{(q - q^{-1})^2} \\ &= FEE + \frac{(q - q^{-1})K^2E + (q^{-1} - q)K^{-2}E}{(q - q^{-1})^2} + \frac{q^{-1}K^2E + qK^{-2}E}{(q - q^{-1})^2} \\ &= FEE + \frac{qK^2E + q^{-1}K^{-2}E}{(q - q^{-1})^2} = CE, \end{aligned}$$

similarly we get

$$\begin{aligned} CF &= F[E, F] + FFE + \frac{q^{-1}FK^2 + qFK^{-2}}{(q - q^{-1})^2} \\ &= FFE + \frac{(q - q^{-1})FK^2 + (q^{-1} - q)FK^{-2}}{(q - q^{-1})^2} + \frac{q^{-1}FK^2 + qFK^{-2}}{(q - q^{-1})^2} \\ &= FFE + \frac{qFK^2 + q^{-1}FK^{-2}}{(q - q^{-1})^2} = FC, \end{aligned}$$

and the relation $CK = KC$ is obvious.

Let us now return to the general theory. Consider a Verma module $M(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$. If $Z \in ZU_q(\mathfrak{g})$ and $v_\lambda \in M(\lambda)$ is the highest weight vector, then $Z \cdot v_\lambda$ is

again a highest weight vector, and therefore must be of the form $\xi_\lambda(Z)v_\lambda$, for some linear map $\xi_\lambda : ZU_q(\mathfrak{g}) \rightarrow \mathbb{K}$. Note that

$$Z \cdot v_\lambda = \mathcal{P}(Z) \cdot v_\lambda = \chi_\lambda(\mathcal{P}(Z))v_\lambda,$$

where χ_λ is the character given by $\chi_\lambda(K_\nu) = q^{(\lambda, \nu)}$ for all $\nu \in \mathbf{P}$, and therefore

$$\xi_\lambda = \chi_\lambda \circ \mathcal{P}.$$

Moreover, since v_λ is a cyclic vector for $M(\lambda)$ it follows that Z acts by multiplication by $\xi_\lambda(Z)$ on all of $M(\lambda)$. It is easy to check that $\xi_\lambda(YZ) = \xi_\lambda(Y)\xi_\lambda(Z)$ for all $Y, Z \in ZU_q(\mathfrak{g})$, which means that ξ_λ is a character.

Definition 2.123. Let $\lambda \in \mathfrak{h}_q^*$. The character $\xi_\lambda : ZU_q(\mathfrak{g}) \rightarrow \mathbb{K}$ defined above is called the central character associated with λ .

From the previous considerations it follows that if $M(\mu) \subset M(\lambda)$ is a submodule then $\xi_\mu = \xi_\lambda$.

We shall now introduce the notion of linkage, which is designed to analyze the structure of Verma modules. For a classical complex semisimple Lie algebra \mathfrak{g} , two Verma modules $M(\lambda)$ and $M(\mu)$ with highest weights $\lambda, \mu \in \mathfrak{h}^*$ have the same central character if and only if λ and μ are in the same orbit of the shifted Weyl group action, in which case they are called W -linked, see [37]. For $U_q(\mathfrak{g})$ the group W needs to be enlarged slightly to take into account the existence of elements in \mathfrak{h}_q^* of order 2.

Definition 2.124. We define $\mathbf{Y}_q = \{\zeta \in \mathfrak{h}_q^* \mid 2\zeta = 0\}$.

If characteristic of the ground field \mathbb{K} is not equal to 2, there is a canonical isomorphism

$$\mathbf{Y}_q \cong \mathbf{Q}^\vee / 2\mathbf{Q}^\vee$$

which associates to the class of $\alpha^\vee \in \mathbf{Q}^\vee$ the character $\zeta_{\alpha^\vee} \in \mathfrak{h}_q^*$ defined by

$$q^{(\zeta_{\alpha^\vee}, \mu)} = (-1)^{(\alpha^\vee, \mu)},$$

for $\mu \in \mathbf{Q}$. Here we are using the q -exponential notation for \mathfrak{h}_q^* from Definition 2.26. This formula is well-defined since the character ζ_{α^\vee} is trivial if $\alpha^\vee \in 2\mathbf{Q}^\vee$. Moreover, if $\zeta \in \mathfrak{h}_q^*$ is any element of order two then we have $q^{(\zeta, \mu)} = \pm 1$ for all $\mu \in \mathbf{P}$, which implies that there exists $\alpha^\vee \in \mathbf{Q}^\vee$ such that $\zeta = \zeta_{\alpha^\vee}$, as given by the above formula.

In the case where the ground field is \mathbb{C} , where we make the identification $\mathfrak{h}_q^* = \mathfrak{h}^* / i\hbar^{-1}\mathbf{Q}^\vee$, the natural identification is $\mathbf{Y}_q \cong \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee / i\hbar^{-1}\mathbf{Q}^\vee$.

Note that if \mathbb{K} is an exponential field and if there exists $z \in \mathbb{K}$ such that $q^z = -1$ then we have $\zeta_{\alpha^\vee} = z\alpha^\vee$. Here we are using the canonical map $\mathfrak{h}^* \rightarrow \mathfrak{h}_q^*$ described in Subsection 2.3.1.

Now we define the extended Weyl group, see Section 8.3.2 in [41]. Note that the action of the Weyl group W on \mathfrak{h}_q^* restricts to \mathbf{Y}_q .

Definition 2.125. The extended Weyl group \hat{W} is defined as the semidirect product

$$\hat{W} = \mathbf{Y}_q \rtimes W$$

with respect to the action of W on \mathbf{Y}_q .

Observe that the extended Weyl group is a finite group. Explicitly, the group structure of \hat{W} is

$$(\zeta, v)(\eta, w) = (\zeta + v\eta, vw)$$

for all $\zeta, \eta \in \mathbf{Y}_q$ and $v, w \in W$.

We define two actions of \hat{W} on \mathfrak{h}_q^* by

$$(\zeta, w)\lambda = w\lambda + \zeta$$

and

$$(\zeta, w).\lambda = w.\lambda + \zeta = w(\lambda + \rho) - \rho + \zeta,$$

for $\lambda \in \mathfrak{h}_q^*$. The latter is called the shifted action of \hat{W} on \mathfrak{h}_q^* .

Definition 2.126. We say that $\mu, \lambda \in \mathfrak{h}_q^*$ are \hat{W} -linked if $\hat{w}.\lambda = \mu$ for some $\hat{w} \in \hat{W}$.

Notice that \hat{W} -linkage is an equivalence relation on \mathfrak{h}_q^* . This should be compared to the usual notion of W -linkage, see [37]. Indeed, the two notions coincide if we restrict our attention to integral weights.

Lemma 2.127. *Two integral weights $\mu, \lambda \in \mathbf{P} \subset \mathfrak{h}_q^*$ are \hat{W} -linked iff they are W -linked.*

Proof. If $\mu, \lambda \in \mathbf{P}$ with $(\zeta, w).\lambda = w.\lambda + \zeta = \mu$ then $\zeta \in \mathbf{P} \subset \mathfrak{h}_q^*$. But then for all $\nu \in \mathbf{P}$, we have $q^{(\zeta, \nu)} \in s^{\mathbb{Z}}$ and also $q^{(\zeta, \nu)} = \pm 1$. Since q is not a root of unity, this implies that $\zeta = 0$. \square

We shall now state and prove the following analogue of Harish-Chandra's Theorem, compare Section 1.10 in [37].

Theorem 2.128. *Two elements $\mu, \lambda \in \mathfrak{h}_q^*$ are \hat{W} -linked iff $\xi_\mu = \xi_\lambda$.*

Proof. Assume first that λ and $\mu = \hat{w}.\lambda$ are linked, where $\hat{w} = (\zeta, w) \in \hat{W}$. In order to show $\xi_{\hat{w}.\lambda} = \xi_\lambda$ it suffices due to Theorem 2.122 to show $\chi_{\hat{w}.\lambda} = \chi_\lambda$, where both sides are viewed as characters on the fixed point subalgebra $\text{im}(\tau)^W$ under the shifted Weyl group action. If

$$K = \sum_{\nu \in \mathbf{P}} c_\nu K_{2\nu} = \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, 2w\nu - 2\nu)} K_{2w\nu}$$

is contained in $\text{im}(\tau)^W$ then we obtain

$$\begin{aligned} \chi_\mu(K) &= \chi_{w.\lambda + \zeta}(K) = \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, 2w\nu - 2\nu)} q^{(w.\lambda + \zeta, 2w\nu)} \\ &= \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, 2w\nu - 2\nu)} q^{(w.\lambda, 2w\nu)} \\ &= \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, 2w\nu - 2\nu)} q^{(w(\lambda + \rho) - \rho, 2w\nu)} \\ &= \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, -2\nu)} q^{(w(\lambda + \rho), 2w\nu)} \\ &= \sum_{\nu \in \mathbf{P}} c_\nu q^{(\rho, -2\nu)} q^{(\lambda + \rho, 2\nu)} \\ &= \sum_{\nu \in \mathbf{P}} c_\nu q^{(\lambda, 2\nu)} = \chi_\lambda(K) \end{aligned}$$

as desired.

Let us now assume that μ and λ are not \hat{W} -linked. This implies that the shifted Weyl group orbits $W.(2\mu)$ and $W.(2\lambda)$ are disjoint. Let us write $2W.\mu \cup 2W.\lambda = \{2\eta_1, \dots, 2\eta_m\} \subset \mathfrak{h}_q^*$. Each element of this set corresponds to a character $\mathbf{P} \rightarrow \mathbb{K}^\times$ sending $\nu \in \mathbf{P}$ to $q^{(2\eta_i, \nu)}$. Therefore Artin's Theorem on linear independence of

characters allows us to find elements $\nu_1, \dots, \nu_n \in \mathbf{P}$ such that the matrix $(q^{(2\eta_i, \nu_j)})$ is invertible, and hence scalars $c_1, \dots, c_n \in \mathbb{K}$ such that

$$\sum_{j=1}^n c_j q^{(\eta_i, 2\nu_j)} = \begin{cases} 1 & \text{if } \eta_i \in W.\mu \\ 0 & \text{if } \eta_i \in W.\lambda. \end{cases}$$

If we write $K = \sum_j c_j K_{2\nu_j}$ then $L = \frac{1}{|W|} \sum_{w \in W} w.K$ is contained in $\text{im}(\tau)^W$. Moreover, since

$$\begin{aligned} \chi_{\eta_i}(w.K) &= \sum_j c_j \chi_{\eta_i}(w.K_{2\nu_j}) \\ &= \sum_j c_j q^{(\rho, 2w\nu_j - 2\nu_j)} q^{(\eta_i, 2w\nu_j)} \\ &= \sum_j c_j q^{(w^{-1}(\eta_i + \rho) - \rho, 2\nu_j)} = \sum_j c_j q^{(w^{-1}.\eta_i, 2\nu_j)} = \chi_{w^{-1}.\eta_i}(K) \end{aligned}$$

we have $\chi_{\eta_i}(L) = 1$ for $\eta_i \in W.\mu$ and $\chi_{\eta_i}(L) = 0$ for $\eta_i \in W.\lambda$. Taking the preimage $Z \in ZU_q(\mathfrak{g})$ of $L \in \text{im}(\tau)^W$ under the Harish-Chandra isomorphism therefore gives $\xi_\lambda(Z) = 1$ and $\xi_\mu(Z) = 0$. Hence $\xi_\mu \neq \xi_\lambda$. \square

Let us conclude this subsection with the following result on the structure of characters of $ZU_q(\mathfrak{g})$.

Proposition 2.129. *Assume that \mathbb{K} is an algebraically closed field. Then any character $\chi : ZU_q(\mathfrak{g}) \rightarrow \mathbb{K}$ is of the form ξ_λ for some $\lambda \in \mathfrak{h}_q^*$.*

Proof. Using Theorem 2.122 we can identify $ZU_q(\mathfrak{g})$ with the W -invariant part A^W of the Laurent polynomial ring $A = \text{im } \tau = \mathbb{K}[K_{2\varpi_1}^{\pm 1}, \dots, K_{2\varpi_N}^{\pm 1}]$, where W is acting by the shifted Weyl group action. It suffices to show that any character $\chi : A^W \rightarrow \mathbb{K}$ is of the form $\chi = \chi_\lambda$ for $\lambda \in \mathfrak{h}_q^*$.

Since W is a finite group the ring extension $A^W \subset A$ is integral, that is, each element of A is the root of a monic polynomial with coefficients in A^W . In fact, given $f \in A$ we may consider $p(x) = \prod_{w \in W} (x - w.f)$.

To proceed further we need to invoke some commutative algebra for integral ring extensions. More precisely, due to Theorem 5.10 in [4] the maximal ideal $\ker(\chi)$ is of the form $\ker(\chi) = A^W \cap \mathfrak{p}$ for some prime ideal \mathfrak{p} of A , and Corollary 5.8 in [4] implies that \mathfrak{p} is a maximal ideal. Since \mathbb{K} is algebraically closed we conclude that $\mathfrak{p} = \ker(\eta)$ for some character $\eta : A \rightarrow \mathbb{K}$ extending χ .

Using once again that \mathbb{K} is algebraically closed, we can find $\lambda \in \mathfrak{h}_q^*$ such that $\eta(K_\mu) = q^{(\mu, \lambda)} = \chi_\lambda(K_\mu)$ for all elements $K_\mu \in A$. Clearly this yields $\chi = \chi_\lambda$ as desired. \square

2.14. Noetherianity. In this section we show that $U_q(\mathfrak{g})$ and some related algebras are Noetherian. We assume throughout that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

2.14.1. Noetherian algebras. Recall that an algebra A is called left (right) Noetherian if it satisfies the ascending chain condition (ACC) for left (right) ideals. The ACC says that any ascending chain $I_1 \subset I_2 \subset I_3 \subset \dots$ of left (right) ideals of A becomes eventually constant, that is, satisfies $I_n = I_{n+1} = I_{n+2} = \dots$ for some $n \in \mathbb{N}$. The algebra A is called Noetherian if it is both left and right Noetherian.

We will have to work with graded and filtered algebras. Let A be an algebra and P an additive abelian semigroup, and assume that there exists a direct sum decomposition

$$A = \bigoplus_{\mu \in P} A_\mu$$

into linear subspaces A_μ such that $1 \in A_0$ and $A_\mu A_\nu \subset A_{\mu+\nu}$ for all $\mu, \nu \in P$. In this case we call A a graded algebra. A left (right) ideal I of a graded algebra A is called graded if $I = \bigoplus_{\mu \in P} I_\mu$ where $I_\mu = A_\mu \cap I$.

Assume now in addition that the abelian semigroup P is ordered, that is, equipped with an order relation \leq such that $\mu \leq \nu$ implies $\mu + \lambda \leq \nu + \lambda$ for all $\mu, \nu, \lambda \in P$. We shall also assume that P has cancellation, so that $\mu + \lambda = \nu + \lambda$ for some $\lambda \in P$ implies $\mu = \nu$. A P -filtration on an algebra A is a family of linear subspaces $\mathcal{F}^\mu(A) \subset A$ indexed by $\mu \in P$ such that $1 \in \mathcal{F}^0(A)$, $\mathcal{F}^\mu(A) \subset \mathcal{F}^\nu(A)$ if $\mu \leq \nu$,

$$\bigcup_{\mu \in P} \mathcal{F}^\mu(A) = A,$$

and $\mathcal{F}^\mu(A)\mathcal{F}^\nu(A) \subset \mathcal{F}^{\mu+\nu}(A)$ for all $\mu, \nu \in P$. In this case we also say that A is a filtered algebra. If A is a filtered algebra, then the associated graded algebra is

$$gr_{\mathcal{F}}(A) = \bigoplus_{\mu \in P} \mathcal{F}^\mu(A)/\mathcal{F}^{<\mu}(A),$$

where

$$\mathcal{F}^{<\mu}(A) = \sum_{\nu < \mu} \mathcal{F}^\nu(A),$$

and we write $\nu < \mu$ if $\nu \leq \mu$ and $\nu \neq \mu$. The associated graded algebra is indeed a graded algebra in a natural way with graded subspaces $gr_{\mathcal{F}}(A)_\mu = \mathcal{F}^\mu(A)/\mathcal{F}^{<\mu}(A)$ labelled by P .

Recall that P is called well-founded if every subset of P contains a minimal element. This is equivalent to the descending chain condition: any infinite sequence $\mu_1 \geq \mu_2 \geq \dots$ in P is eventually constant.

We say that a filtration $\mathcal{F}^\mu(A)$ of an algebra A is locally bounded below if for each nonzero element $a \in A$ the set of all $\mu \in P$ with $a \in \mathcal{F}^\mu(A)$ has a minimal element. Recall also that an algebra A is called a domain if it has no zero-divisors, that is, if $ab = 0$ in A implies $a = 0$ or $b = 0$.

Let us record the following basic fact, compare Appendix I.12 in [15] and Chapter V in [38].

Lemma 2.130. *Let A be a filtered algebra with a filtration $(\mathcal{F}^\mu(A))_{\mu \in P}$ with respect to an abelian semigroup P . Then the following holds.*

- a) *Suppose P is well-founded. If the associated graded algebra $gr_{\mathcal{F}}(A)$ is left (right) Noetherian then A is left (right) Noetherian.*
- b) *Suppose P is totally ordered and the filtration is locally bounded below. If $gr_{\mathcal{F}}(A)$ is a domain then A is a domain.*

Proof. a) We shall consider left ideals only, the proof for right ideals is completely analogous.

Note that if $I \subset A$ is a left ideal then setting $\mathcal{F}^\mu(I) = I \cap \mathcal{F}^\mu(A)$ for $\mu \in P$ defines a filtration of I , with associated graded

$$gr_{\mathcal{F}}(I) = \bigoplus_{\mu \in P} gr_{\mathcal{F}}(I)_\mu = \bigoplus_{\mu \in P} \mathcal{F}^\mu(I)/\mathcal{F}^{<\mu}(I).$$

There is a canonical inclusion of $\mathcal{F}^\mu(I)/\mathcal{F}^{<\mu}(I)$ into $\mathcal{F}^\mu(A)/\mathcal{F}^{<\mu}(A)$, and in this way $gr_{\mathcal{F}}(I)$ becomes a graded left ideal of $gr_{\mathcal{F}}(A)$.

Assume now that $I_1 \subset I_2 \subset \dots$ is an increasing chain of left ideals in A . Then the associated graded left ideals fit into an ascending chain $gr_{\mathcal{F}}(I_1) \subset gr_{\mathcal{F}}(I_2) \subset \dots$. By assumption, this chain stabilizes eventually, so that $gr_{\mathcal{F}}(I_n) = gr_{\mathcal{F}}(I_{n+1}) = \dots$ for some $n \in \mathbb{N}$.

We shall show that in general if $I \subset J$ are left ideals of A such that $gr_{\mathcal{F}}(I) = gr_{\mathcal{F}}(J)$ then $I = J$. This will clearly imply that our original chain $I_1 \subset I_2 \subset \dots$ is eventually constant.

Suppose $I \neq J$. Since P is well-founded, we can find $\mu \in P$ minimal such that $\mathcal{F}^\mu(I) \neq \mathcal{F}^\mu(J)$. Choose $a \in \mathcal{F}^\mu(J) \setminus \mathcal{F}^\mu(I)$. Since $gr_{\mathcal{F}}(I) = gr_{\mathcal{F}}(J)$, there exists $b \in \mathcal{F}^\mu(I)$ such that $b - a \in \mathcal{F}^{<\mu}(J) = \mathcal{F}^{<\mu}(I)$. But then $a = b + (a - b) \in \mathcal{F}^\mu(I)$, contradicting the choice of a . Therefore $I = J$ as claimed.

b) Assume $a, b \in A$ are nonzero elements such that $ab = 0$. Since P is locally bounded below we may pick $\mu, \nu \in P$ minimal such that $a \in \mathcal{F}^\mu(A)$ and $b \in \mathcal{F}^\nu(A)$. The corresponding cosets in $gr_{\mathcal{F}}(A)$ multiply to zero, so that $a + \mathcal{F}^{<\mu}(A) = 0$ or $b + \mathcal{F}^{<\nu}(A) = 0$ in $gr_{\mathcal{F}}(A)$. This means $a \in \mathcal{F}^{<\mu}(A)$ or $b \in \mathcal{F}^{<\nu}(A)$. Since P is totally ordered this implies $a \in \mathcal{F}^\lambda(A)$ for some $\lambda < \mu$ or $b \in \mathcal{F}^\eta(A)$ for some $\eta < \nu$, contradicting the choice of μ and ν . Hence A is a domain. \square

We remark that in the proof of part a) it suffices to assume that $gr_{\mathcal{F}}(A)$ is Noetherian for graded ideals. It follows that a graded algebra A is Noetherian if and only if it is Noetherian for graded ideals. For we may equip A with the filtration \mathcal{F} given by

$$\mathcal{F}^\mu(A) = \bigoplus_{\nu \leq \mu} A_\nu,$$

so that $A \cong gr_{\mathcal{F}}(A)$ as graded algebras. Then the proof of Lemma 2.130 a) implies that A is left Noetherian if $gr_{\mathcal{F}}(A) = A$ is Noetherian for graded left ideals. Conversely, if A is left Noetherian then it is clearly also Noetherian for graded left ideals.

Let A be an algebra and let $\theta \in \text{Aut}(A)$ be an algebra automorphism. Given θ we can form the semigroup crossed product $A \rtimes_{\theta} \mathbb{N}_0$, which has $A \otimes \mathbb{K}[t]$ as underlying vector space, with elements written as $a \rtimes t^m = a \times t^m$, and multiplication given by

$$(a \times t^m)(b \times t^n) = a\theta^m(b) \times t^{m+n}$$

for $a, b \in A$ and $m, n \in \mathbb{N}_0$.

Lemma 2.131. *Let A be an algebra, let $\theta \in \text{Aut}(A)$ and let $A \rtimes_{\theta} \mathbb{N}_0$ be the associated semigroup crossed product.*

- a) *If A is left (right) Noetherian then $A \rtimes_{\theta} \mathbb{N}_0$ is left (right) Noetherian.*
b) *If A is a domain then $A \rtimes_{\theta} \mathbb{N}_0$ is a domain.*

Proof. a) The argument is a variant of the Hilbert Basis Theorem, for a proof see Theorem 1.2.9 in [59].

b) Let $a = \sum_{i=0}^m a_i \times t^i, b = \sum_{j=0}^n b_j \times t^j$ be nonzero elements of $A \rtimes_{\theta} \mathbb{N}_0$, and assume without loss of generality that a_m and b_n are nonzero. Observe that

$$ab = \sum_{i=0}^m \sum_{j=0}^n a_i \theta^i(b_j) \times t^{i+j}.$$

Since θ is an automorphism we have $\theta^k(b_n) \neq 0$ for all $k \in \mathbb{N}_0$. However, $ab = 0$ implies $a_m \theta^m(b_n) = 0$, contradicting our assumption that A is a domain. It follows that $A \rtimes_{\theta} \mathbb{N}_0$ is a domain. \square

Let us call an algebra A a skew-polynomial algebra if it is generated by finitely many elements y_1, \dots, y_m and relations

$$y_i y_j = q_{ij} y_j y_i,$$

where $q_{ij} \in \mathbb{K}^\times$ are invertible scalars for all $1 \leq i, j \leq m$ with $q_{ij} = q_{ji}^{-1}$ and $q_{ii} = 1$ for all i, j .

Lemma 2.132. *Any skew-polynomial algebra is a Noetherian domain.*

Proof. We use induction on the number of generators. For $m = 0$ we have that $A = \mathbb{K}$, so that the assertion is obvious. Assume now that the claim has been proved if $k < m$ for some $m \in \mathbb{N}$, and let y_1, \dots, y_m be generators of A as above. Then the subalgebra $B \subset A$ generated by y_1, \dots, y_{m-1} is a skew-polynomial algebra and hence a Noetherian domain by the inductive hypothesis. Moreover, it is easy to see that $\theta(y_j) = q_{mj}y_j$ defines an algebra automorphism of B . One checks that the corresponding semigroup crossed $B \rtimes_{\theta} \mathbb{N}_0$ is isomorphic to A . Hence the assertion follows from Lemma 2.131. \square

The following result is Proposition I.8.17 in [15], which we reproduce here for the convenience of the reader.

Proposition 2.133. *Let A be an algebra generated by elements u_1, \dots, u_m , and assume that there are scalars $q_{ij} \in \mathbb{K}^\times$ and $\alpha_{ij}^{st}, \beta_{ij}^{st} \in \mathbb{K}$ such that*

$$u_i u_j - q_{ij} u_j u_i = \sum_{s=1}^{j-1} \sum_{t=1}^m \alpha_{ij}^{st} u_s u_t + \beta_{ij}^{st} u_t u_s$$

for all $1 \leq j < i \leq m$. Then A is a Noetherian.

Proof. The idea is to construct a filtration $\mathcal{F}^n(A)$ for $n \in \mathbb{N}_0$ of A such that $gr_{\mathcal{F}}(A)$ is generated by elements y_1, \dots, y_m such that $y_i y_j = q_{ij} y_j y_i$ for all $1 \leq i, j \leq m$.

To this end let

$$d_i = 2^m - 2^{m-i}$$

for $i = 1, \dots, m$, so that $d_1 < d_2 < \dots < d_m$. For $1 < j < m$ we have

$$2^{m-j-1} + 2^{m-j} = (1 + 2) \cdot 2^{m-j-1} < 4 \cdot 2^{m-j-1} < 2^{m-j+1} + 1,$$

and hence

$$\begin{aligned} d_{j-1} + d_m &= 2^m - 2^{m-j+1} + 2^m - 1 \\ &= 2^{m+1} - (2^{m-j+1} + 1) < 2^{m+1} - (2^{m-j-1} + 2^{m-j}) = d_{j+1} + d_j. \end{aligned}$$

We thus get for $i > j > s$ and any t the relation

$$d_s + d_t \leq d_{j-1} + d_m < d_{j+1} + d_j \leq d_i + d_j,$$

using the fact that $d_s \leq d_{j-1}$ and $d_t \leq d_m$ in the first step.

Define $\mathcal{F}^0(A) = \mathbb{K}1$ and let $\mathcal{F}^d(A)$ for $d > 0$ be the linear subspace spanned by all monomials $u_{i_1} \cdots u_{i_r}$ such that $d_{i_1} + \dots + d_{i_r} \leq d$. It is straightforward to check that this defines an \mathbb{N}_0 -filtration on A . Let y_i be the coset of u_i in $gr_{\mathcal{F}}(A)$. Each nonzero homogeneous component of $gr_{\mathcal{F}}(A)$ is spanned by the cosets of those monomials $u_{i_1} \cdots u_{i_r}$ such that $d_{i_1} + \dots + d_{i_r} = d$. In particular, the elements y_1, \dots, y_m generate $gr_{\mathcal{F}}(A)$ as an algebra. Since $u_i u_j - q_{ij} u_j u_i$ for $i > j$ is a linear combination of monomials $u_s u_t$ and $u_t u_s$ with $s < j$, and all such monomials have filtration degree strictly smaller than $d_i + d_j$, we see that

$$y_i y_j - q_{ij} y_j y_i = 0$$

in $gr_{\mathcal{F}}(A)$. That is, $gr_{\mathcal{F}}(A)$ is a quotient of a skew-polynomial algebra, and according to Lemma 2.132 this means that $gr_{\mathcal{F}}(A)$ is Noetherian. Due to Lemma 2.130 this implies that A itself is a Noetherian. \square

2.14.2. Noetherianity of $U_q(\mathfrak{g})$. In this subsection we use the PBW-basis to show that $U_q(\mathfrak{g})$ and a few other algebras we have encountered in our study of quantized universal enveloping algebras so far are Noetherian. These results are originally due to de Concini-Kac, see Section 1 in [20].

Proposition 2.134. *The algebras $U_q(\mathfrak{n}_{\pm}), U_q(\mathfrak{b}_{\pm})$ and $U_q(\mathfrak{g})$ are Noetherian domains.*

Proof. We will prove the claim for $A = U_q(\mathfrak{g})$, reproducing the argument in I.6 of [15]. The other cases are treated in a similar manner. Let us remark that the characteristic zero assumption made in [15] is not needed.

We have the PBW-basis vectors

$$X(\mathbf{t}, \mu, \mathbf{s}) = F_{\beta_n}^{t_n} \cdots F_{\beta_1}^{t_1} K_\mu E_{\beta_n}^{s_n} \cdots E_{\beta_1}^{s_1}$$

where $\mathbf{t} = (t_1, \dots, t_n)$, $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{N}_0^n$. If $\nu = \sum \nu_i \alpha_i \in \mathbf{Q}$ then we define the height of ν by $ht(\nu) = \nu_1 + \cdots + \nu_N \in \mathbb{Z}$. Let us define moreover the height of $X(\mathbf{t}, \mu, \mathbf{s})$ by

$$ht(X(\mathbf{t}, \mu, \mathbf{s})) = (t_1 + s_1)ht(\beta_1) + \cdots + (t_N + s_N)ht(\beta_N),$$

so that the height of this vector is the difference of the heights of its graded pieces in $U_q(\mathfrak{n}_\pm)$. We define the total degree of a PBW-basis vector by

$$d(X(\mathbf{t}, \mu, \mathbf{s})) = (s_n, \dots, s_1, t_n, \dots, t_1, ht(X(\mathbf{t}, \mu, \mathbf{s}))) \in \mathbb{N}_0^{2n+1}.$$

Let us give the additive semigroup $P = \mathbb{N}_0^{2n+1}$ the lexicographical ordering from right to left, so that $e_1 < e_2 < \cdots < e_{2n+1}$, where $e_j \in \mathbb{N}_0^{2n+1}$ denotes the j -th standard basis vector. This turns P into a well-ordered abelian semigroup, in particular P is totally ordered and well-founded. Moreover let $\mathcal{F}^{\mathbf{m}}(A) \subset A$ for $\mathbf{m} \in \mathbb{N}_0^{2n+1}$ be the linear span of all PBW-basis vectors such that $d(X(\mathbf{t}, \mu, \mathbf{s})) \leq (m_1, \dots, m_{2n+1}) = \mathbf{m}$ with respect to the lexicographical ordering.

We claim that the spaces $\mathcal{F}^{\mathbf{m}}(A)$ for $\mathbf{m} \in P$ define a locally bounded below filtration of A . Most of the required properties are straightforward, except that $Y \in \mathcal{F}^{\mathbf{m}}(A)$, $Z \in \mathcal{F}^{\mathbf{n}}(A)$ implies $YZ \in \mathcal{F}^{\mathbf{m}+\mathbf{n}}(A)$. The main point to check here is what happens when monomials in the E_{β_j} or F_{β_j} need to be reordered. Note that the formula obtained in Proposition 2.69 shows that $E_{\beta_i}E_{\beta_j} - q^{(\beta_i, \beta_j)}E_{\beta_j}E_{\beta_i}$ for $i < j$ is a sum of monomials of the same height as $E_{\beta_i}E_{\beta_j}$. According to the definition of the order in P , all these monomials have strictly smaller total degree than $E_{\beta_j}E_{\beta_i}$.

Using this observation we see that the associated graded algebra $gr_{\mathcal{F}}(A)$ is the algebra with generators $E_{\beta_1}, \dots, E_{\beta_n}, K_\mu, F_{\beta_1}, \dots, F_{\beta_n}$ and relations

$$\begin{aligned} K_\mu K_\nu &= K_\nu K_\mu, \\ K_\mu E_{\beta_i} &= q^{(\mu, \beta_i)} E_{\beta_i} K_\mu, \\ K_\mu F_{\beta_i} &= q^{-(\mu, \beta_i)} F_{\beta_i} K_\mu, \\ E_{\beta_i} F_{\beta_j} &= F_{\beta_j} E_{\beta_i}, \\ E_{\beta_i} E_{\beta_j} &= q^{(\beta_i, \beta_j)} E_{\beta_j} E_{\beta_i}, \\ F_{\beta_i} F_{\beta_j} &= q^{(\beta_i, \beta_j)} F_{\beta_j} F_{\beta_i} \end{aligned}$$

for all $\mu, \nu \in \mathbf{P}$ and $1 \leq i \leq n$ or $1 \leq i < j \leq n$, respectively. Therefore $gr_{\mathcal{F}}(A)$ is a skew-polynomial ring in the generators $E_{\beta_1}, \dots, E_{\beta_n}, F_{\beta_1}, \dots, F_{\beta_n}$ and $K_{\varpi_1}, \dots, K_{\varpi_N}$. As such it is a Noetherian domain by Lemma 2.132, and hence the same holds for A by Lemma 2.130. \square

2.14.3. Noetherianity of $\mathcal{O}(G_q)$. We show here that the coordinate algebra $\mathcal{O}(G_q)$ is Noetherian, again following the proof in Section I.8 of [15]. The techniques developed in this context will be used again in the next subsection to obtain Noetherianity of $FU_q(\mathfrak{g})$.

We need some preparations. For any $\lambda \in \mathbf{P}^+$ we fix a linear basis of weight vectors $e_1^\lambda, \dots, e_m^\lambda$ for $V(\lambda)$ with dual basis $e_1^\lambda, \dots, e_m^\lambda \in V(\lambda)^*$. We write ϵ_j for the weight of e_j^λ , and we may assume without loss of generality that the vectors are ordered in a non-ascending order, so that $\epsilon_i > \epsilon_j$ implies $i < j$. Let us denote by $u_{ij}^\lambda = \langle e_i^\lambda | \bullet | e_j^\lambda \rangle$ the corresponding matrix coefficients.

Proposition 2.135. *Let $\mu, \nu \in \mathbf{P}^+$ and $1 \leq i, j \leq m, 1 \leq k, l \leq n$. With the notations as above, there are scalars $\alpha_{rs}^{ijkl}, \beta_{uv}^{ijkl} \in \mathbb{K}$ such that*

$$q^{-(\epsilon_j, \epsilon_l)} u_{ij}^\mu u_{kl}^\nu + \sum_{r=1}^{j-1} \sum_{s=l+1}^m \alpha_{rs}^{ijkl} u_{ir}^\mu u_{ks}^\nu = q^{-(\epsilon_i, \epsilon_k)} u_{kl}^\nu u_{ij}^\mu + \sum_{u=i+1}^m \sum_{v=1}^{k-1} \beta_{uv}^{ijkl} u_{vl}^\nu u_{uj}^\mu$$

in $\mathcal{O}(G_q)$. Moreover $\alpha_{rs}^{ijkl} = 0$ unless $\epsilon_r > \epsilon_j$ and $\epsilon_s < \epsilon_l$. Similarly $\beta_{uv}^{ijkl} = 0$ unless $\epsilon_u < \epsilon_i$ and $\epsilon_v > \epsilon_k$.

Proof. From general properties of the universal R -matrix we obtain

$$f_{(1)} g_{(1)} (\mathcal{R}, f_{(2)} \otimes g_{(2)}) = (\mathcal{R}, f_{(1)} \otimes g_{(1)}) g_{(2)} f_{(2)}$$

for all $f, g \in \mathcal{O}(G_q)$. Inspecting the description of the universal R -matrix in Theorem 2.108 for $f = u_{ij}^\mu$ and $g = u_{kl}^\nu$ yields the desired formula. \square

Theorem 2.136. *The algebra $\mathcal{O}(G_q)$ is Noetherian.*

Proof. Note that the modules $V(\varpi_1), \dots, V(\varpi_N)$ generate all irreducible finite dimensional weight modules of $U_q(\mathfrak{g})$ in the sense that any $V(\mu)$ for $\mu \in \mathbf{P}^+$ is a submodule of a suitable tensor product of the $V(\varpi_j)$. Let us pick bases of weight vectors $e_1^k, \dots, e_{n_k}^k$ for each $V(\varpi_k)$ with dual bases $e_k^1, \dots, e_k^{n_k}$, and write ϵ_j for the weight of e_j^k . The matrix elements $u_{ij}^k = \langle e_k^i | \bullet | e_j^k \rangle$ for $k = 1, \dots, N$ and $1 \leq i, j \leq n_k$ generate $\mathcal{O}(G_q)$ as an algebra.

Let X be the collection of all these matrix elements u_{ij}^k . We shall order the elements of X in a list u_1, \dots, u_m such that the following condition holds: For $u_a = u_{ij}^r, u_b = u_{kl}^s$ we have $b < a$ if either $\epsilon_k < \epsilon_i$, or $\epsilon_k = \epsilon_i$ and $\epsilon_l > \epsilon_j$. According to Proposition 2.135 we then obtain scalars $q_{ij} \in \mathbb{K}^\times$ and $\alpha_{ij}^{st}, \beta_{ij}^{st} \in \mathbb{K}$ such that

$$u_i u_j = q_{ij} u_j u_i + \sum_{s=1}^{j-1} \sum_{t=1}^m \alpha_{ij}^{st} u_s u_t + \beta_{ij}^{st} u_t u_s$$

for all $1 \leq j < i \leq m$. Therefore Lemma 2.133 yields the claim. \square

2.14.4. *Noetherianity of $FU_q(\mathfrak{g})$.* In this subsection we discuss Noetherianity of the locally finite part of $U_q(\mathfrak{g})$. Joseph [41] relies on tricky filtration arguments for this, we shall instead give a proof based on the link between $FU_q(\mathfrak{g})$ and $\mathcal{O}(G_q)$ obtained in Theorem 2.113.

Theorem 2.137. *The algebra $FU_q(\mathfrak{g})$ is Noetherian.*

Proof. From the calculations after Proposition 2.116 it follows that we can identify the opposite algebra of $FU_q(\mathfrak{g})$ with $\mathcal{O}(G_q) = A$, if the latter is equipped with the multiplication

$$f \bullet g = f_{(2)} \hat{S}^{-1}(l^-(f_{(1)})) \rightarrow g$$

where

$$X \rightarrow g = (\hat{S}(X_{(1)}), g_{(1)}) g_{(2)}(X_{(2)}, g_{(3)})$$

is the coadjoint action of $U_q(\mathfrak{g})$ on $\mathcal{O}(G_q)$. Equivalently, according to Lemma 2.111 and the definition of the l -functionals we can write

$$\begin{aligned} f \bullet g &= (l^-(f_{(2)}), g_{(1)}) f_{(3)} g_{(2)} (\hat{S}^{-1}(l^-(f_{(1)})), g_{(3)}) \\ &= (\mathcal{R}^{-1}, f_{(2)} \otimes g_{(1)}) f_{(3)} g_{(2)} (\mathcal{R}^{-1}, f_{(1)} \otimes S(g_{(3)})) \\ &= g_{(1)} f_{(2)} (\mathcal{R}^{-1}, f_{(3)} \otimes g_{(2)}) (\mathcal{R}^{-1}, f_{(1)} \otimes S(g_{(3)})), \end{aligned}$$

using properties of the universal R -matrix in the final step.

We shall construct a certain filtration of A as follows. The ordered semigroup will be

$$P = \{(\mu, \lambda) \in \mathbf{P}^+ \times \mathbf{P} \mid \lambda \text{ is a weight of } V(\mu)\}$$

with the product ordering, namely

$$(\mu, \lambda) \leq (\mu', \lambda') \quad \text{iff} \quad \mu \leq \mu' \text{ and } \lambda \leq \lambda'.$$

Note that this partial order is well-founded.

To define the filtration, we begin with a P -grading on the underlying vector space of A . Let us write $C(\mu) = \text{End}(V(\mu))^* \subset \mathcal{O}(G_q)$ for the space of matrix coefficients of $V(\mu)$. Thus, as a linear space we have

$$A = \mathcal{O}(G_q) = \bigoplus_{\mu \in \mathbf{P}^+} C(\mu)$$

and this determines a \mathbf{P}^+ -grading on A . At the same time we have a \mathbf{P} -grading on the underlying vector space of A given by the right regular action of $U_q(\mathfrak{h})$, namely, $f \in \mathcal{O}(G_q)$ has weight λ with respect to this grading if

$$K_\nu \rightharpoonup f = f_{(1)}(K_\nu, f_{(2)}) = q^{(\nu, \lambda)} f$$

for all $\nu \in \mathbf{P}$. Let us write $\mathcal{O}(G_q)_\lambda$ for the subspace of all vectors of weight λ and $C(\mu)_\lambda = C(\mu) \cap \mathcal{O}(G_q)_\lambda$. We therefore obtain a direct sum decomposition

$$A \cong \bigoplus_{(\mu, \lambda) \in P} C(\mu)_\lambda,$$

and the P -filtration of A is defined by

$$\mathcal{F}^{(\mu, \lambda)}(A) = \bigoplus_{(\mu', \lambda') \leq (\mu, \lambda)} C(\mu')_{\lambda'}.$$

We claim that this is an algebra filtration of A . Suppose $f \in C(\mu_1)_{\lambda_1}$ and $g \in C(\mu_2)_{\lambda_2}$. Note that $f \bullet g \in C(\mu_1)C(\mu_2)$. Since the irreducible components of $V(\mu_1) \otimes V(\mu_2)$ have highest weights less than or equal to $\mu_1 + \mu_2$, we see that

$$f \bullet g \in \bigoplus_{\mu \leq \mu_1 + \mu_2} C(\mu).$$

Moreover, the description of \mathcal{R}^{-1} in the proof of Theorem 2.108 shows that

$$f \bullet g = (\mathcal{R}^{-1}, f_{(2)} \otimes g_{(1)}) f_{(3)} g_{(2)} (\mathcal{R}^{-1}, f_{(1)} \otimes S(g_{(3)}))$$

is a sum of terms whose weights with respect to the right regular action of $U_q(\mathfrak{h})$ are of the form $\lambda_1 + \lambda_2 - \gamma$ with $\gamma \in \mathbf{Q}^+$.

Therefore

$$f \bullet g \in \bigoplus_{(\mu, \lambda) \leq (\mu_1 + \mu_2, \lambda_1 + \lambda_2)} C(\mu)_\lambda = \mathcal{F}^{(\mu_1 + \mu_2, \lambda_1 + \lambda_2)}(A)$$

as required.

We write $B = gr_{\mathcal{F}}(A)$ for the associated graded algebra. We again have $B \cong \bigoplus_{(\mu, \lambda) \in P} C(\mu)_\lambda = \mathcal{O}(G_q)$ as a vector space, and we write $f \circ g$ for the product in B of $f, g \in \mathcal{O}(G_q)$. Then for $f \in C(\mu_1)_{\lambda_1}$ and $g \in C(\mu_2)_{\lambda_2}$, we have that $f \circ g$ is the projection onto $C(\mu_1 + \mu_2)_{\lambda_1 + \lambda_2}$ of $f \bullet g$. In order to give an explicit formula for this multiplication, we introduce the notation $f \cdot g \in C(\mu + \nu)$ for the projection of the usual $\mathcal{O}(G_q)$ -product fg onto the component $C(\mu + \nu)$, where $f \in C(\mu)$ and $g \in C(\nu)$. Using the above formula for $f \bullet g$ and again considering the formula for \mathcal{R}^{-1} from Theorem 2.108 we obtain

$$\begin{aligned} f \circ g &= (\mathcal{R}^{-1}, f_{(2)} \otimes g_{(1)}) f_{(3)} \cdot g_{(2)} (q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)}, f_{(1)} \otimes g_{(3)}) \\ &= g_{(1)} \cdot f_{(2)} (\mathcal{R}^{-1}, f_{(3)} \otimes g_{(2)}) (q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)}, f_{(1)} \otimes g_{(3)}). \end{aligned}$$

Let us calculate this product in terms of matrix coefficients. Let e_1^μ, \dots, e_m^μ be a basis of weight vectors for $V(\mu)$ with dual basis $e_1^\mu, \dots, e_m^\mu \in V(\mu)^*$. We write ϵ_j for the weight of e_j^μ , and assume that the vectors are ordered in a non-ascending order, so that $\epsilon_i > \epsilon_j$ implies $i < j$. Write $u_{ij}^\mu = \langle e_i^\mu | \bullet | e_j^\mu \rangle$ for the corresponding matrix coefficients. Similarly, for $\nu \in \mathbf{P}^+$ we fix a basis e_1^ν, \dots, e_n^ν of $V(\nu)$ with dual basis $e_1^\nu, \dots, e_n^\nu \in V(\nu)^*$ with the same properties. Using this notation one obtains

$$\begin{aligned} u_{ij}^\mu \circ u_{kl}^\nu &= q^{(\epsilon_i, \epsilon_l - \epsilon_k)} u_{ij}^\mu \cdot u_{kl}^\nu + \sum_{r=i+1}^m \sum_{s=1}^{k-1} \alpha_{rs}^{ijkl} u_{rj}^\mu \cdot u_{sl}^\nu \\ &= q^{(\epsilon_i, \epsilon_i - \epsilon_j)} u_{kl}^\nu \cdot u_{ij}^\mu + \sum_{u=1}^{j-1} \sum_{v=l+1}^n \beta_{uv}^{ijkl} u_{kv}^\nu \cdot u_{iu}^\mu \end{aligned}$$

with certain scalars $\alpha_{rs}^{ijkl}, \beta_{uv}^{ijkl} \in \mathbb{K}$. Moreover $\alpha_{rs}^{ijkl} = 0$ unless $\epsilon_r < \epsilon_i$ and $\epsilon_s > \epsilon_k$. Similarly $\beta_{uv}^{ijkl} = 0$ unless $\epsilon_u > \epsilon_j$ and $\epsilon_v < \epsilon_l$.

The above formulas show in particular that the space $C(\mu) \circ C(\nu)$ is contained in $C(\mu) \cdot C(\nu)$. Using an induction argument on the first of these formulas, we also obtain

$$u_{ij}^\mu \circ u_{kl}^\nu = q^{(\epsilon_i, \epsilon_l - \epsilon_k)} u_{ij}^\mu \cdot u_{kl}^\nu + \sum_{r=i+1}^m \sum_{s=1}^{k-1} \gamma_{rs}^{ijkl} u_{rj}^\mu \circ u_{sl}^\nu$$

for certain coefficients $\gamma_{rs}^{ijkl} \in \mathbb{K}$ with $\gamma_{rs}^{ijkl} = 0$ unless $\epsilon_r < \epsilon_i$ and $\epsilon_s > \epsilon_k$. This implies $C(\mu) \cdot C(\nu) \subset C(\mu) \circ C(\nu)$ and thus $C(\mu) \circ C(\nu) = C(\mu) \cdot C(\nu) = C(\mu + \nu)$.

In a similar way, the second formula for the product \circ from above, with the roles of u_{ij}^μ and u_{kl}^ν swapped, yields

$$u_{kl}^\nu \circ u_{ij}^\mu = q^{(\epsilon_j, \epsilon_k - \epsilon_i)} u_{ij}^\mu \cdot u_{kl}^\nu + \sum_{r=i}^m \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^m \delta_{rsuv}^{ijkl} u_{rv}^\mu \circ u_{su}^\nu$$

for certain coefficients $\delta_{rsuv}^{ijkl} \in \mathbb{K}$ with $\delta_{rsuv}^{ijkl} = 0$ unless $\epsilon_u > \epsilon_l$, $\epsilon_v < \epsilon_j$, $\epsilon_r \leq \epsilon_i$ and $\epsilon_s \geq \epsilon_k$. Setting $q_{ijkl} = q^{(\epsilon_j + \epsilon_i, \epsilon_k - \epsilon_i)}$ this yields

$$\begin{aligned} u_{kl}^\nu \circ u_{ij}^\mu - q_{ijkl} u_{ij}^\mu \circ u_{kl}^\nu &= \sum_{r=i}^m \sum_{s=1}^k \sum_{u=1}^{l-1} \sum_{v=j+1}^m \delta_{rsuv}^{ijkl} u_{rv}^\mu \circ u_{su}^\nu \\ &\quad - \sum_{r=i+1}^m \sum_{s=1}^{k-1} q_{ijkl} \gamma_{rs}^{ijkl} u_{rj}^\mu \circ u_{sl}^\nu. \end{aligned}$$

We now wish to show that B satisfies the hypotheses of Proposition 2.133. Consider the matrix coefficients $u_{ij}^k = \langle e_i^k | \bullet | e_j^k \rangle$ for $k = 1, \dots, N$ and $1 \leq i, j \leq n_k$. From the relation $C(\mu) \circ C(\nu) = C(\mu + \nu)$ for all $\mu, \nu \in \mathbf{P}^+$ one sees immediately that these elements generate B as an algebra.

Let X be the collection of all the elements u_{ij}^k . We list the elements of X in an ordered sequence u_1, \dots, u_m such that the following condition holds: For $u_a = u_{ij}^r, u_b = u_{kl}^s$ we have $b < a$ if either $\epsilon_k < \epsilon_i$, or $\epsilon_k = \epsilon_i$ and $\epsilon_l < \epsilon_j$. According to our above considerations we obtain elements $q_{ij} \in \mathbb{K}^\times$ and $\alpha_{ij}^{st} \in \mathbb{K}$ such that

$$u_i \circ u_j = q_{ij} u_i \circ u_j + \sum_{s=1}^{j-1} \sum_{t=1}^m \alpha_{ij}^{st} u_s \circ u_t$$

for all $1 \leq j < i \leq m$. Therefore Proposition 2.133 shows that B is Noetherian.

According to Lemma 2.130 it follows that A is Noetherian. \square

Since $FU_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ is a subalgebra it is immediate from Proposition 2.134 that $FU_q(\mathfrak{g})$ is a domain as well.

Let us remark that Noetherianity of $U_q(\mathfrak{g})$ can be deduced from Theorem 2.137 as follows, independently of Proposition 2.134. Firstly, extend $FU_q(\mathfrak{g})$ with abstract Cartan generators L_μ for $\mu \in \mathbf{P}$, commuting with the elements of $FU_q(\mathfrak{g})$ like the Cartan generators K_μ of $U_q(\mathfrak{g})$. The resulting algebra B is Noetherian by Theorem 2.137 and Lemma 2.131, and hence Noetherianity of $U_q(\mathfrak{g})$ follows by observing that $U_q(\mathfrak{g})$ is naturally a quotient of B .

2.15. Canonical bases. In this section we give a brief summary of the theory of canonical bases. This theory, due to Lusztig [55] and Kashiwara [46], [45], is devoted to studying the $q \rightarrow 0$ limit of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. For our purposes we only need relatively basic aspects of the theory. A thorough exposition can be found in [57], see also [34].

In the literature on canonical bases the deformation parameter q is usually taken to be a transcendental variable over \mathbb{Q} . We will also be interested in the specialization to non-root of unity invertible elements in an arbitrary ground field \mathbb{K} .

2.15.1. Crystal bases. In this subsection we discuss without proofs the existence and uniqueness of crystal bases for integrable $U_q(\mathfrak{g})$ -modules.

We shall work with the following definition of an abstract crystal.

Definition 2.138. Let $I = \{1, \dots, N\}$ be a finite set. A *crystal* is a set B together with maps $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}$ for all $i \in I$ such that the following conditions hold.

- a) $\tilde{e}_i(0) = 0 = \tilde{f}_i(0)$ for all $i \in I$.
- b) For any $i \in I$ and $b \in B$ there exists $n \in \mathbb{N}$ such that $\tilde{e}_i^n(b) = 0 = \tilde{f}_i^n(b)$.
- c) For any $i \in I$ and $b, c \in B$ we have $c = \tilde{f}_i(b)$ iff $\tilde{e}_i(c) = b$.

Given crystals B_1, B_2 , a (strict) morphism from B_1 to B_2 is a map $g : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ such that $g(0) = 0$ and g commutes with all operators \tilde{e}_i, \tilde{f}_i .

In the above definition, the symbol \sqcup stands for disjoint union. For an element $b \in B$ one sets

$$\varepsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n(b) \neq 0\}, \quad \varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n(b) \neq 0\}.$$

Let \mathbf{P} be the free abelian group abstractly generated by elements $\varpi_1, \dots, \varpi_N$. If B is a crystal we define a map $wt : B \rightarrow \mathbf{P}$ by

$$wt(b) = \sum_{i \in I} (\varphi_i(b) - \varepsilon_i(b)) \varpi_i,$$

and refer to $wt(b)$ as the weight of b .

If B_1, B_2 are crystals then the direct sum $B_1 \oplus B_2$ is the crystal with underlying set $B_1 \sqcup B_2$, with the operators $\tilde{e}_i, \tilde{f}_i : B_1 \sqcup B_2 \sqcup \{0\} \rightarrow B_1 \sqcup B_2 \sqcup \{0\}$ induced by the corresponding operators for B_1 and B_2 .

If B_1, B_2 are crystals then the tensor product $B_1 \otimes B_2$ is the crystal with underlying set $B_1 \times B_2$, and elements written as $(b_1, b_2) = b_1 \otimes b_2$, together with the action

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2) \end{cases}$$

where we interpret $b_1 \otimes 0 = 0 = 0 \otimes b_2$ for all $b_1 \in B_1, b_2 \in B_2$. The tensor product $B_1 \otimes B_2$ is again a crystal, and the operation of taking tensor products is associative.

Given a crystal B , an element $b \in B$ is called a highest weight vector if $\tilde{e}_i(b) = 0$ for all i . Similarly, $b \in B$ is called a lowest weight vector if $\tilde{f}_i(b) = 0$ for all i .

For the rest of this subsection we work over the field $\mathbb{K} = \mathbb{Q}(s)$ where s is an indeterminate and $q = s^L$ as before, deviating slightly from most of the literature in which $\mathbb{Q}(q)$ is taken as base field. We fix the semisimple Lie algebra \mathfrak{g} associated with a finite Cartan matrix $A = (a_{ij})$. Our choice of $\mathbb{Q}(s)$ is necessary because we work with the *simply connected* version of $U_q(\mathfrak{g})$; however this does not affect the constructions and arguments in a serious way.

Given an integrable $U_q(\mathfrak{g})$ -module M and $1 \leq i \leq N$, one can write every element $m \in M$ of weight $\lambda \in \mathbf{P}$ uniquely in the form

$$m = \sum_{n \geq 0} F_i^{(n)} \cdot m_n$$

where $m_n \in M_{\lambda+n\alpha_i}$ satisfies $E_i \cdot m_n = 0$, and we recall that $F_i^{(n)} = F_i^n / [n]_{q_i}!$. On a vector $m \in M_\lambda$ written in the above form, the Kashiwara operators are defined by

$$\begin{aligned} \tilde{e}_i(m) &= \sum_{n \geq 1} F_i^{(n-1)} \cdot m_n \\ \tilde{f}_i(m) &= \sum_{n \geq 0} F_i^{(n+1)} \cdot m_n, \end{aligned}$$

and this is extended linearly to all of M . Hence we obtain linear operators $\tilde{e}_i, \tilde{f}_i : M \rightarrow M$ for $i = 1, \dots, N$ in this way.

Consider the algebra \mathcal{A}_0 obtained by localizing the polynomial ring $\mathbb{Q}[s]$ at the maximal ideal generated by s , corresponding to the point $s = 0$ on the affine line. Explicitly, the elements of \mathcal{A}_0 can be written in the form $f(s)/g(s)$ where $f(s), g(s) \in \mathbb{Q}[s]$ and $g(0) \neq 0$.

Let us also recall that β denotes the field automorphism of $\mathbb{Q}(s)$ which maps s to s^{-1} , and define $\mathcal{A}_\infty = \beta(\mathcal{A}_0) \subset \mathbb{Q}(s)$. This can be viewed as a localization at ∞ , which will be more convenient for us than localization at 0.

The following definition of crystal bases corresponds to the notion of basis at ∞ in the sense of Chapter 20 in [57].

Definition 2.139. Let M be an integrable $U_q(\mathfrak{g})$ -module. A crystal basis $(\mathcal{L}, \mathcal{B})$ for M is a free \mathcal{A}_∞ -module $\mathcal{L} \subset M$ such that $\mathbb{Q}(s) \otimes_{\mathcal{A}_\infty} \mathcal{L} = M$, together with a basis \mathcal{B} of the vector space $\mathcal{L}/s^{-1}\mathcal{L}$ over \mathbb{Q} such that the following conditions hold.

- a) For any $\mu \in \mathbf{P}$, the space $\mathcal{L}_\mu = M_\mu \cap \mathcal{L}$ satisfies $\mathbb{Q}(s) \otimes_{\mathcal{A}_\infty} \mathcal{L}_\mu = M_\mu$, and $\mathcal{B}_\mu = \mathcal{B} \cap \mathcal{L}_\mu/s^{-1}\mathcal{L}_\mu$ is a basis of $\mathcal{L}_\mu/s^{-1}\mathcal{L}_\mu$ over \mathbb{Q} .
- b) The Kashiwara operators \tilde{e}_i, \tilde{f}_i on M leave \mathcal{L} invariant and induce on $\mathcal{L}/s^{-1}\mathcal{L}$ operators which leave $\mathcal{B} \sqcup \{0\}$ invariant, such that \mathcal{B} becomes a crystal.

Consider the simple module $M = V(\lambda)$ for $\lambda \in \mathbf{P}^+$ and let $\mathcal{L}(\lambda)$ be the \mathcal{A}_∞ -submodule of $V(\lambda)$ spanned by all vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v_\lambda)$ where $i_1, \dots, i_l \in I$ and $\tilde{f}_1, \dots, \tilde{f}_N$ denote the Kashiwara operators. Moreover let $\mathcal{B}(\lambda)$ be the collection of all nonzero cosets in $\mathcal{L}(\lambda)/s^{-1}\mathcal{L}(\lambda)$ of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_l}(v_\lambda)$. The fact that $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ is a crystal basis for $V(\lambda)$ is crucial in the proof of the following foundational result due to Lusztig and Kashiwara, see [45].

Theorem 2.140. For every integrable $U_q(\mathfrak{g})$ -module M there exists a crystal basis $(\mathcal{L}, \mathcal{B})$. Moreover, if $(\mathcal{L}_1, \mathcal{B}_1), (\mathcal{L}_2, \mathcal{B}_2)$ are crystal bases of M then there exists an automorphism $f : M \rightarrow M$ of $U_q(\mathfrak{g})$ -modules which restricts to an isomorphism of crystals $f : \mathcal{B}_1 \rightarrow \mathcal{B}_2$.

The proof of Theorem 2.140 relies on the *grand loop* argument. We refer to [45] for the details, see also [34] and Chapter 5 in [41]. At the same time, one can construct suitable crystal bases for $U_q(\mathfrak{n}_-)$ and for all Verma modules.

2.15.2. Global bases. In this subsection we describe how to use crystal bases to obtain global bases for $U_q(\mathfrak{g})$ -modules in the case of an arbitrary base field \mathbb{K} provided that $q = s^L \in \mathbb{K}^\times$ is not a root of unity. We follow the exposition in Chapter 6 of [34].

Initially, we shall work over $\mathbb{K} = \mathbb{Q}(s)$ and consider several subrings of this field. Note that we may view $\mathbb{Q}(s)$ as the ring of all fractions $f(s)/g(s)$ with $f(s), g(s) \in \mathbb{Z}[s]$ and $g(s) \neq 0$. Note also that the localization \mathcal{A}_0 of $\mathbb{Q}[s]$ at $s = 0$ can be identified with the localization $\mathbb{Z}[s]_0$ of the ring $\mathbb{Z}[s]$ of polynomial functions with integer coefficients at $s = 0$. That is, elements of \mathcal{A}_0 can be written as fractions $f(s)/g(s)$ where $f(s), g(s) \in \mathbb{Z}[s]$ such that $g(0) \neq 0$. Let us also define $\mathcal{A}_\infty = \beta(\mathcal{A}_0)$, where we recall that $\beta : \mathbb{Q}(s) \rightarrow \mathbb{Q}(s)$ denotes the field automorphism determined by $\beta(s) = s^{-1}$. The ring \mathcal{A}_∞ can be thought of as localization of $\mathbb{Q}[s^{-1}]$ at $s = \infty$. We shall write $\mathcal{A} = \mathbb{Z}[s, s^{-1}]$ as before. Note that $\mathcal{A}_0, \mathcal{A}_\infty, \mathcal{A} \subset \mathbb{Q}(s)$ are naturally subrings.

In the sequel we shall consider various lattices in the sense of the following definition.

Definition 2.141. If $R \subset S$ is a subring and V is a free S -module, then a free R -submodule $\mathcal{L} \subset V$ is called a free R -lattice of V if the canonical map $S \otimes_R \mathcal{L} \rightarrow V$ is an isomorphism.

Assume that V is a finite dimensional $\mathbb{Q}(s)$ -vector space, and let $\mathcal{L}_0, \mathcal{L}_\infty$ and $V_{\mathcal{A}}$ be free \mathcal{A}_0 -, \mathcal{A}_∞ - and \mathcal{A} -lattices of V , respectively. Even if no compatibility between these lattices is assumed a priori, we automatically have the following properties, compare Section 6.1 in [34].

Lemma 2.142. *In the above situation, the canonical map $\mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \rightarrow \mathcal{L}_0$ is an isomorphism of \mathcal{A}_0 -modules. Similarly, the canonical map $\mathcal{A}_\infty \otimes_{\mathbb{Z}[s^{-1}]} (V_{\mathcal{A}} \cap \mathcal{L}_\infty) \rightarrow \mathcal{L}_\infty$ is an isomorphism of \mathcal{A}_∞ -modules.*

Proof. We shall only prove the assertion for \mathcal{L}_0 , the proof for \mathcal{L}_∞ is analogous. Since \mathcal{A}_0 and \mathcal{A} are localizations of $\mathbb{Z}[s]$, we see that the inclusion map $V_{\mathcal{A}} \cap \mathcal{L}_0 \rightarrow \mathcal{L}_0$ induces injective maps $\mathcal{A} \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \rightarrow \mathcal{A} \otimes_{\mathbb{Z}[s]} \mathcal{L}_0$ and $\mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \rightarrow \mathcal{A}_0 \otimes_{\mathbb{Z}[s]} \mathcal{L}_0$. The latter identifies with the canonical map $\mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \rightarrow \mathcal{L}_0$ since $\mathcal{A}_0 \otimes_{\mathbb{Z}[s]} \mathcal{L}_0 \cong \mathcal{A}_0 \otimes_{\mathcal{A}_0} \mathcal{L}_0 \cong \mathcal{L}_0$.

Hence it suffices to show that the map $\mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \rightarrow \mathcal{L}_0$ is surjective. Since $V_{\mathcal{A}} \subset V$ is an \mathcal{A} -lattice any element of V , in particular any element $v \in \mathcal{L}_0$, can be written in the form $\frac{f(s)}{g(s)}u$ where $f(s), g(s) \in \mathbb{Z}[s], g(s) \neq 0$ and $u \in V_{\mathcal{A}}$. That is, there exists a nonzero element $g(s) \in \mathbb{Z}[s]$ such that $g(s)v \in V_{\mathcal{A}}$. Since $V_{\mathcal{A}}$ is an \mathcal{A} -module, upon dividing by a suitable power of s we may assume without loss of generality that $g(0) \neq 0$. Hence we have

$$\frac{1}{g(s)} \otimes g(s)v \in \mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0),$$

and this element maps to v under the canonical map as desired. \square

Let us introduce the notion of a balanced triple.

Definition 2.143. Let V be a finite dimensional $\mathbb{Q}(s)$ -vector space. Moreover let $\mathcal{L}_0 \subset V$ be a free \mathcal{A}_0 -lattice, $V_{\mathcal{A}} \subset V$ a free \mathcal{A} -lattice, and $\mathcal{L}_\infty \subset V$ a free \mathcal{A}_∞ -lattice. If we define

$$\mathcal{L} = \mathcal{L}_0 \cap V_{\mathcal{A}} \cap \mathcal{L}_\infty$$

then $(\mathcal{L}_0, V_{\mathcal{A}}, \mathcal{L}_{\infty})$ is called a balanced triple for V provided the following conditions hold.

- a) \mathcal{L} is a free \mathbb{Z} -lattice for the \mathcal{A}_0 -module \mathcal{L}_0 .
- b) \mathcal{L} is a free \mathbb{Z} -lattice for the \mathcal{A} -module $V_{\mathcal{A}}$.
- c) \mathcal{L} is a free \mathbb{Z} -lattice for the \mathcal{A}_{∞} -module \mathcal{L}_{∞} .

The conditions for $(\mathcal{L}_0, V_{\mathcal{A}}, \mathcal{L}_{\infty})$ to be a balanced triple mean that \mathcal{L} has finite rank, and that the canonical multiplication maps induce isomorphisms

$$\mathcal{A}_0 \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{L}_0, \quad \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{L} \cong V_{\mathcal{A}}, \quad \mathcal{A}_{\infty} \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{L}_{\infty}.$$

Proposition 2.144. *Let V be a finite dimensional $\mathbb{Q}(s)$ -vector space. Moreover let $\mathcal{L}_0 \subset V$ be a free \mathcal{A}_0 -lattice, $V_{\mathcal{A}} \subset V$ a free \mathcal{A} -lattice, and $\mathcal{L}_{\infty} \subset V$ a free \mathcal{A}_{∞} -lattice. Then the following conditions are equivalent.*

- a) $(\mathcal{L}_0, V_{\mathcal{A}}, \mathcal{L}_{\infty})$ is a balanced triple for V .
- b) The canonical map $\mathcal{L} \rightarrow \mathcal{L}_0/s\mathcal{L}_0$ is an isomorphism.
- c) The canonical map $\mathcal{L} \rightarrow \mathcal{L}_{\infty}/s^{-1}\mathcal{L}_{\infty}$ is an isomorphism.

Proof. a) \Rightarrow b) We have canonical isomorphisms

$$\mathcal{L} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathbb{Z} \otimes_{\mathcal{A}_0} \mathcal{A}_0 \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathbb{Z} \otimes_{\mathcal{A}_0} \mathcal{L}_0 \cong \mathcal{A}_0/s\mathcal{A}_0 \otimes_{\mathcal{A}_0} \mathcal{L}_0 \cong \mathcal{L}_0/s\mathcal{L}_0.$$

This yields the claim.

b) \Rightarrow a) We shall first prove by induction that the canonical map $m_k : \mathcal{L} \rightarrow \mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^k \mathcal{L}_{\infty}$ given by $m_k(v) = s^k v$ induces an isomorphism

$$\left(\bigoplus_{k=0}^n \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^n \mathcal{L}_{\infty}$$

for any $n \in \mathbb{N}_0$. For $n = 0$ this is obvious, so assume that the assertion holds for $n - 1$ for some $n > 0$. Then we have a canonical isomorphism

$$\left(\bigoplus_{k=1}^n \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} \cong s\mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^n \mathcal{L}_{\infty}$$

and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \left(\bigoplus_{k=1}^n \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} & \longrightarrow & \left(\bigoplus_{k=0}^n \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} & \longrightarrow & \mathcal{L} \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & s\mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^n \mathcal{L}_{\infty} & \longrightarrow & \mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^n \mathcal{L}_{\infty} & \longrightarrow & \mathcal{L}_0/s\mathcal{L}_0 \end{array}$$

with exact rows. Hence the 5-Lemma shows that the middle vertical arrow is an isomorphism, which yields the inductive step.

As a consequence, we have

$$\left(\bigoplus_{k=0}^{b-a} \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^{b-a} \mathcal{L}_{\infty}$$

for all $b \geq a$, which implies

$$\left(\bigoplus_{k=a}^b \mathbb{Z}s^k \right) \otimes_{\mathbb{Z}} \mathcal{L} \cong s^a \mathcal{L}_0 \cap V_{\mathcal{A}} \cap s^b \mathcal{L}_{\infty}$$

under the canonical map. Since \mathcal{L}_∞ is an \mathcal{A}_∞ -lattice in V we have $\bigcup_{n=0}^\infty s^n \mathcal{L}_\infty = V$, and hence the above yields

$$\begin{aligned}\mathbb{Z}[s] \otimes_{\mathbb{Z}} \mathcal{L} &\cong V_{\mathcal{A}} \cap \mathcal{L}_0, \\ \mathbb{Z}[s, s^{-1}] \otimes_{\mathbb{Z}} \mathcal{L} &\cong V_{\mathcal{A}}, \\ \mathbb{Z}[s^{-1}] \otimes_{\mathbb{Z}} \mathcal{L} &\cong V_{\mathcal{A}} \cap \mathcal{L}_\infty.\end{aligned}$$

The first of these isomorphisms implies

$$\mathcal{A}_0 \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{A}_0 \otimes_{\mathbb{Z}[s]} \mathbb{Z}[s] \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{A}_0 \otimes_{\mathbb{Z}[s]} (V_{\mathcal{A}} \cap \mathcal{L}_0) \cong \mathcal{L}_0,$$

using Lemma 2.142 in the last step. In the same way one obtains

$$\mathcal{A}_\infty \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{A}_\infty \otimes_{\mathbb{Z}[s^{-1}]} \mathbb{Z}[s^{-1}] \otimes_{\mathbb{Z}} \mathcal{L} \cong \mathcal{A}_\infty \otimes_{\mathbb{Z}[s^{-1}]} (V_{\mathcal{A}} \cap \mathcal{L}_\infty) \cong \mathcal{L}_\infty,$$

and we conclude that $(\mathcal{L}_0, V_{\mathcal{A}}, \mathcal{L}_\infty)$ is a balanced triple.

$a) \Leftrightarrow c)$ is proved in the same way. \square

Assume that $(\mathcal{L}_0, V_{\mathcal{A}}, \mathcal{L}_\infty)$ is a balanced triple for the finite dimensional $\mathbb{Q}(s)$ -vector space V , and let $G : \mathcal{L}_\infty/s^{-1}\mathcal{L}_\infty \rightarrow \mathcal{L}$ be the inverse of the isomorphism $\mathcal{L} \cong \mathcal{L}_\infty/s^{-1}\mathcal{L}_\infty$ obtained in Proposition 2.144. If \mathcal{B} is a \mathbb{Z} -basis of $\mathcal{L}_\infty/s^{-1}\mathcal{L}_\infty$ then the vectors $G(b)$ for $b \in \mathcal{B}$ form an \mathcal{A} -basis $G(\mathcal{B})$ of $V_{\mathcal{A}}$ and a $\mathbb{Q}(s)$ -basis of V . Indeed, writing $\mathcal{L} = \bigoplus_{b \in \mathcal{B}} \mathbb{Z}G(b)$ we obtain

$$V_{\mathcal{A}} = \mathcal{A} \otimes_{\mathbb{Z}} \mathcal{L} = \mathcal{A} \otimes_{\mathbb{Z}} \bigoplus_{b \in \mathcal{B}} \mathbb{Z}G(b) = \bigoplus_{b \in \mathcal{B}} \mathcal{A}G(b),$$

and the claim for V follows similarly from $V \cong \mathbb{Q}(s) \otimes_{\mathcal{A}} V_{\mathcal{A}}$. One calls $G(\mathcal{B})$ the global basis associated to the local basis \mathcal{B} .

We continue to work over $\mathbb{K} = \mathbb{Q}(s)$. Recall the definition of the integral form $U_q^{\mathcal{A}}(\mathfrak{g})$ of $U_q(\mathfrak{g})$ from Definition 2.17 and that $U_q^{\mathcal{A}}(\mathfrak{n}_-)$ is the \mathcal{A} -subalgebra of $U_q(\mathfrak{n}_-)$ generated by the divided powers $F_i^{(m)}$ for $i = 1, \dots, N$ and $m \in \mathbb{N}_0$. The automorphism β of $U_q(\mathfrak{g})$ defined in Lemma 2.19 restricts to an automorphism of $U_q^{\mathcal{A}}(\mathfrak{g})$ preserving $U_q^{\mathcal{A}}(\mathfrak{n}_-)$. Moreover, for $\lambda \in \mathbf{P}^+$ we obtain a well-defined \mathbb{Q} -linear automorphism $\beta_{V(\lambda)} : V(\lambda) \rightarrow V(\lambda)$ by setting $\beta_{V(\lambda)}(Y \cdot v_\lambda) = \beta(Y) \cdot v_\lambda$, for any $Y \in U_q(\mathfrak{g})$. Indeed, view $V(\lambda)$ as a module over $U_q(\mathfrak{g})$ with action $X \cdot v = \beta(X) \cdot v$, and write $V(\lambda)^\beta$ for this module. Then $V(\lambda)^\beta$ is irreducible and of highest weight λ , and hence must be isomorphic to $V(\lambda)$ as $U_q(\mathfrak{g})$ -module. The corresponding intertwiner is precisely the desired map $\beta_{V(\lambda)}$.

Let $(\mathcal{L}(\lambda), \mathcal{B}(\lambda))$ be the crystal basis of $V(\lambda)$ as discussed before Theorem 2.140, and let us also write $\mathcal{L}(\lambda) = \mathcal{L}_\infty(\lambda)$. If we define $\mathcal{L}_0(\lambda) = \beta_{V(\lambda)}(\mathcal{L}(\lambda))$, then $\mathcal{L}_0(\lambda)$ is a free \mathcal{A}_0 -lattice of $V(\lambda)$. Moreover set

$$V(\lambda)_{\mathcal{A}} = U_q^{\mathcal{A}}(\mathfrak{g}) \cdot v_\lambda = U_q^{\mathcal{A}}(\mathfrak{n}_-) \cdot v_\lambda.$$

Then we have $\beta_{V(\lambda)}(V(\lambda)_{\mathcal{A}}) = V(\lambda)_{\mathcal{A}}$.

Theorem 2.145. *Let $\lambda \in \mathbf{P}^+$. With the notation as above, $(\mathcal{L}_0(\lambda), V(\lambda)_{\mathcal{A}}, \mathcal{L}_\infty(\lambda))$ is a balanced triple for $V(\lambda)$.*

The proof of Theorem 2.145 can be found in [45], see also Chapter 6 of [34] and Section 6.2 in [41]. According to the discussion after Proposition 2.144 we obtain global basis elements $G(b)$ for $b \in \mathcal{B}(\lambda)$ such that

$$V(\lambda)_{\mathcal{A}} = \bigoplus_{b \in \mathcal{B}(\lambda)} \mathcal{A}G(b),$$

and the elements $G(b)$ also form a basis of $V(\lambda)$ as a $\mathbb{Q}(s)$ -vector space.

Let us now consider the case where \mathbb{K} is arbitrary and $q = s^L \in \mathbb{K}^\times$ is not a root of unity. We write $U_q(\mathfrak{g})$ for the quantized universal enveloping algebra over

\mathbb{K} . Then the canonical ring homomorphism $\mathcal{A} = \mathbb{Z}[s, s^{-1}] \rightarrow \mathbb{K}$ induces an \mathcal{A} -linear map from the \mathcal{A} -module $V(\lambda)_{\mathcal{A}}$ as defined above into the \mathbb{K} -vector space $V(\lambda)$, the irreducible highest weight module of $U_q(\mathfrak{g})$ corresponding to $\lambda \in \mathbf{P}^+$. The image of the resulting \mathbb{K} -linear map $\iota_\lambda : \mathbb{K} \otimes_{\mathcal{A}} V(\lambda)_{\mathcal{A}} \rightarrow V(\lambda)$ is a nonzero submodule of $V(\lambda)$, and hence ι_λ is surjective by the irreducibility of $V(\lambda)$. Conversely, $\mathbb{K} \otimes_{\mathcal{A}} V(\lambda)_{\mathcal{A}}$ is clearly an integrable $U_q(\mathfrak{g})$ -module, and hence a quotient of $V(\lambda)$ by Theorem 2.43 and Theorem 2.97. It follows that ι_λ is in fact an isomorphism.

We thus obtain the following result.

Theorem 2.146. *Let \mathbb{K} be a field and assume $q = s^L \in \mathbb{K}^\times$ is not a root of unity. For any $\lambda \in \mathbf{P}^+$ the elements $G(b)$ for $b \in \mathcal{B}(\lambda)$ form a basis of $V(\lambda)$ as a \mathbb{K} -vector space.*

2.16. Separation of Variables. In this section we prove a key result on the structure of the locally finite part of $U_q(\mathfrak{g})$, originally due to Joseph and Letzter [43], see also Section 7.3 in [41]. We follow the approach developed by Baumann [8], [10]. Throughout this section we assume that $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

2.16.1. Based modules. The key part of the proof of separation of variables further below relies on some results involving canonical bases. We shall collect these facts here, and refer to [57] for the proofs. In this subsection we work over $\mathbb{K} = \mathbb{Q}(s)$ with $q = s^L$ as before.

Let us first introduce the notion of a based module, see Section 27.1 in [57]. By definition, an involution on a $U_q(\mathfrak{g})$ -module M is a \mathbb{Q} -linear automorphism $\beta_M : M \rightarrow M$ such that

$$\beta_M(X \cdot m) = \beta(X) \cdot \beta_M(m)$$

for all $X \in U_q(\mathfrak{g})$ and $m \in M$, where β is the bar involution of $U_q(\mathfrak{g})$ as in Lemma 2.19. For instance, if $M = V(\lambda)$ for $\lambda \in \mathbf{P}^+$ and $v_\lambda \in V(\lambda)$ is a highest weight vector then the map β_M defined by $\beta_M(X \cdot v_\lambda) = \beta(X) \cdot v_\lambda$ for $X \in U_q(\mathfrak{g})$, as discussed before Theorem 2.145, defines an involution of M .

Definition 2.147. An integrable $U_q(\mathfrak{g})$ -module M with involution β_M together with a $\mathbb{Q}(s)$ -basis B is called a based module if the following conditions hold.

- a) $B \cap M_\mu$ is a basis of M_μ for any $\mu \in \mathbf{P}$.
- b) The \mathcal{A} -submodule $M_{\mathcal{A}}$ generated by B is stable under $U_q^{\mathcal{A}}(\mathfrak{g})$.
- c) We have $\beta_M(b) = b$ for all $b \in B$.
- d) The \mathcal{A}_∞ -submodule \mathcal{L}_M generated by B together with the image \mathcal{B} of B in $\mathcal{L}_M/s^{-1}\mathcal{L}_M$ forms a crystal basis for M .

A morphism of based modules from (M, B_M) to (N, B_N) is a $U_q(\mathfrak{g})$ -linear map $f : M \rightarrow N$ such that $f(b) \in B_N \cup \{0\}$ for all $b \in B_M$ and $B \cap \ker(f)$ is a basis of the kernel $\ker(f)$. In this case the kernel of f naturally inherits the structure of a based module.

For any $\lambda \in \mathbf{P}^+$, the simple module $V(\lambda)$, equipped with the involution fixing v_λ , together with the corresponding global basis as in Theorem 2.146 is a based module. In this way simple integrable $U_q(\mathfrak{g})$ -modules can be viewed as based modules.

Let us now sketch the construction of a suitable based module structure on the tensor product $M \otimes N$ of two based modules M, N , see Chapter 27.3 in [57]. One first defines an involution $\beta_{M \otimes N}$ on $M \otimes N$ out of the involutions β_M and β_N and the quasi- R -matrix for $U_q(\mathfrak{g})$. Note here that the ordinary tensor product of the involutions β_M and β_N is not compatible with the $U_q(\mathfrak{g})$ -module structure on $M \otimes N$ in general. The construction of an appropriate basis for $M \otimes N$ is then characterized by the following result, see Theorem 24.3.3 in [57].

Theorem 2.148. *Let (M, B) and (N, C) be based modules. Moreover let \mathcal{L} be the $\mathbb{Z}[s^{-1}]$ -submodule of $M \otimes N$ generated by all elements $b \otimes c$ for $(b, c) \in B \times C$. Then for any $(b, c) \in B \times C$ there exists a unique element $b \diamond c \in \mathcal{L}$ such that $\beta_{M \otimes N}(b \diamond c) = b \otimes c$ and $b \diamond c - b \otimes c \in s^{-1}\mathcal{L}$. Moreover, the elements $b \diamond c$ form a basis $B \diamond C$ of $M \otimes N$ which turn the latter into a based module with involution $\beta_{M \otimes N}$.*

The basis of the tensor product $M \otimes N$ as in Theorem 2.148 will also be referred to as the canonical basis.

Recall that if M is a finite dimensional $U_q(\mathfrak{g})$ -module then the dual module is the dual space M^* with the $U_q(\mathfrak{g})$ -module structure defined by $(X \cdot f)(m) = f(\hat{S}(X) \cdot m)$ for all $m \in M$. We have $(M \otimes N)^* \cong N^* \otimes M^*$ naturally if M, N are finite dimensional.

Assume that $N = V(\mu)$ for some $\mu \in \mathbf{P}^+$ with highest weight vector v_μ . Then $M = V(\mu)^*$ is an irreducible $U_q(\mathfrak{g})$ -module with lowest weight $-\mu$, and we denote by $v^\mu \in V(\mu)^*$ the lowest weight vector satisfying $v^\mu(v_\mu) = 1$. Since $V(\mu)$ is a simple $U_q(\mathfrak{g})$ -module there exists a unique $U_q(\mathfrak{g})$ -linear map $ev_\mu : V(\mu)^* \otimes V(\mu) \rightarrow \mathbb{K}$ such that $ev_\mu(v^\mu \otimes v_\mu) = 1$.

For $\mu, \lambda \in \mathbf{P}^+$ let $p_{\mu\lambda} : V(\mu) \otimes V(\lambda) \rightarrow V(\mu + \lambda)$ be the unique $U_q(\mathfrak{g})$ -linear map satisfying $p_{\mu\lambda}(v_\mu \otimes v_\lambda) = v_{\mu+\lambda}$. Similarly, let $i_{\mu\lambda} : V(\mu + \lambda) \rightarrow V(\mu) \otimes V(\lambda)$ be the unique $U_q(\mathfrak{g})$ -linear map satisfying $i_{\mu\lambda}(v_{\mu+\lambda}) = v_\mu \otimes v_\lambda$. By construction we then have $p_{\mu\lambda}i_{\mu\lambda} = \text{id}$. The transpose of $p_{\mu\lambda}$ determines a $U_q(\mathfrak{g})$ -linear map $p_{\mu\lambda}^* : V(\mu + \lambda)^* \rightarrow (V(\mu) \otimes V(\lambda))^* \cong V(\lambda)^* \otimes V(\mu)^*$ satisfying $p_{\mu\lambda}^*(v^{\mu+\lambda}) = v^\lambda \otimes v^\mu$. Similarly, the transpose of $i_{\mu\lambda}$ determines a $U_q(\mathfrak{g})$ -linear map $i_{\mu\lambda}^* : V(\lambda)^* \otimes V(\mu)^* \cong (V(\mu) \otimes V(\lambda))^* \rightarrow V(\mu + \lambda)^*$ satisfying $i_{\mu\lambda}^*(v^\lambda \otimes v^\mu) = v^{\mu+\lambda}$.

Let us define $T_\lambda : V(\mu + \lambda)^* \otimes V(\mu + \lambda) \rightarrow V(\mu)^* \otimes V(\mu)$ as the composition

$$V(\mu + \lambda)^* \otimes V(\mu + \lambda) \xrightarrow{p_{\lambda\mu}^* \otimes i_{\lambda\mu}} V(\mu)^* \otimes V(\lambda)^* \otimes V(\lambda) \otimes V(\mu) \xrightarrow{ev_\lambda} V(\mu)^* \otimes V(\mu).$$

Notice that T_λ maps $v^{\mu+\lambda} \otimes v_{\mu+\lambda}$ to $v^\mu \otimes v_\mu$. Since the vector $v^\mu \otimes v_\mu$ generates $V(\mu)^* \otimes V(\mu)$ as a $U_q(\mathfrak{g})$ -module we see that T_λ is surjective. We also observe that $T_\lambda \circ T_{\lambda'} = T_{\lambda+\lambda'}$ for any $\lambda, \lambda' \in \mathbf{P}^+$, as a map from $V(\mu + \lambda + \lambda')^* \otimes V(\mu + \lambda + \lambda')$ to $V(\mu)^* \otimes V(\mu)$.

The following result is a translation of Proposition 27.3.5 in [57].

Proposition 2.149. *The map T_λ is a morphism of based modules.*

For any $\nu \in \mathbf{P}^+$ we shall identify $V(\nu) \cong V(\nu)^{**}$ such that v_ν is mapped to the linear form which evaluates v^ν to 1, and consider the corresponding $U_q(\mathfrak{g})$ -linear isomorphism

$$(V(\nu)^* \otimes V(\nu))^* \cong V(\nu)^* \otimes V(\nu)^{**} \cong V(\nu)^* \otimes V(\nu).$$

Then the transpose of T_λ identifies with a $U_q(\mathfrak{g})$ -linear map $\delta_\lambda : V(\mu)^* \otimes V(\mu) \rightarrow V(\mu + \lambda)^* \otimes V(\mu + \lambda)$ satisfying

$$\delta_\lambda(v^\mu \otimes v_\mu)(v^{\mu+\lambda} \otimes v_{\mu+\lambda}) = 1.$$

More precisely, we have

$$\begin{aligned} \delta_\lambda(v^\mu \otimes v_\mu) &= v^{\mu+\lambda} \otimes v_{\mu+\lambda} \\ &= i_{\lambda\mu}^*(v^\mu \otimes v^\lambda) \otimes p_{\lambda\mu}(v_\lambda \otimes v_\mu) \\ &= (i_{\lambda\mu}^* \otimes p_{\lambda\mu})(\text{id} \otimes ev_\lambda^* \otimes \text{id})(v^\mu \otimes v_\mu) \end{aligned}$$

with these conventions.

Consider the partial order on \mathbf{P}^+ given by declaring $\mu \preceq \nu$ if $\nu - \mu \in \mathbf{P}^+$. Note that \mathbf{P}^+ with this partial order is a lattice, in particular, every pair $\mu, \nu \in \mathbf{P}^+$ has a least upper bound.

The following Lemma is a reformulation of Lemma 5 in [10], which is based in turn on Proposition 25.1.10 in [57].

Lemma 2.150. *Let $\mu \in \mathbf{P}^+$. For each vector y in the canonical basis of the based module $V(\mu)^* \otimes V(\mu)$ there exists a greatest element $\lambda(y)$ in the lattice \mathbf{P}^+ with respect to the partial order \preceq such that $T_{\lambda(y)}(y) \neq 0$.*

Note that the weight $\lambda(y)$ in Lemma 2.150 is uniquely determined by y , due to the fact that greatest elements in partially ordered sets are necessarily unique.

2.16.2. *Further preliminaries.* In this subsection we collect some additional preparations for the proof of the main theorem presented in the next subsection. Throughout this subsection the ground field \mathbb{K} is arbitrary and $q = s^L \in \mathbb{K}^\times$ is not a root of unity.

Let us first discuss a certain filtration of $\mathcal{O}(G_q)$. We define

$$ht(\mu) = \mu_1 + \cdots + \mu_N$$

if $\mu = \mu_1\alpha_1 + \cdots + \mu_N\alpha_N \in \mathbf{P}$. For $\mu \in \mathbf{Q}$ this agrees with the height as in the proof of Proposition 2.134, however, we allow here arbitrary $\mu \in \mathbf{P}$. It can be shown that $ht(\mu) \in \frac{1}{2}\mathbb{N}_0$ for all $\mu \in \mathbf{P}^+$, compare Table 1 in Section 13.2 of [35].

We obtain an algebra filtration of $A = \mathcal{O}(G_q)$ indexed by $\frac{1}{2}\mathbb{N}_0$ by setting

$$\mathcal{F}^m(A) = \bigoplus_{ht(\mu) \leq m} V(\mu)^* \otimes V(\mu)$$

for $m \in \frac{1}{2}\mathbb{N}_0$. We call this filtration the height filtration of $\mathcal{O}(G_q)$.

The associated graded algebra $E = gr_{\mathcal{F}}(\mathcal{O}(G_q))$ is canonically isomorphic to $\mathcal{O}(G_q)$ as a left and right $U_q(\mathfrak{g})$ -module, and thus also as a $U_q(\mathfrak{g})$ -module with the coadjoint action from Section 2.12. We will always view $\mathcal{O}(G_q)$ and E as a $U_q(\mathfrak{g})$ -module with this action in the sequel.

Let us write

$$E = \bigoplus_{\mu \in \mathbf{P}^+} E(\mu)$$

with $E(\mu) = V(\mu)^* \otimes V(\mu)$. Using the maps $p_{\mu\lambda}, i_{\mu\lambda}$ introduced before Proposition 2.149 we can write the product $x \cdot y \in E$ of elements $x, y \in E$ as follows.

Lemma 2.151. *With the notation as above, the multiplication of E restricts to linear maps $E(\mu) \otimes E(\lambda) \rightarrow E(\mu + \lambda)$ for $\mu, \lambda \in \mathbf{P}^+$, explicitly given by the formula*

$$\langle f | \bullet | v \rangle \cdot \langle g | \bullet | w \rangle = \langle i_{\lambda\mu}^*(f \otimes g) | \bullet | p_{\lambda\mu}(w \otimes v) \rangle$$

for $f \in V(\mu)^*, v \in V(\mu), g \in V(\lambda)^*, w \in V(\lambda)$.

Moreover, right multiplication with the quantum trace $t_\lambda \in E(\lambda)$ identifies with the map δ_λ defined above, up to multiplication by $q^{(2\rho, \lambda)}$.

Proof. Using the identification of E with $\mathcal{O}(G_q)$ as vector spaces and the pairing between $U_q(\mathfrak{g})$ and $\mathcal{O}(G_q)$ we have

$$\begin{aligned} (X, \langle f | \bullet | v \rangle \cdot \langle g | \bullet | w \rangle) &= (g \otimes f)(i_{\lambda\mu}(X \cdot p_{\lambda\mu}(w \otimes v))) \\ &= (X, \langle i_{\lambda\mu}^*(f \otimes g) | \bullet | p_{\lambda\mu}(w \otimes v) \rangle) \end{aligned}$$

for $X \in U_q(\mathfrak{g})$, since multiplication in E only takes into account the highest weight component of the product in $\mathcal{O}(G_q)$.

Next, recall that in our present conventions we identify $V(\lambda)^{**}$ with $V(\lambda)$ such that highest weight vector $v_\lambda \in V(\lambda)$ pairs with $v^\lambda \in V(\lambda)^*$ to give 1. This

means, more generally, that a vector $v \in V(\lambda)$ corresponds to the form on $V(\lambda)^*$ given by evaluation on $q^{(-2\rho, \lambda)} K_{2\rho} \cdot v$. It follows that the quantum trace $t_\lambda \in V(\lambda)^* \otimes V(\lambda)$ from Definition 2.118 evaluates on $f \otimes v \in V(\lambda)^* \otimes V(\lambda)$ as $t_\lambda(f \otimes v) = q^{(-2\rho, \lambda)} ev_\lambda(f \otimes v)$. That is, we have $ev_\lambda^*(1) = q^{(2\rho, \lambda)} t_\lambda$, and combining this with the computations at the end of the previous subsection yields the claim. \square

Let us next review some facts about graded vector spaces. Recall that if A, B are vector spaces filtered by \mathbb{N}_0 then a linear map $f : A \rightarrow B$ is called a morphism of filtered vector spaces if $f(\mathcal{F}^n(A)) \subset \mathcal{F}^n(B)$ for all n . In this case f induces a map $gr(f) : gr(A) \rightarrow gr(B)$ of the associated graded vector spaces. Moreover, f is an isomorphism provided $gr(f)$ is an isomorphism.

If A, B are vector spaces filtered by \mathbb{N}_0 we obtain a filtration of $A \otimes B$ by setting

$$\mathcal{F}^n(A \otimes B) = \sum_{k+l=n} \mathcal{F}^k(A) \otimes \mathcal{F}^l(B).$$

Moreover, the canonical map $gr(A) \otimes gr(B) \rightarrow gr(A \otimes B)$ is an isomorphism in this case.

Combining these observations we obtain the following basic fact.

Lemma 2.152. *Let H, Z and A be \mathbb{N}_0 -filtered vector spaces and let $m : H \otimes Z \rightarrow A$ be a morphism of filtered vector spaces. If the induced map $gr(m) : gr(H) \otimes gr(Z) \rightarrow gr(A)$ is an isomorphism, then m is an isomorphism as well.*

Finally, we need a result on the structure of certain mapping spaces.

Lemma 2.153. *Let $\mu, \nu, \lambda \in \mathbf{P}^+$. Then the linear map*

$$\phi : \text{Hom}_{U_q(\mathfrak{g})}(V(\lambda) \otimes V(\nu)^*, V(\mu)) \rightarrow V(\mu)_{\lambda-\nu}$$

given by $\phi(f) = f(v_\lambda \otimes v^\nu)$ is injective and

$$\text{im}(\phi) = \{v \in V(\mu)_{\lambda-\nu} \mid E_i^{(\nu, \alpha_i^\vee)+1} \cdot v = 0 \text{ for all } i = 1, \dots, N\}.$$

Proof. From weight considerations we see that ϕ is well-defined, that is, $f(v_\lambda \otimes v^\nu)$ is indeed contained in $V(\mu)_{\lambda-\nu}$ for $f \in \text{Hom}_{U_q(\mathfrak{g})}(V(\lambda) \otimes V(\nu)^*, V(\mu))$. Since $v_\lambda \otimes v^\nu$ is a cyclic vector for $V(\lambda) \otimes V(\nu)^*$ it follows that ϕ is injective.

In order to determine the image of ϕ let us abbreviate

$$U = \{v \in V(\mu)_{\lambda-\nu} \mid E_i^{(\nu, \alpha_i^\vee)+1} \cdot v = 0 \text{ for all } i = 1, \dots, N\}.$$

We observe that $E_i^k \cdot (v_\lambda \otimes v^\nu) = v_\lambda \otimes E_i^k \cdot v^\nu$ for all $k \in \mathbb{N}_0$, hence the smallest power of E_i killing $v_\lambda \otimes v^\nu$ agrees with the smallest power of E_i killing v^ν . This is determined by the weight of ν with respect to $U_{q_i}(\mathfrak{g}_i) \subset U_q(\mathfrak{g})$, and equals $E_i^{(\nu, \alpha_i^\vee)+1}$. It follows that $\text{im}(\phi)$ is indeed contained in U .

To prove the reverse inclusion $U \subset \text{im}(\phi)$ we construct a linear map

$$\psi : U \rightarrow \text{Hom}_{U_q(\mathfrak{g})}(V(\lambda) \otimes V(\nu)^*, V(\mu)) \cong \text{Hom}_{U_q(\mathfrak{g})}(V(\lambda), \text{Hom}(V(\nu)^*, V(\mu)))$$

as follows. Let $u \in U$ be given and define a $U_q(\mathfrak{n}_+)$ -linear map $T(u) : N(-\nu) \rightarrow V(\mu)$ by $T(u)(X \cdot v_{-\nu}) = X \cdot u$ for $X \in U_q(\mathfrak{n}_+)$. Here $N(-\nu)$ is the universal lowest weight module with lowest weight $-\nu$ generated by the lowest weight vector $v_{-\nu}$. Since $E_i^{(\nu, \alpha_i^\vee)+1} \cdot u = 0$ for all $i = 1, \dots, N$, it follows from Theorem 2.97 that $T(u)$ factorizes through $V(\nu)^*$. Moreover, for any $v \in V(\nu)^*$ we have

$$\begin{aligned} (K_\eta \rightarrow T(u))(v) &= q^{(\eta, \nu) + (\eta, \lambda - \nu)} T(u)(v) = q^{(\eta, \lambda)} T(u)(v), \\ (E_i \rightarrow T(u))(v) &= E_i \cdot T(u)(K_i^{-1} \cdot v) - T(u)(E_i K_i^{-1} \cdot v) = 0 \end{aligned}$$

for $\eta \in \mathbf{P}$ and $i = 1, \dots, N$. We conclude that $T(u) \in \text{Hom}(V(\nu)^*, V(\mu))$ is a highest weight vector of weight λ . Since $\text{Hom}(V(\nu)^*, V(\mu))$ is integrable, Theorem 2.97 shows that setting $\psi(u)(v_\lambda) = T(u)$ determines a well-defined element

$\psi(u)$ of $\text{Hom}_{U_q(\mathfrak{g})}(V(\lambda), \text{Hom}(V(\nu)^*, V(\mu)))$. Using the canonical identification with $\text{Hom}_{U_q(\mathfrak{g})}(V(\lambda) \otimes V(\nu)^*, V(\mu))$ it is straightforward to check that $\phi(\psi(u)) = u$, and we thus conclude $\text{im}(\phi) = U$ as desired. \square

As a consequence of Lemma 2.153 we see that $\text{Hom}_{U_q(\mathfrak{g})}(\text{End}(V(\lambda)), V(\mu))$ for $\mu, \lambda \in \mathbf{P}^+$ can be identified with the subspace of $V(\mu)_0$ consisting of all vectors v satisfying $E_i^{(\lambda, \alpha_i^\vee)+1} \cdot v = 0$ for $i = 1, \dots, N$. If the coefficients m_1, \dots, m_N in $\lambda = \sum_{i=1}^N m_i \varpi_i$ are sufficiently large the latter condition becomes vacuous. In particular, we have $[\text{End}(V(\lambda)) : V(\mu)] = \dim(V(\mu)_0)$ for all λ of the form $\lambda = \mu + \eta$ for $\eta \in \mathbf{P}^+$.

2.16.3. Separation of Variables. We shall now prove the main result of this section, originally due to Joseph and Letzter [43]. As mentioned above, we follow closely the approach by Baumann [8], [10].

Theorem 2.154 (Separation of Variables). *Assume $q \in \mathbb{K}^\times$ is not a root of unity. There exists a linear subspace $\mathbb{H} \subset FU_q(\mathfrak{g})$, invariant under the adjoint action, such that the multiplication map $\mathbb{H} \otimes ZU_q(\mathfrak{g}) \rightarrow FU_q(\mathfrak{g})$ is an isomorphism. Moreover, for any $\mu \in \mathbf{P}^+$ we have $[\mathbb{H} : V(\mu)] = \dim(V(\mu)_0)$ for the multiplicity of the isotypical component of type μ of \mathbb{H} .*

Proof. Due to Theorem 2.116 and the remarks following it, it suffices to find a linear subspace $H \subset \mathcal{O}(G_q)$, invariant under the coadjoint action, such that the multiplication map $H \otimes \mathcal{O}(G_q)^{G_q} \rightarrow \mathcal{O}(G_q)$ is an isomorphism.

As explained in the proof of Theorem 2.120, the algebra $Z = \mathcal{O}(G_q)^{G_q}$ has a linear basis consisting of the quantum traces t_λ for $\lambda \in \mathbf{P}^+$, and can be identified with the polynomial algebra $\mathbb{K}[t_{\varpi_1}, \dots, t_{\varpi_N}]$. In view of Lemma 2.151 it will be convenient to rescale t_λ and work with $\theta_\lambda = q^{(2\rho, \lambda)} t_\lambda$ instead, and to identify $Z \cong \mathbb{K}[\theta_{\varpi_1}, \dots, \theta_{\varpi_N}]$. If \mathcal{F} denotes the height filtration of $\mathcal{O}(G_q)$ discussed in the previous subsection, then each θ_λ can also be viewed as an element of the associated graded algebra $E = gr_{\mathcal{F}}(\mathcal{O}(G_q))$ in a natural way. We have $\theta_\lambda \theta_\eta = \theta_{\lambda+\eta}$ in $gr_{\mathcal{F}}(\mathcal{O}(G_q))$ for all $\lambda, \eta \in \mathbf{P}^+$, and $Z = gr_{\mathcal{F}}(\mathcal{O}(G_q)^{G_q})$ can be identified with the polynomial algebra $\mathbb{K}[\theta_{\varpi_1}, \dots, \theta_{\varpi_N}]$ as well.

Let $\mu \in \mathbf{P}^+$ and recall the notation $E(\mu) = V(\mu)^* \otimes V(\mu) \subset E$ introduced before Lemma 2.151. According to Lemma 2.151, the map $\delta_\lambda : V(\mu)^* \otimes V(\mu) \rightarrow V(\lambda + \mu)^* \otimes V(\lambda + \mu)$ obtained by the transposition of T_λ identifies with the map $m_\lambda : E(\mu) \rightarrow E(\lambda + \mu)$ given by $m_\lambda(x) = x \cdot \theta_\lambda$ for any $\lambda \in \mathbf{P}^+$. We note that m_λ is injective since T_λ is surjective.

Let us momentarily work over $\mathbb{K} = \mathbb{Q}(s)$. With the identification $E(\mu) \cong V(\mu)^* \otimes V(\mu)^{**} \cong (V(\mu)^* \otimes V(\mu))^*$ used in the discussion after Proposition 2.149, we obtain a linear basis $B(\mu)$ of $E(\mu)$ dual to the canonical basis of $V(\mu)^* \otimes V(\mu)$. If we set

$$B = \bigcup_{\mu \in \mathbf{P}^+} B(\mu),$$

then Proposition 2.149 shows that m_λ restricts to an injective map $m_\lambda : B \rightarrow B$ for all $\lambda \in \mathbf{P}^+$.

If x is an element of the canonical basis of $V(\mu)^* \otimes V(\mu)$ let us write x^\vee for the dual element of B , determined by the relation $x^\vee(y) = \delta_{xy}$ for all canonical basis elements y . We define a map $\epsilon : B \rightarrow B$ by stipulating $\epsilon(x^\vee) = T_{\lambda(x)}(x)^\vee$, where $\lambda(x) \in \mathbf{P}^+$ is the weight determined in Lemma 2.150. Then we have

$$x^\vee(y) = \delta_{xy} = \delta_{T_{\lambda(x)}(x), T_{\lambda(x)}(y)} = T_{\lambda(x)}(x)^\vee(T_{\lambda(x)}(y)) = m_{\lambda(x)}(T_{\lambda(x)}(x)^\vee)(y)$$

for any y , so that $x^\vee = m_{\lambda(x)}(\epsilon(x^\vee)) = \epsilon(x^\vee) \cdot \theta_{\lambda(x)}$. Let us define

$$\begin{aligned} B_{\mathcal{H}} &= \{\epsilon(b) \mid b \in B\} \\ B_{\mathcal{Z}} &= \{\theta_\lambda \mid \lambda \in \mathbf{P}^+\}. \end{aligned}$$

Note that since $\epsilon(\epsilon(b)) = \epsilon(b)$ for all $b \in B$ we can write $B_{\mathcal{H}} = \{b \in B \mid \epsilon(b) = b\}$. Now assume $x^\vee = y^\vee \cdot \theta_\lambda$ for some $y^\vee \in B_{\mathcal{H}}$ and $\theta_\lambda \in B_{\mathcal{Z}}$. Then $1 = x^\vee(x) = m_\lambda(y^\vee)(x) = y^\vee(T_\lambda(x))$ implies that $T_\lambda(x) = y$ is nonzero. Due to Lemma 2.150 we thus have $\lambda \preceq \lambda(x)$. Since $y^\vee \in B_{\mathcal{H}}$ we also get $T_{\lambda+\mu}(x) = T_\mu(y) = 0$ for any nonzero element $\mu \in \mathbf{P}^+$. This implies $\lambda = \lambda(x)$ and hence $y = T_{\lambda(x)}(x)$, or equivalently $y^\vee = \epsilon(x^\vee)$. Summarizing this discussion, we conclude that multiplication in E induces a bijection $m : B_{\mathcal{H}} \times B_{\mathcal{Z}} \rightarrow B$.

Let $E_{\mathcal{A}} \subset E$ be the \mathcal{A} -linear span of the basis B . Similarly, let $\mathcal{H}_{\mathcal{A}} \subset E_{\mathcal{A}}$ the \mathcal{A} -linear span of $B_{\mathcal{H}}$, and let $\mathcal{Z}_{\mathcal{A}} \subset E$ be the \mathcal{A} -linear span of $B_{\mathcal{Z}}$. From our above considerations we conclude that multiplication in E induces an isomorphism

$$\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{Z}_{\mathcal{A}} \rightarrow E_{\mathcal{A}}$$

of \mathcal{A} -modules.

Now let \mathbb{K} be again an arbitrary field and $q = s^L \in \mathbb{K}^\times$ not a root of unity. We write $\mathcal{O}(G_q)$ for the \mathbb{K} -algebra of matrix coefficients, and let $Z = \mathcal{O}(G_q)^{G_q} \subset \mathcal{O}(G_q)$ be the invariant subspace with respect to the coadjoint action. As before we denote by $E = gr_{\mathcal{F}}(\mathcal{O}(G_q))$ the associated graded algebra for the height filtration of $\mathcal{O}(G_q)$. The canonical bases from Theorem 2.148 determine \mathbb{K} -bases for the $U_q(\mathfrak{g})$ -modules $V(\mu)^* \otimes V(\mu)$, and we write $B_{\mathbb{K}}$ for the dual \mathbb{K} -basis of $\mathcal{O}(G_q)$, and hence of E , which is obtained in this way. Mapping the elements $b \in B \subset E_{\mathcal{A}}$ to the corresponding basis vectors of $B_{\mathbb{K}}$ in E yields a \mathbb{K} -algebra isomorphism $E_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K} \cong E$ by Lemma 2.151.

Let $H \subset \mathcal{O}(G_q)$ be the \mathbb{K} -linear span of the image of $B_{\mathcal{H}}$ in $\mathcal{O}(G_q)$. If $\mathcal{H} = gr_{\mathcal{F}}(H) \subset E$ denotes the associated graded of H and $\mathcal{Z} = gr_{\mathcal{F}}(Z) \subset E$ the associated graded of Z , then we have $\mathcal{Z}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K} \cong \mathcal{Z}$ and $\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K} \cong \mathcal{H}$ under the identification $E_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{K} \cong E$. Hence the isomorphism $\mathcal{H}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{Z}_{\mathcal{A}} \rightarrow E_{\mathcal{A}}$ of \mathcal{A} -modules induces an isomorphism

$$\mathcal{H} \otimes \mathcal{Z} \rightarrow E$$

of \mathbb{K} -vector spaces, given by multiplication in E . Therefore, we may apply Lemma 2.152 to conclude that multiplication in $\mathcal{O}(G_q)$ induces an isomorphism

$$H \otimes \mathcal{O}(G_q)^{G_q} = H \otimes Z \rightarrow \mathcal{O}(G_q)$$

as required.

In order to determine the multiplicity of $V(\mu)$ inside H observe that $[H : V(\mu)] = [gr_{\mathcal{F}}(H) : V(\mu)] = [\mathcal{H} : V(\mu)]$. Thanks to the decomposition $B_{\mathcal{H}} \times B_{\mathcal{Z}} \cong B$ described above, for any $\lambda \in \mathbf{P}^+$ the canonical basis vectors $b \in B(\lambda)$ which belong to submodules of $E(\lambda)$ of highest weight $\mu \in \mathbf{P}^+$ can all be written in the form $b = \epsilon(b) \cdot \theta_\nu$ for some $\nu \in \mathbf{P}^+$, where $\epsilon(b) \in B_{\mathcal{H}}$ belongs to a submodule of \mathcal{H} of highest weight μ . It follows that the multiplicity $[\mathcal{H} : V(\mu)]$ is given by the maximal value of $[\text{End}(V(\lambda)) : V(\mu)]$ as λ runs through \mathbf{P}^+ . According to the remarks after Lemma 2.153, we thus obtain $[\mathcal{H} : V(\mu)] = \dim(V(\mu))_0$. This finishes the proof. \square

We remark that each basis element of the space $\mathbb{H} \subset FU_q(\mathfrak{g})$ corresponding to $H \subset \mathcal{O}(G_q)$ constructed in the proof of Theorem 2.154 is contained in a subspace of the form $U_q(\mathfrak{g}) \rightarrow K_{2\lambda}$ for some $\lambda \in \mathbf{P}^+$. Note finally that since $ZU_q(\mathfrak{g}) \subset FU_q(\mathfrak{g})$ commutes pointwise with all elements of $FU_q(\mathfrak{g})$, Theorem 2.154 shows that multiplication also induces a $U_q(\mathfrak{g})$ -linear isomorphism $ZU_q(\mathfrak{g}) \otimes \mathbb{H} \cong FU_q(\mathfrak{g})$.

3. COMPLEX SEMISIMPLE QUANTUM GROUPS

In this chapter we introduce our main object of study, namely complex semisimple quantum groups. We complement the discussion with some background material on locally compact quantum groups in general, and on compact quantum groups arising from q -deformations in particular.

Throughout this chapter we work over the complex numbers \mathbb{C} . If \mathcal{H} is a Hilbert space we write $\mathbb{L}(\mathcal{H})$ for the algebra of bounded operators on \mathcal{H} , and denote by $\mathbb{K}(\mathcal{H})$ the algebra of compact operators. If A is a C^* -algebra we write $M(A)$ for the multiplier algebra of A in the sense of C^* -algebras; this needs to be distinguished from the algebraic multiplier algebra $\mathcal{M}(A)$ in chapter 1. By slight abuse of notation, we use the symbol \otimes to denote algebraic tensor products, tensor products of Hilbert spaces, minimal tensor products of C^* -algebras, or spatial tensor products of von Neumann algebras. It should always be clear from the context which tensor product is used. If X is a subset of a Banach space B we write $[X] \subset B$ for the closed linear span of X . For general background on C^* -algebras and von Neumann algebras we refer to [61].

3.1. Locally compact quantum groups. In this section we review some basic definitions and facts from the theory of locally compact quantum groups [54].

3.1.1. *Hopf C^* -algebras.* Let us start with basic definitions and constructions related to Hopf C^* -algebras.

Definition 3.1. A Hopf C^* -algebra is a C^* -algebra H together with an injective nondegenerate $*$ -homomorphism $\Delta : H \rightarrow M(H \otimes H)$ such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & M(H \otimes H) \\ \downarrow & & \downarrow \text{id} \otimes \Delta \\ M(H \otimes H) & \xrightarrow{\Delta \otimes \text{id}} & M(H \otimes H \otimes H) \end{array}$$

is commutative and $[\Delta(H)(1 \otimes H)] = H \otimes H = [(H \otimes 1)\Delta(H)]$.

Comparing Definition 3.1 with the algebraic definition of a multiplier Hopf algebra in Definition 1.6, we note that the density conditions in the former can be thought of as a replacement of the requirement that the Galois maps are isomorphisms in the latter.

If H is a Hopf C^* -algebra we write H^{cop} for the Hopf- C^* -algebra obtained by equipping H with the opposite comultiplication $\Delta^{\text{cop}} = \sigma\Delta$.

A unitary corepresentation of a Hopf- C^* -algebra H on a Hilbert space \mathcal{E} is a unitary $X \in M(H \otimes \mathbb{K}(\mathcal{E}))$ satisfying

$$(\Delta \otimes \text{id})(X) = X_{13}X_{23}$$

in $M(H \otimes H \otimes \mathbb{K}(\mathcal{E}))$, where we are using leg-numbering notation as described after Definition 2.103. More generally, we may define a corepresentation of H with values in a C^* -algebra A as a unitary $X \in M(H \otimes A)$ satisfying $(\Delta \otimes \text{id})(X) = X_{13}X_{23}$.

A universal dual of H is a Hopf- C^* -algebra \hat{H} together with a unitary corepresentation $\mathcal{X} \in M(H \otimes \hat{H})$ of H with values in \hat{H} satisfying the following universal property: for every Hilbert space \mathcal{E} and every unitary corepresentation $X \in M(H \otimes \mathbb{K}(\mathcal{E}))$ there exists a unique nondegenerate $*$ -homomorphism $\pi_X : \hat{H} \rightarrow \mathbb{L}(\mathcal{E})$ such that $(\text{id} \otimes \pi_X)(\mathcal{X}) = X$.

We shall be exclusively interested in Hopf C^* -algebras arising from locally compact quantum groups.

3.1.2. *The definition of locally compact quantum groups.* The theory of locally compact quantum groups has been axiomatized by Kustermans and Vaes [54].

Let ϕ be a normal, semifinite and faithful weight on a von Neumann algebra M . We use the standard notation

$$\mathcal{M}_\phi^+ = \{x \in M_+ | \phi(x) < \infty\}, \quad \mathcal{N}_\phi = \{x \in M | \phi(x^*x) < \infty\}$$

and write M_*^+ for the space of positive normal linear functionals on M . Assume that $\Delta : M \rightarrow M \otimes M$ is a normal unital $*$ -homomorphism. The weight ϕ is called left invariant with respect to Δ if

$$\phi((\omega \otimes \text{id})\Delta(x)) = \phi(x)\omega(1)$$

for all $x \in \mathcal{M}_\phi^+$ and $\omega \in M_*^+$. Similarly, a normal semifinite faithful weight ψ is right-invariant if

$$\psi((\text{id} \otimes \omega)\Delta(x)) = \psi(x)\omega(1)$$

for all $x \in \mathcal{M}_\phi^+$ and $\omega \in M_*^+$.

Definition 3.2. A locally compact quantum group G is given by a von Neumann algebra $L^\infty(G)$ together with a normal unital $*$ -homomorphism $\Delta : L^\infty(G) \rightarrow L^\infty(G) \otimes L^\infty(G)$ satisfying the coassociativity relation

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta,$$

and normal semifinite faithful weights ϕ and ψ on $L^\infty(G)$ which are left and right invariant, respectively. The weights ϕ and ψ are called the left and right Haar weights of G .

Our notation for locally compact quantum groups is intended to make clear how ordinary locally compact groups can be viewed as quantum groups. Indeed, if G is a locally compact group, then the algebra $L^\infty(G)$ of essentially bounded measurable functions on G together with the comultiplication $\Delta : L^\infty(G) \rightarrow L^\infty(G) \otimes L^\infty(G)$ given by

$$\Delta(f)(s, t) = f(st)$$

defines a locally compact quantum group. The weights ϕ and ψ are given by integration with respect to left and right Haar measures, respectively.

For a general locally compact quantum group G the notation $L^\infty(G)$ is purely formal. Similar remarks apply to the C^* -algebras $C_f^*(G)$, $C_r^*(G)$ and $C_0^f(G)$, $C_0^r(G)$ associated to G that we discuss below. It is convenient to view all of them as different incarnations of the quantum group G .

Let G be a locally compact quantum group and let $\Lambda : \mathcal{N}_\phi \rightarrow L^2(G)$ be the GNS-construction for the weight ϕ . Throughout we shall only consider quantum groups for which $L^2(G)$ is a separable Hilbert space. One obtains a unitary $W_G = W$ on $L^2(G) \otimes L^2(G)$ such that

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))$$

for all $x, y \in \mathcal{N}_\phi$. This unitary is multiplicative, which means that W satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

From W one can recover the von Neumann algebra $L^\infty(G)$ as the strong closure of the algebra $(\text{id} \otimes \mathbb{L}(L^2(G))_*)(W)$ where $\mathbb{L}(L^2(G))_*$ denotes the space of normal linear functionals on $\mathbb{L}(L^2(G))$. Moreover one has

$$\Delta(x) = W^*(1 \otimes x)W$$

for all $x \in M$. The algebra $L^\infty(G)$ has an antipode which is an unbounded, σ -strong* closed linear map S given by $S(\text{id} \otimes \omega)(W) = (\text{id} \otimes \omega)(W^*)$ for $\omega \in \mathbb{L}(L^2(G))_*$. Moreover there is a polar decomposition $S = R\tau_{-i/2}$ where R is an

antiautomorphism of $L^\infty(G)$ called the unitary antipode and (τ_t) is a strongly continuous one-parameter group of automorphisms of $L^\infty(G)$ called the scaling group. The unitary antipode satisfies $\sigma(R \otimes R)\Delta = \Delta R$.

The group-von Neumann algebra $\mathcal{L}(G)$ of the quantum group G is the strong closure of the algebra $(\mathbb{L}(L^2(G))_* \otimes \text{id})(W)$ with the comultiplication $\hat{\Delta} : \mathcal{L}(G) \rightarrow \mathcal{L}(G) \otimes \mathcal{L}(G)$ given by

$$\hat{\Delta}(y) = \hat{W}^*(1 \otimes y)\hat{W},$$

where $\hat{W} = \Sigma W^* \Sigma$ and $\Sigma \in \mathbb{L}(L^2(G) \otimes L^2(G))$ is the flip map. It defines a locally compact quantum group \hat{G} which is called the dual of G . The GNS construction for the left invariant weight $\hat{\phi}$ of the dual quantum group can be identified with a map $\hat{\Lambda} : \mathcal{N}_{\hat{\phi}} \rightarrow L^2(G)$ such that we have $\mathcal{L}(G) = L^\infty(\hat{G})$.

We will mainly work with the C^* -algebras associated to the locally compact quantum group G . The algebra $[(\text{id} \otimes \mathbb{L}(L^2(G))_*)(W)]$ is a strongly dense C^* -subalgebra of $L^\infty(G)$ which we denote by $C_0^r(G)$. Dually, the algebra $[(\mathbb{L}(L^2(G))_* \otimes \text{id})(W)]$ is a strongly dense C^* -subalgebra of $\mathcal{L}(G)$ which we denote by $C_r^*(G)$. These algebras are called the reduced algebra of continuous functions vanishing at infinity on G and the reduced group C^* -algebra of G , respectively. One has $W \in M(C_0^r(G) \otimes C_r^*(G))$. Restriction of the comultiplications on $L^\infty(G)$ and $\mathcal{L}(G)$ turns $C_0^r(G)$ and $C_r^*(G)$ into Hopf C^* -algebras.

For every locally compact quantum group G there exists a universal dual $C_f^*(G)$ of $C_0^r(G)$ and a universal dual $C_0^f(G)$ of $C_r^*(G)$, respectively [52]. We call $C_f^*(G)$ the maximal group C^* -algebra of G and $C_0^f(G)$ the maximal algebra of continuous functions on G vanishing at infinity. Since $L^2(G)$ is assumed to be separable the C^* -algebras $C_0^f(G)$, $C_0^r(G)$ and $C_f^*(G)$, $C_r^*(G)$ are separable. The quantum group G is called compact if $C_0^f(G)$ is unital, and it is called discrete if $C_f^*(G)$ is unital. In the compact case we also write $C^f(G)$ and $C^r(G)$ instead of $C_0^f(G)$ and $C_0^r(G)$, respectively.

In general, we have a surjective morphism $\hat{\pi} : C_f^*(G) \rightarrow C_r^*(G)$ of Hopf- C^* -algebras associated to the left regular corepresentation $W \in M(C_0^r(G) \otimes C_r^*(G))$. Similarly, there is a surjective morphism $\pi : C_0^f(G) \rightarrow C_0^r(G)$. We will call the quantum group G amenable if $\hat{\pi} : C_f^*(G) \rightarrow C_r^*(G)$ is an isomorphism and coamenable if $\pi : C_0^f(G) \rightarrow C_0^r(G)$ is an isomorphism. If G is amenable or coamenable, respectively, we also write $C^*(G)$ or $C_0(G)$ for the corresponding C^* -algebras. For more information on amenability for locally compact quantum groups see [11].

3.2. Algebraic quantum groups. The analytical theory of locally compact quantum groups simplifies considerably if one restricts attention to examples that are essentially determined algebraically. This is the case for the class of *algebraic quantum group* in the sense of van Daele [74]. The concept of an algebraic quantum group is a variant of the notion of a regular multiplier Hopf algebra with integrals in which one adds $*$ -structures.

3.2.1. The definition of algebraic quantum groups. Recall the notion of a regular multiplier Hopf algebra with integrals from Definitions 1.6 and 1.9. Let us now introduce the notion of a multiplier Hopf $*$ -algebra [74].

Definition 3.3. A multiplier Hopf $*$ -algebra is a regular multiplier Hopf algebra H which is equipped with a $*$ -structure such that $\Delta : H \rightarrow \mathcal{M}(H \otimes H)$ is a $*$ -homomorphism.

Let H be a multiplier Hopf $*$ -algebra. Then the counit $\epsilon : H \rightarrow \mathbb{C}$ is a $*$ -homomorphism and the antipode $S : H \rightarrow H$ satisfies

$$S(S(f^*)^*) = f$$

for all $f \in H$, see Section 5 in [73]. Let us remark that the regularity condition in the definition of a regular multiplier Hopf algebra is in fact automatic in the $*$ -algebraic situation.

Definition 3.4. An algebraic quantum group is a multiplier Hopf $*$ -algebra $H = \mathfrak{C}_c^\infty(G)$ such that there exists a positive left invariant integral $\phi : H \rightarrow \mathbb{C}$ and a positive right invariant integral $\psi : H \rightarrow \mathbb{C}$. We will also refer to the virtual object G as an algebraic quantum group.

Here a linear functional $\omega : H \rightarrow \mathbb{C}$ is called positive if $\omega(f^*f) \geq 0$ for all $f \in H$. We note that (positive) left/right invariant integrals are always unique up to a (positive) scalar, see Section 3 in [74].

In a similar way as in the definition of a locally compact quantum group, our notation $H = \mathfrak{C}_c^\infty(G)$ is meant to suggest that H should be thought of as an algebra of compactly supported smooth functions on an underlying object G , and by slight abuse of language which we will sometimes also refer to the latter as an algebraic quantum group. In contrast to the situation for locally compact quantum groups the situation is not quite as clean here; for instance, if G is a Lie group then the algebra $C_c^\infty(G)$ is typically not a multiplier Hopf algebra.

The duality theory for a regular multiplier Hopf algebra with integrals H discussed in Section 1.3 is compatible with the positivity requirement for Haar functionals, see [74]. In particular, if H is an algebraic quantum group and we consider the $*$ -structure on \hat{H} defined by

$$(x^*, f) = \overline{(x, S(f)^*)}$$

for $f \in H$ and $x \in \hat{H}$, then the dual \hat{H} is an algebraic quantum group as well. When using the notation $H = \mathfrak{C}_c^\infty(G)$ we will write either $\hat{H} = \mathfrak{D}(G)$, which we refer to as the group algebra of G , or $\hat{H} = \mathfrak{C}_c^\infty(\hat{G})$, where \hat{G} is called the Pontrjagin dual of G .

One has the following version of Theorem 1.14.

Theorem 3.5 (Biduality Theorem for algebraic quantum groups). *Let H be an algebraic quantum group. Then the dual of \hat{H} is isomorphic to H as an algebraic quantum group.*

Let us note that for an algebraic quantum group H , the modular element $\delta \in M(H)$ of Subsection 1.3.1, is a positive element, see [53]. We write $\hat{\delta} \in M(\hat{H})$ for the modular element of the dual \hat{H} .

3.2.2. Algebraic quantum groups on the Hilbert space level. In this subsection we explain how to associate a locally compact quantum group to any algebraic quantum group. A detailed exposition of this is material can be found in the work of Kustermans and van Daele [51], [53].

Let G be an algebraic quantum group and let $\phi : \mathfrak{C}_c^\infty(G) \rightarrow \mathbb{C}$ be a left invariant integral. We write $L^2(G)$ for the Hilbert space completion of $\mathfrak{C}_c^\infty(G)$ with respect to the scalar product

$$\langle f, g \rangle = \phi(f^*g),$$

and we let $\Lambda : \mathfrak{C}_c^\infty(G) \rightarrow L^2(G)$ be the GNS map. Then one can define a unitary operator W on $L^2(G) \otimes L^2(G)$ by

$$W(\Lambda(f) \otimes \Lambda(g)) = \Lambda(S^{-1}(g_{(1)})f) \otimes \Lambda(g_{(2)}),$$

using the inverse of the antipode S of $\mathfrak{C}_c^\infty(G)$. The inverse of W is given by

$$W^*(\Lambda(f) \otimes \Lambda(g)) = \Lambda(g_{(1)}f) \otimes \Lambda(g_{(2)}),$$

which formally agrees with the definition given in the case of locally compact quantum groups. It is straightforward to check that W is multiplicative, that is, we have the pentagon relation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ as in Section 3.1.

Note that the action of $\mathfrak{C}_c^\infty(G)$ on itself by left multiplication induces a $*$ -homomorphism $\lambda : \mathfrak{C}_c^\infty(G) \rightarrow \mathbb{L}(L^2(G))$, explicitly given by

$$\lambda(f)(\Lambda(g)) = \Lambda(fg)$$

for $f, g \in \mathfrak{C}_c^\infty(G)$. To see this we first claim that the left regular action of $\mathfrak{C}_c^\infty(G)$ on $\mathfrak{C}_c^\infty(G) \subset L^2(G)$ can be written in the form

$$\lambda(f) = (\text{id} \otimes \omega_{f_{(1)}^*, S(f_{(2)})\chi})(W),$$

where $\chi \in \mathfrak{C}_c^\infty(G)$ is any element satisfying $\phi(\chi) = 1$, and $\omega_{h,k}(T) = \langle \Lambda(h), T\Lambda(k) \rangle$ for all $h, k \in \mathfrak{C}_c^\infty(G)$. Indeed, we have

$$\begin{aligned} (\text{id} \otimes \omega_{f_{(1)}^*, S(f_{(2)})\chi})(W)\Lambda(h) &= \Lambda(S^{-1}(S(f_{(3)})\chi_{(1)})h)\langle f_{(1)}^*, S(f_{(2)})\chi_{(2)} \rangle \\ &= \Lambda(S^{-1}(S(f_{(3)})\chi_{(1)})h)\phi(f_{(1)}S(f_{(2)})\chi_{(2)}) \\ &= \Lambda(S^{-1}(\chi_{(1)})fh)\phi(\chi_{(2)}) \\ &= \Lambda(fh)\phi(\chi) = \lambda(f)\Lambda(h) \end{aligned}$$

for all $h \in \mathfrak{C}_c^\infty(G)$. In particular, $\lambda(f)$ extends naturally to a bounded operator on $L^2(G)$. It is then straightforward to check that λ yields in fact a faithful $*$ -representation of $\mathfrak{C}_c^\infty(G)$ on $L^2(G)$.

Moreover we have

$$\Delta(f) = W^*(1 \otimes f)W$$

for all $f \in \mathfrak{C}_c^\infty(G)$, where we identify f with $\lambda(f) \in \mathbb{L}(L^2(G))$. Indeed, one computes

$$\begin{aligned} (W^*(1 \otimes f)W)(\Lambda(g) \otimes \Lambda(h)) &= (W^*(1 \otimes f))(\Lambda(S^{-1}(h_{(1)})g) \otimes \Lambda(h_{(2)})) \\ &= W^*(\Lambda(S^{-1}(h_{(1)})g) \otimes \Lambda(fh_{(2)})) \\ &= \Lambda(f_{(1)}g) \otimes \Lambda(f_{(2)}h) \\ &= \Delta(f)(\Lambda(g) \otimes \Lambda(h)) \end{aligned}$$

for all $g, h \in \mathfrak{C}_c^\infty(G)$.

We shall next identify the dual multiplier Hopf algebra $\mathfrak{D}(G)$ inside $\mathbb{L}(L^2(G))$. Recall from Section 1.3 that $\mathfrak{C}_c^\infty(G)$ is linked with its dual $\mathfrak{D}(G)$ by Fourier transform. We will also write $\mathfrak{D}(G) = \mathfrak{C}_c^\infty(\hat{G})$ for the dual. We shall sometimes refer to $\mathfrak{D}(G)$ as the group algebra of the algebraic quantum group G .

Lemma 3.6. *Let G be an algebraic quantum group. The Fourier transform $\mathcal{F} : \mathfrak{C}_c^\infty(G) \rightarrow \mathfrak{D}(G)$ given by $\mathcal{F}(f)(h) = \phi(hf)$ induces an isometric linear isomorphism $L^2(G) \rightarrow L^2(\hat{G})$.*

Proof. For $f, g, h \in \mathfrak{C}_c^\infty(G)$ we have

$$\begin{aligned} (\mathcal{F}(f)^*\mathcal{F}(g), h) &= (\mathcal{F}(f)^*, h_{(1)})\phi(h_{(2)}g) \\ &= (\mathcal{F}(f)^*, h_{(1)}g_{(2)}S^{-1}(g_{(1)}))\phi(h_{(2)}g_{(3)}) \\ &= (\mathcal{F}(f)^*, S^{-1}(g_{(1)}))\phi(hg_{(2)}) \\ &= \overline{\phi(g_{(1)}^*f)}\phi(hg_{(2)}) \\ &= \phi(f^*g_{(1)})(\mathcal{F}(g_{(2)}), h). \end{aligned}$$

Hence we obtain

$$\hat{\phi}(\mathcal{F}(f)^*\mathcal{F}(g)) = \phi(f^*g_{(1)})\hat{\phi}(\mathcal{F}(g_{(2)})) = \phi(f^*g)$$

for all $f, g \in \mathfrak{C}_c^\infty(G)$. This shows that \mathcal{F} extends to an isometric isomorphism with respect to the canonical scalar products. \square

Using the Fourier transform from Lemma 3.6 we can transport the left regular representation of $\mathfrak{C}_c^\infty(\hat{G}) = \mathfrak{D}(G)$ on $L^2(\hat{G})$ to $L^2(G)$ as follows. Define a linear map $\hat{\lambda}$ from $\mathfrak{D}(G)$ to the space of linear endomorphisms of $\mathfrak{C}_c^\infty(G)$, viewed as a subspace of $L^2(G)$, by the formula

$$\hat{\lambda}(x)\Lambda(f) = (\hat{S}(x), f_{(1)})\Lambda(f_{(2)}) = (x, S^{-1}(f_{(1)}))\Lambda(f_{(2)}),$$

for $x \in \mathfrak{D}(G)$, $f \in \mathfrak{C}_c^\infty(G)$. Then for all $h \in \mathfrak{C}_c^\infty(G)$ we have

$$\begin{aligned} (\mathcal{F}(\hat{\lambda}(x)\Lambda(f)), h) &= (x, S^{-1}(f_{(1)}))\phi(hf_{(2)}) \\ &= (x, S^{-1}(S(h_{(1)})h_{(2)}f_{(1)}))\phi(h_{(3)}f_{(2)}) \\ &= (x, h_{(1)})\phi(h_{(2)}f) \\ &= (x, h_{(1)})(\mathcal{F}(f), h_{(2)}) \\ &= (x\mathcal{F}(f), h), \end{aligned}$$

which means that $\mathcal{F}\hat{\lambda}(x)\mathcal{F}^{-1}$ corresponds to the GNS-representation of $\mathfrak{C}_c^\infty(\hat{G})$ on $L^2(\hat{G})$. In particular, we obtain a faithful $*$ -representation $\hat{\lambda} : \mathfrak{D}(G) \rightarrow \mathbb{L}(L^2(G))$ using the above construction.

In terms of the multiplicative unitary W , the comultiplication $\hat{\Delta}$ for $\mathfrak{D}(G)$ is determined by the formula

$$\hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}$$

where $\hat{W} = \Sigma W^* \Sigma$, and we identify x with $\hat{\lambda}(x) \in \mathbb{L}(L^2(G))$.

Inspecting the above formulas we see that we obtain Hopf C^* -algebras $C_0^r(G)$ and $C_r^*(G) = C_0^r(\hat{G})$ inside $\mathbb{L}(L^2(G))$ by taking the closures of $\lambda(\mathfrak{C}_c^\infty(G))$ and $\hat{\lambda}(\mathfrak{D}(G))$. These algebras identify with the legs of the multiplicative unitary W . Moreover, these constructions are compatible with the multiplier Hopf algebra structures of $\mathfrak{C}_c^\infty(G)$ and $\mathfrak{D}(G)$, respectively. In a similar way one obtains von Neumann algebras $L^\infty(G)$ and $\mathcal{L}(G)$ with comultiplications by taking the weak closures of $\lambda(\mathfrak{C}_c^\infty(G))$ and $\hat{\lambda}(\mathfrak{D}(G))$, compare the constructions in Section 3.1.

The key result due to Kustermans and van Daele is that these operator algebras define a locally compact quantum group with multiplicative unitary \hat{W} , see Section 6 in [51]. Let us phrase this as follows.

Theorem 3.7. *Let G be an algebraic quantum group. With the notation as above, the left/right invariant integrals on $\mathfrak{C}_c^\infty(G)$ extend to left/right invariant weights on $L^\infty(G)$. In particular, G canonically defines a locally compact quantum group.*

To conclude this subsection, let us verify that the unitary $\hat{W} = \Sigma W^* \Sigma$ indeed corresponds to the fundamental multiplicative unitary for \hat{G} under Fourier transform. For any $f, g \in \mathfrak{C}_c^\infty(G)$, we have

$$\begin{aligned} (\mathcal{F} \otimes \mathcal{F})(\Sigma W^* \Sigma)(\Lambda(f) \otimes \Lambda(g)) &= (\mathcal{F} \otimes \mathcal{F})(\Lambda(f_{(2)}) \otimes \Lambda(f_{(1)}g)) \\ &= \hat{\Lambda}(\mathcal{F}(f_{(2)})) \otimes \hat{\Lambda}(\mathcal{F}(f_{(1)}g)). \end{aligned}$$

Now, for any $h, k \in \mathfrak{C}_c^\infty(G)$ we can compute

$$\begin{aligned} (\mathcal{F}(f_{(2)}) \otimes \mathcal{F}(f_{(1)}g), h \otimes k) &= \phi(hf_{(2)})\phi(kf_{(1)}g) \\ &= \phi(h_{(3)}f_{(2)})\phi(kS(h_{(1)})h_{(2)}f_{(1)}g) \\ &= \phi(kS(h_{(1)})g)\phi(h_{(2)}f) \\ &= (\mathcal{F}(g), kS(h_{(1)}))(\mathcal{F}(f), h_{(2)}) \\ &= (\mathcal{F}(g)_{(1)}, S(h_{(1)}))(\mathcal{F}(f), h_{(2)})(\mathcal{F}(g)_{(2)}, k) \\ &= (\hat{S}^{-1}(\mathcal{F}(g)_{(1)})\mathcal{F}(f) \otimes \mathcal{F}(g)_{(2)}, h \otimes k), \end{aligned}$$

and therefore the previous calculation gives

$$\begin{aligned} (\mathcal{F} \otimes \mathcal{F})(\Sigma W^* \Sigma)(\Lambda(f) \otimes \Lambda(g)) &= \hat{\Lambda}(\mathcal{F}(f_{(2)})) \otimes \hat{\Lambda}(\mathcal{F}(f_{(1)}g)) \\ &= (\hat{\Lambda} \otimes \hat{\Lambda})(\hat{S}^{-1}(\mathcal{F}(g)_{(1)})\mathcal{F}(f) \otimes \mathcal{F}(g)_{(2)}) \\ &= \hat{W}(\hat{\Lambda}(\mathcal{F}(f)) \otimes \hat{\Lambda}(\mathcal{F}(g))). \end{aligned}$$

This yields the claim.

3.2.3. Compact quantum groups. In this subsection we briefly sketch the theory of compact quantum groups. For more information we refer to [76], [58] and [48].

There are various ways in which the concept of a compact quantum group can be defined. In our set-up, it is convenient to consider compact quantum groups as a special case of algebraic quantum groups as in Definition 3.4. Historically, the development took place in the opposite order, in fact, the invention of algebraic quantum groups was strongly motivated by the theory of compact quantum groups and attempts to generalize it, see [74].

Definition 3.8. A compact quantum group is an algebraic quantum group H such that the underlying algebra of H is unital.

We shall also write $H = \mathfrak{C}^\infty(K)$ in this case and refer to K as a compact quantum group. Note that the comultiplication is a $*$ -homomorphism $\Delta : H \rightarrow H \otimes H$, so that H is in particular a Hopf $*$ -algebra.

Moreover, by definition there exists a positive left invariant integral $\phi : \mathfrak{C}^\infty(K) \rightarrow \mathbb{C}$ and a positive right invariant integral $\psi : \mathfrak{C}^\infty(K) \rightarrow \mathbb{C}$ such that $\phi(1) = 1 = \psi(1)$. We have

$$\phi(f) = \phi(f)\psi(1) = \psi((\text{id} \otimes \phi)\Delta(f)) = (\psi \otimes \phi)\Delta(f) = \psi(f)\phi(1) = \psi(f)$$

for all $f \in H$, so that in fact $\phi = \psi$. We refer to this left and right invariant functional as the Haar state of $\mathfrak{C}^\infty(K)$.

In particular, due to Proposition 1.10 the Hopf algebra $\mathfrak{C}^\infty(K)$ is cosemisimple. That is, we can write

$$\mathfrak{C}^\infty(K) \cong \bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})^*$$

as a direct sum of simple matrix coalgebras. Moreover, this isomorphism can be chosen so that the standard matrix coefficient functionals $u_{ij}^\lambda \in M_{n_\lambda}(\mathbb{C})^*$ satisfy $(u_{ij}^\lambda)^* = S(u_{ji}^\lambda)$ for all $\lambda \in \Lambda$. Equivalently, the matrix $u^\lambda = (u_{ij}^\lambda) \in M_{n_\lambda}(\mathfrak{C}^\infty(K))$ is unitary for all $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$ there exists a unique positive invertible matrix $F_\lambda \in M_{n_\lambda}(\mathbb{C})$ such that $S^2(u^\lambda) = F_\lambda u^\lambda F_\lambda^{-1}$ and $\text{tr}(F_\lambda) = \text{tr}(F_\lambda^{-1})$. Here we write $S^2(u^\lambda)$ for the matrix obtained by applying S^2 entrywise to u^λ , and we consider the unnormalized standard trace tr on $M_{n_\lambda}(\mathbb{C})$.

If we fix matrix coefficients u_{ij}^λ as above then we have the Schur orthogonality relations

$$\phi(u_{ij}^\beta (u_{kl}^\gamma)^*) = \delta_{\beta\gamma} \delta_{ik} \frac{(F_\beta)_{lj}}{\text{tr}(F_\beta)}, \quad \phi((u_{ij}^\beta)^* u_{kl}^\gamma) = \delta_{\beta\gamma} \delta_{jl} \frac{(F_\beta^{-1})_{ki}}{\text{tr}(F_\beta)},$$

compare for instance Chapter 11 in [48].

The Schur orthogonality relations imply that the dual algebraic quantum group $\mathfrak{D}(K)$ can be written as a direct sum

$$\mathfrak{D}(K) = \bigoplus_{\lambda \in \Lambda} M_{n_\lambda}(\mathbb{C})$$

of matrix algebras, such that the pairing (x, f) for $x \in \mathfrak{D}(K)$, $f \in \mathfrak{C}^\infty(K)$ is given by evaluation in each component. As usual, we write $X \rightarrow h = (X, h_{(2)})h_{(1)}$ and

$h \leftarrow X = (X, h_{(1)})h_{(2)}$ for the natural left and right actions of $\mathfrak{D}(K)$ on $\mathfrak{C}_c^\infty(K)$. If we define the element

$$F = \bigoplus_{\lambda \in \Lambda} F_\lambda \in \mathfrak{D}(K),$$

where the matrices F_λ are those from the Schur orthogonality formulas above, then the Haar state satisfies

$$\phi(fg) = \phi(g(F \rightarrow f \leftarrow F)) = \phi((F^{-1} \rightarrow g \leftarrow F^{-1})f)$$

for all $f, g \in \mathfrak{C}^\infty(K)$.

We obtain a basis of $\mathfrak{D}(K)$ consisting of the functionals ω_{ij}^μ defined by

$$(\omega_{ij}^\mu, u_{kl}^\eta) = \delta_{\mu\eta} \delta_{ik} \delta_{jl}.$$

With this notation, the multiplicative unitary $W \in \mathcal{M}(\mathfrak{C}^\infty(K) \otimes \mathfrak{D}(K))$ can be written as

$$W = \sum_{\mu, i, j} u_{ij}^\mu \otimes \omega_{ij}^\mu.$$

Indeed, we compute

$$\begin{aligned} \sum_{\mu, i, j} (\lambda(u_{ij}^\mu) \otimes \hat{\lambda}(\omega_{ij}^\mu)) (\Lambda(f) \otimes \Lambda(g)) &= \sum_{\mu, i, j} (\omega_{ij}^\mu, S^{-1}(g_{(1)})) \Lambda(u_{ij}^\mu f) \otimes \Lambda(g_{(2)}) \\ &= \Lambda(S^{-1}(g_{(1)})f) \otimes \Lambda(g_{(2)}) \end{aligned}$$

for all $f, g \in \mathfrak{C}^\infty(K)$. In a similar way one obtains

$$W^{-1} = \sum_{\mu, i, j} S(u_{ij}^\mu) \otimes \omega_{ij}^\mu = \sum_{\mu, i, j} u_{ij}^\mu \otimes \hat{S}^{-1}(\omega_{ij}^\mu)$$

for the inverse of W .

Positive left and right invariant Haar functionals for $\mathfrak{D}(K)$ are given by

$$\hat{\phi}(x) = \sum_{\mu \in \Lambda} \text{tr}(F_\mu) \text{tr}(F_\mu^{-1}x), \quad \hat{\psi}(x) = \sum_{\mu \in \Lambda} \text{tr}(F_\mu) \text{tr}(F_\mu x),$$

respectively. Note here that the positive matrices $F_\lambda \in M_{n_\lambda}(\mathbb{C})$ are naturally elements of $\mathfrak{D}(K)$, and that $F_\lambda^{\pm 1}x$ is contained in $M_{n_\lambda}(\mathbb{C}) \subset \mathfrak{D}(K)$ for any $x \in \mathfrak{D}(K)$. We note the formulas

$$\hat{\phi}(xy) = \hat{\phi}(FyF^{-1}x), \quad \hat{\psi}(xy) = \hat{\psi}(F^{-1}yFx),$$

for all $x, y \in \mathfrak{D}(K)$. We remark finally that $\hat{\delta} = F^2$ is the modular element of the dual quantum group $\mathfrak{D}(K)$. In particular, $\mathfrak{D}(K)$ is unimodular iff $F = 1$.

3.2.4. The Drinfeld double of algebraic quantum groups. In this subsection we discuss the Drinfeld double construction in the framework of algebraic quantum groups. The Drinfeld double of regular multiplier Hopf algebras was already treated in Section 1.4. Here we shall explain how to incorporate $*$ -structures in the construction, and also approach it from the dual point of view.

Let K and L be algebraic quantum groups with group algebras $\mathfrak{D}(K)$ and $\mathfrak{D}(L)$, respectively. Recall from Section 1.4 that in order to form the Drinfeld double $\mathfrak{D}(K) \bowtie \mathfrak{D}(L)$ one needs a skew-pairing $\tau : \mathfrak{D}(K) \times \mathfrak{D}(L) \rightarrow \mathbb{C}$. We shall say that the skew-pairing τ is unitary if

$$\tau(x^*, y^*) = \overline{\tau^{-1}(x, y)}$$

for all $x \in \mathfrak{D}(K), y \in \mathfrak{D}(L)$, where τ^{-1} denotes the convolution inverse of τ as in Section 1.4.

Let us also introduce the notion of a unitary bicharacter.

Definition 3.9. Let K and L be algebraic quantum groups. A (unitary) bicharacter for K, L is a (unitary) invertible element $U \in \mathcal{M}(\mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L))$ such that

$$(\Delta_K \otimes \text{id})(U) = U_{13}U_{23}, \quad (\text{id} \otimes \Delta_L)(U) = U_{13}U_{12}$$

and

$$(\epsilon_K \otimes \text{id})(U) = 1, \quad (\text{id} \otimes \epsilon_L)(U) = 1.$$

In the same way as in the discussion of universal R -matrices in Section 2.11 one checks that a bicharacter U satisfies

$$(S_K \otimes \text{id})(U) = U^{-1} = (\text{id} \otimes S_L^{-1})(U).$$

The notion of a bicharacter is dual to the concept of a skew-pairing in the following sense.

Proposition 3.10. *Assume that K and L are algebraic quantum groups. If $U \in \mathcal{M}(\mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L))$ is a bicharacter then $\tau_U : \mathfrak{D}(K) \times \mathfrak{D}(L) \rightarrow \mathbb{C}$ given by*

$$\tau_U(x, y) = (x \otimes y, U^{-1})$$

is a skew-pairing. Every skew-pairing $\mathfrak{D}(K) \times \mathfrak{D}(L) \rightarrow \mathbb{C}$ arises in this way from a bicharacter. Moreover U is unitary iff the skew-pairing τ_U is unitary.

Proof. Let us sketch the argument. If U is a bicharacter then one can check in the same way as in the proof of Lemma 2.105 that τ_U yields a skew-pairing.

To see that every skew-pairing arises from a bicharacter let $\tau : \mathfrak{D}(K) \times \mathfrak{D}(L) \rightarrow \mathbb{C}$ be given. Using regularity one checks that the formulas

$$\begin{aligned} (x \otimes y, U_\tau^{-1}(f \otimes g)) &= \tau(x_{(2)}, y_{(2)})(x_{(1)}, f)(y_{(1)}, g) \\ (x \otimes y, (f \otimes g)U_\tau^{-1}) &= (x_{(2)}, f)(y_{(2)}, g)\tau(x_{(1)}, y_{(1)}) \end{aligned}$$

determine an invertible multiplier $U_\tau \in \mathcal{M}(\mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L))$. Moreover, one checks that this multiplier satisfies the conditions in Definition 3.9. The skew-pairing τ is reobtained by applying the above construction to U_τ .

For the last claim use the relation $(S_K \otimes S_L)(U) = U$ to compute

$$\tau_U(x^*, y^*) = (x^* \otimes y^*, U^{-1}) = \overline{(x \otimes y, (S_K \otimes S_L)(U^{-1})^*)} = \overline{(x \otimes y, (U^{-1})^*)}$$

and

$$\overline{\tau_U^{-1}(x, y)} = \overline{(x \otimes y, U)}.$$

Comparing these expressions yields the assertion. \square

Let K and L be algebraic quantum groups and assume $U \in \mathcal{M}(\mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L))$ is a bicharacter. As in Section 1.4 we can then construct the Drinfeld double $\mathfrak{D}(K) \bowtie \mathfrak{D}(L)$ using the skew-pairing τ_U from Proposition 3.10. Recall that this is the regular multiplier Hopf algebra

$$\mathfrak{D}(K \bowtie L) = \mathfrak{D}(K) \bowtie \mathfrak{D}(L),$$

equipped with the tensor product comultiplication and the multiplication given by

$$\begin{aligned} (x \bowtie f)(y \bowtie g) &= x \tau_U(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau_U(\hat{S}_K(y_{(3)}), f_{(3)})g \\ &= x \tau_U(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau_U(y_{(3)}, \hat{S}_L^{-1}(f_{(3)}))g. \end{aligned}$$

The counit of $\mathfrak{D}(K \bowtie L)$ is given by

$$\hat{\epsilon}_{K \bowtie L}(x \bowtie f) = \hat{\epsilon}_K(x)\hat{\epsilon}_L(f)$$

for $x \in \mathfrak{D}(K), f \in \mathfrak{D}(L)$. The antipode of $\mathfrak{D}(K \bowtie L)$ is defined by

$$\begin{aligned} \hat{S}_{K \bowtie L}(x \bowtie f) &= (1 \bowtie \hat{S}_L(f))(\hat{S}_K(x) \bowtie 1) \\ &= \tau_U(\hat{S}(x_{(3)}), \hat{S}_L(f_{(3)}))\hat{S}_K(x_{(2)}) \bowtie \hat{S}_L(f_{(2)})\tau_U(\hat{S}_K(x_{(1)}), f_{(1)}). \end{aligned}$$

If U is unitary we can define a $*$ -structure on $\mathfrak{D}(K \bowtie L)$ by

$$(x \bowtie f)^* = \tau_U(x_{(1)}^*, f_{(1)}^*)x_{(2)}^* \bowtie f_{(2)}^* \tau_U^{-1}(x_{(3)}^*, f_{(3)}^*) = (1 \bowtie f^*)(x^* \bowtie 1).$$

To check antimultiplicativity of this $*$ -structure one computes

$$\begin{aligned} ((x \bowtie f)(y \bowtie g))^* &= (x\tau_U(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau_U^{-1}(y_{(3)}, f_{(3)})g)^* \\ &= \overline{\tau_U(y_{(1)}, f_{(1)})(xy_{(2)} \bowtie f_{(2)}g)^* \tau_U^{-1}(y_{(3)}, f_{(3)})} \\ &= \tau_U^{-1}(y_{(1)}^*, f_{(1)}^*)(xy_{(2)} \bowtie f_{(2)}g)^* \tau_U(y_{(3)}^*, f_{(3)}^*) \\ &= \tau_U^{-1}(y_{(1)}^*, f_{(1)}^*)(1 \bowtie g^*)\tau_U(y_{(2)}^*, f_{(2)}^*)(y_{(3)}^* \bowtie f_{(3)}^*) \\ &\quad \times \tau_U^{-1}(y_{(4)}^*, f_{(4)}^*)(x^* \bowtie 1)\tau_U(y_{(5)}^*, f_{(5)}^*) \\ &= (1 \bowtie g^*)(y^* \bowtie f^*)(x^* \bowtie 1) \\ &= (1 \bowtie g^*)(y^* \bowtie 1)(1 \bowtie f^*)(x^* \bowtie 1) \\ &= (y \bowtie g)^*(x \bowtie f)^*, \end{aligned}$$

where we use

$$\overline{\tau_U(x, f)} = \tau_U^{-1}(x^* \otimes f^*).$$

For involutivity note that

$$(x \bowtie f)^{**} = ((1 \bowtie f^*)(x^* \bowtie 1))^* = x \bowtie f$$

using that $*$ is antimultiplicative. Similarly, to check that $\hat{\Delta}_{K \bowtie L}$ is a $*$ -homomorphism one calculates

$$\begin{aligned} \hat{\Delta}_{K \bowtie L}((x \bowtie f)^*) &= ((1 \bowtie f_{(1)}^*) \otimes (1 \bowtie f_{(2)}^*))((x_{(1)}^* \bowtie 1) \otimes (x_{(2)}^* \bowtie 1)) \\ &= (x_{(1)}^* \bowtie f_{(1)}^*) \otimes (x_{(2)}^* \bowtie f_{(2)}^*) \\ &= \hat{\Delta}_{K \bowtie L}(x \bowtie f)^*. \end{aligned}$$

Finally, a left Haar integral for $\mathfrak{D}(K \bowtie L)$ is given by

$$\hat{\phi}_{K \bowtie L}(x \bowtie f) = \hat{\phi}_K(x)\hat{\phi}_L(f),$$

where $\hat{\phi}_K$ and $\hat{\phi}_L$ are left Haar integrals for $\mathfrak{D}(K)$ and $\mathfrak{D}(L)$, respectively. Similarly, a right Haar integral for $\mathfrak{D}(K \bowtie L)$ is given by the tensor product of right Haar integrals for $\mathfrak{D}(K)$ and $\mathfrak{D}(L)$.

If L is a discrete quantum group then it is not hard to check that the resulting functionals are positive provided one starts from positive integrals for K and L , respectively. For the question of positivity in general see [21].

From general theory, we obtain the dual multiplier Hopf algebra $\mathfrak{C}_c^\infty(K \bowtie L)$ of $\mathfrak{D}(K \bowtie L)$. Sometimes this is referred to as the *Drinfeld codouble*, but we shall call $\mathfrak{C}_c^\infty(K \bowtie L)$ the algebra of functions on the Drinfeld double $K \bowtie L$. Explicitly, the structure of $\mathfrak{C}_c^\infty(K \bowtie L)$ looks as follows.

Proposition 3.11. *Let K and L be algebraic quantum groups and assume that $U \in \mathcal{M}(\mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L))$ is a unitary bicharacter. Then the algebra*

$$\mathfrak{C}_c^\infty(K \bowtie L) = \mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L),$$

equipped with the comultiplication

$$\Delta_{K \bowtie L} = (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L)$$

is an algebraic quantum group. Moreover, the counit of $\mathfrak{C}_c^\infty(K \bowtie L)$ is the tensor product counit $\epsilon_{K \bowtie L} = \epsilon_K \otimes \epsilon_L$, and the antipode is given by

$$S_{K \bowtie L}(f \otimes x) = U^{-1}(S_K(f) \otimes S_L(x))U = (S_K \otimes S_L)(U(f \otimes x)U^{-1})$$

The algebraic quantum group $\mathfrak{C}_c^\infty(K \bowtie L)$ is dual to the double $\mathfrak{D}(K) \bowtie \mathfrak{D}(L)$, the latter being constructed with respect to the skew-pairing τ_U corresponding to U .

Proof. Note first that $\text{ad}(U)$ is conjugation with the bicharacter U and σ denotes the flip map in the above formula for the comultiplication.

Let us abbreviate $G = K \bowtie L$. In order to prove the Proposition we shall show that the dual of $\mathfrak{D}(G) = \mathfrak{D}(K) \bowtie \mathfrak{D}(L)$ can be identified as stated.

Firstly, using $\hat{\phi}_G(x \bowtie f) = \hat{\phi}_K(x)\hat{\phi}_L(f)$ we compute

$$\begin{aligned} \hat{\mathcal{F}}_G(y \bowtie g)(x \bowtie f) &= \hat{\phi}_G(x\tau_U(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}\tau_U^{-1}(y_{(3)}, f_{(3)})g) \\ &= \tau_U(y_{(1)}, f_{(1)})\hat{\phi}_K(xy_{(2)})\hat{\phi}_L(f_{(2)}g)\tau_U(\hat{S}_K(y_{(3)}), f_{(3)}) \\ &= \tau_U(y_{(1)}, \hat{S}_L^{-1}(g_{(1)}))\hat{\phi}_K(xy_{(2)})\hat{\phi}_L(f_{(1)}g_{(2)})\tau_U(\hat{S}_K(y_{(3)}), f_{(2)}) \\ &= \tau_U(y_{(1)}, \hat{S}_L^{-1}(g_{(1)}))\hat{\phi}_K(xy_{(2)})\hat{\phi}_L(f_{(1)}g_{(2)})\tau_U(y_{(3)}, S_L^{-1}(f_{(2)}g_{(3)}S_L(g_{(4)})) \\ &= \tau_U(y_{(1)}, \hat{S}_L^{-1}(g_{(1)}))\hat{\phi}_K(xy_{(2)})\hat{\phi}_L(fg_{(2)})\tau_U(y_{(3)}, g_{(3)}\hat{\delta}_L^{-1}) \\ &= (x \otimes f, \tau_U^{-1}(y_{(1)}, g_{(1)})\hat{\mathcal{F}}_K(y_{(2)}) \otimes \hat{\mathcal{F}}_L(g_{(2)})\tau_U(y_{(3)}, g_{(3)}\hat{\delta}_L^{-1})) \end{aligned}$$

for all $x, y \in \mathfrak{D}(K), f, g \in \mathfrak{D}(L)$. Here $\hat{\delta}_L$ denotes the modular element of $\mathfrak{D}(L)$. This implies

$$\hat{\mathcal{F}}_G(\tau_U(y_{(1)}, g_{(1)})y_{(2)} \bowtie g_{(2)}\tau_U^{-1}(y_{(3)}, g_{(3)}\hat{\delta}_L^{-1})) = \hat{\mathcal{F}}_K(y) \otimes \hat{\mathcal{F}}_L(g).$$

In particular, we can identify the underlying vector space of the dual quantum group of $\mathfrak{D}(G)$ with the space $\mathfrak{C}_c^\infty(G) = \mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L)$ such that the canonical pairing between $\mathfrak{D}(G)$ and $\mathfrak{C}_c^\infty(G)$ becomes

$$(y \bowtie g, f \otimes x) = (y, f)(g, x)$$

for all $y \in \mathfrak{D}(K), f \in \mathfrak{C}_c^\infty(K), g \in \mathfrak{D}(L), x \in \mathfrak{C}_c^\infty(L)$. Moreover, from the definition of the comultiplication in $\mathfrak{D}(G)$ it is clear that the algebra structure of the dual multiplier Hopf algebra $\mathfrak{C}_c^\infty(G) = \mathfrak{C}_c^\infty(K) \otimes \mathfrak{C}_c^\infty(L)$ is the tensor product algebra structure.

Let us next identify the comultiplication of $\mathfrak{C}_c^\infty(G)$. By general theory, this is determined by

$$\begin{aligned} ((y \bowtie g) \otimes (z \bowtie h), \Delta_G(f \otimes x)) &= (y\tau_U(z_{(1)}, g_{(1)})z_{(2)} \bowtie g_{(2)}\tau_U^{-1}(z_{(3)}, g_{(3)})h, f \otimes x) \\ &= (y \otimes \tau_U(z_{(1)}, g_{(1)})z_{(2)} \bowtie g_{(2)} \otimes \tau_U^{-1}(z_{(3)}, g_{(3)})h, (\Delta_K \otimes \Delta_L)(f \otimes x)) \\ &= (y \otimes (z_{(1)} \otimes g_{(1)}, U^{-1})z_{(2)} \bowtie g_{(2)} \otimes (z_{(3)} \otimes g_{(3)}, U)h, (\Delta_K \otimes \Delta_L)(f \otimes x)) \\ &= (y \otimes z \otimes g \otimes h, (\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L)(f \otimes x)) \\ &= ((y \bowtie g) \otimes (z \bowtie h), (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L)(f \otimes x)), \end{aligned}$$

so that we obtain

$$\Delta_G = (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L)$$

as claimed. Remark that it is immediate from unitarity of U that $\Delta_G : \mathfrak{C}_c^\infty(G) \rightarrow \mathcal{M}(\mathfrak{C}_c^\infty(G) \otimes \mathfrak{C}_c^\infty(G))$ is indeed an essential $*$ -algebra homomorphism.

The formula $\epsilon_G = \epsilon_K \otimes \epsilon_L$ for the counit of $\mathfrak{C}_c^\infty(G)$ can also be deduced from duality. Alternatively, we may check the counit property of ϵ_G directly and compute

$$\begin{aligned} (\epsilon_G \otimes \text{id})\Delta_G &= (\epsilon_K \otimes \epsilon_L \otimes \text{id} \otimes \text{id})(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\ &= (\epsilon_K \otimes \text{id} \otimes \epsilon_L \otimes \text{id})(\text{id} \otimes \text{ad}((\text{id} \otimes \epsilon_L)(U)) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\ &= (\epsilon_K \otimes \text{id} \otimes \epsilon_L \otimes \text{id})(\Delta_K \otimes \Delta_L) \\ &= (\text{id} \otimes \epsilon_K \otimes \text{id} \otimes \epsilon_L)(\Delta_K \otimes \Delta_L) \\ &= (\text{id} \otimes \epsilon_K \otimes \text{id} \otimes \epsilon_L)(\text{id} \otimes \text{ad}((\epsilon_K \otimes \text{id})(U)) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\ &= (\text{id} \otimes \text{id} \otimes \epsilon_K \otimes \epsilon_L)(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\ &= (\text{id} \otimes \epsilon_G)\Delta_G, \end{aligned}$$

using the relations $(\epsilon_K \otimes \text{id})(U) = \text{id}$ and $(\text{id} \otimes \epsilon_L)(U) = \text{id}$.

To verify the formula for the antipode we check

$$\begin{aligned}
m_G(S_G \otimes \text{id})\Delta_G &= m_G(S_G \otimes \text{id})(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\
&= (m_K \otimes m_L)(S_K \otimes \text{id} \otimes S_L \otimes \text{id})\text{ad}(U_{13}U_{23})(\Delta_K \otimes \Delta_L) \\
&= (m_K \otimes m_L)(S_K \otimes \text{id} \otimes S_L \otimes \text{id})\text{ad}((\Delta_K \otimes \text{id})(U)_{123})(\Delta_K \otimes \Delta_L) \\
&= m_L(S_L \otimes \text{id})(\epsilon_K \otimes \Delta_L) \\
&= \epsilon_K \otimes \epsilon_L,
\end{aligned}$$

using the antipode axioms for K and L , respectively. A similar computation shows

$$\begin{aligned}
m_G(\text{id} \otimes S_G)\Delta_G &= m_G(\text{id} \otimes S_G)(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(U) \otimes \text{id})(\Delta_K \otimes \Delta_L) \\
&= (m_K \otimes m_L)(\text{id} \otimes S_K \otimes \text{id} \otimes S_L)\text{ad}(U_{24}U_{23})(\Delta_K \otimes \Delta_L) \\
&= (m_K \otimes m_L)(\text{id} \otimes S_K \otimes \text{id} \otimes S_L)\text{ad}((\text{id} \otimes \Delta_L)(U)_{234})(\Delta_K \otimes \Delta_L) \\
&= m_K(\text{id} \otimes S_K)(\Delta_K \otimes \epsilon_L) \\
&= \epsilon_K \otimes \epsilon_L.
\end{aligned}$$

According to [21] and the duality theory for algebraic quantum groups, there exists a positive left invariant functional and a positive right invariant functional on $\mathfrak{C}_c^\infty(G)$. Hence $\mathfrak{C}_c^\infty(G)$ is again an algebraic quantum group. \square

Explicitly, the left Haar functional for $G = K \bowtie L$ are obtained by combining the left Haar functional for K and a twisted version of the left Haar functional for L , depending on the modular properties of the quantum groups and the pairing involved. We refer to [6] for a detailed analysis.

In the special case of the Drinfeld double of a compact quantum group we will write down an explicit formula for a (left and right) invariant integral further below.

3.3. Compact semisimple quantum groups. Let \mathfrak{g} be a semisimple Lie algebra. In order to pass from the algebraic theory of quantized universal enveloping algebras developed in Chapter 2 to the analytical setting one has to introduce a $*$ -structure on $U_q(\mathfrak{g})$ and $\mathcal{O}(G_q)$.

In the case of $U_q(\mathfrak{g})$ we shall work with the $*$ -structure defined as follows.

Lemma 3.12. *Assume $q \in \mathbb{C}^\times$ is real and $q \neq \pm 1$. Then there is a unique $*$ -structure on $U_q(\mathfrak{g})$ satisfying*

$$E_i^* = K_i F_i, \quad F_i^* = E_i K_i^{-1}, \quad K_\lambda^* = K_\lambda$$

for $i = 1, \dots, N$ and $\lambda \in \mathbf{P}$. The comultiplication Δ is a $*$ -homomorphism for this $*$ -structure.

Proof. This is done in the same way as for the definition of the algebra antiautomorphism τ in Lemma 2.16, note that $*$ acts in the same way on generators. The only difference is that $*$ is extended anti-linearly, whereas τ is extended linearly to general elements. \square

Definition 3.13. We will write $U_q^{\mathbb{R}}(\mathfrak{k})$ to signify $U_q(\mathfrak{g})$ with the Hopf $*$ -structure of the lemma above.

Here, $U_q^{\mathbb{R}}(\mathfrak{k})$ should be viewed as the universal enveloping algebra of the *complexification* of the Lie algebra \mathfrak{k} of the compact real form K of the simply connected group G corresponding to \mathfrak{g} .

We have the following compatibility of the \mathcal{R} -matrix with the $*$ -structure.

Lemma 3.14. *The universal R -matrix of $U_q(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k})$ satisfies*

$$(\hat{S} \otimes \hat{S})(\mathcal{R}^*) = \mathcal{R}_{21},$$

where \mathcal{R}_{21} is obtained from \mathcal{R} by flipping the tensor factors.

Proof. Due to Theorem 2.108 the universal R -matrix is the product of the Cartan part $q^{\sum_{i,j=1}^N B_{ij}(H_i \otimes H_j)}$ and the nilpotent part $\prod_{\alpha \in \Delta^+} \exp_{q_\alpha}((q_\alpha - q_\alpha^{-1})(E_\alpha \otimes F_\alpha))$. Since the antipode and the $*$ -structure are both antimultiplicative, it suffices to show that applying the $*$ -structure followed by $\hat{S} \otimes \hat{S}$ to each individual factor of these elements switches legs in the tensor product. For the Cartan part this is obvious since $\hat{S} \otimes \hat{S}$ introduces two minus signs which cancel out, and the $*$ -structure leaves the Cartan generators H_k fixed.

For the nilpotent factors note that

$$\begin{aligned} (\hat{S} \otimes \hat{S})((E_i \otimes F_i)^*) &= (\hat{S} \otimes \hat{S})(K_i F_i \otimes E_i K_i^{-1}) \\ &= (-K_i F_i K_i^{-1}) \otimes (-K_i E_i K_i^{-1}) \\ &= F_i \otimes E_i \end{aligned}$$

for all $i = 1, \dots, N$. This yields the claim for all factors corresponding to simple roots.

For the factors corresponding to arbitrary positive roots we use Theorem 2.58 and the relations

$$\begin{aligned} \hat{S}(E_i^*) &= \hat{S}(K_i F_i) = (-K_i F_i) K_i^{-1} = -q^{-(\alpha_i, \alpha_i)} F_i = -q_i^{-2} F_i \\ \hat{S}(F_i^*) &= \hat{S}(E_i K_i^{-1}) = K_i (-E_i K_i^{-1}) = -q^{(\alpha_i, \alpha_i)} E_i = -q_i^2 E_i. \end{aligned}$$

More precisely, we compute

$$q_i^2 \mathcal{T}_i(\hat{S}(E_i^*)) = -\mathcal{T}_i(F_i) = E_i K_i^{-1} = q_i^2 K_i^{-1} E_i = -\hat{S}((K_i F_i)^*) = \hat{S}(\mathcal{T}_i(E_i)^*)$$

and

$$\begin{aligned} q^{a_{ij}} \mathcal{T}_i(\hat{S}(E_j^*)) &= -q_i^{a_{ij}} q_j^{-2} \mathcal{T}_i(F_j) \\ &= -q_i^{a_{ij}} q_j^{-2} \sum_{k=0}^{-a_{ij}} (-1)^k q_i^{-k} F_i^{(-a_{ij}-k)} F_j F_i^{(k)} \\ &= \sum_{k=0}^{-a_{ij}} (-1)^{-a_{ij}-k} q_i^{-a_{ij}-k} (-1)^{-a_{ij}+1} q_i^{2a_{ij}} q_j^{-2} F_i^{(-a_{ij}-k)} F_j F_i^{(k)} \\ &= \sum_{k=0}^{-a_{ij}} (-1)^{-a_{ij}-k} q_i^{-a_{ij}-k} \hat{S}((E_i^{(-a_{ij}-k)} E_j E_i^{(k)})^*) \\ &= \sum_{k=0}^{-a_{ij}} (-1)^k q_i^k \hat{S}((E_i^{(k)} E_j E_i^{(-a_{ij}-k)})^*) = \hat{S}(\mathcal{T}_i(E_j)^*) \end{aligned}$$

for $i \neq j$. Similarly, one obtains

$$q^{-a_{ij}} \mathcal{T}_i(\hat{S}(F_j^*)) = \hat{S}(\mathcal{T}_i(F_j)^*)$$

for all i, j . An induction argument then yields

$$(\hat{S} \otimes \hat{S})((E_\alpha \otimes F_\alpha)^*) = F_\alpha \otimes E_\alpha$$

for all positive roots α . □

Recall the definition of $\mathcal{O}(G_q)$ from Section 2.10. By construction, there exists a canonical bilinear pairing between $U_q^{\mathbb{R}}(\mathfrak{k})$ and $\mathcal{O}(G_q)$. We shall introduce a $*$ -structure on $\mathcal{O}(G_q)$ by stipulating

$$(x, f^*) = \overline{(\hat{S}^{-1}(x)^*, f)}$$

for all $x \in U_q^{\mathbb{R}}(\mathfrak{k})$. In this way $\mathcal{O}(G_q)$ becomes a Hopf $*$ -algebra.

Definition 3.15. We write $\mathfrak{C}^\infty(K_q)$ to denote $\mathcal{O}(G_q)$ with the above $*$ -structure.

The canonical pairing between $U_q^{\mathbb{R}}(\mathfrak{k})$ and $\mathfrak{C}^\infty(K_q)$ satisfies

$$(xy, f) = (x, f_{(1)})(y, f_{(2)}), \quad (x, fg) = (x_{(2)}, f)(x_{(1)}, g)$$

and

$$(\hat{S}(x), f) = (x, S^{-1}(f)), \quad (\hat{S}^{-1}(x), f) = (x, S(f))$$

for $f, g \in \mathfrak{C}^\infty(K_q)$, $x, y \in U_q^{\mathbb{R}}(\mathfrak{k})$, and the compatibility with the $*$ -structures is given by

$$(x, f^*) = \overline{(\hat{S}^{-1}(x)^*, f)}, \quad (x^*, f) = \overline{(x, S(f)^*)}.$$

In addition,

$$(x, 1) = \hat{\epsilon}(x), \quad (1, f) = \epsilon(f),$$

for $f \in \mathfrak{C}^\infty(K_q)$ and $x \in U_q^{\mathbb{R}}(\mathfrak{k})$. We may summarize this by saying that the canonical pairing is a skew-pairing of the Hopf $*$ -algebras $U_q^{\mathbb{R}}(\mathfrak{k})$ and $\mathfrak{C}^\infty(K_q)$.

We will also need to make use of the skew-pairing between $\mathfrak{C}^\infty(K_q)$ and $U_q^{\mathbb{R}}(\mathfrak{k})$ defined by

$$(f, x) = (\hat{S}(x), f) = (x, S^{-1}(f))$$

for $f \in \mathfrak{C}^\infty(K_q)$ and $x \in U_q^{\mathbb{R}}(\mathfrak{k})$. Then

$$(f^*, x) = (\hat{S}(x), f^*) = \overline{(x^*, f)} = \overline{(\hat{S}(\hat{S}(x)^*), f)} = \overline{(f, \hat{S}(x)^*)},$$

so that this pairing is again compatible with the $*$ -structures. As in Subsection 1.3.2, we stress that our skew-pairings $U_q^{\mathbb{R}}(\mathfrak{k}) \times \mathfrak{C}^\infty(K_q) \rightarrow \mathbb{C}$ and $\mathfrak{C}^\infty(K_q) \times U_q^{\mathbb{R}}(\mathfrak{k}) \rightarrow \mathbb{C}$ need to be distinguished, and in particular $(f, x) \neq (x, f)$ in general.

Recall from Definition 2.102 that the Hopf algebra $\mathfrak{C}^\infty(K_q)$ is defined as the span of the matrix coefficients of the irreducible integrable representations $V(\mu)$ of $U_q^{\mathbb{R}}(\mathfrak{k})$ for $\mu \in \mathbf{P}^+$. In particular, it is cosemisimple and so admits a left and right invariant integral ϕ given by projection onto the coefficient of the trivial corepresentation, see Proposition 1.10. In order to see that $\mathfrak{C}^\infty(K_q)$ defines a compact quantum group K_q it remains to check that ϕ is positive.

Following the discussion in Section 11.3 of [48], this is equivalent to proving that the irreducible integrable representations $V(\mu)$ of $U_q^{\mathbb{R}}(\mathfrak{k})$ are all unitarizable. Here, a representation V is called unitarizable if it admits a positive definite sesquilinear form which is invariant in the sense that

$$\langle X \cdot v, w \rangle = \langle v, X^* \cdot w \rangle$$

for all $X \in U_q^{\mathbb{R}}(\mathfrak{k})$ and all $v, w \in V$.

Let us explain how this can be done. First, we claim that each module $V(\mu)$ for $\mu \in \mathbf{P}^+$ can be equipped with an essentially unique invariant sesquilinear form $\langle \cdot, \cdot \rangle$. Indeed, let $\overline{V(\mu)^*}$ be the $U_q(\mathfrak{g})$ -module defined on the conjugate vector space of the dual $V(\mu)^*$ by setting $(X \cdot \bar{f})(v) = \bar{f}(X^* \cdot v)$ for all $v \in V(\mu)$. Then $\overline{V(\mu)^*}$ is an irreducible highest weight module of highest weight μ , and hence $\overline{V(\mu)^*} \cong V(\mu)$. In particular, there exists a unique hermitian sesquilinear form $\langle \cdot, \cdot \rangle$ on $V(\mu)$ such that $\langle v_\mu, v_\mu \rangle = 1$.

Proposition 3.16. *Let $q \neq 1$ be a strictly positive real number. For each $\mu \in \mathbf{P}^+$ the hermitian form $\langle \cdot, \cdot \rangle$ on $V(\mu)$ constructed above is positive definite.*

Proof. We shall use a continuity argument in the parameter q . In order to emphasize the dependence on q we write $V(\mu)_q$ for the irreducible highest weight module of highest weight μ associated with q . Then $V(\mu)_q = V(\mu)_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$ where $V(\mu)_{\mathcal{A}}$ is the integral form of $V(\mu)$ and $s \in \mathcal{A}$ acts on \mathbb{C} such that $s^L = q$. Let us write $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_q$ for the hermitian sesquilinear form on $V(\mu)_q$ satisfying $\langle v_\mu, v_\mu \rangle_q = 1$.

Choosing a free \mathcal{A} -basis of $V(\mu)_{\mathcal{A}}$ we see that the forms $\langle \cdot, \cdot \rangle_q$ can be viewed as a continuous family of hermitian forms depending on $q \in (0, \infty)$ on a fixed vector space.

Recall from Proposition 2.25 that the specialisation $U_1(\mathfrak{g})$ of $U_q^{\mathcal{A}}(\mathfrak{g})$ at 1 maps onto the classical universal enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} over \mathbb{C} . This map is compatible with $*$ -structures if we consider the $*$ -structure on $U(\mathfrak{g})$ given by

$$E_i^* = F_i, \quad F_i^* = E_i, \quad H_i^* = H_i$$

for $i = 1, \dots, N$, corresponding to the compact real form or \mathfrak{g} .

The representation $V(\mu)_1$ of $U_q^{\mathcal{A}}(\mathfrak{g})$ at $q = 1$ correspond to the irreducible highest weight representation of weight μ of the classical Lie algebra \mathfrak{g} . In particular, the sesquilinear form $\langle \cdot, \cdot \rangle_1$ is positive definite. By continuity, we conclude that $\langle \cdot, \cdot \rangle_q$ is positive definite for all $q \in (0, \infty)$. \square

Let us specify the element $F \in \mathfrak{D}(K_q)$ which appears in the Schur orthogonality relations, see Subsection 3.2.3.

Lemma 3.17. *We have $F = K_{-2\rho}$.*

Proof. Let us fix an orthonormal weight basis (e_i) for $V(\mu)$ and let $M = (M_{ij})$ denote the matrix of $K_{-2\rho}$ acting on this basis, so that $M_{ij} = \langle e_i, K_{-2\rho} \cdot e_j \rangle = \langle K_{-2\rho}, u_{ij}^\mu \rangle$. We will show that M satisfies the defining properties of F_μ from Subsection 3.2.3. Clearly, M is positive. If $w_0 \in W$ is the longest element of the Weyl group then $w_0\rho = -\rho$ and so by the Weyl group invariance of the set of weights of $V(\mu)$ we have

$$\text{tr}(M) = \sum_{\nu \in \mathbf{P}(V(\mu))} q^{(-2\rho, \nu)} = \sum_{\nu \in \mathbf{P}(V(\mu))} q^{(2\rho, w_0\nu)} = \text{tr}(M^{-1}),$$

where the sum is over all weights of $V(\mu)$ counted with multiplicities. Finally, using Lemma 2.7 we have for any $X \in U_q^{\mathbb{R}}(\mathfrak{k})$,

$$\begin{aligned} (X, S^2(u_{ij}^\mu)) &= (\hat{S}^{-2}(X), u_{ij}^\mu) \\ &= (K_{-2\rho} X K_{2\rho}, u_{ij}^\mu) \\ &= \left(X, \sum_{k,l} (K_{-2\rho}, u_{ik}^\mu) u_{kl}^\mu (K_{2\rho}, u_{lj}^\mu) \right). \end{aligned}$$

It follows that $S^2(u^\mu) = M u^\mu M^{-1}$. This completes the proof. \square

We shall write $C(K_q)$ for the Hopf C^* -algebra of functions on K_q obtained as the completion of $\mathfrak{C}^\infty(K_q)$ in the GNS-representation of the left Haar weight. Let us remark that K_q is coamenable, so that there is no need to distinguish between maximal and reduced algebras of functions in this case, see Corollary 5.1 in [7].

To conclude this section, let us explain how the classical maximal torus T of K appears also as a quantum subgroup of K_q . Let $U_q^{\mathbb{R}}(\mathfrak{t})$ be Hopf algebra $U_q(\mathfrak{h})$ equipped with the $*$ -structure induced from $U_q^{\mathbb{R}}(\mathfrak{k})$. Define $\mathfrak{C}^\infty(T) \subset U_q^{\mathbb{R}}(\mathfrak{t})^*$ to be the restriction of elements of $\mathfrak{C}^\infty(K_q) \subset U_q^{\mathbb{R}}(\mathfrak{k})^*$ to $U_q^{\mathbb{R}}(\mathfrak{t})$. In light of definition

2.102, the image of this restriction map is the set of linear combination of characters of the form

$$\chi_\lambda(K_\mu) = q^{\mu, \lambda}$$

with $\lambda \in \mathbf{P}$. We therefore have a canonical isomorphism of $\mathfrak{C}^\infty(T)$ with the group algebra $\mathbb{C}[\mathbf{P}]$ which is isomorphic to the polynomials on the maximal torus T of K .

This homomorphism induces a surjection $C(K_q) \rightarrow C(T)$ on the C^* -level.

3.4. Complex semisimple quantum groups. In this section we define our main objects of study, namely q -deformations of complex semisimple Lie groups. The construction of these quantum groups relies on the Drinfeld double construction. We also discuss some variants of the quantum groups we will be interested in, related to connected components and certain central extensions. Throughout this section we fix a positive deformation parameter $q = e^h$ different from 1.

3.4.1. *The definition of complex quantum groups.* The definition of complex semisimple quantum groups is a special case of the Drinfeld double construction explained in Section 3.2.

Definition 3.18. Let G be a simply connected semisimple complex Lie group and let K be a maximal compact subgroup. The complex semisimple quantum group G_q is the Drinfeld double

$$G_q = K_q \bowtie \hat{K}_q$$

of K_q and its Pontrjagin dual \hat{K}_q with respect to the canonical bicharacter $W \in \mathcal{M}(\mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(K_q))$ given by the multiplicative unitary of K_q .

Let us explicitly write down some of the structure maps underlying the quantum group G_q , based on the general discussion in Section 3.2. By definition, G_q is the algebraic quantum group with underlying algebra

$$\mathfrak{C}_c^\infty(G_q) = \mathfrak{C}^\infty(K_q \bowtie \hat{K}_q) = \mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(K_q),$$

equipped with the comultiplication

$$\Delta_{G_q} = (\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(W) \otimes \text{id})(\Delta \otimes \hat{\Delta}).$$

Note that we can identify

$$\mathfrak{C}_c^\infty(G_q) = \bigoplus_{\mu \in \mathbf{P}^+} \mathfrak{C}^\infty(K_q) \otimes \mathbb{L}(V(\mu)),$$

so that the algebraic multiplier algebra of $\mathfrak{C}_c^\infty(G_q)$ is

$$\mathfrak{C}^\infty(G_q) = \mathcal{M}(\mathfrak{C}_c^\infty(G_q)) = \prod_{\mu \in \mathbf{P}^+} \mathfrak{C}^\infty(K_q) \otimes \mathbb{L}(V(\mu)).$$

The counit of $\mathfrak{C}^\infty(G_q)$ is the tensor product counit $\epsilon_G = \epsilon \otimes \hat{\epsilon}$, and the antipode is given by

$$S_{G_q}(f \otimes x) = W^{-1}(S(f) \otimes \hat{S}(x))W = (S \otimes \hat{S})(W(f \otimes x)W^{-1}).$$

Like its classical counterpart, the quantum group G_q is unimodular [63].

Proposition 3.19. *A positive left and right invariant integral on $\mathfrak{C}_c^\infty(G_q)$ is given by*

$$\phi_{G_q}(f \otimes x) = \phi(f)\hat{\psi}(x),$$

for $f \in \mathfrak{C}^\infty(K_q)$, $x \in \mathfrak{D}(K_q)$.

Proof. It follows from the end of Subsection 3.2.3 that the right invariant Haar integral $\hat{\psi}$ on $\mathfrak{D}(K_q)$ is given by

$$\hat{\psi}(x) = \hat{\phi}(F^2x)$$

for all $x \in \mathfrak{D}(K_q)$, where $F = K_{-2\rho}$ is the invertible element obtained from the Schur orthogonality relations, see Lemma 3.17. Moreover, observe that

$$\sum_{\mu,i,j} (F^{-1} \rightharpoonup u_{ij}^\mu \leftarrow F^{-1}) \otimes F\omega_{ij}^\mu F = \sum_{\mu,i,j} u_{ij}^\mu \otimes \omega_{ij}^\mu = W$$

due to the dual basis property of W . This implies

$$\begin{aligned} (\phi \otimes \hat{\phi})(W(f \otimes x)) &= \sum_{\mu,i,j} \phi((F^{-1} \rightharpoonup u_{ij}^\mu \leftarrow F^{-1})f) \hat{\phi}(F\omega_{ij}^\mu Fx) \\ &= \sum_{\mu,i,j} \phi(fu_{ij}^\mu) \hat{\phi}(F^2x\omega_{ij}^\mu) \\ &= \sum_{\mu,i,j} \phi(fu_{ij}^\mu) \hat{\psi}(x\omega_{ij}^\mu) \\ &= (\phi \otimes \hat{\psi})(f \otimes x)W \end{aligned}$$

for all $f \in \mathfrak{C}^\infty(K_q)$, $x \in \mathfrak{D}(K_q)$.

Since $(\text{id} \otimes \hat{\Delta})(W) = W_{13}W_{12}$ we thus obtain

$$\begin{aligned} (\text{id} \otimes \phi_G)\Delta_G(f \otimes x) &= (\text{id} \otimes \phi_G)(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(W) \otimes \text{id})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\text{id} \otimes \phi \otimes \text{id} \otimes \hat{\psi})\text{ad}(W_{23})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\text{id} \otimes \phi \otimes \text{id} \otimes \hat{\psi})\text{ad}(W_{24}^{-1}(\text{id} \otimes \hat{\Delta})(W)_{234})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\text{id} \otimes \phi \otimes \text{id} \otimes \hat{\phi})\text{ad}((\text{id} \otimes \hat{\Delta})(W)_{234})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\text{id} \otimes \phi \otimes \hat{\phi})\text{ad}(W_{23})(\Delta \otimes \text{id})(f \otimes x) \\ &= (\text{id} \otimes \phi \otimes \hat{\psi})(\Delta \otimes \text{id})(f \otimes x) \\ &= (\phi \otimes \hat{\psi})(f \otimes x) = \phi_G(f \otimes x) \end{aligned}$$

for all $f \in \mathfrak{C}_c^\infty(K_q)$, $x \in \mathfrak{D}(K_q)$ as desired. Hence ϕ_G is left invariant. Similarly, since $(\Delta \otimes \text{id})(W) = W_{13}W_{23}$ we obtain

$$\begin{aligned} (\phi_G \otimes \text{id})\Delta_G(f \otimes x) &= (\phi_G \otimes \text{id})(\text{id} \otimes \sigma \otimes \text{id})(\text{id} \otimes \text{ad}(W) \otimes \text{id})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \text{id} \otimes \hat{\psi} \otimes \text{id})\text{ad}(W_{23})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \text{id} \otimes \hat{\psi} \otimes \text{id})\text{ad}(W_{13}^{-1}(\Delta \otimes \text{id})(W)_{123})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \text{id} \otimes \hat{\phi} \otimes \text{id})\text{ad}((\Delta \otimes \text{id})(W)_{123})(\Delta \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \hat{\phi} \otimes \text{id})\text{ad}(W_{12})(\text{id} \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \hat{\psi} \otimes \text{id})(\text{id} \otimes \hat{\Delta})(f \otimes x) \\ &= (\phi \otimes \hat{\psi})(f \otimes x) = \phi_G(f \otimes x), \end{aligned}$$

which means that ϕ_G is also right invariant. \square

The dual $\mathfrak{D}(G_q)$ of $\mathfrak{C}_c^\infty(G_q)$ in the sense of algebraic quantum groups is given by

$$\mathfrak{D}(G_q) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q),$$

equipped with the tensor product comultiplication and the multiplication given by

$$\begin{aligned} (x \bowtie f)(y \bowtie g) &= x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(\hat{S}(y_{(3)}), f_{(3)})g \\ &= x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(y_{(3)}, S^{-1}(f_{(3)}))g, \end{aligned}$$

using the natural skew-pairing between $\mathfrak{D}(K_q)$ and $\mathfrak{C}^\infty(K_q)$.

It is important to note that when relating $\mathfrak{C}_c^\infty(G_q)$ with $\mathfrak{D}(G_q)$ one should work with the pairing

$$(f \otimes x, y \bowtie g) = (f, y)(x, g),$$

for $f, g \in \mathfrak{C}^\infty(K_q), x, y \in \mathfrak{D}(K_q)$, where we recall that we write $(x, g) = (g, \hat{S}(x))$ for $x \in \mathfrak{D}(K_q)$ and $g \in \mathfrak{C}^\infty(K_q)$. The appearance of the antipode in this context is due to the fact that the multiplication of $\mathfrak{D}(G_q)$ is supposed to be the transpose of the opposite multiplication on $\mathfrak{C}_c^\infty(G_q)$, so that one has to flip the roles of $\mathfrak{C}^\infty(K_q)$ and $\mathfrak{D}(K_q)$ in the second tensor factor of the pairing.

The antipode of $\mathfrak{D}(G_q)$ is defined by

$$\begin{aligned} S(x \bowtie f) &= (1 \bowtie S(f))(\hat{S}(x) \bowtie 1) \\ &= (\hat{S}(x_{(3)}), S(f_{(3)}))\hat{S}(x_{(2)}) \bowtie S(f_{(2)})(\hat{S}(x_{(1)}), f_{(1)}), \\ &= (x_{(3)}, f_{(3)})\hat{S}(x_{(2)}) \bowtie S(f_{(2)})(\hat{S}(x_{(1)}), f_{(1)}), \end{aligned}$$

and the $*$ -structure on $\mathfrak{D}(G_q)$ is given by

$$(x \bowtie f)^* = (1 \bowtie f^*)(x^* \bowtie 1) = (x_{(1)}^*, f_{(1)}^*)x_{(2)}^* \bowtie f_{(2)}^*(\hat{S}(x_{(3)}^*), f_{(3)}^*).$$

A left Haar integral for $\mathfrak{D}(G_q)$ is given by

$$\hat{\phi}_{G_q}(x \bowtie f) = \hat{\phi}(x)\phi(f)$$

where $\hat{\phi}$ and ϕ are left Haar integrals for $\mathfrak{D}(K_q)$ and $\mathfrak{C}^\infty(K_q)$, respectively. Similarly, a right Haar integral for $\mathfrak{D}(G_q)$ is given by the tensor product of right Haar integrals for $\mathfrak{D}(K_q)$ and $\mathfrak{C}^\infty(K_q)$.

Since G_q is an algebraic quantum group, the general theory outlined in Section 3.2 implies that it defines a locally compact quantum group. Explicitly, the Hopf C^* -algebra of functions on G_q is given by

$$C_0(G_q) = C(K_q) \otimes C^*(K_q).$$

Note that since the compact quantum group K_q is both amenable and coamenable there is no need to distinguish between maximal and reduced versions of $C(K_q)$ and $C^*(K_q)$ here. The quantum group G_q is coamenable as well because $\epsilon \otimes \hat{\epsilon}$ is a bounded counit for $C_0(G_q) = C_0^r(G_q)$, so that $C_0^f(G_q) \cong C_0^r(G_q)$, compare Theorem 3.1 in [11].

The full and reduced Hopf C^* -algebras $C_f^*(G_q)$ and $C_r^*(G_q)$ may be constructed using the general theory of locally compact quantum groups, or more directly as completions of $\mathfrak{D}(G_q)$. We remark that the maximal and reduced group C^* -algebras of G_q are not isomorphic, that is, the quantum group G_q is not amenable.

It is not apparent from Definition 3.18 why G_q should be understood as a quantum deformation of the complex Lie group G . Nevertheless, this is indeed the case, and it is a basic instance of the quantum duality principle, see [25]. From this point of view, the discrete part \hat{K}_q of the Drinfeld double corresponds to the subgroup AN in the Iwasawa decomposition $G = KAN$ of G . In fact, the group G can be viewed as the classical double of the Poisson-Lie group K with its standard Poisson structure, see [49]. As we shall discuss further below, the deformation aspect of the theory of complex quantum groups is most visible if one works with algebras of polynomial functions instead. This is in fact the starting point taken by Podleś and Woronowicz in [63].

3.4.2. The connected component of the identity. Although the classical group G is connected, the quantization G_q defined above behaves more like an almost connected group. Specifically, we shall see here that each of the quantum groups G_q

admits a quotient to a finite classical group. This has implications in the representation theory of G_q . In particular, it leads to a finite group of one-dimensional representations.

Recall that we are using G to denote the connected, simply connected Lie group associated to the complex semisimple Lie algebra \mathfrak{g} . The centre Z of G is a finite subgroup of the maximal compact torus T in K . As such, we can identify Z also as a quantum subgroup of the compact quantum group K_q , via the surjective morphism of Hopf $*$ -algebras $\pi_Z : \mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(T) \rightarrow \mathfrak{C}^\infty(Z)$.

There is a canonical isomorphism between Z and the quotient group $\mathbf{P}^\vee/\mathbf{Q}^\vee$, where

$$\mathbf{P}^\vee = \{\mu \in \mathfrak{h}^* \mid (\mu, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \mathbf{Q}\}$$

denotes the coweight lattice associated to the root system of \mathfrak{g} . We can explicitly realize $\mathbf{P}^\vee/\mathbf{Q}^\vee$ as a central quantum subgroup of K_q as follows. Consider the finite subgroup $(i\hbar^{-1}\mathbf{P}^\vee)/(i\hbar^{-1}\mathbf{Q}^\vee)$ of \mathfrak{h}_q^* . The corresponding algebra characters $K_{i\hbar^{-1}\gamma}$ with $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$ have the property that if $f = \langle v' \mid \bullet \mid v \rangle \in \text{End}(V(\mu))^*$ is a matrix coefficient of any simple module $V(\mu)$, with v a vector of weight ν then

$$(K_{i\hbar^{-1}\gamma}, f) = q^{i\hbar^{-1}(\gamma, \nu)} \langle v', v \rangle = e^{2\pi i(\gamma, \nu)} \epsilon(f) = e^{2\pi i(\gamma, \mu)} \epsilon(f),$$

since $\mu - \nu \in \mathbf{Q}$. That is to say, $K_{i\hbar^{-1}\gamma}$ evaluates on any matrix coefficient in $\text{End}(V(\mu))^*$ as a constant multiple $e^{2\pi i(\gamma, \mu)}$ of the counit. It follows that

$$K_{i\hbar^{-1}\gamma} \rightharpoonup f = f_{(1)}(K_{i\hbar^{-1}\gamma}, f_{(2)}) = (K_{i\hbar^{-1}\gamma}, f_{(1)})f_{(2)} = f \leftarrow K_{i\hbar^{-1}\gamma}$$

for all $f \in \mathfrak{C}^\infty(K_q)$, and thus that $K_{i\hbar^{-1}\gamma}$ is a central group-like element in $\mathcal{M}(\mathfrak{D}(K_q))$. The subalgebra of $\mathcal{M}(\mathfrak{D}(K_q))$ spanned by these elements is a Hopf $*$ -algebra isomorphic to the group algebra $\mathfrak{D}(\mathbf{P}^\vee/\mathbf{Q}^\vee) \cong \mathfrak{D}(Z)$. The above calculation also implies that the projection $\pi_Z : \mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(Z)$ satisfies

$$\pi_Z(f_{(1)}) \otimes f_{(2)} = \pi_Z(f_{(2)}) \otimes f_{(1)}$$

for all $f \in \mathfrak{C}^\infty(K_q)$.

Proposition 3.20. *The linear map $\mathfrak{D}(G_q) \rightarrow \mathfrak{D}(\hat{Z})$ defined by*

$$\hat{\epsilon} \otimes \pi_Z : \mathfrak{D}(G_q) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(Z) \cong \mathfrak{D}(\hat{Z})$$

is a surjective morphism of algebraic quantum groups.

Proof. Let $x \bowtie f, y \bowtie g \in \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q)$. Using the above property of π_Z , we have

$$\begin{aligned} (\hat{\epsilon} \otimes \pi_Z)((x \bowtie f)(y \bowtie g)) &= (y_{(1)}, f_{(1)}) (\hat{S}(y_{(3)}), f_{(3)}) \hat{\epsilon}(xy_{(2)}) \pi_Z(f_{(2)}g) \\ &= (y_{(1)}, f_{(1)}) (\hat{S}(y_{(3)}), f_{(2)}) \hat{\epsilon}(x) \hat{\epsilon}(y_{(2)}) \pi_Z(f_{(3)}) \pi_Z(g) \\ &= (y_{(1)} \hat{S}(y_{(2)}), f_{(1)}) \hat{\epsilon}(x) \pi_Z(f_{(2)}) \pi_Z(g) \\ &= \hat{\epsilon}(y) \hat{\epsilon}(x) \pi_Z(f) \pi_Z(g) \\ &= (\hat{\epsilon} \otimes \pi_Z)(x \bowtie f) (\hat{\epsilon} \otimes \pi_Z)(y \bowtie g). \end{aligned}$$

The map $\hat{\epsilon} \otimes \pi_Z$ clearly respects the coproduct, and is therefore a morphism of multiplier Hopf-algebras. It is surjective since the quotient map $\mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(Z)$ is surjective, and it respects the involution since

$$(\hat{\epsilon} \otimes \pi_Z)((x \bowtie f)^*) = ((\hat{\epsilon} \otimes \pi_Z)(1 \bowtie f^*)) ((\hat{\epsilon} \otimes \pi_Z)(x^* \bowtie 1)) = \overline{\hat{\epsilon}(x)} \pi_Z(f)^*$$

for all $x \bowtie f \in \mathfrak{D}(G_q)$. \square

Since the elements $K_{i\hbar^{-1}\gamma}$ for $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$ are group-like in $\mathfrak{D}(Z) = \mathfrak{C}^\infty(\hat{Z})$, we can use the morphism of Proposition 3.20 to define a finite group of algebra characters of $\mathfrak{D}(G_q)$, indexed by $Z \cong \mathbf{P}^\vee/\mathbf{Q}^\vee$, as follows.

Definition 3.21. For each $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$, we define a character $\hat{\epsilon}_\gamma : \mathfrak{D}(G_q) \rightarrow \mathbb{C}$ by

$$\hat{\epsilon}_\gamma(x \bowtie f) = (K_{i\hbar^{-1}\gamma}, (\hat{\epsilon} \otimes \pi_Z)(x \bowtie f)) = \hat{\epsilon}(x)(K_{i\hbar^{-1}\gamma}, f).$$

These are non-degenerate $*$ -characters, so they correspond to one-dimensional unitary representations of the quantum group G_q . Note that we have $(\hat{\epsilon}_\gamma \otimes \hat{\epsilon}_{\gamma'}) \circ \hat{\Delta} = \hat{\epsilon}_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \mathbf{P}^\vee/\mathbf{Q}^\vee$.

We shall denote by PK_q the quotient quantum group $PK_q = K_q/Z$. Explicitly, PK_q is defined via its algebra of functions

$$\begin{aligned} \mathfrak{C}^\infty(PK_q) &= \{f \in \mathfrak{C}^\infty(K_q) \mid (\text{id} \otimes \pi_Z)\Delta(f) = f \otimes 1\} \\ &= \bigoplus_{\mu \in \mathbf{P}^+ \cap \mathbf{Q}} \text{End}(V(\mu))^*. \end{aligned}$$

Geometrically, this corresponds to the projective version of K_q , which is a quantization of the group of adjoint type associated with \mathfrak{k} . The dual multiplier Hopf algebra is

$$\mathfrak{D}(PK_q) = \bigoplus_{\mu \in \mathbf{P}^+ \cap \mathbf{Q}} \text{End}(V(\mu)),$$

and the obvious projection $\hat{\pi} : \mathfrak{D}(K_q) \rightarrow \mathfrak{D}(PK_q)$ defines a morphism of algebraic quantum groups.

Let us define the “connected component” G_q^0 of G_q to be the quantum double $G_q^0 = K_q \bowtie \widehat{PK}_q$. That is,

$$\mathfrak{C}_c^\infty(G_q^0) = \mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(PK_q)$$

with the coproduct twisted by the bicharacter $U = (\text{id} \otimes \hat{\pi})(W_{K_q})$, where $W_{K_q} \in \mathcal{M}(\mathfrak{C}^\infty(K_q) \otimes \mathfrak{D}(K_q))$ is the fundamental multiplicative unitary for K_q and $\hat{\pi} : \mathfrak{D}(K_q) \rightarrow \mathcal{M}(\mathfrak{D}(PK_q))$ is the projection homomorphism. The dual algebraic quantum group $\mathfrak{D}(G_q^0) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(PK_q)$ satisfies

$$\mathfrak{D}(G_q^0) = \{u \in \mathfrak{D}(G_q) \mid \hat{\epsilon}_\gamma(u) = \hat{\epsilon}_{G_q}(u) \text{ for all } \gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee\},$$

which is to say that the one-dimensional representations of G_q from Definition 3.21 all become trivial upon restriction to the quantum subgroup G_q^0 .

3.5. Polynomial functions. Our next aim is to describe the Hopf $*$ -algebra of holomorphic and antiholomorphic polynomial functions on the complex quantum group G_q . We keep our general assumptions from the previous section.

Using the universal R -matrix of $U_q(\mathfrak{g})$ we obtain a skew-pairing r^{-1} between $\mathcal{O}(G_q)$ and $\mathcal{O}(G_q)$ given by

$$r^{-1}(f, g) = (\mathcal{R}^{-1}, f \otimes g),$$

see Section 2.11. We also set $r(f, g) = (\mathcal{R}, f \otimes g) = r^{-1}(S(f) \otimes g)$. The functional r will be referred to as the universal r -form on $\mathcal{O}(G_q)$.

Starting from the universal r -form, we may form the Hopf algebraic quantum double of $\mathcal{O}(G_q)$. More precisely, we shall consider the Hopf algebra

$$\mathcal{O}^{\mathbb{R}}(G_q) = \mathcal{O}(G_q) \bowtie \mathcal{O}(G_q),$$

with underlying vector space $\mathcal{O}(G_q) \otimes \mathcal{O}(G_q)$ together with the tensor product coalgebra structure and the multiplication

$$(f \bowtie g)(h \bowtie k) = (\mathcal{R}^{-1}, h_{(1)} \otimes g_{(1)})fh_{(2)} \bowtie g_{(2)}k(\mathcal{R}, h_{(3)} \otimes g_{(3)}).$$

Lemma 3.22. *The algebra $\mathcal{O}^{\mathbb{R}}(G_q) = \mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)$ becomes a Hopf $*$ -algebra with the $*$ -structure*

$$(f \bowtie g)^* = g^* \bowtie f^*.$$

Proof. Due to the relation $(\hat{S} \otimes \hat{S})(\mathcal{R}^*) = \mathcal{R}_{21}$ obtained in Lemma 3.14 the universal r -form on $\mathcal{O}(G_q) = \mathcal{O}^{\mathbb{R}}(K_q)$ is real, that is,

$$(\mathcal{R}, f^* \otimes g^*) = \overline{(\mathcal{R}, g \otimes f)}.$$

We compute

$$\begin{aligned} ((f \bowtie g)(h \bowtie k))^* &= \overline{(\mathcal{R}^{-1}, h_{(1)} \otimes g_{(1)})(k^* g_{(2)}^* \bowtie h_{(2)}^* f^*)} \overline{(\mathcal{R}, h_{(3)} \otimes g_{(3)})} \\ &= \overline{(\mathcal{R}^{-1}, g_{(1)}^* \otimes h_{(1)}^*)(k^* g_{(2)}^* \bowtie h_{(2)}^* f^*)} \overline{(\mathcal{R}, g_{(3)}^* \otimes h_{(3)}^*)} \\ &= \overline{(k^* \bowtie h^*)(g^* \bowtie f^*)} \\ &= (h \bowtie k)^*(f \bowtie g)^* \end{aligned}$$

using

$$\begin{aligned} (\mathcal{R}^{-1}, g^* \otimes h^*) &= (\mathcal{R}, S^{-1}(g^*) \otimes h^*) \\ &= (\mathcal{R}, S(g)^* \otimes h^*) \\ &= \overline{(\mathcal{R}, h \otimes S(g))} = \overline{(\mathcal{R}^{-1}, h \otimes g)}. \end{aligned}$$

Hence $\mathcal{O}^{\mathbb{R}}(G_q)$ is indeed a $*$ -algebra with the above $*$ -structure. It is clear that the comultiplication on $\mathcal{O}^{\mathbb{R}}(G_q)$ is a $*$ -homomorphism. \square

The Hopf $*$ -algebra $\mathcal{O}^{\mathbb{R}}(G_q)$ should be thought of as a quantum version of the algebra of polynomial functions on the real Lie group underlying G . The two copies of $\mathcal{O}(G_q)$ inside $\mathcal{O}^{\mathbb{R}}(G_q)$ may be interpreted as holomorphic and antiholomorphic polynomials.

We recall from Lemma 2.111 that the l -functionals on $\mathcal{O}(G_q)$, given by

$$l^+(f)(h) = (\mathcal{R}, h \otimes f), \quad l^-(f)(h) = (\mathcal{R}^{-1}, f \otimes h)$$

for $f \in \mathcal{O}(G_q)$, satisfy $\hat{\Delta}(l^\pm(f)) = l^\pm(f_{(1)}) \otimes l^\pm(f_{(2)})$ and $l^\pm(fg) = l^\pm(f)l^\pm(g)$ for $f, g \in \mathcal{O}(K_q)$.

Lemma 3.23. *For $f \in \mathcal{C}^\infty(K_q)$ we have*

$$l^\pm(f)^* = l^\mp(f^*)$$

in $U_q^{\mathbb{R}}(\mathfrak{k})$.

Proof. We compute

$$\begin{aligned} (l^+(f)^*, h) &= \overline{(l^+(f), S(h)^*)} \\ &= \overline{(\mathcal{R}, S(h)^* \otimes f)} \\ &= (\mathcal{R}, f^* \otimes S(h)) \\ &= (\mathcal{R}^{-1}, f^* \otimes h) \\ &= (l^-(f^*), h). \end{aligned}$$

Similarly,

$$\begin{aligned} (l^-(f)^*, h) &= \overline{(l^-(f), S(h)^*)} \\ &= \overline{(\mathcal{R}^{-1}, f \otimes S(h)^*)} \\ &= \overline{(\mathcal{R}, f \otimes h^*)} \\ &= (\mathcal{R}, h \otimes f^*) \\ &= (l^+(f^*), h) \end{aligned}$$

as claimed. \square

The following result should be viewed as a quantum analogue of the fact that complex-valued polynomial functions on the group G embed into the algebra of smooth functions on G , compare [33].

Proposition 3.24. *The linear map $i : \mathcal{O}^{\mathbb{R}}(G_q) \rightarrow \mathfrak{C}^{\infty}(G_q) = \mathcal{M}(\mathfrak{C}_c^{\infty}(G_q))$ given by*

$$i(f \bowtie g) = f_{(1)}g_{(1)} \otimes l^-(f_{(2)})l^+(g_{(2)})$$

is a nondegenerate homomorphism of multiplier Hopf $$ -algebras.*

Proof. Let $f, g, h, k \in \mathcal{O}(G_q)$. Recall from Lemma 2.105 that we have the relation $(\mathcal{R}, f_{(1)} \otimes g_{(1)})g_{(2)}f_{(2)} = f_{(1)}g_{(1)}(\mathcal{R}, f_{(2)} \otimes g_{(2)})$ and hence

$$(\mathcal{R}^{-1}, f_{(1)} \otimes g_{(1)})f_{(2)}g_{(2)} = g_{(1)}f_{(1)}(\mathcal{R}^{-1}, f_{(2)} \otimes g_{(2)}).$$

Moreover, from the calculation

$$\begin{aligned} & (\mathcal{R}, f_{(1)} \otimes g_{(1)}) (l^+(g_{(2)})l^-(f_{(2)}), k) \\ &= (\mathcal{R}, f_{(1)} \otimes g_{(1)}) (\mathcal{R}, k_{(1)} \otimes g_{(2)}) (\mathcal{R}^{-1}, f_{(2)} \otimes k_{(2)}) \\ &= (\mathcal{R}, k_{(1)}f_{(1)} \otimes g) (\mathcal{R}^{-1}, f_{(2)} \otimes k_{(2)}) \\ &= (\mathcal{R}^{-1}, f_{(1)} \otimes k_{(1)}) (\mathcal{R}, f_{(2)}k_{(2)} \otimes g) \\ &= (\mathcal{R}^{-1}, f_{(1)} \otimes k_{(1)}) (\mathcal{R}, f_{(2)} \otimes g_{(2)}) (\mathcal{R}, k_{(2)} \otimes g_{(1)}) \\ &= (l^-(f_{(1)})l^+(g_{(1)}), k) (\mathcal{R}, f_{(2)} \otimes g_{(2)}) \end{aligned}$$

we obtain the exchange relation

$$(\mathcal{R}, f_{(1)} \otimes g_{(1)})l^+(g_{(2)})l^-(f_{(2)}) = l^-(f_{(1)})l^+(g_{(1)})(\mathcal{R}, f_{(2)} \otimes g_{(2)}).$$

With these formulas, we compute

$$\begin{aligned} & i((f \bowtie g)(h \bowtie k)) \\ &= (\mathcal{R}^{-1}, h_{(1)} \otimes g_{(1)}) i(fh_{(2)} \bowtie g_{(2)}k) (\mathcal{R}, h_{(3)} \otimes g_{(3)}) \\ &= (\mathcal{R}^{-1}, h_{(1)} \otimes g_{(1)}) f_{(1)}h_{(2)}g_{(2)}k_{(1)} \otimes l^-(f_{(2)})l^-(h_{(3)})l^+(g_{(3)})l^+(k_{(2)}) (\mathcal{R}, h_{(4)} \otimes g_{(4)}) \\ &= f_{(1)}g_{(1)}h_{(1)}k_{(1)} (\mathcal{R}^{-1}, h_{(2)} \otimes g_{(2)}) \otimes (\mathcal{R}, h_{(3)} \otimes g_{(3)}) l^-(f_{(2)})l^+(g_{(4)})l^-(h_{(4)})l^+(k_{(2)}) \\ &= f_{(1)}g_{(1)}h_{(1)}k_{(1)} \otimes l^-(f_{(2)})l^+(g_{(2)})l^-(h_{(2)})l^+(k_{(2)}) \\ &= i(f \bowtie g)i(h \bowtie k). \end{aligned}$$

We conclude that i is an algebra homomorphism.

To check the coalgebra homomorphism property it is enough to consider elements of the form $f \bowtie 1$ and $1 \bowtie g$. Using Lemma 2.111 we compute

$$\begin{aligned} \Delta_{G_q}i(f \bowtie 1) &= \Delta_{G_q}(f_{(1)} \otimes l^-(f_{(2)})) \\ &= (\text{id} \otimes \sigma \otimes \text{id})(f_{(1)} \otimes W(f_{(2)} \otimes l^-(f_{(3)}))W^{-1} \otimes l^-(f_{(4)})) \\ &= (f_{(1)} \otimes l^-(f_{(2)})) \otimes (f_{(3)} \otimes l^-(f_{(4)})) \\ &= (i \otimes i)(f_{(1)} \bowtie 1 \otimes f_{(2)} \bowtie 1) \\ &= (i \otimes i)\Delta(f \bowtie 1) \end{aligned}$$

for $f \in \mathcal{O}(G_q)$, using

$$\begin{aligned}
 (W(f_{(1)} \otimes l^-(f_{(2)})), x \otimes h) &= (W, x_{(1)} \otimes h_{(1)})(f_{(1)} \otimes l^-(f_{(2)}), x_{(2)} \otimes h_{(2)}) \\
 &= (h_{(1)}, x_{(1)})(f_{(1)}, x_{(2)})(l^-(f_{(2)}), h_{(2)}) \\
 &= (h_{(1)}f_{(1)}, x)(\mathcal{R}^{-1}, f_{(2)} \otimes h_{(2)}) \\
 &= (\mathcal{R}^{-1}, f_{(1)} \otimes h_{(1)})(f_{(2)}h_{(2)}, x) \\
 &= (l^-(f_{(1)}), h_{(1)})(f_{(2)}, x_{(1)})(h_{(2)}, x_{(2)}) \\
 &= (f_{(2)} \otimes l^-(f_{(1)}), x_{(1)} \otimes h_{(1)})(W, x_{(2)} \otimes h_{(2)}) \\
 &= ((f_{(2)} \otimes l^-(f_{(1)}))W, x \otimes h).
 \end{aligned}$$

In the same way one checks $\Delta_{G_q} i(1 \bowtie g) = (i \otimes i)\Delta(1 \bowtie g)$ for $g \in \mathcal{O}(G_q)$.

For the $*$ -compatibility notice that we have

$$\begin{aligned}
 i((f \otimes g)^*) &= i(g^* \otimes f^*) \\
 &= g_{(1)}^* f_{(1)}^* \otimes l^-(g_{(2)}^*) l^+(f_{(2)}^*) \\
 &= (f_{(1)} g_{(1)})^* \otimes l^+(g_{(2)})^* l^-(f_{(2)})^* \\
 &= i(f \otimes g)^*,
 \end{aligned}$$

due to Lemma 3.23. □

3.6. The quantized universal enveloping algebra of a complex group. In this section we introduce the quantized universal enveloping algebra of the complex quantum group G_q and discuss some related constructions and results. We keep our general assumptions from the previous section.

Recall that the group algebra $\mathfrak{D}(G_q)$ can be identified with the Drinfeld double

$$\mathfrak{D}(G_q) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q)$$

with respect to the canonical pairing between $\mathfrak{D}(K_q)$ and $\mathfrak{C}^\infty(K_q)$.

Proposition 3.25. *The linear map $\iota : \mathfrak{D}(G_q) \rightarrow \mathcal{M}(\mathfrak{D}(K_q) \otimes \mathfrak{D}(K_q))$ given by*

$$\iota(x \bowtie f) = \hat{\Delta}(x)(l^-(f_{(1)}) \otimes l^+(f_{(2)}))$$

is an injective essential algebra homomorphism.

Proof. We compute

$$\begin{aligned}
 \iota((x \bowtie f)(y \bowtie g)) &= \iota(x(y_{(1)}, f_{(1)})y_{(2)} \bowtie f_{(2)}(\hat{S}(y_{(3)}), f_{(3)})g) \\
 &= (y_{(1)}, f_{(1)})\hat{\Delta}(xy_{(2)})(\hat{S}(y_{(3)}), f_{(4)})(l^-(f_{(2)}g_{(1)}) \otimes l^+(f_{(3)}g_{(2)})) \\
 &= \hat{\Delta}(x)(y_{(1)}, f_{(1)})\hat{\Delta}(y_{(2)})(\hat{S}(y_{(3)}), f_{(4)})(l^-(f_{(2)})l^-(g_{(1)}) \otimes l^+(f_{(3)})l^+(g_{(2)})) \\
 &= \hat{\Delta}(x)(l^-(f_{(1)}) \otimes l^+(f_{(2)}))\hat{\Delta}(y)(l^-(g_{(1)}) \otimes l^+(g_{(2)})) \\
 &= \iota(x \bowtie f)\iota(y \bowtie g)
 \end{aligned}$$

for $x \bowtie f, y \bowtie g \in \mathfrak{D}(G_q)$, using

$$\begin{aligned}
 &((y_{(1)}, f_{(1)})\hat{\Delta}(y_{(2)})(l^-(f_{(2)}) \otimes l^+(f_{(3)})), h \otimes k) \\
 &= (y_{(1)}, f_{(1)})(y_{(2)}, k_{(1)}h_{(1)})(\mathcal{R}^{-1}, f_{(2)} \otimes h_{(2)})(\mathcal{R}, k_{(2)} \otimes f_{(3)}) \\
 &= (y, k_{(1)}h_{(1)}f_{(1)})(\mathcal{R}^{-1}, f_{(2)} \otimes h_{(2)})(\mathcal{R}, k_{(2)} \otimes f_{(3)}) \\
 &= (\mathcal{R}^{-1}, f_{(1)} \otimes h_{(1)})(\mathcal{R}, k_{(1)} \otimes f_{(2)})(y, f_{(3)}k_{(2)}h_{(2)}) \\
 &= (\mathcal{R}^{-1}, f_{(1)} \otimes h_{(1)})(\mathcal{R}, k_{(1)} \otimes f_{(2)})(y_{(1)}, k_{(2)}h_{(2)})(y_{(2)}, f_{(3)}) \\
 &= ((l^-(f_{(1)}) \otimes l^+(f_{(2)}))\hat{\Delta}(y_{(1)})(y_{(2)}, f_{(3)}), h \otimes k).
 \end{aligned}$$

Hence ι is an algebra homomorphism, and it is evident that ι is essential.

We next show that ι is injective. For this, note that the image of ι belongs to the space

$$M = \{m \in \mathcal{M}(\mathfrak{D}(K_q) \otimes \mathfrak{D}(K_q)) \mid (z \otimes 1)m \in \mathfrak{D}(K_q) \otimes \mathfrak{D}(K_q) \text{ for all } z \in \mathfrak{D}(K_q)\}$$

Note that the Galois map on $\mathfrak{D}(K_q) \otimes \mathfrak{D}(K_q)$ defined by

$$x \otimes y \mapsto x\hat{S}(y_{(1)}) \otimes y_{(2)}$$

extends to a linear isomorphism $\gamma : M \rightarrow M$. It suffices therefore to prove the injectivity of the composition

$$i : \mathfrak{D}(G_q) \rightarrow \mathfrak{D}(G_q) \xrightarrow{\iota} M \xrightarrow{\gamma} M,$$

where the first map is the linear isomorphism sending $x \bowtie f$ to $(1 \bowtie f)(x \bowtie 1)$. We calculate

$$i(x \bowtie f) = l^-(f_{(1)})l^+(S(f_{(2)})) \otimes l^+(f_{(3)})x = I(f_{(1)}) \otimes l^+(f_{(2)})x.$$

where $I : \mathcal{O}(G_q) \rightarrow FU_q(\mathfrak{g})$ is the isomorphism of Proposition 2.116.

Consider then an element in the kernel of i . We can write it as a finite linear combination

$$\sum_{\mu \in \mathbf{P}^+} \sum_{j,k} x_{jk}^\mu \bowtie u_{jk}^\mu,$$

where $u_{jk}^\mu = \langle e_j^\mu \mid \bullet \mid e_k^\mu \rangle$ are the matrix coefficients with respect to an orthonormal weight basis (e_j^μ) of $V(\mu)$, and x_{jk}^μ are some elements of $\mathfrak{D}(K_q)$. Then by assumption we have

$$i\left(\sum_{\mu \in \mathbf{P}^+} \sum_{j,k} x_{jk}^\mu \bowtie u_{jk}^\mu\right) = \sum_{\mu \in \mathbf{P}^+} \sum_{j,r,k} I(u_{jr}^\mu) \otimes l^+(u_{rk}^\mu)x_{jk}^\mu = 0.$$

Since I is an isomorphism and the u_{jr}^μ are linearly independent, this implies that for all fixed μ, j, r ,

$$\sum_k l^+(u_{rk}^\mu)x_{jk}^\mu = 0.$$

Let us write ϵ_k for the weight of e_k^μ . From the form of the universal R -matrix, see Theorem 2.108, we see that $l^+(u_{k'k}^\mu) = 0$ unless $\epsilon_k \geq \epsilon_{k'}$. Moreover, $l^+(u_{kk}^\mu)$ is invertible in $U_q(\mathfrak{g})$, while $l^+(u_{k'k}^\mu) = 0$ if $\epsilon_{k'} = \epsilon_k$ but $k' \neq k$.

Fix μ and j and suppose $x_{jk}^\mu \neq 0$ for some k . Choose k' with $\epsilon_{k'}$ maximal amongst all ϵ_k where $x_{jk}^\mu \neq 0$. In this case we have $l^+(u_{k'k}^\mu)x_{jk}^\mu = 0$ for all k except $k = k'$ and hence $x_{jk'}^\mu = 0$, a contradiction. It follows that $x_{jk}^\mu = 0$ for all μ, j, k , as desired. \square

Let us now give the definition of the quantized universal enveloping algebra of a complex group.

Definition 3.26. The quantized universal enveloping algebra of the real Lie algebra underlying \mathfrak{g} is

$$U_q^{\mathbb{R}}(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k}) \bowtie \mathfrak{C}^\infty(K_q),$$

equipped with the standard Hopf $*$ -algebra structure.

Explicitly, the multiplication in $U_q^{\mathbb{R}}(\mathfrak{g})$ is given by

$$(X \bowtie f)(Y \bowtie g) = X(Y_{(1)}, f_{(1)})Y_{(2)} \bowtie f_{(2)}(\hat{S}(Y_{(3)}), f_{(3)})g,$$

for $X, Y \in U_q^{\mathbb{R}}(\mathfrak{k})$ and $f, g \in \mathfrak{C}^\infty(K_q)$ and the $*$ -structure is given by

$$(X \bowtie f)^* = (1 \bowtie f^*)(X^* \bowtie 1) = (X^*_{(1)}, f^*_{(1)})X^*_{(2)} \bowtie f^*_{(2)}(\hat{S}(X^*_{(3)}), f^*_{(3)}).$$

We view $U_q^{\mathbb{R}}(\mathfrak{g})$ as a substitute of the universal enveloping algebra of the real Lie algebra \mathfrak{g} . Note that $U_q^{\mathbb{R}}(\mathfrak{g}) \subset \mathcal{M}(\mathfrak{D}(G_q))$ naturally.

The following result is originally due to Krähmer [50] and Arano [3].

Lemma 3.27. *The linear map $\iota : U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by*

$$\iota(X \bowtie f) = \hat{\Delta}(X)(l^-(f_{(1)}) \otimes l^+(f_{(2)}))$$

is an injective algebra homomorphism, and its image is

$$\begin{aligned} \iota(U_q^{\mathbb{R}}(\mathfrak{g})) &= (FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g})) \\ &= (1 \otimes \hat{S}^{-1}(FU_q(\mathfrak{g})))\hat{\Delta}(U_q(\mathfrak{g})) \\ &= \hat{\Delta}(U_q(\mathfrak{g}))(FU_q(\mathfrak{g}) \otimes 1) \\ &= \hat{\Delta}(U_q(\mathfrak{g}))(1 \otimes \hat{S}^{-1}(FU_q(\mathfrak{g}))). \end{aligned}$$

In particular, we have an algebra isomorphism

$$U_q^{\mathbb{R}}(\mathfrak{g}) \cong FU_q(\mathfrak{g}) \rtimes U_q(\mathfrak{g})$$

where $U_q(\mathfrak{g})$ acts on $FU_q(\mathfrak{g})$ via the adjoint action.

Proof. Since $U_q^{\mathbb{R}}(\mathfrak{g}) \subset \mathcal{M}(\mathfrak{D}(G_q))$ it follows from Proposition 3.25 that ι defines an injective algebra homomorphism $U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$. This fact was proved in a slightly different way in [50].

For the second claim compare Section 4 in [3]. We have

$$\begin{aligned} \iota((1 \bowtie f)(X \bowtie 1)) &= (l^-(f_{(1)}) \otimes l^+(f_{(2)}))\hat{\Delta}(X) \\ &= (l^-(f_{(1)})l^+(S(f_{(2)}))l^+(f_{(3)}) \otimes l^+(f_{(4)}))\hat{\Delta}(X) \\ &= (I(f_{(1)}) \otimes 1)\hat{\Delta}(l^+(f_{(2)}))\hat{\Delta}(X) \subset (FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g})), \end{aligned}$$

where I is the isomorphism from Proposition 2.116. This proves the inclusion $\iota(U_q^{\mathbb{R}}(\mathfrak{g})) \subset (FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g}))$. Conversely, for $X = I(f) \in FU_q(\mathfrak{g})$ and $Y \in U_q(\mathfrak{g})$ we have

$$\begin{aligned} (I(f) \otimes 1)\hat{\Delta}(Y) &= (l^-(f_{(1)})l^+(S(f_{(2)})) \otimes 1)\hat{\Delta}(Y) \\ &= (l^-(f_{(1)}) \otimes l^+(f_{(2)}))(l^+(S(f_{(4)})) \otimes l^+(S(f_{(3)})))\hat{\Delta}(X) \subset \iota(U_q^{\mathbb{R}}(\mathfrak{g})). \end{aligned}$$

Using the relations $X \otimes 1 = (1 \otimes \hat{S}^{-1}(X_{(3)}))(X_{(1)} \otimes X_{(2)})$ and $1 \otimes \hat{S}^{-1}(X) = (X_{(3)} \otimes 1)(\hat{S}^{-1}(X_{(2)}) \otimes \hat{S}^{-1}(X_{(1)}))$, and the fact that $FU_q(\mathfrak{g})$ satisfies $\hat{\Delta}(FU_q(\mathfrak{g})) \subset U_q(\mathfrak{g}) \otimes FU_q(\mathfrak{g})$ we obtain

$$(FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g})) = (1 \otimes \hat{S}^{-1}(FU_q(\mathfrak{g})))\hat{\Delta}(U_q(\mathfrak{g})).$$

The remaining equalities follow from the fact that $\iota(U_q^{\mathbb{R}}(\mathfrak{g}))$ is an algebra, combined with

$$\begin{aligned} (Y \otimes 1)\hat{\Delta}(X) &= X_{(3)}\hat{S}^{-1}(X_{(2)})YX_{(1)} \otimes X_{(4)} \\ &= X_{(2)}\hat{S}^{-1}(X_{(1)}) \rightarrow Y \otimes X_{(3)} \subset \hat{\Delta}(U_q(\mathfrak{g}))(FU_q(\mathfrak{g}) \otimes 1) \end{aligned}$$

and

$$\begin{aligned} (1 \otimes \hat{S}^{-1}(Y))\hat{\Delta}(\hat{S}^{-1}(X)) &= \hat{S}^{-1}(X_{(4)}) \otimes \hat{S}^{-1}(X_{(3)})X_{(2)}\hat{S}^{-1}(Y)\hat{S}^{-1}(X_{(1)}) \\ &= \hat{S}^{-1}(X_{(3)}) \otimes \hat{S}^{-1}(X_{(2)})\hat{S}^{-1}(X_{(1)}) \rightarrow Y \\ &\subset \hat{\Delta}(U_q(\mathfrak{g}))(1 \otimes \hat{S}^{-1}(FU_q(\mathfrak{g}))) \end{aligned}$$

for $Y \in FU_q(\mathfrak{g})$ and $X \in U_q(\mathfrak{g})$.

Finally, we have a linear isomorphism $FU_q(\mathfrak{g}) \rtimes U_q(\mathfrak{g}) \rightarrow (FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g}))$ given by $\gamma(X \rtimes Y) = (X \otimes 1)\hat{\Delta}(Y)$, and since

$$\begin{aligned} \gamma((X \rtimes Y)(X' \rtimes Y')) &= \gamma(XY_{(1)}X'\hat{S}(Y_{(2)}) \rtimes Y_{(3)}Y') \\ &= XY_{(1)}X'Y'_{(1)} \otimes Y_{(2)}Y'_{(2)} \\ &= (X \otimes 1)\hat{\Delta}(Y)(X' \otimes 1)\hat{\Delta}(Y') = \gamma(X \rtimes Y)\gamma(X' \rtimes Y') \end{aligned}$$

the map γ is compatible with multiplication. \square

We remark that ι is not a homomorphism of coalgebras; in fact there is no bialgebra structure on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ for which ι becomes a homomorphism of coalgebras. Let us also point out that ι is not a $*$ -homomorphism in a natural way.

Using Lemma 3.27 we can determine the centre of $U_q^{\mathbb{R}}(\mathfrak{g})$.

Lemma 3.28. *The centre $ZU_q^{\mathbb{R}}(\mathfrak{g})$ of $U_q^{\mathbb{R}}(\mathfrak{g})$ is isomorphic to $ZU_q(\mathfrak{g}) \otimes ZU_q(\mathfrak{g})$.*

Proof. Lemma 3.27 shows that $ZU_q(\mathfrak{g}) \otimes ZU_q(\mathfrak{g}) \subset Z(\text{im}(\iota))$, where $\iota : U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ is as above.

Conversely, any element $X \in \text{im}(\iota)$ which commutes with $\text{im}(\iota)$ must in particular commute with the elements $K_{2\mu} \otimes 1$ and $1 \otimes K_{-2\mu}$ for $\mu \in \mathbf{P}^+$. It follows that X has weight 0 with respect to the diagonal action of $U_q(\mathfrak{h})$ on both left and right tensor factors. Hence X commutes with $U_q(\mathfrak{h})FU_q(\mathfrak{g}) = U_q(\mathfrak{g})$ in both tensor factors, and therefore is contained in $ZU_q(\mathfrak{g}) \otimes ZU_q(\mathfrak{g})$. \square

We have a bilinear pairing between $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $\mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)$ given by

$$(X \otimes Y, f \bowtie g) = (X, g)(Y, f).$$

Using this pairing, elements of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ can be viewed as linear functionals on $\mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)$. For $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$ and $g \bowtie h \in \mathcal{O}^{\mathbb{R}}(G_q)$ we compute

$$\begin{aligned} (\iota(X \bowtie f), g \bowtie h) &= (X_{(1)}l^-(f_{(1)}) \otimes X_{(2)}l^+(f_{(2)}), g \bowtie h) \\ &= (X_{(1)}, h_{(1)})(\mathcal{R}^{-1}, f_{(1)} \otimes h_{(2)})(X_{(2)}, g_{(1)})(\mathcal{R}, g_{(2)} \otimes f_{(2)}) \\ &= (X_{(2)}, g_{(1)})(X_{(1)}, h_{(1)})(\mathcal{R}^{-1}, g_{(2)} \otimes S^{-1}(f_{(2)}))(\mathcal{R}, S^{-1}(f_{(1)}) \otimes h_{(2)}) \\ &= (X, g_{(1)}h_{(1)})(l^-(g_{(2)})l^+(h_{(2)}), S^{-1}(f)) \\ &= (X \bowtie f, g_{(1)}h_{(1)} \otimes l^-(g_{(2)})l^+(h_{(2)})) \\ &= (X \bowtie f, \iota(g \bowtie h)) \end{aligned}$$

for $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$ and $g \bowtie h \in \mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)$.

3.7. Parabolic quantum subgroups. In this section we describe briefly how to obtain quantum analogues of parabolic subgroups in complex semisimple Lie groups. We continue to use the notation introduced in previously.

Recall that $\Sigma = \{\alpha_1, \dots, \alpha_N\}$ denotes the set of simple roots, and let $S \subset \Sigma$ be a subset. We obtain a corresponding Hopf $*$ -subalgebra $U_q^{\mathbb{R}}(\mathfrak{k}_S) \subset U_q^{\mathbb{R}}(\mathfrak{k})$ generated by all K_λ for $\lambda \in \mathbf{P}$ together with the generators E_i, F_i for $\alpha_i \in S$. Notice that $U_q^{\mathbb{R}}(\mathfrak{k}_\Sigma) = U_q^{\mathbb{R}}(\mathfrak{k})$ and $U_q^{\mathbb{R}}(\mathfrak{k}_\emptyset) = U_q^{\mathbb{R}}(\mathfrak{t})$. The inclusion $U_q^{\mathbb{R}}(\mathfrak{k}_S) \rightarrow U_q^{\mathbb{R}}(\mathfrak{k})$ induces a map $U_q^{\mathbb{R}}(\mathfrak{k})^* \rightarrow U_q^{\mathbb{R}}(\mathfrak{k}_S)^*$, and we denote by $\mathfrak{C}^\infty(K_{q,S})$ the Hopf $*$ -algebra obtained as the image of $\mathfrak{C}^\infty(K_q)$ under this map. In this way, $K_{q,S}$ is a closed quantum subgroup of K_q . Notice that $K_{q,\Sigma} = K_q$, and that $K_{q,\emptyset} = T$ is the classical maximal torus inside K_q .

We define the parabolic quantum subgroup $P_q \subset G_q$ associated to the set S as the Drinfeld double

$$\mathfrak{C}_c^\infty(P_q) = \mathfrak{C}_c^\infty(K_{q,S} \bowtie \hat{K}_q) = \mathfrak{C}^\infty(K_{q,S}) \otimes \mathfrak{D}(K_q)$$

with the coproduct induced from $\mathfrak{C}_c^\infty(G_q)$. That is, in the formula for the comultiplication of $\mathfrak{C}_c^\infty(G_q)$ one has to replace W by $(\pi_S \otimes \text{id})(W) \in \mathcal{M}(\mathfrak{C}^\infty(K_{q,S}) \otimes \mathfrak{D}(K_q))$, where $\pi_S : \mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(K_{q,S})$ is the canonical quotient map.

In a similar way we construct the quantized universal enveloping algebras of the corresponding real Lie groups. More precisely, we define the Hopf $*$ -algebra $U_q^{\mathbb{R}}(\mathfrak{p})$ associated to the parabolic quantum subgroup P_q by

$$U_q^{\mathbb{R}}(\mathfrak{p}) = U_q^{\mathbb{R}}(\mathfrak{k}_S) \bowtie \mathfrak{C}^\infty(K_q)$$

where the pairing between $U_q^{\mathbb{R}}(\mathfrak{k}_S)$ and $\mathfrak{C}^\infty(K_q)$ used in the definition of the double is induced from the canonical pairing between $U_q^{\mathbb{R}}(\mathfrak{k})$ and $\mathfrak{C}^\infty(K_q)$. By construction, we have a canonical inclusion homomorphism $U_q^{\mathbb{R}}(\mathfrak{p}) \rightarrow U_q^{\mathbb{R}}(\mathfrak{g})$.

The quantum Borel subgroup $B_q \subset G_q$ is defined to be the parabolic subgroup corresponding to $S = \emptyset$. Explicitly,

$$\mathfrak{C}_c^\infty(B_q) = \mathfrak{C}_c^\infty(T \bowtie \hat{K}_q) = \mathfrak{C}^\infty(T) \otimes \mathfrak{D}(K_q)$$

with the comultiplication twisted by the bicharacter as explained above. The corresponding quantized universal enveloping algebra is

$$U_q^{\mathbb{R}}(\mathfrak{b}) = U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(K_q).$$

In the study of principal series representations we will only work with the Borel quantum subgroup and its quantized universal enveloping algebra.

4. CATEGORY \mathcal{O}

In this chapter we study some aspects of the representation theory of quantized universal enveloping algebras with applications to the theory of Yetter-Drinfeld modules and complex quantum groups. Our main goal is a proof of the Verma module annihilator Theorem, following the work of Joseph and Letzter. We refer to [28] for a survey of the ideas involved in the proof and background.

In this chapter, unless explicitly stated otherwise, we shall work over $\mathbb{K} = \mathbb{C}$ and assume that $1 \neq q = e^{\hbar}$ is positive. We shall also use the notation $\hbar = \frac{\hbar}{2\pi}$. Recall that in this situation, we have the identification $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$, see Subsection 2.3.1.

4.1. The definition of category \mathcal{O} . In this section we introduce category \mathcal{O} for $U_q(\mathfrak{g})$, compare Section 4.1.4 in [41] and [1], [37].

We start with the following definition, compare [1].

Definition 4.1. A left module M over $U_q(\mathfrak{g})$ is said to belong to category \mathcal{O} if

- a) M is finitely generated as a $U_q(\mathfrak{g})$ -module.
- b) M is a weight module, that is, a direct sum of its weight spaces M_λ for $\lambda \in \mathfrak{h}_q^*$.
- c) The action of $U_q(\mathfrak{n}_+)$ on M is locally nilpotent.

Morphisms in category \mathcal{O} are all $U_q(\mathfrak{g})$ -linear maps.

Note that category \mathcal{O} is closed under taking submodules and quotient modules. Specifically, finite generation passes to submodules by Noetherianity of $U_q(\mathfrak{g})$, see Proposition 2.134, and submodules and quotients of weight modules are again weight modules. Local nilpotency is obvious in either case.

Due to finite generation, any module M in category \mathcal{O} satisfies $\dim M_\lambda < \infty$ for all $\lambda \in \mathfrak{h}_q^*$. We define the formal character of M by setting

$$\text{ch}(M) = \sum_{\lambda \in \mathfrak{h}_q^*} \dim(M_\lambda) e^\lambda,$$

here the expression on the right hand side is interpreted as a formal sum.

More generally, we will consider formal sums of the form $\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda) e^\lambda$ where $f : \mathfrak{h}_q^* \rightarrow \mathbb{Z}$ is any integer valued function whose support lies in a finite union of sets of the form $\nu - \mathbf{Q}^+$ with $\nu \in \mathfrak{h}_q^*$. For such formal sums, there is a well-defined convolution product given by

$$\left(\sum_{\lambda \in \mathfrak{h}_q^*} f(\lambda) e^\lambda \right) \left(\sum_{\mu \in \mathfrak{h}_q^*} g(\mu) e^\mu \right) = \sum_{\lambda, \mu \in \mathfrak{h}_q^*} f(\lambda) g(\mu) e^{\lambda+\mu}.$$

In particular, let us define

$$p = \prod_{\beta \in \Delta^+} \left(\sum_{m=0}^{\infty} e^{-m\beta} \right) = \sum_{\nu \in \mathbf{Q}^+} P(\nu) e^{-\nu},$$

where P is Kostant's partition function,

$$P(\nu) = |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1\beta_1 + \dots + r_n\beta_n = \nu\}|.$$

If $\mu \in \mathfrak{h}_q^*$ and $M = M(\mu)$ is the Verma module with highest weight μ then Proposition 2.68 immediately shows that the character of M is given by

$$\text{ch}(M(\mu)) = e^\mu p.$$

4.1.1. *Category \mathcal{O} is Artinian.* In this subsection we discuss finiteness properties of category \mathcal{O} .

Let us first show that every module M in \mathcal{O} admits a finite filtration by highest weight modules, compare Section 1.2 in [37]. Recall that a $U_q(\mathfrak{g})$ -module M is called a highest weight module of highest weight λ if there exists a primitive vector v_λ in M which generates M . Recall also that we have a partial order on \mathfrak{h}_q^* defined by $\lambda \leq \mu$ if and only if $\mu - \lambda \in \mathbf{Q}^+$, see Section 2.3.

Lemma 4.2. *Let M be a nonzero module in \mathcal{O} . Then M has a finite filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that each subquotient M_j/M_{j-1} for $1 \leq j \leq n$ is a highest weight module.

Proof. According to condition a) in Definition 4.1 we see that M is generated by finitely many weight vectors. Consider the $U_q(\mathfrak{n}_+)$ -module V generated by a finite generating set of weight vectors, which by condition c) is finite dimensional.

We prove the claim by induction on $\dim(V)$, the case $\dim(V) = 1$ being trivial. For the inductive step pick a weight in \mathfrak{h}_q^* which is maximal among the weights appearing in V . Then we can find a corresponding primitive vector $v \in M$ and obtain an associated submodule $M_0 = U_q(\mathfrak{g}) \cdot v \subset M$. The quotient M/M_0 is again finitely generated by the $U_q(\mathfrak{n}_+)$ -module V/Cv . We can therefore apply our inductive hypothesis to obtain a filtration of the desired type for M/M_0 , and pulling this filtration back to M yields the assertion. \square

Recall that a module M over a ring R is called Noetherian (Artinian) if every ascending chain $M_1 \subset M_2 \subset M_3 \subset \cdots$ of submodules (every descending chain $M_1 \supset M_2 \supset \cdots$ of submodules) becomes stationary, that is, if there exists $n \in \mathbb{N}$ such that $M_n = M_{n+1} = M_{n+2} = \cdots$.

We show that every module in \mathcal{O} is Artinian and Noetherian, compare Section 1.11 in [37].

Theorem 4.3. *Every module M in category \mathcal{O} is Artinian and Noetherian. Moreover $\dim_{U_q(\mathfrak{g})}(M, N) < \infty$ for all $M, N \in \mathcal{O}$.*

Proof. According to Lemma 4.2, any nonzero module M in \mathcal{O} has a finite filtration with subquotients given by highest weight modules. Hence it suffices to treat the case where $M = M(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is a Verma module.

Consider the finite dimensional subspace $V = \bigoplus_{\hat{w} \in \hat{W}} M(\lambda)_{\hat{w}\lambda}$ of $M = M(\lambda)$, where we recall that \hat{W} is the extended Weyl group defined in Section 2.13. Assume that $N_1 \subset N_2$ is a proper inclusion of submodules of M . Then $ZU_q(\mathfrak{g})$ acts on N_1, N_2 and N_2/N_1 by the central character ξ_λ . Since $N_2 \subset M$ the module N_2/N_1 contains a primitive vector v_μ of some weight $\mu \leq \lambda$. Therefore $\xi_\mu = \xi_\lambda$. According to Theorem 2.128 this implies $\mu = w.\lambda$ for some $w \in \hat{W}$. We conclude $N_2 \cap V \neq 0$ and $\dim(N_2 \cap V) > \dim(N_1 \cap V)$. Since V is finite dimensional this means that any strictly ascending or descending chain of submodules will have length at most $\dim(V)$. In particular, every module M in \mathcal{O} is both Artinian and Noetherian.

Any $U_q(\mathfrak{g})$ -linear map $M \rightarrow N$ is determined by its values on a finite generating set of weight vectors. Since the weight spaces of N are finite dimensional we conclude $\dim_{U_q(\mathfrak{g})}(M, N) < \infty$ for all $M, N \in \mathcal{O}$. \square

Of course, Noetherianity of modules in \mathcal{O} follows also from the fact that $U_q(\mathfrak{g})$ is Noetherian, see Theorem 2.134.

Due to Theorem 4.3 one can apply Jordan-Hölder theory to category \mathcal{O} . More precisely, every module $M \in \mathcal{O}$ has a decomposition series $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that all subquotients M_{j+1}/M_j are simple highest weight modules.

Moreover, the number of subquotients isomorphic to $V(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is independent of the decomposition series and will be denoted by $[M : V(\lambda)]$. Note that this gives the formula

$$\text{ch}(M) = \sum_{\lambda \in \mathfrak{h}_q^*} [M : V(\lambda)] \text{ch}(V(\lambda))$$

for the character of M , with only finitely many nonzero terms on the right hand side.

In particular, for any $\mu \in \mathfrak{h}_q^*$ the character of the Verma module $M(\mu)$ can be written as

$$\text{ch}(M(\mu)) = \sum_{\lambda \in \mathfrak{h}_q^*} [M(\mu) : V(\lambda)] \text{ch}(V(\lambda)).$$

Note that $[M(\mu) : V(\mu)] = 1$, and moreover $[M(\mu) : V(\lambda)] = 0$ unless $\lambda \leq \mu$ and λ is \hat{W} -linked to μ , due to Theorem 2.128. It follows that the character of the simple module $V(\mu)$ can be expressed in the form

$$\text{ch}(V(\mu)) = \sum_{\lambda \in W \cdot \mu} m_\lambda \text{ch}(M(\lambda))$$

for certain integers m_λ . For $\mu \in \mathbf{P}^+$ these coefficients are given as follows.

Proposition 4.4. *Let $\mu \in \mathbf{P}^+$. Then we have*

$$\text{ch}(V(\mu)) = \sum_{w \in W} (-1)^{l(w)} \text{ch}(M(w \cdot \mu)).$$

Proof. The above discussion shows that

$$\text{ch}(V(\mu)) = \sum_{w \in W} m_{w \cdot \mu} \text{ch}(M(w \cdot \mu))$$

for some integers $m_{w \cdot \mu}$. We need to show that $m_{w \cdot \mu} = (-1)^{l(w)}$.

Recall that $\text{ch}(M(\mu)) = e^\mu p$ where $p = \prod_{\beta \in \Delta^+} (1 + e^{-\beta} + e^{-2\beta} + \dots)$. We introduce the formal sum

$$q = \prod_{\beta \in \Delta^+} (e^{\beta/2} - e^{-\beta/2}) = e^\rho \prod_{\beta \in \Delta^+} (1 - e^{-\beta}),$$

and note that the right hand expression gives $pq = e^\rho$. Therefore, we obtain

$$\text{ch}(V(\mu))q = \sum_{w \in W} m_{w \cdot \mu} e^{w \cdot \mu} pq = \sum_{w \in W} m_{w \cdot \mu} e^{w \mu}.$$

Consider the Weyl group action on finite formal sums defined by $we^\lambda = e^{w\lambda}$ for $w \in W$ and $\lambda \in \mathfrak{h}_q^*$. Note that q is alternating with respect to this action, meaning that $wq = (-1)^{l(w)}q$ for all $w \in W$. Moreover, $\text{ch}(V(\mu))$ is W -invariant by Proposition 2.42. Therefore, $\text{ch}(V(\mu))q$ is alternating, and hence $m_{w \cdot \mu} = (-1)^{l(w)}m_\mu$ for all $w \in W$. Since $m_\mu = 1$, the result follows. \square

As a consequence, the weight spaces of the irreducible module $V(\mu)$ for $\mu \in \mathbf{P}^+$ have the same dimensions as in the classical case, and the dimension of $V(\mu)$ is given by the Weyl dimension formula.

4.1.2. *Duality.* The appropriate duality operation in category \mathcal{O} is different from the usual duality for modules over Hopf algebras defined in terms of the antipode.

We define duals in category \mathcal{O} in the same way as we did for general weight modules in Section 2.3. Namely, if M is a module in category \mathcal{O} we let the dual of M be the $U_q(\mathfrak{g})$ -module

$$M^\vee = \bigoplus_{\lambda \in \mathfrak{h}_q^*} \text{Hom}(M_\lambda, \mathbb{C}),$$

with the left $U_q(\mathfrak{g})$ -module structure given by

$$(X \cdot f)(v) = f(\tau(X) \cdot v),$$

where τ is the involution from Definition 2.16. Notice that we have a canonical isomorphism $M^{\vee\vee} \cong M$ since all weight spaces are finite dimensional and τ is involutive. If $0 \rightarrow K \rightarrow M \rightarrow Q \rightarrow 0$ is an exact sequence of modules in category \mathcal{O} then the dual sequence $0 \rightarrow Q^\vee \rightarrow M^\vee \rightarrow K^\vee \rightarrow 0$ is again exact.

Let us show that the dual M^\vee of a module M in category \mathcal{O} is again in category \mathcal{O} . It is clear that M^\vee is a weight module such that $(M^\vee)_\lambda = \text{Hom}(M_\lambda, \mathbb{C})$. Local nilpotency of the action of $U_q(\mathfrak{n}_+)$ follows from the fact that the elements in M^\vee are supported on only finitely many weight spaces.

To see that M^\vee is finitely generated we argue as follows, see [37]. Assume M^\vee is not finitely generated. Then we can find a strictly increasing infinite sequence $0 = U_0 \subset U_1 \subset \dots \subset M^\vee$ of finitely generated submodules $U_k \subset M^\vee$. Setting $Q_j = M^\vee/U_j$ we obtain a corresponding infinite sequence $M^\vee = M^\vee/0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \dots$ of quotient modules of M^\vee and surjective module maps. By exactness of the duality functor this in turn leads to a strictly decreasing infinite sequence of submodules of M . As we have seen in the proof of Theorem 4.3 this is impossible.

In summary, we conclude that sending M to M^\vee defines a contravariant involutive self-equivalence of category \mathcal{O} .

4.1.3. Dominant and antidominant weights. In this subsection we discuss the notion of dominant and antidominant weights in analogy to the classical theory, see Chapter 10 in [37]. There are some new features in the quantum setting due to the exponentiation in the Cartan part of the quantized universal enveloping algebra.

Recall from Definition 2.125 that the extended Weyl group is defined as $\hat{W} = \mathbf{Y}_q \rtimes W$, where \mathbf{Y}_q denotes the subgroup of elements of order at most 2 in \mathfrak{h}_q^* . Under the identification $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ we have $\mathbf{Y}_q = \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee$. The extended Weyl group acts on \mathfrak{h}_q^* by

$$(\zeta, w)\lambda = w\lambda + \zeta,$$

for $\zeta \in \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee$, $w \in W$ and $\lambda \in \mathfrak{h}_q^*$, and also by the shifted action

$$(\zeta, w) \cdot \lambda = w \cdot \lambda + \zeta = w(\lambda + \rho) - \rho + \zeta.$$

Two elements of \mathfrak{h}_q^* are \hat{W} -linked if they lie in the same orbit of the shifted \hat{W} -action.

Following the notation in [37], we define

$$\Delta_{[\lambda]} = \{\alpha \in \Delta \mid q_\alpha^{(\lambda, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}\}.$$

Here we are writing $q_\alpha^{(\lambda, \mu)}$ to denote $q^{(\lambda, d_\alpha \mu)}$ for $\mu \in \mathbf{P}$. Note that $\Delta_{[\mu]} = \Delta_{[\lambda]}$ if $\mu \in \lambda + \mathbf{P}$, so we may equivalently characterize $\Delta_{[\lambda]}$ by

$$\Delta_{[\lambda]} = \{\alpha \in \Delta \mid q_\alpha^{(\lambda + \rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}\}.$$

Next we define

$$\hat{W}_{[\lambda]} = \{\hat{w} \in \hat{W} \mid \hat{w}\lambda - \lambda \in \mathbf{Q}\}.$$

Note that if $\mu = \lambda + \nu$ with $\nu \in \mathbf{P}$ then for any $\hat{w} = (\zeta, w) \in \hat{W}_{[\lambda]}$ we have

$$(\zeta, w)\mu - \mu = (w\lambda + \zeta - \lambda) + (w\nu - \nu) = ((\zeta, w)\lambda - \lambda) + (w\nu - \nu) \in \mathbf{Q}.$$

It follows that $\hat{W}_{[\mu]} = \hat{W}_{[\lambda]}$ whenever $\mu \in \lambda + \mathbf{P}$, and therefore

$$\hat{W}_{[\lambda]} = \{\hat{w} \in W \mid \hat{w} \cdot \lambda - \lambda \in \mathbf{Q}\}.$$

We define $W_{[\lambda]}$ as the image of $\hat{W}_{[\lambda]}$ under the canonical projection $\hat{W} \rightarrow W$. Note that $\hat{W}_{[\lambda]} \cap (\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee \rtimes \{e\})$ is trivial, so the projection map $\hat{W}_{[\lambda]} \rightarrow W_{[\lambda]}$ is in fact an isomorphism. Explicitly, we have

$$\begin{aligned} W_{[\lambda]} &= \{w \in W \mid w\lambda - \lambda \in \mathbf{Q} + \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee\} \\ &= \{w \in W \mid w \cdot \lambda - \lambda \in \mathbf{Q} + \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee\}. \end{aligned}$$

In the next proposition, we write E for the \mathbb{R} -span of the root system $\mathbf{\Delta} \subset \mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ we write $\lambda = \operatorname{Re}(\lambda) + i\operatorname{Im}(\lambda)$ where $\operatorname{Re}(\lambda), \operatorname{Im}(\lambda)$ belong to E . Similarly, we decompose $\lambda \in \mathfrak{h}_q^*$, under the understanding that $\operatorname{Im}(\lambda)$ is only defined modulo $\hbar^{-1}\mathbf{Q}^\vee$.

Proposition 4.5. *Let $\lambda \in \mathfrak{h}_q^*$. Then $\mathbf{\Delta}_{[\lambda]}$ is a root system in its \mathbb{R} -span $E(\lambda) \subset E$, where $E(\lambda)$ is equipped with the inner product induced from E . Moreover $W_{[\lambda]}$ is the Weyl group of $\mathbf{\Delta}_{[\lambda]}$.*

Proof. In order to verify that $\mathbf{\Delta}_{[\lambda]}$ is a root system in $E(\lambda)$ it is enough to check $s_\beta \mathbf{\Delta}_{[\lambda]} \subset \mathbf{\Delta}_{[\lambda]}$ for all $\beta \in \mathbf{\Delta}_{[\lambda]}$. So let $\alpha \in \mathbf{\Delta}_{[\lambda]}$ and observe that $(s_\beta \alpha)^\vee = s_\beta \alpha^\vee$ since the action of s_β on E is isometric. Therefore

$$q_\alpha^{(\lambda, (s_\beta \alpha)^\vee)} = q_\alpha^{(\lambda, s_\beta \alpha^\vee)} = q_\alpha^{(\lambda, \alpha^\vee) - (\lambda, \beta^\vee)(\beta, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}$$

as desired, using that both α and β are contained in $\mathbf{\Delta}_{[\lambda]}$. Since $q_\alpha = q_{s_\beta \alpha}$ this shows $s_\beta \alpha \in \mathbf{\Delta}_{[\lambda]}$.

In order to prove that $W_{[\lambda]}$ is the Weyl group of $\mathbf{\Delta}_{[\lambda]}$ we proceed in a similar way as in Section 3.4 in [37]. We show first that $\alpha \in \mathbf{\Delta}$ satisfies $\alpha \in \mathbf{\Delta}_{[\lambda]}$ iff $s_\alpha \in W_{[\lambda]}$. Indeed, we have $s_\alpha \cdot \lambda - \lambda = -(\lambda + \rho, \alpha^\vee)\alpha$ in \mathfrak{h}_q^* , and the condition $q_\alpha^{(\lambda + \rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}$ is equivalent to $-(\lambda + \rho, \alpha^\vee)\alpha \in \mathbf{Q}$ in $\mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$. Thus, the Weyl group of $\mathbf{\Delta}_{[\lambda]}$ is contained in $W_{[\lambda]}$, and to complete the proof it suffices to show that $W_{[\lambda]}$ is generated by the reflections it contains.

We first consider the case $\lambda \in E$. We introduce the affine Weyl group $W_a = \mathbf{Q} \rtimes W$, where \mathbf{Q} acts upon E by translations and W by the standard Weyl group action. Note that an element $w \in W$ belongs to $W_{[\lambda]}$ if and only if $(\beta, w)\lambda = \lambda$ for some $\beta \in \mathbf{Q}$. By Theorem 4.8 of [36] the subgroup of W_a which fixes any $\lambda \in E$ is generated by its reflections, and the result follows.

Now let $\lambda \in \mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ be arbitrary, and lift it to an element $\lambda_0 + i\lambda_1 \in \mathfrak{h}^*$ with $\lambda_0, \lambda_1 \in E$. An element $w \in W$ belongs to $W_{[\lambda]}$ if and only if

$$w\lambda_0 - \lambda_0 \in \mathbf{Q} \quad \text{and} \quad w\lambda_1 - \lambda_1 \in \frac{1}{2}\hbar^{-1}\mathbf{Q}^\vee.$$

By the previous case, the first condition is equivalent to $w \in W_{[\lambda_0]}$, which is the Weyl group of the root system $\mathbf{\Delta}_{[\lambda_0]}$ in the subspace $E(\lambda_0)$.

Consider the rescaled coroot system

$$\mathbf{\Delta}' = \{\frac{1}{2}\hbar^{-1}\alpha^\vee \mid \alpha \in \mathbf{\Delta}_{[\lambda_0]}\} \subset E(\lambda_0).$$

It is a root system with the same Weyl group $W_{[\lambda_0]}$. We write \mathbf{Q}' for the lattice it generates and $W'_a = \mathbf{Q}' \rtimes W_{[\lambda_0]}$ for the associated affine Weyl group. Let us decompose λ_1 as

$$\lambda_1 = \lambda'_1 + \lambda''_1 \in E(\lambda_0) \oplus E(\lambda_0)^\perp.$$

Since $W_{[\lambda_0]}$ fixes $E(\lambda_0)^\perp$, we see that if $w \in W_{[\lambda_0]}$ then the above condition on λ_1 is equivalent to

$$w\lambda_1 - \lambda_1 = w\lambda'_1 - \lambda'_1 \in \frac{1}{2}\hbar^{-1}\mathbf{Q}^\vee \cap E(\lambda_0) = \mathbf{Q}',$$

and therefore $(\gamma, w)\lambda'_1 = \lambda'_1$ for some $\gamma \in \mathbf{Q}'$. As before, the subgroup of W'_a fixing λ'_1 is generated by its reflections. Thus $W_{[\lambda]}$ is generated by its reflections, and the result follows. \square

Let us now introduce the concept of dominant and antidominant weights. We recall that we use the convention $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 4.6. Let $\lambda \in \mathfrak{h}_q^*$. Then

- a) λ is dominant if $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \notin \pm q_\alpha^{-\mathbb{N}}$ for all $\alpha \in \Delta^+$.
- b) λ is antidominant if $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{N}}$ for all $\alpha \in \Delta^+$.

We remark that $-\rho$ is both dominant and antidominant. Note that $\lambda \in \mathfrak{h}_q^*$ is not dominant iff there exists some $\alpha \in \Delta^+$ such that $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}}$, or equivalently, such that $(\lambda + \rho, \alpha^\vee) = -m + \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ for some $m \in \mathbb{N}$, where we are using the notation $\hbar_\alpha = d_\alpha \hbar$. Similarly, $\lambda \in \mathfrak{h}_q^*$ is not antidominant iff there exists some $\alpha \in \Delta^+$ such that $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$, or equivalently, such that $(\lambda + \rho, \alpha^\vee) = m + \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ for some $m \in \mathbb{N}$.

We remark that the terminology introduced in Definition 4.6 is in tension with standard terminology used for dominant integral weights. More precisely, $\lambda \in \mathbf{P}$ is dominant in the sense of Definition 4.6 iff $(\lambda + \rho, \alpha^\vee) \geq 0$ for all $\alpha \in \Delta^+$, whereas λ is dominant integral if $(\lambda, \alpha^\vee) \geq 0$ for all $\alpha \in \Delta^+$. We shall refer to weights in \mathbf{P}^+ as dominant integral weights, which means that being dominant and integral is not the same thing as being dominant integral.

Let us introduce some further notation for reflections in the extended Weyl group $\hat{W} = \mathbf{Y}_q \rtimes W$, where we recall that $\mathbf{Y}_q = \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee / i\hbar^{-1}\mathbf{Q}^\vee$. If $k \in \mathbb{Z}$ and $\alpha \in \Delta_{[\lambda]}$, we write $s_{k,\alpha} = (\frac{k}{2}i\hbar^{-1}\alpha^\vee, s_\alpha) \in \hat{W}$. Explicitly, this element acts on $\nu \in \mathfrak{h}_q^*$ by

$$s_{k,\alpha} \cdot \nu = s_\alpha \cdot \nu + \frac{k}{2}i\hbar^{-1}\alpha^\vee.$$

Note that $s_{k,\alpha} = s_{k+2m,\alpha}$ for all $m \in \mathbb{Z}$, so we shall frequently parametrize $s_{k,\alpha}$ by $k \in \mathbb{Z}_2$.

In the sequel we will use the notation $\lambda \leq \mu$ in $\mathfrak{h}_q^* / \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ to say that $\mu - \lambda$ is congruent to an element of \mathbf{Q}^+ modulo $\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$. In other words, this corresponds to the natural order relation in the quotient $\mathfrak{h}_q^* / \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ of \mathfrak{h}_q^* .

Proposition 4.7. Let $\lambda \in \mathfrak{h}_q^*$. Then λ is antidominant if and only if one of the following equivalent conditions hold.

- a) $(\lambda + \rho, \alpha) \leq 0$ in $\mathbb{C} / \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ for all $\alpha \in \Delta_{[\lambda]}^+$, that is, the real part of $(\lambda + \rho, \alpha)$ is non-positive, and the imaginary part of $(\lambda + \rho, \alpha)$ is contained in $\frac{1}{2}\hbar^{-1}\mathbb{Z}$.
- b) $\lambda \leq s_\alpha \cdot \lambda$ in $\mathfrak{h}_q^* / \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ for all $\alpha \in \Delta_{[\lambda]}^+$.
- c) $\lambda \leq w \cdot \lambda$ in $\mathfrak{h}_q^* / \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ for all $w \in W_{[\lambda]}$.
- d) If $\hat{w} \in \hat{W}_{[\lambda]}$ satisfies $\hat{w} \cdot \lambda \leq \lambda$ in \mathfrak{h}_q^* then $\hat{w} = e$.

Proof. Recall that $\alpha \in \Delta_{[\lambda]}$ if and only if $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}$. Therefore λ is antidominant if and only if for every $\alpha \in \Delta_{[\lambda]}^+$ we have in fact $q_\alpha^{(\lambda+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}_0}$. This is equivalent to condition a).

Now we prove the equivalence of the four listed conditions.

a) \Leftrightarrow b) For any root $\alpha \in \Delta^+$ we have

$$s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho = \lambda - (\lambda + \rho, \alpha^\vee)\alpha = \lambda - (\lambda + \rho, \alpha)\alpha^\vee.$$

This shows the desired equivalence.

c) \Rightarrow b) is trivial.

b) \Rightarrow c) We use induction on the length of $w \in W_{[\lambda]}$. For $w = e$ the claim obviously holds. Hence assume $w = vs_i$ for some $v \in W_{[\lambda]}$ with $l(v) < l(w)$ and s_i the reflection associated to some simple root α_i in $\Delta_{[\lambda]}$. Note that, in this case, $w\alpha < 0$. We have

$$\lambda - w \cdot \lambda = (\lambda - v \cdot \lambda) + v(\lambda - s_i \cdot \lambda),$$

where in the second term v is acting by the unshifted action. The first term satisfies $\lambda - v.\lambda \leq 0$ by our inductive hypothesis. For the second term we obtain

$$v(\lambda - s_i.\lambda) = w(s_i.\lambda - \lambda) = -(\lambda + \rho, \alpha_i^\vee)w\alpha_i \leq 0$$

as well.

$c) \Rightarrow d)$ Assume $\hat{w} = (\beta^\vee, w)$ satisfies $\hat{w}.\lambda \leq \lambda$. We have

$$\hat{w}.\lambda - \lambda = w.\lambda - \lambda + \frac{1}{2}i\hbar^{-1}\beta^\vee,$$

so that $w.\lambda \leq \lambda$ in $\mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$. Hence condition $c)$ implies $w = e$. Now $(\beta^\vee, e).\lambda = \lambda + \frac{1}{2}i\hbar^{-1}\beta^\vee \leq \lambda$ yields $\beta^\vee = 0$.

$d) \Rightarrow a)$ Fix $\alpha \in \Delta_{[\lambda]}^+$ and consider the affine reflection $s_{k,\alpha} = (k\alpha^\vee, s_\alpha)$. Since $s_{k,\alpha} \in \hat{W}_{[\lambda]}$ we automatically have

$$-(\lambda + \rho, \alpha^\vee)\alpha + \frac{k}{2}i\hbar^{-1}\alpha^\vee = s_{k,\alpha}.\lambda - \lambda \in \mathbf{Q}$$

inside $\mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$, that is, $(\lambda + \rho, \alpha^\vee)\alpha \in \mathbf{Q} \subset \mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$. In particular, $(\text{Im}(\lambda) + \rho, \alpha) \in \frac{1}{2}i\hbar^{-1}\mathbb{Z}$. If $(\text{Re}(\lambda) + \rho, \alpha^\vee) > 0$ then we get $s_{k,\alpha}.\lambda < \lambda$ for suitable choice of k , which is impossible by $d)$. Hence $(\text{Re}(\lambda) + \rho, \alpha^\vee) \leq 0$.

This finishes the proof. \square

An analogous result holds for dominant weights.

Proposition 4.8. *Let $\lambda \in \mathfrak{h}_q^*$. Then λ is dominant if and only if one of the following equivalent conditions hold.*

- a) $(\lambda + \rho, \alpha) \geq 0$ in $\mathbb{C}/\frac{1}{2}i\hbar^{-1}\mathbb{Z}$ for all $\alpha \in \Delta_{[\lambda]}^+$, that is, the real part of $(\lambda + \rho, \alpha)$ is non-negative, and the imaginary part of $(\lambda + \rho, \alpha)$ is contained in $\frac{1}{2}\hbar^{-1}\mathbb{Z}$.*
- b) $\lambda \geq s_\alpha.\lambda$ in $\mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ for all $\alpha \in \Delta_{[\lambda]}^+$.*
- c) $\lambda \geq w.\lambda$ in $\mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$ for all $w \in W_{[\lambda]}$.*
- d) If $\hat{w} \in \hat{W}_{[\lambda]}$ satisfies $\hat{w}.\lambda \geq \lambda$ in \mathfrak{h}_q^* then $\hat{w} = e$.*

Proof. This is a direct translation of the proof of Proposition 4.7. \square

4.2. Submodules of Verma modules. In this section we examine the existence of submodules of Verma modules. We shall discuss in particular an analogue of Verma's Theorem and a necessary and sufficient condition for the irreducibility of Verma modules.

The socle $\text{Soc}(M)$ of a module M is the sum of all simple submodules of M . Recall from Theorem 2.134 that the algebra $U_q(\mathfrak{n}_-)$ is Noetherian without zero-divisors. Since $M(\lambda)$ for $\lambda \in \mathfrak{h}_q^*$ is a free $U_q(\mathfrak{n}_-)$ -module, the following result follows from general facts, see Section 4.1 in [37].

Lemma 4.9. *Let $\lambda \in \mathfrak{h}_q^*$. Any two nonzero submodules of $M(\lambda)$ have a nonzero intersection.*

Proof. Assume $M, N \subset M(\lambda)$ are nonzero submodules. Then $M \supset U_q(\mathfrak{n}_-)m \cdot v_\lambda$ and $N \supset U_q(\mathfrak{n}_-)n \cdot v_\lambda$ for some nonzero elements $m, n \in U_q(\mathfrak{n}_-)$. Since $U_q(\mathfrak{n}_-)$ is Noetherian and has no zero divisors, the left ideals $U_q(\mathfrak{n}_-)n$ and $U_q(\mathfrak{n}_-)m$ must have nontrivial intersection, see Section 4.1 in [37]. \square

We recall that $V(\mu)$ denotes the unique simple quotient of the Verma module $M(\mu)$.

Lemma 4.10. *Let $\lambda \in \mathfrak{h}_q^*$. Then the socle of $M(\lambda)$ is of the form*

$$\text{Soc}(M(\lambda)) \cong V(\mu) \cong M(\mu)$$

for a unique weight $\mu \leq \lambda$.

Proof. According to Lemma 4.9, the socle of $M(\lambda)$ must be simple, therefore of the form $V(\mu)$ for some $\mu \in \mathfrak{h}_q^*$. From the weight space structure of $M(\lambda)$ it is immediate that $\mu \leq \lambda$. Moreover $V(\mu) \subset M(\lambda)$ is necessarily a Verma module since $M(\lambda)$ is free as a $U_q(\mathfrak{n}_-)$ -module. Hence $V(\mu) \cong M(\mu)$. \square

Let us remark that since $M(\lambda)$ has finite length it is not hard to check that $\text{Soc}(M(\lambda)) = M(\mu) \subset V$ for any submodule $V \subset M(\lambda)$.

We shall now study more general Verma submodules of $M(\lambda)$. Recall the affine reflections s_{k, α_i} defined before Proposition 4.7.

Lemma 4.11. *Let $\lambda \in \mathfrak{h}_q^*$ and assume $s_{k, \alpha_i} \cdot \lambda \leq \lambda$ for some $1 \leq i \leq N$ and $k \in \mathbb{Z}_2$. Then there exists an embedding of $M(s_{k, \alpha_i} \cdot \lambda)$ into $M(\lambda)$.*

Proof. Note that

$$\lambda - s_{k, \alpha_i} \cdot \lambda = (\lambda + \rho, \alpha_i^\vee) \alpha_i - \frac{k}{2} i \hbar^{-1} \alpha_i^\vee,$$

so the assumption $s_{k, \alpha_i} \cdot \lambda \leq \lambda$ implies

$$\text{Re}(\lambda + \rho, \alpha_i^\vee) \in \mathbb{N}_0,$$

$$\text{Im}(\lambda, \alpha_i^\vee) = \text{Im}(\lambda + \rho, \alpha_i^\vee) \equiv \frac{k}{2} \hbar^{-1} d_i^{-1} \pmod{\hbar^{-1} d_i^{-1} \mathbb{Z}}.$$

Put $m = \text{Re}(\lambda + \rho, \alpha_i^\vee) = \text{Re}(\lambda, \alpha_i^\vee) + 1$. We claim that the vector $F_i^m v_\lambda$ is primitive. Indeed, for $j \neq i$ we have

$$E_j F_i^m v_\lambda = F_i^m E_j v_\lambda = 0,$$

while for $j = i$, Lemma 2.36 gives

$$\begin{aligned} E_i F_i^m \cdot v_\lambda &= F_i^m E_i v_\lambda + [m]_q F_i^{m-1} \frac{q_i^{-m+1} K_i - q_i^{m-1} K_i^{-1}}{q_i - q_i^{-1}} \cdot v_\lambda \\ &= [m]_q \frac{q^{-d_i \text{Re}(\lambda, \alpha_i^\vee)} q^{(\lambda, \alpha_i)} - q^{d_i \text{Re}(\lambda, \alpha_i^\vee)} q^{-(\lambda, \alpha_i)}}{q_i - q_i^{-1}} F_i^{m-1} \cdot v_\lambda \\ &= [m]_q \frac{q^{-i \text{Im}(\lambda, \alpha_i)} - q^{i \text{Im}(\lambda, \alpha_i)}}{q_i - q_i^{-1}} F_i^{m-1} \cdot v_\lambda \\ &= 0. \end{aligned}$$

The weight of $F_i^m \cdot v_\lambda$ is

$$\lambda - \text{Re}(\lambda + \rho, \alpha_i^\vee) \alpha_i = s_i \cdot \lambda + \text{Im}(\lambda + \rho, \alpha_i^\vee) \alpha_i = s_{k, \alpha_i} \cdot \lambda \in \mathfrak{h}_q^*.$$

Hence we obtain an embedding as desired. \square

Let us recall some definitions related to the Weyl group W , see Appendix A.1 in [41]. For $w \in W$ set

$$S(w) = \{\alpha \in \Delta^+ \mid w\alpha \in \Delta^-\} = \Delta^+ \cap w^{-1} \Delta^-.$$

If $u, v \in W$ and $\alpha \in \Delta^+$ we write $u \stackrel{\alpha}{\leftarrow} v$ or simply $u \leftarrow v$ if $u = s_\alpha v$ and $l(u) = l(v) - 1$. In this case one has $\alpha \in S(v^{-1})$. The Bruhat order on W is defined by saying that $u \leq v$ if $u = v$ or if there is a chain $u = w_1 \leftarrow \cdots \leftarrow w_n = v$. As usual we write $u < v$ if $u \leq v$ and $u \neq v$.

Lemma 4.12. *Let $\mu \in \mathbf{P}^+$ and assume $u, v \in W$ are such that $u \leq v$ for the Bruhat order. Then there exists an embedding of $M(v \cdot \mu)$ into $M(u \cdot \mu)$.*

Proof. We use induction on the length of v . The case $l(v) = 0$, corresponding to $v = e$ is trivial. For the inductive step assume $s_i v < v$ for some i . Then

$s_i \cdot (v \cdot \mu) - v \cdot \mu = s_i v(\mu + \rho) - v(\mu + \rho) = -(v(\mu + \rho), \alpha_i^\vee) = -(\mu + \rho, v^{-1} \alpha_i^\vee) > 0$, which means $v \cdot \mu \leq s_i \cdot (v \cdot \mu)$. Hence $M(v \cdot \mu)$ is a submodule of $M(s_i v \cdot \mu)$ according to Lemma 4.11.

Properties of the Bruhat order imply that we have either $u \leq s_i v$ or $s_i u \leq s_i v$, see for instance Proposition A.1.7 in [41].

In the first case we obtain an embedding $M(s_i v, \mu) \subset M(u, \mu)$ by our inductive hypothesis since $s_i v$ has length $l(v) - 1$. Combining this with the embedding $M(v, \mu) \subset M(s_i v, \mu)$ obtained above yields the claim in this situation.

In the second case we may assume $s_i u < u$, since otherwise we are again in the previous case. Consider the embedding $\pi : M(v, \mu) \rightarrow M(s_i v, \mu) \rightarrow M(s_i u, \mu)$ obtained from the inductive hypothesis. Let x be a highest weight vector for $M(v, \mu)$. Note that $K_i \cdot x = q^{(\alpha_i, v, \mu)} x$ with

$$(\alpha_i, v, \mu) = (v^{-1}(\alpha_i), \mu + \rho) - (\alpha_i, \rho) < 0.$$

Therefore, repeated application of Lemma 2.36 shows that $E_i^r F_i^r \cdot x$ is a nonzero multiple of x for all $r \in \mathbb{N}_0$. On the other hand, let y be a highest weight vector for $M(s_i u, \mu)$, and note from the proof of Lemma 4.11 that a highest weight vector of the submodule $M(u, \mu) \subset M(s_i u, \mu)$ is given by $F_i^m y$ for some $m \in \mathbb{N}$. Since $\pi(x) \in U_q(\mathfrak{n}_-) \cdot y$, repeated application of the Serre relations shows that for $r \gg 0$ we have $F_i^r \pi(x) \in U_q(\mathfrak{n}_-) F_i^m \cdot y = M(u, \mu)$. Therefore, $\pi(E_i^r F_i^r \cdot x) \in M(u, \mu)$ and hence π maps $M(v, \mu)$ into $M(u, \mu)$, as desired. \square

Definition 4.13. Let $\mu, \lambda \in \mathfrak{h}_q^*$. We say that $\mu \uparrow \lambda$ if $\mu = \lambda$ or if there exists a chain of positive roots $\alpha_1, \dots, \alpha_r \in \Delta^+$ and $k_1, \dots, k_r \in \mathbb{Z}$ such that

$$\mu = s_{k_1, \alpha_1} \cdots s_{k_r, \alpha_r} \cdot \lambda < s_{k_2, \alpha_2} \cdots s_{k_r, \alpha_r} \cdot \lambda < \cdots < s_{k_r, \alpha_r} \cdot \lambda < \lambda.$$

We say that μ is *strongly linked* to λ if $\mu \uparrow \lambda$.

Note that $s_{k, \alpha} \cdot \nu < \nu$ for some $k \in \mathbb{Z}$ if and only if

$$q_\alpha^{(\nu + \rho, \alpha^\vee)} \in \pm q_\alpha^\mathbb{N}.$$

We are now ready to prove the following analogue of Verma’s Theorem, compare section 4.4.9 in [41] and Section 4.7 in [37].

Theorem 4.14. *Let $\mu, \lambda \in \mathfrak{h}_q^*$. If μ is strongly linked to λ then $M(\mu) \subset M(\lambda)$, in particular $[M(\lambda) : V(\mu)] \neq 0$.*

Proof. In order to prove the claim it is enough to consider the case $\mu = s_{k, \alpha} \cdot \lambda$ for some $\alpha \in \Delta^+$ with $q_\alpha^{(\lambda + \rho, \alpha^\vee)} \in \pm q_\alpha^\mathbb{N}$.

For $\alpha \in \Delta^+$ and $n \in \mathbb{N}$ consider the sets

$$\Lambda_{n, \alpha} = \{ \lambda \in \mathfrak{h}_q^* \mid q_\alpha^{(\lambda + \rho, \alpha^\vee)} = \pm q_\alpha^n \}$$

and

$$X_{n, \alpha} = \{ \lambda \in \mathfrak{h}_q^* \mid \text{Hom}_{U_q(\mathfrak{g})}(M(\lambda - n\alpha), M(\lambda)) \neq 0 \}.$$

Note that $X_{n, \alpha} \subset \Lambda_{n, \alpha}$. Our aim is to show $X_{n, \alpha} = \Lambda_{n, \alpha}$.

We begin by showing that $\mathbf{P} \cap \Lambda_{n, \alpha} \subset X_{n, \alpha}$. Let $\lambda \in \mathbf{P} \cap \Lambda_{n, \alpha}$, and choose $w \in W$ such that $\mu = w^{-1} s_\alpha \cdot \lambda \in \mathbf{P}^+$. Then also $w^{-1} s_\alpha \cdot \lambda + \rho \in \mathbf{P}^+$ and

$$(w^{-1} s_\alpha \cdot \lambda + \rho, w^{-1} \alpha^\vee) = (w^{-1} s_\alpha (\lambda + \rho), w^{-1} \alpha^\vee) = -(\lambda + \rho, \alpha^\vee) = -n,$$

which means $w^{-1} \alpha \in \Delta^-$. This implies $w > s_\alpha w$, so we obtain $M(s_\alpha \cdot \lambda) = M(w, \mu) \subset M(s_\alpha w, \mu) = M(\lambda)$ by Lemma 4.12. Hence $\lambda \in X_{n, \alpha}$, and we conclude $\mathbf{P} \cap \Lambda_{n, \alpha} \subset X_{n, \alpha}$.

To complete the proof, we use a Zariski density argument. If $B = \mathbb{C}[x_1, \dots, x_d]/I$ is a commutative algebra then the algebraic variety $\text{Spm}(B)$ is the set of all points $x = (x_1, \dots, x_d) \in \mathbb{C}^d$ such that $f(x_1, \dots, x_d) = 0$ for all $f \in I$. Observe that \mathfrak{h}_q^* can be identified with the variety $\text{Spm}(B)$ of $B = U_q(\mathfrak{h})$ such that $\lambda \in \mathfrak{h}_q^*$ corresponds to the point of $\text{Spm}(B)$ given by the character χ_λ on B . Clearly, $\Lambda_{n, \alpha}$ is a Zariski closed subspace.

We claim that $X_{n,\alpha} \subset \mathfrak{h}_q^*$ is also Zariski closed. Indeed, for each $\gamma \in \mathbf{Q}^+$ fix a basis $Y_1^\gamma, \dots, Y_r^\gamma$ of $U_q(\mathfrak{n}_-)_{-\gamma}$. From the defining relations of $U_q(\mathfrak{g})$ it is clear that we can find Laurent polynomials p_{ij}^k in the generators K_1, \dots, K_N such that

$$[E_k, Y_i^\gamma] = \sum_j Y_j^{\gamma-\alpha_k} p_{ij}^k.$$

Now consider $Y \in U_q(\mathfrak{n}_-)_{-\gamma}$ and write $Y = \sum c_i Y_i^\gamma$. Then we have

$$E_k Y \cdot v_\lambda = [E_k, Y] \cdot v_\lambda = \sum_{i,j} c_i Y_j^{\gamma-\alpha_k} p_{ij}^k \cdot v_\lambda = \sum_{i,j} c_i \chi_\lambda(p_{ij}^k) Y_j^{\gamma-\alpha_k} \cdot v_\lambda$$

Since $M(\lambda)$ is a free $U_q(\mathfrak{n}_-)$ -module it follows that $Y \cdot v_\lambda$ is the highest weight vector for a submodule $M(\lambda - \gamma) \subset M(\lambda)$ iff $\sum_i c_i \chi_\lambda(p_{ij}^k) = 0$ for all k, j . Thus $\lambda \in X_{n,\alpha}$ if and only if there exists a nontrivial kernel of the linear map

$$(c_i)_i \mapsto \left(\sum_i c_i \chi_\lambda(p_{ij}^k) \right)_{j,k},$$

whose coefficients are polynomial in the character χ_λ . The existence of such a kernel is determined by the vanishing of the determinants of minors of the corresponding matrix $(\chi_\lambda(p_{ij}^k))_{(j,k),i}$. Applying this to $\gamma = n\alpha$ we conclude that $X_{n,\alpha}$ is an algebraic subset of $\text{Spm}(B)$.

According to our above considerations $X_{n,\alpha} \cap \mathbf{P} = \Lambda_{n,\alpha} \cap \mathbf{P}$. Since $\mathbf{P} \cap \Lambda_{n,\alpha} \subset \Lambda_{n,\alpha}$ is Zariski dense we conclude $X_{n,\alpha} = \Lambda_{n,\alpha}$ as desired. \square

As a consequence of Theorem 4.14 we obtain the following characterization of simple Verma modules, compare Section 17.4 in [18].

Theorem 4.15. *Let $\lambda \in \mathfrak{h}_q^*$. The Verma module $M(\lambda)$ is simple iff λ is antidominant.*

Proof. If λ is not antidominant there exists an $\alpha \in \Delta^+$ such that $(\lambda + \rho, \alpha^\vee) \in m + \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$ for some $m \in \mathbb{N}$, where as before we use the notation $\hbar_\alpha = d_\alpha \hbar$. In this case $s_{k,\alpha} \cdot \lambda < \lambda$ for suitable k , and according to Theorem 4.14 there exists an embedding $M(s_{k,\alpha} \cdot \lambda) \hookrightarrow M(\lambda)$. In particular, $M(\lambda)$ is not irreducible.

Conversely, assume that λ is antidominant. Then due to Proposition 4.7 the weight λ is minimal in its $\hat{W}_{[\lambda]}$ -orbit. By Harish-Chandra's Theorem 2.128, all irreducible subquotients of $M(\lambda)$ must be of the form $V(\mu)$ for some $\mu \leq \lambda$ in the $\hat{W}_{[\lambda]}$ -orbit of λ . We conclude $\mu = \lambda$, and hence $M(\lambda)$ is irreducible. \square

Let us also characterize projective Verma modules, compare Section 8.2 in [37].

Proposition 4.16. *Let $\lambda \in \mathfrak{h}_q^*$. Then the following conditions are equivalent.*

- a) $M(\lambda)$ is a projective object in \mathcal{O} .
- b) If $\text{Hom}_{U_q(\mathfrak{g})}(M(\lambda), M(\mu)) \neq 0$ for $\mu \geq \lambda$ then $\lambda = \mu$.
- c) λ is dominant.

Proof. a) \Rightarrow b) Assume $f : M(\lambda) \rightarrow M(\mu)$ is a nonzero homomorphism and $\mu > \lambda$. Note that f is injective and set $M = M(\mu)/f(I(\lambda))$, where $I(\lambda)$ is the maximal proper submodule in $M(\lambda)$. Then f induces an embedding $V(\lambda) \subset M$, and hence a surjection $M^\vee \rightarrow V(\lambda)$. Since $M(\lambda)$ is projective we get a nonzero map $M(\lambda) \rightarrow M^\vee$ lifting the canonical projection $M(\lambda) \rightarrow V(\lambda)$, or equivalently, a nonzero map $M \rightarrow M(\lambda)^\vee$. This induces a nonzero map $M(\mu) \rightarrow M(\lambda)^\vee$. This is impossible, since the μ -weight space of $M(\lambda)^\vee$ is trivial.

b) \Rightarrow c) Assume λ is not dominant. Then according to Proposition 4.8 there exists some $\mu > \lambda$ such that $\lambda \uparrow \mu$, and hence an embedding $M(\lambda) \rightarrow M(\mu)$ by Theorem 4.14. This contradicts condition b).

$c) \Rightarrow a)$ Assume that $\pi : M \rightarrow N$ is a surjective morphism in \mathcal{O} and let $f : M(\lambda) \rightarrow N$ be given. We may assume that f is nonzero. Moreover, since the centre acts by ξ_λ on $M(\lambda)$ the same is true for the action of $ZU_q(\mathfrak{g})$ on $f(M(\lambda))$. Using primary decomposition we may assume that $Z - \xi_\lambda(Z)$ acts nilpotently on M and N for all $Z \in ZU_q(\mathfrak{g})$.

Let $v \in M$ be a vector of weight λ such that $\pi(v) = f(v_\lambda)$ and let $V = U_q(\mathfrak{n}_+) \cdot v \subset M$ be the $U_q(\mathfrak{n}_+)$ -submodule of M generated by v . Since M is in \mathcal{O} the space V is finite dimensional. Choose a primitive vector $v_\mu \in V$, for some weight $\mu \geq \lambda$. Then $\xi_\lambda = \xi_\mu$, from which we conclude that μ and λ are \hat{W} -linked. Due to dominance of λ this means $\mu = \lambda$. Hence v itself is a primitive vector, which means that there exists a unique homomorphism $F : M(\lambda) \rightarrow M$ such that $F(v_\lambda) = v$. Clearly F is a lift for f as desired. \square

4.3. The Shapovalov determinant. In this section we study the Shapovalov form for $U_q(\mathfrak{g})$. This form and its determinant are important tools in the study of Verma modules. We refer to [32] for the analysis of Shapovalov forms in greater generality.

Let us recall from Lemma 2.16 that the involutive algebra anti-automorphism τ of $U_q(\mathfrak{g})$ keeps $U_q(\mathfrak{h})$ pointwise fixed and interchanges $U_q(\mathfrak{b}_+)$ and $U_q(\mathfrak{b}_-)$. Recall also that the Harish-Chandra map is the linear map $\mathcal{P} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ given by $\hat{\epsilon} \otimes \text{id} \otimes \hat{\epsilon}$ with respect to the triangular decomposition $U_q(\mathfrak{g}) \cong U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+)$, see Proposition 2.14.

We shall call the bilinear form $\text{Sh} : U_q(\mathfrak{n}_-) \times U_q(\mathfrak{n}_-) \rightarrow U_q(\mathfrak{h})$ defined by

$$\text{Sh}(X, Y) = \mathcal{P}(\tau(X)Y)$$

the Shapovalov form for $U_q(\mathfrak{g})$. One checks that $\mathcal{P}\tau = \mathcal{P}$, and using $\tau^2 = \text{id}$ this implies immediately that Sh is symmetric. Moreover we have $\text{Sh}(X, Y) = 0$ if X, Y are of different weight.

We denote the restriction of Sh to the weight space $U_q(\mathfrak{n}_-)_{-\mu}$ for $\mu \in \mathbf{Q}^+$ by Sh_μ . In the following, we will use the determinant of the form Sh_μ as follows. Fix a basis x_1, \dots, x_{n_μ} of $U_q(\mathfrak{n}_-)_{-\mu}$, and denote by $\det(\text{Sh}_\mu)$ the determinant of the matrix $(\text{Sh}_\mu(x_i, x_j)) \in M_{n_\mu}(U_q(\mathfrak{h}))$. This only depends on the choice of basis up to multiplication by a nonzero scalar in \mathbb{C} , which we shall disregard in the following discussion. Moreover, by inspecting the definition of τ and the commutation relations for E_i and F_j , one sees that the Shapovalov determinant $\det(\text{Sh}_\mu)$ is in fact contained in the subalgebra of $U_q(\mathfrak{h})$ generated by the elements $K_j^{\pm 1}$ for $j = 1, \dots, N$.

Let $\lambda \in \mathfrak{h}_q^*$. The contravariant bilinear form on $M(\lambda)$ is defined as the unique bilinear form $\text{Sh}^\lambda : M(\lambda) \times M(\lambda) \rightarrow \mathbb{C}$ satisfying $\text{Sh}^\lambda(v_\lambda, v_\lambda) = 1$ and

$$\text{Sh}^\lambda(\tau(X) \cdot v, w) = \text{Sh}^\lambda(v, X \cdot w)$$

for all $v, w \in M(\lambda)$ and $X \in U_q(\mathfrak{g})$. Different weight spaces of $M(\lambda)$ are orthogonal with respect to the form, that is, we have $\text{Sh}^\lambda(M(\lambda)_\mu, M(\lambda)_\nu) = 0$ for $\mu \neq \nu$. Notice that

$$\text{Sh}^\lambda(X \cdot v_\lambda, Y \cdot v_\lambda) = \chi_\lambda(\text{Sh}(X, Y))$$

for all $X, Y \in U_q(\mathfrak{n}_-)$, where we recall that $\chi_\lambda(K_\mu) = q^{(\lambda, \mu)}$.

The radical of Sh^λ is the subspace $R^\lambda \subset M(\lambda)$ consisting of all u satisfying $\text{Sh}^\lambda(u, v) = 0$ for all $v \in M(\lambda)$.

Lemma 4.17. *Let $\lambda \in \mathfrak{h}_q^*$. The radical of Sh^λ agrees with the maximal proper submodule $I(\lambda)$ of $M(\lambda)$.*

Proof. From the invariance property of Sh^λ it is clear that the radical R^λ is a submodule of $M(\lambda)$. It is also clear that the highest weight vector v_λ is not contained in

R^λ , and hence $R^\lambda \subset I(\lambda)$. Assume that the quotient $M(\lambda)/R^\lambda$ is not simple. Then $M(\lambda)/R^\lambda$ must contain a primitive vector, that is, a nonzero vector v of weight $\lambda - \mu$ for some $\mu \in \mathbf{Q}^+ \setminus \{0\}$ which is annihilated by $U_q(\mathfrak{n}_+) \cap \ker(\hat{\epsilon})$. This implies

$$\text{Sh}^\lambda(v, U_q(\mathfrak{n}_-) \cdot v_\lambda) = \text{Sh}^\lambda(U_q(\mathfrak{n}_+) \cdot v, v_\lambda) = 0$$

and hence $v = 0$ by nondegeneracy of the induced form on $M(\lambda)/R^\lambda$. This is a contradiction, so that $M(\lambda)/R^\lambda$ must be simple. Hence we obtain $R^\lambda = I(\lambda)$ as claimed. \square

Instead of working with the form $\text{Sh}(X, Y)$ it is sometimes technically more convenient to consider the modified form $\mathcal{S} : U_q(\mathfrak{n}_-) \times U_q(\mathfrak{n}_-) \rightarrow U_q(\mathfrak{h})$ given by

$$\mathcal{S}(X, Y) = \mathcal{P}(\Omega(X)Y),$$

where Ω is the algebra anti-homomorphism from Lemma 2.15. As before, if $\nu \in \mathbf{Q}^+$ we write \mathcal{S}_ν for the restriction of \mathcal{S} to $U_q(\mathfrak{h})_{-\nu}$. The determinant of Sh_ν differs from the one of \mathcal{S}_ν only by an invertible element of $U_q(\mathfrak{h})$.

Lemma 4.18. *Let $\nu \in \mathbf{Q}^+$ and $X, Y \in U_q(\mathfrak{n}_-)_{-\nu}$. Then*

$$\mathcal{S}_\nu(X, Y) \in \sum_{\substack{\beta, \gamma \in \mathbf{Q}^+ \\ \beta + \gamma = \nu}} \mathbb{C}K_{\beta - \gamma}.$$

Proof. We may assume without loss of generality that both X and Y are monomials in the generators F_1, \dots, F_N . Let us proceed by induction on the height $ht(\nu) = \nu_1 + \dots + \nu_N$ of $\nu = \nu_1\alpha_1 + \dots + \nu_N\alpha_N$. Note that for $ht(\nu) = 0$ the claim is obvious.

Fix $m \geq 0$ and assume that the claim holds for all terms of height m . Let $X, Y \in U_q(\mathfrak{n}_-)_{-\nu}$ be monomials in F_1, \dots, F_N with $ht(\nu) = m + 1$, and write $X = F_i Z$ for suitable i and $Z \in U_q(\mathfrak{n}_-)_{-\nu + \alpha_i}$. Let r be the total number of factors F_i appearing in the monomial Y . If $1 \leq k \leq r$ then for the k -th occurrence of F_i in Y we decompose $Y = Y_k^{(a)} F_i Y_k^{(b)}$, where $Y_k^{(a)} Y_k^{(b)}$ are monomials in the generators F_1, \dots, F_N containing a total number of factors F_i one less than for Y . We obtain

$$\mathcal{S}_\nu(X, Y) = \sum_{k=1}^r \mathcal{P}(\Omega(Z)Y_k^{(a)}[E_i, F_i]Y_k^{(b)}) \in \sum_{k=1}^r (\mathbb{C}K_i + \mathbb{C}K_i^{-1}) \mathcal{P}(\Omega(Z)Y_k^{(a)}Y_k^{(b)}).$$

The latter is contained in

$$(\mathbb{C}K_i + \mathbb{C}K_i^{-1}) \sum_{\beta, \gamma \in \mathbf{Q}^+, \beta + \gamma = \nu - \alpha_i} \mathbb{C}K_{\beta - \gamma} = \sum_{\beta, \gamma \in \mathbf{Q}^+, \beta + \gamma = \nu} \mathbb{C}K_{\beta - \gamma}$$

according to our inductive hypothesis. This yields the claim. \square

Let us enumerate the positive roots of \mathfrak{g} again as β_1, \dots, β_n . Recall that the Kostant partition function $P : \mathbf{Q} \rightarrow \mathbb{N}_0$ is given by

$$P(\nu) = |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1\beta_1 + \dots + r_n\beta_n = \nu\}|.$$

Lemma 4.19. *We have an equality of weights*

$$\dim(U_q(\mathfrak{n}_-)_{-\nu})\nu = \sum_{j=1}^n \sum_{m=1}^{\infty} P(\nu - m\beta_j)\beta_j$$

for any $\nu \in \mathbf{Q}^+$.

Proof. According to the PBW-Theorem 2.67, the space $U_q(\mathfrak{n}_-)_{-\nu}$ is spanned by all PBW-vectors $F_{\beta_1}^{r_1} \dots F_{\beta_n}^{r_n}$ such that $r_1\beta_1 + \dots + r_n\beta_n = \nu$. Therefore,

$$\dim(U_q(\mathfrak{n}_-)_{-\nu})\nu = \sum_{\substack{(r_1, \dots, r_n) \in \mathbb{N}_0^n \\ \text{with } r_1\beta_1 + \dots + r_n\beta_n = \nu}} r_1\beta_1 + \dots + r_n\beta_n.$$

Separating the coefficients of β_j in this sum gives

$$\begin{aligned} & \sum_{j=1}^n \sum_{m=1}^{\infty} |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1\beta_1 + \dots + r_n\beta_n = \nu \text{ with } r_j = m\}| m\beta_j \\ &= \sum_{j=1}^n \sum_{m=1}^{\infty} |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1\beta_1 + \dots + r_n\beta_n = \nu \text{ with } r_j \geq m\}| \beta_j, \\ &= \sum_{j=1}^n \sum_{m=1}^{\infty} |\{(r_1, \dots, r_n) \in \mathbb{N}_0^n \mid r_1\beta_1 + \dots + r_n\beta_n = \nu - m\beta_j\}| \beta_j, \\ &= \sum_{j=1}^n \sum_{m=1}^{\infty} P(\nu - m\beta_j)\beta_j, \end{aligned}$$

as claimed. □

We will also need a result from commutative algebra. Let $B \cong \mathbb{C}[x_1, \dots, x_d]/I$ be a finitely generated commutative algebra over \mathbb{C} . Recall that the algebraic variety $\text{Spm}(B)$ of B is the set of all points $p = (p_1, \dots, p_d)$ in \mathbb{C}^d such that $f(p_1, \dots, p_d) = 0$ for all $f \in I$. The elements of B can be viewed as polynomial functions on $\text{Spm}(B)$, and we write b_p for the value of $b \in B$ at $p \in \text{Spm}(B)$.

If $y(x) \in B[x]$ is a polynomial we write $y(0) \in B$ for its image under evaluation at 0, and we use the same notation for matrices with entries in $B[x]$.

The following two Lemmas are taken from [32].

Lemma 4.20. *Let B be an integral domain and $Y \in M_n(B[x])$ for some $n \in \mathbb{N}$. Then there exist $0 \leq k \leq n$, matrices $M_1, M_2 \in M_n(B)$ with nonzero determinants, a matrix $M \in M_n(B[x])$ and a nonzero element $b \in B$ such that*

$$M_1 Y M_2 = xM + bD_k$$

where $D_k = \text{diag}(1, \dots, 1, 0, \dots, 0)$ is the diagonal matrix projecting onto the first k coordinates.

Proof. Let $\text{Quot}(B)$ be the field of fractions of B . By linear algebra there exist invertible matrices $C_1, C_2 \in M_n(\text{Quot}(B))$ such that

$$C_1 Y(0) C_2 = D_k$$

for some $0 \leq k \leq n$. Let b_1, b_2 be nonzero elements such that $M_1 = b_1 C_1$ and $M_2 = b_2 C_2$ are contained in $M_n(B)$. Write $Y = Y(0) + xY'$ for suitable $Y' \in M_n(B(x))$. If we set $b = b_1 b_2$ and $M = M_1 Y' M_2$ then the claim follows. □

The rank of a matrix Y with entries in an integral domain B is denoted by $\text{rk}(Y)$; by definition this is the usual rank of Y where Y is viewed as a matrix over the field of fractions $\text{Quot}(B)$.

Lemma 4.21. *Let B be an integral domain which is at the same time a finitely generated commutative algebra over \mathbb{C} , and let $n \in \mathbb{N}$. If there exists $0 \leq r \leq n$ such that $Y(x) = (y_{ij}(x)) \in M_n(B[x])$ satisfies $\text{rk}(Y(0)_p) \leq r$ for all points p in a nonempty Zariski-open subset of the affine variety $\text{Spm}(B)$ of B , then $\det(Y(x)) = x^{n-r}b$ for some $b \in B[x]$.*

Proof. According to Lemma 4.20 there exists $0 \leq k \leq n$ such that $M_1 Y M_2 = xM + bD_k$ for matrices M_1, M_2 and $b \in B$ such that $\det(M_1), \det(M_2)$ are nonzero. Let V be a non-empty Zariski-open set of $\text{Spm}(B)$ such that $\det(M_1)_p \neq 0, \det(M_2)_p \neq 0, b_p \neq 0$ and $\text{rk}(Y(0)_p) \leq r$ for all $p \in V$. This exists by assumption since $\text{Spm}(B)$ is irreducible. Since $\text{rk}(Y(0)_p) \leq r$ for $p \in V$ we get $k \leq r$. Hence

$$\det(M_1) \det(Y) \det(M_2) = x^{n-r}c$$

for some $c \in B[x]$. Since $\det(M_1), \det(M_2) \in B$ and B is an integral domain we conclude $\det(Y) \in x^{n-r}B[x]$. \square

For the following result compare Theorem 1.9 in [20] and Theorem 8.1 in [32].

Theorem 4.22. *For $\nu \in \mathbf{Q}^+$, the Shapovalov determinant $\det(\text{Sh}_\nu)$ of $U_q(\mathfrak{g})$ is given, up to multiplication by an invertible element of $U_q(\mathfrak{h})$, by*

$$\det(\text{Sh}_\nu) = \prod_{\beta \in \Delta^+} \prod_{m=1}^{\infty} (q_\beta^{2(\rho, \beta^\vee)} K_\beta - q_\beta^{2m} K_\beta^{-1})^{P(\nu - m\beta)},$$

where P is the Kostant partition function.

Proof. Note first that if $\nu \in \mathbf{Q}^+$ is fixed, then for any $\beta \in \Delta^+$ the expression $P(\nu - m\beta)$ is zero for large values of m . Therefore the right hand side of the asserted formula is in fact a finite product.

As explained before Lemma 4.18, it suffices to prove the claim for the form \mathcal{S}_ν instead of Sh_ν .

Let D denote the dimension of $U_q(\mathfrak{n}_-)_-\nu$ and fix a basis x_1, \dots, x_D for this space. From Lemma 4.18 we have

$$\mathcal{S}_\nu(x_i, x_j) \in \sum_{\beta, \gamma \in \mathbf{Q}^+, \beta + \gamma = \nu} \mathbb{C}K_{\beta - \gamma},$$

and so the determinant $\det(\mathcal{S}_\nu)$, which is an order D polynomial in these terms, is contained in the linear subspace

$$L_\nu = \sum_{\beta, \gamma \in \mathbf{Q}^+, \beta + \gamma = D\nu} \mathbb{C}K_{\beta - \gamma} \subset U_q(\mathfrak{h}).$$

For all $1 \leq j \leq n$ and $m \in \mathbb{N}$ the polynomials

$$q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1}$$

have mutually distinct irreducible factors. According to Lemma 4.19, the double product in the theorem is contained in L_ν as well. Therefore it suffices to show $\det(\mathcal{S}_\nu) \neq 0$ and that each term $(q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1})^{P(\nu - m\beta_j)}$ divides $\det(\mathcal{S}_\nu)$.

Let $V \subset \mathfrak{h}_q^*$ be the affine subvariety of \mathfrak{h}_q^* defined by $\det(\mathcal{S}_\nu) = 0$. Then $\lambda \in V$ if and only if

$$\chi_\lambda(\det(\text{Sh}(x_i, x_j)_{ij})) = \det(\text{Sh}^\lambda(x_i \cdot v_\lambda, x_j \cdot v_\lambda)_{ij}) = 0.$$

By Lemma 4.17 this implies that $M(\lambda)$ is not simple, so by Theorem 4.15 λ is not antidominant. Thus

$$\begin{aligned} \chi_\lambda(q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1}) &= q_{\beta_j}^{2(\rho, \beta_j^\vee)} q^{(\lambda, \beta_j)} - q_{\beta_j}^{2m} q^{-(\lambda, \beta_j)} \\ &= q^{-(\lambda, \beta_j)} \left(q_{\beta_j}^{2(\rho + \lambda, \beta_j^\vee)} - q_{\beta_j}^{2m} \right) = 0 \end{aligned}$$

for some $\beta_j \in \Delta^+$ and some $m \in \mathbb{N}$. It follows that V is a subset of the union of varieties defined by the zero sets of the polynomials

$$q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1}$$

for $j = 1, \dots, n$ and $m \in \mathbb{N}$.

Using that $\det(\mathcal{S}_\nu)$ is contained in L_ν , and hence a linear combination of terms of the form $K_{2\beta - D\nu}$ for $\beta \in \mathbf{Q}^+$, we conclude that there exists an element $f \in \mathbb{C}[K_1^{\pm 1}, \dots, K_N^{\pm 1}]$ which is invertible on V such that

$$\det(\mathcal{S}_\nu) = f \prod_{j=1}^n \prod_{m=1}^{\infty} (q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1})^{N(\nu, \beta_j, m)},$$

for certain exponents $N(\nu, \beta_j, m) \in \mathbb{N}_0$. It follows in particular that $\det(\mathcal{S}_\nu)$ is nonzero. In order to finish the proof it suffices to show $N(\nu, \beta_j, m) = P(\nu - m\beta_j)$ for all $j = 1, \dots, n$ and $m \in \mathbb{N}$.

Fix $1 \leq j \leq n$ and $m \in \mathbb{N}$, and choose $w \in W$ and $1 \leq i \leq N$ such that $\beta_j = w\alpha_i$. Consider the algebra $A = \mathbb{C}[K_{w\alpha_1}^{\pm 1}, \dots, K_{w\alpha_N}^{\pm 1}]$. Moreover set

$$B = \mathbb{C}[K_{w\alpha_1}^{\pm 1}, \dots, K_{w\alpha_{i-1}}^{\pm 1}, K_{w\alpha_i}, K_{w\alpha_{i+1}}^{\pm 1}, \dots, K_{w\alpha_N}^{\pm 1}].$$

If we put $x = q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1}$ then $A = B[x]$. Let $Y = (\mathcal{S}_\nu(x_i, x_j)) \in M_D(A) = M_D(B[x])$, and write $Y(0) \in M_D(B)$ for the image of Y under the map induced by evaluation at 0.

If $\lambda \in \mathfrak{h}_q^*$ is an element in the variety of $A/(x) \cong B$ then we have

$$\chi_\lambda(q_{\beta_j}^{2(\rho, \beta_j^\vee)} K_{\beta_j} - q_{\beta_j}^{2m} K_{\beta_j}^{-1}) = 0.$$

By a calculation similar to that above, this implies

$$(\lambda + \rho, \beta_j^\vee) = m + \frac{k}{2} i \hbar_{\beta_j}^{-1}$$

for some $k \in \mathbb{Z}$, and so $s_{k, \beta_j} \cdot \lambda = \lambda - m\beta_j < \lambda$. According to Theorem 4.14 there exists a submodule $M(\lambda - m\beta_j) \subset M(\lambda)$. This submodule is contained in $I(\lambda)$, and hence the radical of \mathcal{S}_ν evaluated at λ contains a subspace of dimension

$$\dim(M(\lambda - m\beta_j)_{\lambda - \nu}) = \dim(U_q(\mathfrak{n}_-)_{-\nu + m\beta_j}) = P(\nu - m\beta_j).$$

In other words, the rank of $Y(0)$ evaluated at a point p of the variety of $B = A/(x)$ satisfies $\text{rk}(Y(0)_p) \leq D - P(\nu - m\beta_j)$. Hence $\det(Y) = x^{P(\nu - m\beta_j)} b$ for some $b \in B$ according to Lemma 4.21. In particular, $x^{P(\nu - m\beta_j)}$ is a factor of $\det(\mathcal{S}_\nu)$. This finishes the proof. \square

4.4. Jantzen filtration and the BGG Theorem. In this section we discuss the Jantzen filtration and the BGG Theorem for $U_q(\mathfrak{g})$, giving us information about the composition factors of Verma modules.

We shall first formulate the Jantzen filtration. Consider $B = \mathbb{C}[T, T^{-1}]$, that is, the algebra of Laurent polynomials with coefficients in \mathbb{C} . Informally, we will think of T as s^t where $s \in \mathbb{C}$ such that $s^L = q$ and t is an indeterminate. The ring B is a principal ideal domain, and we write \mathbb{L} for its field of quotients. Then \mathbb{L} and $q \in \mathbb{L}^\times$ satisfy our general assumptions in the discussion of the quantized universal enveloping algebra in Chapter 2. Let us write $U_q^\mathbb{L}(\mathfrak{g})$ for the quantized universal enveloping algebra defined over \mathbb{L} . If $\lambda \in \mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ is a weight with respect to the ground field \mathbb{C} , we define a character $\chi_{\lambda T} : U_q^\mathbb{L}(\mathfrak{h}) \rightarrow \mathbb{L}$ by the formula

$$\chi(K_\mu) = q^{(\mu, \lambda)} T^{L(\mu, \rho)} = q^{(\mu, \lambda + \rho t)},$$

where the second equality is obtained by formally writing $T = s^t$.

Let \mathcal{M} be a free module of finite rank over B . We write $M = \mathcal{M} \otimes_B \mathbb{C}$ for the vector space induced from the evaluation homomorphism $\epsilon : B \rightarrow \mathbb{C}, \epsilon(f) = f(1)$. If $S : \mathcal{M} \times \mathcal{M} \rightarrow B$ is a symmetric B -bilinear form let us set

$$\mathcal{M}^i = \{x \in \mathcal{M} \mid S(x, \mathcal{M}) \subset (T - 1)^i B\},$$

and similarly $M^i = \mathcal{M}^i \otimes_B \mathbb{C}$ for all i . For any $x \in B$ let $v(x) \in \mathbb{N}_0$ be the largest number n such that x can be written in the form $x = y(T - 1)^n$ for some $y \in B$.

With this notation in place we have the following result, which is a special case of the Lemma in Section 5.6 of [37].

Lemma 4.23. *Let \mathcal{M} be a free module of finite rank over $B = \mathbb{C}[T, T^{-1}]$. Assume that $S : \mathcal{M} \times \mathcal{M} \rightarrow B$ is a nondegenerate symmetric B -bilinear form on \mathcal{M} , and*

let $\det(S)$ be its determinant with respect to some basis of \mathcal{M} . Then the following holds.

- a) We have $v(\det(S)) = \sum_{i>0} \dim_{\mathbb{C}}(M^i)$.
- b) For each $i \geq 0$ the map $S_i : \mathcal{M}^i \times \mathcal{M}^i \rightarrow B$ given by

$$S_i(x, y) = (T - 1)^{-i} S(x, y)$$

is well-defined and induces a nondegenerate B -bilinear form on M^i/M^{i+1} with values in \mathbb{C} .

Proof. We follow the treatment in [37], adapted to the special case at hand for the convenience of the reader.

a) Assume that \mathcal{M} is free of rank r . The dual module $\mathcal{M}^* = \text{Hom}_B(\mathcal{M}, B)$ is again free of rank r , as is the submodule $\mathcal{M}^\vee \subset \mathcal{M}^*$ consisting of all linear functionals of the form x^\vee where $x^\vee(y) = S(x, y)$ for $x \in \mathcal{M}$. Using the structure theory for modules over principal ideal domains we find a basis e_1, \dots, e_r of \mathcal{M} and elements $d_1, \dots, d_r \in B$ such that $d_i e^i$ is a basis of \mathcal{M}^\vee , where e^1, \dots, e^r of \mathcal{M}^* is the dual basis defined by $e^i(e_j) = \delta_{ij}$. We then obtain $\det(S) = d_1 \cdots d_r$ up to an invertible scalar. Moreover, there is another basis f_1, \dots, f_r of \mathcal{M} such that $f_i^\vee = d_i e^i$. If $f = \sum a_j f_j \in \mathcal{M}$ and $n_j = v(d_j)$ we get $f \in M^i$ iff $v(S(e_j, f)) \geq i$ for all j iff $v(a_j) \geq i - n_j$ for all j . Hence M^i is spanned by the elements f_j for which $i \leq n_j$ and the $(T - 1)^{i-n_j} f_j$ for which $i > n_j$. In particular $M^i = \mathcal{M}^i \otimes_B \mathbb{C} = 0$ for sufficiently large i . Moreover we obtain

$$v(\det(S)) = \sum_{j=1}^r n_j = \sum_{i>0} |\{j | i \leq n_j\}| = \sum_{i>0} \dim(M^i)$$

as desired.

b) By construction, the form S_i takes values in $B \subset \mathbb{L}$. We need to check that it descends to a form on M^i . For this consider $x = (T - 1)y \in \mathcal{M}^i$ and compute

$$S_i(x, \mathcal{M}^i) \subset (T - 1)^{-i} S(x, \mathcal{M}^i) \subset (T - 1)^{-i+1} (\mathcal{M}, \mathcal{M}^i) \subset (T - 1)B,$$

so that $S_i(x, \mathcal{M}^i) = 0$ in $B \otimes_B \mathbb{C} = \mathbb{C}$. Hence S^i induces a bilinear form $M^i \times M^i \rightarrow \mathbb{C}$. In order to compute the radical $R^i \subset M^i$ of the latter notice that $M^{i+1} \subset R^i$. Moreover, the coset of f_j as above is contained in R^i iff $i < n_j$. Since the f_j with $i \leq n_j$ form a basis of M^i we see that the form is nondegenerate on M^i/M^{i+1} , so that $R^i = M^{i+1}$. \square

If M is a $U_q(\mathfrak{g})$ -module then a bilinear form on M will be called contravariant if

$$(\tau(X) \cdot m, n) = (m, X \cdot n)$$

for all $x \in U_q(\mathfrak{g})$ and $m, n \in M$.

Theorem 4.24 (Jantzen filtration). *Let $\lambda \in \mathfrak{h}_q^*$. Then there exist a filtration*

$$M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 \supset M(\lambda)^2 \supset \dots$$

of $M(\lambda)$ with $M(\lambda)^i = 0$ for i large, such that the following conditions hold.

- a) *Each nonzero quotient $M(\lambda)^i/M(\lambda)^{i+1}$ has a nondegenerate contravariant bilinear form.*
- b) *$M(\lambda)^1 = I(\lambda)$ is the maximal submodule of $M(\lambda)$.*
- c) *(Jantzen Sum Formula) The formal characters of the modules $M(\lambda)^i$ satisfy*

$$\sum_{i>0} \text{ch}(M(\lambda)^i) = \sum_{\substack{\alpha \in \Delta^+, k \in \mathbb{Z}_2 \\ s_{k,\alpha} \cdot \lambda < \lambda}} \text{ch}(M(s_{k,\alpha} \cdot \lambda)).$$

Proof. We work over $B = \mathbb{C}[T, T^{-1}]$ and its field of quotients \mathbb{L} as above. Given $\lambda \in \mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$ consider the character χ_{λ_T} where $\lambda_T = \lambda + t\rho$. We claim that $M(\lambda_T)$ is a simple $U_q^{\mathbb{L}}(\mathfrak{g})$ -module. Indeed, by Theorem 2.128 any submodule of $M(\lambda_T)$ must have highest weight in the same \hat{W} -linkage class as λ_T . But if $\hat{w} = (\zeta, w) \in \hat{W}$ is nontrivial then

$$\hat{w} \cdot (\lambda + t\rho) - (\lambda + t\rho) = (w\lambda + w\rho - \lambda - \rho + \zeta) + t(w\rho - \rho) \notin \mathbf{Q},$$

so the \hat{W} -linkage class of λ_T is trivial. Hence the unique contravariant bilinear form on $M(\lambda_T)$ with values in \mathbb{L} such that $(v_\lambda, v_\lambda) = 1$ is nondegenerate.

Using the PBW Theorem 2.67 we obtain from the B -form $U_q^B(\mathfrak{g}) = B \otimes U_q(\mathfrak{g})$ a B -form $\mathcal{M}(\lambda_T)$ of the Verma module $M(\lambda_T)$ in which all weight subspaces are free of finite rank. For $\mu \in \mathbf{Q}^+$ define

$$\mathcal{M}(\lambda_T)_{\lambda_T - \mu}^i = \{x \in \mathcal{M}(\lambda_T)_{\lambda_T - \mu} \mid (x, \mathcal{M}(\lambda_T)_{\lambda_T - \mu}) \subset (T - 1)^i B\}$$

and

$$\mathcal{M}(\lambda_T)^i = \sum_{\mu \in \mathbf{Q}^+} \mathcal{M}(\lambda_T)_{\lambda_T - \mu}^i.$$

It is straightforward to check that $\mathcal{M}(\lambda_T)^i$ is a $U_q^B(\mathfrak{g})$ -submodule of $\mathcal{M}(\lambda_T)$, and that these submodules form a decreasing filtration.

Setting $T = 1$ yields a decreasing filtration $M(\lambda)^i = \mathcal{M}(\lambda_T)^i / (T - 1)\mathcal{M}(\lambda_T)^i$ of $M(\lambda)$ such that $M^0(\lambda) = M(\lambda)$. Thanks to Lemma 4.23 the quotients of this filtration acquire nondegenerate contravariant bilinear forms. Since $M(\lambda)$ has finite length we see that $M(\lambda)^i = 0$ for $i \gg 0$. This proves part *a*).

For part *b*) we only need to observe that the contravariant form on $M(\lambda)/M(\lambda)^1$ is nondegenerate, and therefore $M(\lambda)^1 = I(\lambda)$ by Lemma 4.17.

It remains to prove the Jantzen Sum Formula in part *c*). Consider $\nu \in \mathbf{Q}^+$ and denote the determinant of the Shapovalov form of $\mathcal{M}(\lambda_T)$ on the $\lambda_T - \nu$ weight space by $\det(\text{Sh}_\nu)$. Since B contains \mathbb{C} , the formula for the Shapovalov determinant in Theorem 4.22 continues to hold over $U_q^B(\mathfrak{g})$. More precisely, we obtain

$$\chi_{\lambda_T}(\det(\text{Sh}_\nu)) = \prod_{\alpha \in \Delta^+} \prod_{m=1}^{\infty} (q^{2(\lambda + \rho + t\rho, \alpha)} - q^{m(\alpha, \alpha)})^{P(\nu - m\alpha)},$$

up to an invertible element of B . Applying v to the term

$$x = q^{2(\lambda + \rho + t\rho, \alpha)} - q^{m(\alpha, \alpha)} = q^{2(\lambda + \rho, \alpha)} T^{2L(\rho, \alpha)} - q^{m(\alpha, \alpha)}$$

gives 0 unless $q^{m(\alpha, \alpha)} = q^{2(\lambda + \rho, \alpha)}$, or equivalently, unless $(\lambda + \rho, \alpha^\vee) = m$ modulo $\frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$, where as usual $\hbar_\alpha = d_\alpha \hbar$. In this case we get $v(x) = 1$. The contribution of the $(\lambda - \nu)$ -weight space to the valuation is therefore $P(\nu - (\lambda + \rho, \alpha^\vee)\alpha)$, where P denotes Kostant's partition function. Using Lemma 4.23 we obtain

$$\begin{aligned} \sum_{i>0} \text{ch}(M(\lambda)^i) &= \sum_{\nu \in \mathbf{Q}^+} \sum_{\substack{\alpha \in \Delta^+, k \in \mathbb{Z}_2 \\ s_{k, \alpha} \cdot \lambda < \lambda}} P(\nu - (\lambda + \rho, \alpha^\vee)\alpha) e^{\lambda - \nu} \\ &= \sum_{\substack{\alpha \in \Delta^+, k \in \mathbb{Z}_2 \\ s_{k, \alpha} \cdot \lambda < \lambda}} \sum_{\nu \in \mathbf{Q}^+} P(\nu) e^{\lambda - (\lambda + \rho, \alpha^\vee)\alpha - \nu} \\ &= \sum_{\substack{\alpha \in \Delta^+, k \in \mathbb{Z}_2 \\ s_{k, \alpha} \cdot \lambda < \lambda}} \text{ch}(M(s_{k, \alpha} \cdot \lambda)) \end{aligned}$$

as desired. □

Notice that the Jantzen filtration reduces to the trivial filtration $M(\lambda) = M(\lambda)^0 \supset M(\lambda)^1 = 0$ in the case that $M(\lambda)$ is simple. The Jantzen sum formula does not provide any information in this case.

Let $\mu, \lambda \in \mathfrak{h}_q^*$. Recall that μ is strongly linked to λ , written $\mu \uparrow \lambda$, if $\mu = \lambda$ or if there exists a chain of positive roots $\alpha_1, \dots, \alpha_r \in \Delta^+$ and $k_1, \dots, k_r \in \mathbb{Z}_2$ such that

$$\mu = s_{k_1, \alpha_1} \cdots s_{k_r, \alpha_r} \cdot \lambda < s_{k_2, \alpha_2} \cdots s_{k_r, \alpha_r} \cdot \lambda < \cdots < s_{k_r, \alpha_r} \cdot \lambda < \lambda.$$

Using the Jantzen filtration we shall now prove an analogue of the BGG Theorem, compare Section 5.1 in [37].

Theorem 4.25 (The BGG Theorem). *Let $\mu, \lambda \in \mathfrak{h}_q^*$. Then $[M(\lambda) : V(\mu)] \neq 0$ iff μ is strongly linked to λ .*

Proof. From Theorem 4.14 we know that $\mu \uparrow \lambda$ implies $[M(\lambda) : V(\mu)] \neq 0$.

For the converse let us use the Jantzen filtration and induction on the number k of weights μ linked to λ satisfying $\mu \leq \lambda$. If $k = 1$ then λ is minimal in its linkage class, so there is nothing to prove. Assume now that the claim is proved for all weights and some k . Suppose $[M(\lambda) : V(\mu)] > 0$ for $\mu < \lambda$. This means $[M(\lambda)^1 : V(\mu)] > 0$ for the first step $M(\lambda)^1$ in the Jantzen filtration of $M(\lambda)$. The sum formula in Theorem 4.24 forces $[M(s_{k, \alpha} \cdot \lambda) : V(\mu)] > 0$ for some $\alpha \in \Delta_{[\lambda]}^+, k \in \mathbb{Z}_2$. By our inductive hypothesis, there exist $\alpha_1, \dots, \alpha_r \in \Delta^+, k_1, \dots, k_r \in \mathbb{Z}_2$ such that

$$\mu = s_{k_1, \alpha_1} \cdots s_{k_r, \alpha_r} s_{k, \alpha} \cdot \lambda < s_{k_2, \alpha_2} \cdots s_{k_r, \alpha_r} s_{k, \alpha} \cdot \lambda < \cdots < s_{k_r, \alpha_r} s_{k, \alpha} \cdot \lambda < s_{k, \alpha} \cdot \lambda.$$

Appending this chain with $s_{k, \alpha} \cdot \lambda < \lambda$ yields $\mu \uparrow \lambda$ as desired. \square

4.5. The PRV determinant. In this section we discuss the Parthasarathy-Ranga Rao-Varadarajan determinants of $U_q(\mathfrak{g})$. In the classical setting they were introduced in [62]. For more information on the quantum case we refer to [41].

4.5.1. *The spaces $F \text{Hom}(M, N)$.* Before discussing the PRV-determinants we will need a couple of preparations regarding the spaces $F \text{Hom}(M, N)$, which we now describe.

Let M, N be $U_q(\mathfrak{g})$ -modules. The space of \mathbb{C} -linear maps $\text{Hom}(M, N)$ becomes a $U_q(\mathfrak{g})$ -module by setting

$$(X \rightarrow T)(m) = X_{(1)} \cdot T(\hat{S}(X_{(2)}) \cdot m)$$

for $T \in \text{Hom}(M, N)$, $X \in U_q(\mathfrak{g})$ and $m \in M$. We will refer to this as the adjoint action of $U_q(\mathfrak{g})$. We denote by $F \text{Hom}(M, N)$ the locally finite part of $\text{Hom}(M, N)$ with respect to the adjoint action.

We can also define a $U_q(\mathfrak{g})$ -bimodule structure on $\text{Hom}(M, N)$ by

$$(Y \cdot T \cdot Z)(m) = Y \cdot T(Z \cdot m).$$

for $Y, Z \in U_q(\mathfrak{g})$, $T \in \text{Hom}(M, N)$, $m \in M$. By restriction, $F \text{Hom}(M, N)$ becomes an $FU_q(\mathfrak{g})$ -bimodule. Indeed, notice that

$$\begin{aligned} X \rightarrow (Y \cdot T \cdot Z)(m) &= X_{(1)} Y \cdot T(Z \hat{S}(X_{(2)}) \cdot m) \\ &= (X_{(1)} \rightarrow Y) \cdot (X_{(2)} \rightarrow T)((X_{(3)} \rightarrow Z) \cdot m) \end{aligned}$$

for all $m \in M$. Hence if $Y, Z \in FU_q(\mathfrak{g})$ and $T \in F \text{Hom}(M, N)$, then $Y \cdot T \cdot Z$ is again contained in $F \text{Hom}(M, N)$.

The $FU_q(\mathfrak{g})$ -bimodule structure on $F \text{Hom}(M, N)$ is compatible with the adjoint $U_q(\mathfrak{g})$ -action in the following sense.

Definition 4.26. Let V be an $FU_q(\mathfrak{g})$ -bimodule, with left and right actions denoted by $X \cdot v$ and $v \cdot X$ for $X \in FU_q(\mathfrak{g})$ and $v \in V$, respectively. We say that a left

module structure of $U_q(\mathfrak{g})$ on V , written $X \rightarrow v$ for $X \in U_q(\mathfrak{g})$ and $v \in V$, is compatible if

$$(X_{(1)} \rightarrow v) \cdot X_{(2)} = X \cdot v$$

for all $X \in FU_q(\mathfrak{g})$ and $v \in V$. Equipped with such a structure, V will be called a $U_q(\mathfrak{g})$ -compatible $FU_q(\mathfrak{g})$ -bimodule.

Note that the formula in Definition 4.26 makes sense since $FU_q(\mathfrak{g})$ is a left coideal, see Lemma 2.112. Morally, the compatibility condition can be written as

$$X \rightarrow v = X_{(1)} \cdot v \cdot \hat{S}(X_{(2)});$$

however, the right hand side of this formula might not be well-defined, even if we consider $X \in FU_q(\mathfrak{g})$. Nevertheless, inspired by this formula, we shall always refer to a compatible action of $U_q(\mathfrak{g})$ on an $FU_q(\mathfrak{g})$ -bimodule as the adjoint action.

Next, we introduce similar structures associated to the linear dual $(M \otimes N)^*$, where again M and N are in category \mathcal{O} . Recall the algebra antiautomorphism and coalgebra automorphism τ of $U_q(\mathfrak{g})$ from Lemma 2.16. We introduce a $U_q(\mathfrak{g})$ -module structure on $(M \otimes N)^*$ by

$$(X \rightarrow \varphi)(m \otimes n) = \varphi(\hat{S}(X_{(2)}) \cdot m \otimes \tau(X_{(1)}) \cdot n)$$

for $X \in U_q(\mathfrak{g})$, $\varphi \in (M \otimes N)^*$ and $m \otimes n \in M \otimes N$. Again, we refer to this as the adjoint action of $U_q(\mathfrak{g})$ on $(M \otimes N)^*$. We also have a $U_q(\mathfrak{g})$ -bimodule structure on $(M \otimes N)^*$ given by

$$(Y \cdot \varphi \cdot Z)(m \otimes n) = \varphi(Z \cdot m \otimes \tau(Y) \cdot n),$$

for $Y, Z \in U_q(\mathfrak{g})$.

We denote by $F((M \otimes N)^*)$ the locally finite part of $(M \otimes N)^*$ with respect to the adjoint action of $U_q(\mathfrak{g})$ defined above. The following lemma shows that by restricting the above actions, $F((M \otimes N)^*)$ becomes a $U_q(\mathfrak{g})$ -compatible $FU_q(\mathfrak{g})$ -bimodule in the sense of Definition 4.26.

Lemma 4.27. *Let M, N be in category \mathcal{O} . Then there exists an isomorphism of $FU_q(\mathfrak{g})$ -bimodules*

$$F \operatorname{Hom}(M, N^\vee) \xrightarrow{\cong} F((M \otimes N)^*),$$

which is $U_q(\mathfrak{g})$ -linear for the adjoint actions denoted by \rightarrow above. Explicitly, the isomorphism is the restriction of the natural map $\gamma : \operatorname{Hom}(M, N^*) \rightarrow (M \otimes N)^*$ defined by

$$\gamma(T)(m \otimes n) = T(m)(n)$$

for $T \in \operatorname{Hom}(M, N^*)$ and $m \in M, n \in N$.

Proof. To begin with, let us show that $F \operatorname{Hom}(M, N^\vee) = F \operatorname{Hom}(M, N^*)$. Here, we are equipping N^* with the obvious extension of the $U_q(\mathfrak{g})$ -action on N^\vee , namely

$$(X \cdot f)(n) = f(\tau(X) \cdot n), \quad f \in N^*, X \in U_q(\mathfrak{g}), n \in N.$$

Let $T \in F \operatorname{Hom}(M, N^*)$. For $K \in U_q(\mathfrak{h})$, we have

$$K \cdot (T(m)) = (K_{(1)} \rightarrow T)(K_{(2)} \cdot m).$$

Since T is locally finite and $m \in M$ is contained in a finite sum of weight spaces, we see that $T(m)$ generates a finite dimensional $U_q(\mathfrak{h})$ -submodule of N^* and so must belong to N^\vee .

Now consider the linear map $\gamma : \text{Hom}(M, N^*) \rightarrow (M \otimes N)^*$ defined above. General facts about tensor products imply that γ is a linear isomorphism. Since

$$\begin{aligned} (X \rightarrow \gamma(T))(m \otimes n) &= \gamma(T)(\hat{S}(X_{(2)}) \cdot m \otimes \tau(X_{(1)}) \cdot n) \\ &= T(\hat{S}(X_{(2)}) \cdot m)(\tau(X_{(1)}) \cdot n) \\ &= (X_{(1)} \cdot T(\hat{S}(X_{(2)}) \cdot m))(n) \\ &= \gamma(X \rightarrow T)(m \otimes n) \end{aligned}$$

for $X \in U_q(\mathfrak{g})$, $m \in M, n \in N$ we see that γ is $U_q(\mathfrak{g})$ -linear and so restricts to an isomorphism on the locally finite parts.

Finally, for any $Y, Z \in U_q(\mathfrak{g})$, $T \in \text{Hom}(M, N^*)$ and $m \otimes n \in M \otimes N$ we have

$$\begin{aligned} (Y \cdot \gamma(T) \cdot Z)(m \otimes n) &= \gamma(T)(Z \cdot m \otimes \tau(Y) \cdot n) \\ &= T(Z \cdot m)(\tau(Y) \cdot n) \\ &= (Y \cdot T(Z \cdot m))(n) \\ &= \gamma(Y \cdot T \cdot Z)(m \otimes n). \end{aligned}$$

Since $F \text{Hom}(M, N^\vee)$ is invariant for the left and right actions of $FU_q(\mathfrak{g})$, we deduce that the same is true for $F((M \otimes N)^*)$, and $\gamma : F \text{Hom}(M, N^\vee) \rightarrow F((M \otimes N)^*)$ is an isomorphism of $FU_q(\mathfrak{g})$ -bimodules. \square

Consider the automorphism $\theta = \tau \hat{S} = \hat{S}^{-1} \tau$. It is an algebra homomorphism and a coalgebra antihomomorphism. Note also that θ is involutive, since $\theta^2 = \tau \hat{S} \hat{S}^{-1} \tau = \tau^2 = \text{id}$.

Lemma 4.28. *Let M, N be in category \mathcal{O} . Then the flip map $M \otimes N \rightarrow N \otimes M$ induces a linear isomorphism*

$$\alpha : F((M \otimes N)^*) \rightarrow F((N \otimes M)^*)$$

such that

$$\alpha(Y \cdot \varphi \cdot Z) = \tau(Z) \cdot \alpha(\varphi) \cdot \tau(Y), \quad \alpha(X \rightarrow \varphi) = \theta(X) \rightarrow \alpha(\varphi)$$

for all $Y, Z \in FU_q(\mathfrak{g})$ and $X \in U_q(\mathfrak{g})$.

Similarly, there exists a linear isomorphism

$$\beta : F \text{Hom}(M, N) \rightarrow F \text{Hom}(N^\vee, M^\vee)$$

such that

$$\beta(Y \cdot T \cdot Z) = \tau(Z) \cdot \beta(T) \cdot \tau(Y), \quad \beta(X \rightarrow T) = \theta(X) \rightarrow \beta(T)$$

for all $Y, Z \in FU_q(\mathfrak{g})$ and $X \in U_q(\mathfrak{g})$.

Proof. We define $\alpha : (M \otimes N)^* \rightarrow (N \otimes M)^*$ by $\alpha(\varphi)(n \otimes m) = \varphi(m \otimes n)$. Clearly α is a linear isomorphism. To check that this isomorphism is compatible with the locally finite parts we compute

$$\begin{aligned} \theta(X) \rightarrow \alpha(\varphi)(n \otimes m) &= \alpha(\varphi)(\hat{S}(\theta(X_{(1)})) \cdot n \otimes \tau(\theta(X_{(2)})) \cdot m) \\ &= \alpha(\varphi)(\tau(X_{(1)}) \cdot n \otimes \hat{S}(X_{(2)}) \cdot m) \\ &= \varphi(\hat{S}(X_{(2)}) \cdot m \otimes \tau(X_{(1)}) \cdot n) \\ &= \alpha(X \rightarrow \varphi)(n \otimes m). \end{aligned}$$

Hence α restricts to an isomorphism $F((M \otimes N)^*) \rightarrow F((N \otimes M)^*)$. Moreover we have

$$\begin{aligned} \alpha(Y \cdot \varphi \cdot Z)(n \otimes m) &= \varphi(Z \cdot m \otimes \tau(Y) \cdot n) \\ &= \alpha(\varphi)(\tau(Y) \cdot n \otimes Z \cdot m) \\ &= (\tau(Z) \cdot \alpha(\varphi) \cdot \tau(Y))(n \otimes m) \end{aligned}$$

for all $Y, Z \in FU_q(\mathfrak{g})$.

In order to prove the second assertion it suffices to construct an isomorphism $\beta : F \text{Hom}(M, N^\vee) \rightarrow F \text{Hom}(N, M^\vee)$ with the claimed properties. This can be done by combining the map α from the first part of the proof with the isomorphism γ from Lemma 4.27. \square

4.5.2. *Conditions for $F \text{Hom}(M, N) = 0$.* In this subsection we use Gelfand-Kirillov dimension as a tool to derive sufficient conditions for $F \text{Hom}(M, N) = 0$ for certain modules $M, N \in \mathcal{O}$. For background on the Gelfand-Kirillov dimension we refer to [14].

Let A be an algebra over \mathbb{C} and let $V \subset A$ be a linear subspace. We write V^n for the linear subspace generated by all n -fold products $a_1 \cdots a_n$ with $a_1, \dots, a_n \in V$, in addition we set $V^0 = \mathbb{C}$. Assume that A is finitely generated and that V is a finite dimensional generating subspace, so that $\bigcup_{n=0}^\infty V^n = A$. Moreover let M be a finitely generated A -module with generating set M^0 , so that $\bigcup_{n=0}^\infty M^n = M$ where $M^n = V^n \cdot M^0$. Then the Gelfand-Kirillov dimension of M over A is defined by

$$d_A(M) = \limsup_{n \rightarrow \infty} \frac{\log(\dim(M^n))}{\log(n)}.$$

It is not hard to check that this does not depend on the choice of the generating subspaces V and M^0 .

Let us collect some general facts regarding the GK dimension.

Lemma 4.29. *Let A be a finitely generated algebra.*

- a) *If $0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$ is an extension of finitely generated A -modules then $d_A(K) \leq d_A(E)$ and $d_A(Q) \leq d_A(E)$.*
- b) *If $I \subset A$ is a left ideal containing an element $a \in I$ which is not a right zero divisor then $d_A(A/I) \leq d_A(A) - 1$.*

Proof. a) Pick a generating subspace K^0 for K and extend it to a generating subspace E^0 for E . Then $K^n \subset E^n$ for all n and hence $\dim(K^n) \leq \dim(E^n)$. Similarly, if $\pi : E \rightarrow Q$ denotes the quotient map then $Q^0 = \pi(E^0)$ is a generating subspace for Q , and we have $\dim(Q^n) \leq \dim(E^n)$ for all n . This yields the claim.

b) We follow the proof of Theorem 3.4 in [14]. Without loss of generality we may assume that $d_A(A)$ is finite. Let $V \subset A$ be a finite dimensional generating subspace containing 1. For each n let $D_n \subset V^n$ be a complement of $V^n \cap I$. If $\pi : A \rightarrow A/I$ is the quotient map then we get $\pi(D_n) = \pi(V^n)$.

We claim that $D_n + D_n a + \cdots + D_n a^m$ is a direct sum for all $m \in \mathbb{N}$. Indeed, from $x_0 + x_1 a + \cdots + x_m a^m = 0$ with $x_j \in D_n$ for all j we get $\pi(x_0) = 0$ because $a \in I$. Hence $x_0 = 0$, and since a is not a right zero divisor we deduce $x_1 + x_2 a + \cdots + x_m a^m = 0$. Therefore the assertion follows by induction.

Next observe $D_n + D_n a + \cdots + D_n a^n \subset V^{2n}$ so that $\dim(V^{2n}) \geq n \dim(D_n) = n \dim(\pi(V^n))$. Since $W = V^2$ is again a generating subspace for A and $W^n = V^{2n}$ we conclude

$$d_A(A) = \limsup_{n \rightarrow \infty} \frac{\log(\dim(V^{2n}))}{\log(n)} \geq \limsup_{n \rightarrow \infty} \frac{\log(n \dim(\pi(V^n)))}{\log(n)} = 1 + d_A(A/I)$$

as desired. \square

We will be interested in the case that $A = U_q(\mathfrak{n}_-)$ or $A = U_q(\mathfrak{b})$ and M is a $U_q(\mathfrak{g})$ -module contained in \mathcal{O} , viewed as an A -module. Note that $d_A(M)$ will be the same for both choices of A , and we will write $d(M) = d_A(M)$ in either case.

In the next lemma, recall that w_0 denotes the longest element of the Weyl group.

Lemma 4.30. *Let $A = U_q(\mathfrak{n}_-)$.*

- a) *If $\lambda \in \mathfrak{h}_q^*$ then $d(M(\lambda)) = l(w_0)$.*
- b) *If Q is a proper quotient of a Verma module then $d(Q) < l(w_0)$.*
- c) *If M is in \mathcal{O} and $\mu \in \mathbf{P}^+$ we have $d(M \otimes V(\mu)) = d(M)$.*

Proof. a) Every Verma module is a free A -module of rank 1. Hence the claim can be obtained by invoking a filtration of A as in the proof of Proposition 2.134, taking into account Satz 5.5 in [14].

b) This follows again from the fact that every Verma module is a free A -module of rank 1 combined with A having no zero-divisors and part b) of Lemma 4.29.

c) We use that M , being in \mathcal{O} , is finitely generated as an A -module. If $V \subset A$ is the linear span of 1 and the generators F_1, \dots, F_N then V is a generating subspace of A . Let M^0 be a finite dimensional generating subspace of M over A . Then $M^0 \otimes V(\mu)$ is a generating subspace for $M \otimes V(\mu)$, and we have

$$V^n \cdot (M^0 \otimes V(\mu)) \subset V^n \cdot M^0 \otimes V(\mu) = M^n \otimes V(\mu),$$

which implies $d(M \otimes V(\mu)) \leq d(M)$. If $v^\mu \in V(\mu)$ is a lowest weight vector then $M \otimes v^\mu$ is isomorphic to M as an A -module, so that $d(M \otimes V(\mu)) \geq d(M)$. Hence we get $d(M \otimes V(\mu)) = d(M)$ as desired. \square

Lemma 4.31. *For M, N in \mathcal{O} , the following are equivalent:*

- a) $F \operatorname{Hom}(M, N) = 0$.
- b) $\operatorname{Hom}_{U_q(\mathfrak{g})}(V(\mu) \otimes M, N) = 0$ for every $\mu \in \mathbf{P}^+$.
- c) $\operatorname{Hom}_{U_q(\mathfrak{g})}(M, V(\mu) \otimes N) = 0$ for every $\mu \in \mathbf{P}^+$.

Proof. The map $\kappa : \operatorname{Hom}(M, N) \otimes V(\mu)^* \rightarrow \operatorname{Hom}(V(\mu) \otimes M, N)$ defined by

$$\kappa(T \otimes f)(v \otimes m) = f(v)T(m)$$

is a $U_q(\mathfrak{g})$ -linear isomorphism

$$\begin{aligned} (X \rightarrow \kappa(T \otimes f))(v \otimes m) &= f(\hat{S}(X_{(3)}) \cdot v) X_{(1)} \cdot T(\hat{S}(X_{(2)}) \cdot m) \\ &= (X_{(1)} \rightarrow T)(m) (X_{(2)} \rightarrow f)(v). \end{aligned}$$

Therefore, if $\operatorname{Hom}(M, N)$ contains a finite dimensional $U_q(\mathfrak{g})$ -submodule of highest weight μ then $\operatorname{Hom}(V(\mu) \otimes M, N)$ contains a trivial submodule, and conversely. An element of $\operatorname{Hom}(V(\mu) \otimes M, N)$ spans a trivial submodule if and only if it is $U_q(\mathfrak{g})$ -linear. This proves that a) and b) are equivalent.

A similar argument proves the equivalence of a) and c). \square

Note that $V(\mu) \otimes M$ can be replaced by $M \otimes V(\mu)$ in b) and c) of the above lemma, since they are isomorphic as $U_q(\mathfrak{g})$ -modules.

Proposition 4.32. *Let M, N be in category \mathcal{O} .*

- a) *If $d(M) < d(K)$ for every simple submodule K of N we have $F \operatorname{Hom}(M, N) = 0$.*
- b) *If $d(N) < d(Q)$ for every simple quotient Q of M we have $F \operatorname{Hom}(M, N) = 0$.*

Proof. a) By Lemma 4.31, it suffices to show $\operatorname{Hom}_{U_q(\mathfrak{g})}(M \otimes V(\mu), N) = 0$ for all $\mu \in \mathbf{P}^+$. Let $f : M \otimes V(\mu) \rightarrow N$ be a nonzero $U_q(\mathfrak{g})$ -linear map. According to part c) of Lemma 4.30 we have $d(M \otimes V(\mu)) = d(M)$, and we have $d(f(M \otimes V(\mu))) \leq d(M \otimes V(\mu)) = d(M) < d(K)$ for all simple submodules $K \subset N$ by assumption. On the other hand, if $K \subset f(M \otimes V(\mu))$ is a simple submodule then

$d(K) \leq d(f(M \otimes V(\mu)))$ by Lemma 4.29. This is a contradiction, thus giving the claim.

b) In a similar fashion it is enough to show $\text{Hom}_{U_q(\mathfrak{g})}(M, V(\mu) \otimes N) = 0$ for all $\mu \in \mathbf{P}^+$. Let $f : M \rightarrow V(\mu) \otimes N$ be a nonzero $U_q(\mathfrak{g})$ -linear map. Using $V(\mu) \otimes N \cong N \otimes V(\mu)$ we have $d(N) = d(V(\mu) \otimes N)$ according to part c) of Lemma 4.30, and hence $d(V(\mu) \otimes N) < d(Q) \leq d(f(M))$ for every simple quotient Q of $f(M)$ by assumption. In addition, $d(f(M)) \leq d(V(\mu) \otimes N)$ by Lemma 4.29 because $f(M) \subset V(\mu) \otimes N$ is a submodule. Again we obtain a contradiction. \square

As a consequence we arrive at the following result.

Proposition 4.33. *Let $\lambda \in \mathfrak{h}_q^*$. If Q is a proper quotient of a Verma module then $F \text{Hom}(Q, M(\lambda)) = 0$. Similarly, if $M(\lambda)$ is simple then $F \text{Hom}(M(\lambda), Q) = 0$.*

Proof. If Q is a proper quotient of a Verma module then $d(Q) < l(w_0)$ according to Lemma 4.30. In particular, we have $d(Q) < d(K)$ for the unique simple submodule $K = \text{Soc}(M(\lambda)) \subset M(\lambda)$, see Lemma 4.10. Hence the first assertion follows from Proposition 4.32 a). Similarly, the second claim follows from Proposition 4.32 b). \square

4.5.3. *Multiplicities in $F \text{Hom}(M, N)$.* If V is any $U_q(\mathfrak{g})$ -module and $\lambda \in \mathfrak{h}_q^*$, then $V \otimes M(\lambda)$ is a free $U_q(\mathfrak{n}_-)$ -module. To see this, let V_τ denote V equipped with the trivial action of $U_q(\mathfrak{n}_-)$ and observe that the isomorphism $V_\tau \otimes M(\lambda) \cong V \otimes M(\lambda)$ defined by

$$v \otimes Y \cdot v_\lambda \mapsto Y_{(1)} \cdot v \otimes Y_{(2)} \cdot v_\lambda,$$

for $v \in V$, $Y \in U_q(\mathfrak{n}_-)$ is $U_q(\mathfrak{n}_-)$ -linear.

Lemma 4.34. *Let $\lambda \in \mathfrak{h}_q^*$ and $\nu \in \mathbf{P}^+$. Then there exists a finite decreasing filtration*

$$V(\nu) \otimes M(\lambda) = M_0 \supset M_1 \supset \dots \supset M_k \supset 0$$

with quotients isomorphic to $M(\lambda + \gamma)$, where γ runs over all weights of $V(\nu)$ counted with multiplicity.

Proof. We can filter $V(\nu) = V_0 \supset V_1 \supset \dots \supset V_k \supset 0$ as a $U_q(\mathfrak{b}_+)$ -module by one-dimensional quotients \mathbb{C}_γ where $U_q(\mathfrak{b}_+)$ acts on \mathbb{C}_γ by the character χ_γ . Using the above observations, tensoring this filtration with $M(\lambda)$ yields the claim. \square

Recall that if M is an integrable $U_q(\mathfrak{g})$ -module and $\nu \in \mathbf{P}^+$ we write $[M : V(\nu)]$ for the multiplicity of $V(\nu)$ in M .

Proposition 4.35. *Let $\lambda, \eta \in \mathfrak{h}_q^*$ and $\nu \in \mathbf{P}^+$. The the following properties hold.*

a) *We always have*

$$[F((M(\lambda) \otimes M(\eta))^*) : V(\nu)] = \dim(V(\nu)_{\eta-\lambda}).$$

b) *If $M(\eta)$ is simple then*

$$[F \text{Hom}(M(\lambda), M(\eta)) : V(\nu)] = \dim(V(\nu)_{\eta-\lambda}).$$

c) *If $M(\lambda)$ is projective then*

$$[F \text{Hom}(M(\lambda), M(\eta)) : V(\nu)] = \dim(V(\nu)_{\eta-\lambda}).$$

Proof. a) Recall that the appropriate $U_q(\mathfrak{g})$ -module structure on $(M(\lambda) \otimes M(\eta))^*$ is given by

$$(X \rightarrow \varphi)(m \otimes n) = \varphi(\hat{S}(X_{(2)}) \cdot m \otimes \tau(X_{(1)}) \cdot n)$$

for $X \in U_q(\mathfrak{g})$, $\varphi \in (M(\lambda) \otimes M(\eta))^*$ and $m \otimes n \in M(\lambda) \otimes M(\eta)$. This can be rewritten as

$$(X \rightarrow \varphi)(m \otimes n) = \varphi(\hat{S}(X) \cdot (m \otimes n)),$$

if we define the $U_q(\mathfrak{g})$ -action on $M(\lambda) \otimes M(\eta)$ by

$$X \cdot (m \otimes n) = X_{(1)} \cdot m \otimes \hat{S}\tau(X_{(2)}) \cdot n.$$

Similarly, let us define a $U_q(\mathfrak{g})$ -action on $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by

$$X \cdot (Y \otimes Z) = X_{(1)}Y \otimes \hat{S}\tau(X_{(2)})Z$$

for $X, Y, Z \in U_q(\mathfrak{g})$. Using the triangular decomposition, see Theorem 2.14, one can check that this action induces an isomorphism

$$\begin{aligned} U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} (U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_+)) &\cong U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \\ X \otimes (Y \otimes Z) &\mapsto X \cdot (Y \otimes Z) \end{aligned}$$

of left $U_q(\mathfrak{g})$ -modules.

We obtain isomorphisms of left $U_q(\mathfrak{g})$ -modules as follows

$$\begin{aligned} (M(\lambda) \otimes M(\eta))^* &\cong \text{Hom}_{U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_+)}(U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}), (\mathbb{C}_\lambda \otimes \mathbb{C}_\eta)^*) \\ &\cong \text{Hom}_{U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_+)}(U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} (U_q(\mathfrak{b}_+) \otimes U_q(\mathfrak{b}_+)), (\mathbb{C}_\lambda \otimes \mathbb{C}_\eta)^*) \\ &\cong \text{Hom}_{U_q(\mathfrak{h})}(U_q(\mathfrak{g}), (\mathbb{C}_\lambda \otimes \mathbb{C}_\eta)^*), \end{aligned}$$

where the Hom spaces are spaces of morphisms of right modules. In the last line, the right $U_q(\mathfrak{h})$ -action on $\mathbb{C}_\lambda \otimes \mathbb{C}_\eta$ is given by

$$((\phi \otimes \psi) \cdot H)(v \otimes w) = (\phi \otimes \psi)(\chi_\lambda(H_{(1)})v \otimes \chi_\eta(\hat{S}\tau(H_{(2)}))w) = \chi_{\lambda-\eta}(H)(\phi \otimes \psi)(v \otimes w),$$

so that the second factor is isomorphic to $\mathbb{C}_{\lambda-\mu}^*$ as a right $U_q(\mathfrak{h})$ -module. Thus

$$(M(\lambda) \otimes M(\eta))^* \cong (U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\lambda-\mu})^*.$$

We now have

$$\begin{aligned} \text{Hom}_{U_q(\mathfrak{g})}(V(\nu), (U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\lambda-\eta})^*) &\cong \text{Hom}_{U_q(\mathfrak{g})}(V(\nu) \otimes (U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\lambda-\eta}), \mathbb{C}) \\ &\cong \text{Hom}_{U_q(\mathfrak{g})}((U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\lambda-\eta}) \otimes V(\nu), \mathbb{C}) \\ &\cong \text{Hom}_{U_q(\mathfrak{g})}(U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{h})} \mathbb{C}_{\lambda-\eta}, V(\nu)^*) \\ &\cong \text{Hom}_{U_q(\mathfrak{h})}(\mathbb{C}_{\lambda-\eta}, V(\nu)^*) \\ &\cong \text{Hom}_{U_q(\mathfrak{h})}(\mathbb{C}_{\eta-\lambda}, V(\nu)). \end{aligned}$$

This yields the first claim.

b) If $M(\eta)$ is simple we have $M(\eta)^\vee \cong V(\eta) \cong M(\eta)$. Hence Lemma 4.27 yields

$$F((M(\lambda) \otimes M(\eta))^*) \cong F((M(\lambda) \otimes M(\eta)^\vee)^*) \cong F \text{Hom}(M(\lambda), M(\eta)),$$

and part a) implies $[F \text{Hom}(M(\lambda), M(\eta)) : V(\nu)] = \dim(V(\nu))_{\eta-\lambda}$ for all $\nu \in \mathbf{P}^+$.

c) Using Frobenius reciprocity we obtain

$$\begin{aligned} [F \text{Hom}(M(\lambda), M(\eta)) : V(\nu)] &= \dim \text{Hom}_{U_q(\mathfrak{g})}(V(\nu), \text{Hom}(M(\lambda), M(\eta))) \\ &= \dim \text{Hom}_{U_q(\mathfrak{g})}(V(\nu) \otimes M(\lambda), M(\eta)) \\ &= \dim \text{Hom}_{U_q(\mathfrak{g})}(M(\lambda), V(\nu)^* \otimes M(\eta)). \end{aligned}$$

By Lemma 4.34, the module $V(\nu)^* \otimes M(\eta)$ admits a finite length filtration with quotients isomorphic to $M(\eta + \gamma)$, where γ runs over all weights of $V(\nu)^*$ counted with multiplicity. If $M(\lambda)$ is projective, this yields a corresponding filtration of $\text{Hom}_{U_q(\mathfrak{g})}(M(\lambda), V(\nu)^* \otimes M(\eta))$, and therefore

$$[F \text{Hom}(M(\lambda), M(\eta)) : V(\nu)] = \sum_{\gamma \in \mathbf{P}(V(\nu)^*)} \dim \text{Hom}_{U_q(\mathfrak{g})}(M(\lambda), M(\eta + \gamma)),$$

where $\mathbf{P}(V(\nu)^*)$ denotes the weights of $V(\nu)^*$ counted with multiplicity. Using Proposition 4.16 we have

$$\dim \operatorname{Hom}_{U_q(\mathfrak{g})}(M(\lambda), M(\eta + \gamma)) = \delta_{\lambda, \eta + \gamma}.$$

We therefore conclude that

$$[F \operatorname{Hom}(M(\lambda), M(\eta)) : V(\nu)] = \dim(V(\nu)_{\lambda - \eta}^*) = \dim(V(\nu)_{\eta - \lambda}).$$

□

Note that Proposition 4.35 implies in particular

$$[F((M(\lambda) \otimes M(\eta))^*) : V(\nu)] = 0$$

for all $\nu \in \mathbf{P}^+$ if $\eta - \lambda$ is not contained in \mathbf{P} . That is, we have $F((M(\lambda) \otimes M(\eta))^*) = 0$ in this case.

4.5.4. *Hilbert-Poincaré series for the locally finite part of $U_q(\mathfrak{g})$.* In this subsection we collect further auxiliary considerations needed for the description of the PRV determinant. We follow closely the treatment in [41].

Let us first recall that the height of a weight $\mu \in \mathbf{P}$ is defined by $ht(\mu) = c_1 + \dots + c_N \in \frac{1}{2}\mathbb{Z}$ if $\mu = c_1\alpha_1 + \dots + c_N\alpha_N$, and that $ht(\mu) \geq 0$ for $\mu \in \mathbf{P}^+$. For our purposes below it will be notationally convenient to work with the degree of μ defined by $\deg(\mu) = 2ht(\mu) = ht(2\mu)$ instead. The advantage of \deg over ht is that the former takes only integral values for $\mu \in \mathbf{P}$.

We define an \mathbb{N}_0 -filtration of the vector space $FU_q(\mathfrak{g})$ by

$$\mathcal{F}^k(FU_q(\mathfrak{g})) = \bigoplus_{\substack{\mu \in \mathbf{P}^+ \\ \deg(\mu) \leq k}} U_q(\mathfrak{g}) \rightarrow K_{2\mu}$$

for $k \in \mathbb{N}_0$, and refer to this as the degree filtration of $FU_q(\mathfrak{g})$. In fact, this is essentially the filtration of $\mathcal{O}(G_q)$ used in the proof of Theorem 2.154, transported to $FU_q(\mathfrak{g})$ via the isomorphism from Proposition 2.116. We note that the degree filtration is not compatible with the algebra structure of $FU_q(\mathfrak{g})$ in an obvious way.

The associated graded components of the degree filtration are

$$gr_k(FU_q(\mathfrak{g})) = \mathcal{F}^k(FU_q(\mathfrak{g})) / \mathcal{F}^{k-1}(FU_q(\mathfrak{g})),$$

observing that $\mathcal{F}^{-1}(FU_q(\mathfrak{g})) = 0$. We note that the graded components $gr_k(FU_q(\mathfrak{g}))$ are finite dimensional for all $k \in \mathbb{N}_0$. Using the isomorphism $FU_q(\mathfrak{g}) \cong \mathbb{H} \otimes ZU_q(\mathfrak{g})$ in Theorem 2.154 we obtain an induced \mathbb{N}_0 -filtration of the vector space $\mathbb{H} \cong \mathbb{H} \otimes 1 \subset FU_q(\mathfrak{g})$. We shall write

$$\mathbb{H}_k = (\mathcal{F}^k(FU_q(\mathfrak{g})) \cap \mathbb{H}) / (\mathcal{F}^{k-1}(FU_q(\mathfrak{g})) \cap \mathbb{H})$$

for the corresponding graded components.

As for any filtration with finite dimensional graded components, we may define associated Hilbert-Poincaré series as the generating function of the corresponding graded dimensions as follows.

Definition 4.36. Let $\mu \in \mathbf{P}^+$. With the notation as above, the μ -Hilbert-Poincaré series of $FU_q(\mathfrak{g})$ and \mathbb{H} are the formal power series defined by

$$h_\mu^{FU_q(\mathfrak{g})}(z) = \sum_{k=0}^{\infty} [gr_k(FU_q(\mathfrak{g})) : V(\mu)] z^k, \quad h_\mu^{\mathbb{H}}(z) = \sum_{k=0}^{\infty} [\mathbb{H}_k : V(\mu)] z^k,$$

respectively.

Recall from the proof of Theorem 2.113 that $U_q(\mathfrak{g}) \rightarrow K_{2\lambda}$ is isomorphic to $\text{End}(V(-w_0\lambda))^* \cong (V(\lambda)^* \otimes V(\lambda))^* \cong V(\lambda) \otimes V(\lambda)^* \cong \text{End}(V(\lambda))$ as a $U_q(\mathfrak{g})$ -module, where we are using the fact that $V \otimes W \cong W \otimes V$ for any finite dimensional $U_q(\mathfrak{g})$ -modules V, W . Therefore we can also write

$$h_\mu(z) = \sum_{\lambda \in \mathbf{P}^+} [\text{End}(V(\lambda)) : V(\mu)] z^{\text{deg}(\lambda)}.$$

In the remainder of this subsection we shall derive a formula for the formal derivative of $h_\mu^{\mathbb{H}}(z)$ at $z = 1$, presented in Proposition 4.42 further below.

Recall that if M is in category \mathcal{O} and $M = M_0 \supset M_1 \supset \dots \supset M_n = 0$ a filtration with simple subquotients as in Lemma 4.2, then $[M : V(\mu)]$ is the number of subquotients M_i/M_{i+1} isomorphic to $V(\mu)$.

Lemma 4.37. *Let $\mu, \lambda \in \mathbf{P}^+$. Then*

$$[\text{End}(V(\lambda)) : V(\mu)] = \sum_{v, w \in W} (-1)^{l(v)+l(w)} P(v.\lambda - w.\lambda + \mu),$$

where P denotes Kostant's partition function.

Proof. If $V(-\lambda) \cong V(\lambda)^*$ denotes the integrable simple $U_q(\mathfrak{g})$ -module with lowest weight $-\lambda$, we have $\text{End}(V(\lambda)) \cong V(\lambda) \otimes V(-\lambda) \cong V(-\lambda) \otimes V(\lambda)$. Using Lemma 4.34 and Proposition 4.4 we therefore obtain

$$\begin{aligned} [\text{End}(V(\lambda)) : V(\mu)] &= [V(-\lambda) \otimes V(\lambda) : V(\mu)] \\ &= \sum_{w \in W} (-1)^{l(w)} [V(-\lambda) \otimes M(w.\lambda) : V(\mu)] \\ &= \sum_{w \in W} \sum_{\nu \in \mathbf{P}(V(-\lambda))} (-1)^{l(w)} [M(w.\lambda + \nu) : V(\mu)], \end{aligned}$$

where in the last line the sum is over the weights ν of $V(-\lambda)$ counted with multiplicities. Since $\mu \in \mathbf{P}^+$ we have $[M(w.\lambda + \nu) : V(\mu)] = 1$ if $\nu = \mu - w.\lambda$ and $[M(w.\lambda + \nu) : V(\mu)] = 0$ otherwise by Theorem 2.43, so that we arrive at

$$\begin{aligned} [\text{End}(V(\lambda)) : V(\mu)] &= \sum_{w \in W} (-1)^{l(w)} \dim(V(-\lambda)_{\mu - w.\lambda}) \\ &= \sum_{w \in W} (-1)^{l(w)} \dim(V(\lambda)_{w.\lambda - \mu}) \\ &= \sum_{v, w \in W} (-1)^{l(v)+l(w)} P(v.\lambda - w.\lambda + \mu) \end{aligned}$$

as claimed, using that $\dim(V(-\lambda)_\nu) = \dim(V(\lambda)_{-\nu})$ for all weights ν and Proposition 4.4 again. \square

Let us write the group ring $\mathbb{C}[\mathbf{P}]$ of the abelian group \mathbf{P} additively, with basis e^λ for $\lambda \in \mathbf{P}$.

Definition 4.38. We define

$$\begin{aligned} Q^{F U_q(\mathfrak{g})}(z) &= \sum_{w \in W} (-1)^{l(w)} e^{w.0} \prod_{j=1}^N \frac{1}{1 - z^{\text{deg}(\varpi_j)} e^{w\varpi_j - \varpi_j}}, \\ Q^{\mathbb{H}}(z) &= \sum_{w \in W} (-1)^{l(w)} e^{w.0} \prod_{j=1}^N \frac{1 - z^{\text{deg}(\varpi_j)}}{1 - z^{\text{deg}(\varpi_j)} e^{w\varpi_j - \varpi_j}}, \end{aligned}$$

both viewed as elements of the function field $\mathbb{F}(z)$, where \mathbb{F} is the field of quotients of the integral domain $\mathbb{C}[\mathbf{P}]$.

We note that expanding the geometric series in Definition 4.38 we can write

$$Q^{FU_q(\mathfrak{g})}(z) = \sum_{\lambda \in \mathbf{P}^+} \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \lambda - \lambda} z^{\deg(\lambda)},$$

$$Q^{\mathbb{H}}(z) = \sum_{\lambda \in \mathbf{P}^+} \sum_{w \in W} (-1)^{l(w)} e^{w \cdot \lambda - \lambda} z^{\deg(\lambda)} \prod_{j=1}^N (1 - z^{\deg(\varpi_j)}),$$

as formal power series with coefficients in $\mathbb{C}[\mathbf{P}]$.

In the next lemma, we will need to consider power series with coefficients belonging to infinite formal series in the e^λ of the type introduced in Section 4.1. That is, we will be considering formal sums of the form

$$f(z) = \sum_{n=0}^{\infty} \sum_{\lambda \in \mathfrak{h}_q^*} f_n(\lambda) e^\lambda z^n,$$

where for each fixed n the support of $f_n : \mathfrak{h}_q^* \rightarrow \mathbb{Z}$ is contained in a finite union of sets of the form $\nu - \mathbf{Q}^+$ for $\nu \in \mathfrak{h}_q^*$. As in Section 4.1, such series admit a well-defined convolution product.

Lemma 4.39. *Let $\mu \in \mathbf{P}^+$ and $h_\mu(z), h_\mu^{\mathbb{H}}(z)$ be the Poincaré-Hilbert series for $FU_q(\mathfrak{g})$ and \mathbb{H} , respectively. Then*

- a) $h_\mu^{FU_q(\mathfrak{g})}(z)$ is the coefficient of e^0 in $\text{ch}(V(\mu))Q^{FU_q(\mathfrak{g})}(z)$.
- b) $h_\mu^{\mathbb{H}}(z)$ is the coefficient of e^0 in $\text{ch}(V(\mu))Q^{\mathbb{H}}(z)$.

Here $Q^{FU_q(\mathfrak{g})}(z)$ and $Q^{\mathbb{H}}(z)$ are viewed as formal power series.

Proof. a) Recall from Section 4.1 that we defined the formal series

$$p = \text{ch}(M(0)) = \sum_{\nu \in \mathbf{P}} P(\nu) e^{-\nu}.$$

Using Proposition 4.4 we can write

$$\begin{aligned} \text{ch}(V(\mu))Q^{FU_q(\mathfrak{g})}(z) &= \sum_{w \in W} (-1)^{l(w)} p e^{w \cdot \mu} Q^{FU_q(\mathfrak{g})}(z) \\ &= \sum_{\lambda \in \mathbf{P}^+} \sum_{v, w \in W} (-1)^{l(v)+l(w)} p e^{w \cdot \mu + v \cdot \lambda - \lambda} z^{\deg(\lambda)} \\ &= \sum_{\lambda \in \mathbf{P}^+} \sum_{w \in W} (-1)^{l(w)} \text{ch}(V(\lambda)) e^{w(\mu + \rho) - \rho - \lambda} z^{\deg(\lambda)} \end{aligned}$$

For each $n \in \mathbb{N}_0$, the coefficient of z^n in the above series is a finite formal sum, so we may consider the action of the Weyl group upon it, given by $w(e^\lambda) = e^{w \cdot \lambda}$. Moreover, the e^0 term in each summand is invariant under the Weyl group action, so the coefficient of e^0 in the above sum is the same as the coefficient of e^0 in

$$\begin{aligned} &\sum_{\lambda \in \mathbf{P}^+} \sum_{w \in W} (-1)^{l(w)} \text{ch}(V(\lambda)) e^{\mu + \rho - w^{-1}(\rho + \lambda)} z^{\deg(\lambda)} \\ &= \sum_{\lambda \in \mathbf{P}^+} \sum_{w \in W} (-1)^{l(w)} \text{ch}(V(\lambda)) e^{\mu - w^{-1} \cdot \lambda} z^{\deg(\lambda)} \\ &= \sum_{\lambda \in \mathbf{P}^+} \sum_{v, w \in W} (-1)^{l(v)+l(w)} p e^{v \cdot \lambda} e^{\mu - w^{-1} \cdot \lambda} z^{\deg(\lambda)} \\ &= \sum_{\nu \in \mathbf{P}} \sum_{\lambda \in \mathbf{P}^+} \sum_{v, w \in W} (-1)^{l(v)+l(w)} P(\nu) e^{v \cdot \lambda - w^{-1} \cdot \lambda + \mu - \nu} z^{\deg(\lambda)}, \end{aligned}$$

where in the second equality we have again used Proposition 4.4. Replacing w by w^{-1} in the sum and using Lemma 4.37, we calculate the coefficient of e^0 to be

$$\begin{aligned} \sum_{\lambda \in \mathbf{P}^+} \sum_{v, w \in W} (-1)^{l(v)+l(w)} P(v.\lambda - w.\lambda + \mu) z^{\deg(\lambda)} \\ = \sum_{\lambda \in \mathbf{P}^+} [\text{End}(V(\lambda)) : V(\mu)] z^{\deg(\lambda)}. \end{aligned}$$

From the formula immediately after Definition 4.36, this is $h_\mu(z)$.

b) Using Theorem 2.120 it is easy to see that the centre $ZU_q(\mathfrak{g})$ has Hilbert-Poincaré series

$$h^{ZU_q(\mathfrak{g})}(z) = \prod_{j=1}^N \left(\sum_{k=0}^{\infty} z^{k \deg(\varpi_j)} \right)$$

with respect to the degree filtration of $FU_q(\mathfrak{g})$, and therefore

$$h^{ZU_q(\mathfrak{g})}(z) \prod_{j=1}^N (1 - z^{\deg(\varpi_j)}) = e^0.$$

Due to Theorem 2.154 we therefore get

$$h_\mu^{\mathbb{H}}(z) = h_\mu^{\mathbb{H}}(z) h^{ZU_q(\mathfrak{g})}(z) \prod_{j=1}^N (1 - z^{\deg(\varpi_j)}) = h_\mu^{FU_q(\mathfrak{g})}(z) \prod_{j=1}^N (1 - z^{\deg(\varpi_j)}).$$

Hence the assertion follows from a). □

Lemma 4.40. *The derivative of the rational function $Q^{\mathbb{H}}(z)$ defined above satisfies*

$$(\partial_z Q^{\mathbb{H}})(1) = \sum_{j=1}^N \frac{e^{-\alpha_j}}{1 - e^{-\alpha_j}} \deg(\varpi_j) = \sum_{j=1}^N \sum_{n=1}^{\infty} e^{-n\alpha_j} \deg(\varpi_j).$$

Proof. For every $\lambda \in \mathbf{P}^+$ the stabilizer over λ with respect to the action of W is generated by the simple reflections it contains, see Theorem 1.12 (a) of [36]. In particular, the stabilizer of ϖ_j is the subgroup generated by all s_i for $i \neq j$. Hence for each $w \in W$ with $l(w) > 1$ there are at least two fundamental weights ϖ_j for which $w\varpi_j \neq \varpi_j$. As a consequence, the product

$$\prod_{j=1}^N \frac{1 - z^{\deg(\varpi_j)}}{1 - z^{\deg \varpi_j} e^{w\varpi_j - \varpi_j}}$$

vanishes at $z = 1$ to at least order 2 when $l(w) > 1$.

It follows that only the summands with $l(w) = 1$ in the definition of $Q^{\mathbb{H}}(z)$ contribute to $(\partial_z Q^{\mathbb{H}})(1)$. Using $s_i.0 = -\alpha_i$ and $s_i\varpi_j = \varpi_j - \delta_{ij}\alpha_i$ we therefore obtain

$$(\partial_z Q^{\mathbb{H}})(1) = \sum_{j=1}^N \frac{e^{-\alpha_j}}{1 - e^{-\alpha_j}} \deg(\varpi_j),$$

as claimed. Expanding the geometric series on the right hand side of this formula yields the second equality. □

Lemma 4.41. *Let $\mu \in \mathbf{P}^+$ and $n \in \mathbb{N}$. Then we have*

$$\sum_{j=1}^N \dim(V(\mu)_{n\alpha_j}) \deg(\varpi_j) = \frac{1}{2} \sum_{\alpha \in \Delta^+} \dim(V(\mu)_{n\alpha}) \deg(\alpha).$$

Proof. We begin with the case where \mathfrak{g} is a simple Lie algebra, so that the root system Δ is irreducible. In this case, any two roots of the same length are conjugate under the Weyl group action, see Lemma C in Section 10.4 of [35], so the decomposition of Δ into W -orbits is given by $\Delta = \Delta_s \cup \Delta_l$, where Δ_s is the set of short roots, meaning roots of minimal length, and Δ_l is its complement, the possibly empty set of long roots. We shall show that the contributions of all roots of a given length on both sides of the claimed formula agree.

More precisely, denote by Δ_s^+ and Δ_l^+ the sets of short and long roots in Δ^+ , respectively, and let $\Sigma_s = \Delta_s^+ \cap \Sigma$ and $\Sigma_l = \Delta_l^+ \cap \Sigma$ be the sets of short simple roots and long simple roots. Since the dimension of weight spaces is constant on Weyl group orbits we have $\dim(V(\mu)_{n\alpha_j}) = \dim(V(\mu)_{n\alpha})$ for any $\alpha_j \in \Sigma_s$ and $\alpha \in \Delta_s^+$, or $\alpha_j \in \Sigma_l$ and $\alpha \in \Delta_l^+$. Hence it suffices to show

$$\sum_{\alpha_j \in \Sigma_s} \varpi_j = \frac{1}{2} \sum_{\alpha \in \Delta_s^+} \alpha, \quad \sum_{\alpha_j \in \Sigma_l} \varpi_j = \frac{1}{2} \sum_{\alpha \in \Delta_l^+} \alpha.$$

For the first equality, write ρ_s for the right hand side and notice that a simple reflection s_j satisfies $s_j \rho_s = \rho_s - \alpha_j$ if $\alpha_j \in \Sigma_s$, and $s_j \rho_s = \rho_s$ otherwise. Hence $(\alpha_j^\vee, \rho_s) = 1$ for $\alpha_j \in \Sigma_s$ and $(\alpha_j^\vee, \rho_s) = 0$ otherwise. However, the same relations characterize the sum on the left hand side.

The second equality can be proved in the same way, or by using the first formula and the fact that the sum of all fundamental weights equals the half-sum of all positive roots.

Finally, if Δ is not irreducible, then the two sums in the Lemma can be decomposed into sums corresponding to each of the irreducible components, and the result follows. \square

Assembling the above results we obtain the following formula.

Proposition 4.42. *Let $\mu \in \mathbf{P}^+$. Then we have*

$$(\partial_z h_\mu^{\mathbb{H}})(1) = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{\alpha \in \Delta^+} \dim(V(\mu)_{n\alpha}) \deg(\alpha).$$

Proof. According to Lemma 4.39 we know that $(\partial_z h_\mu^{\mathbb{H}})(1)$ equals the coefficient of e^0 in $\text{ch}(V(\mu))(\partial_z Q^{\mathbb{H}})(1)$. Due to Lemma 4.40 this coefficient is given by

$$\sum_{j=1}^N \sum_{n=1}^{\infty} \dim(V(\mu)_{n\alpha_j}) \deg(\varpi_j).$$

Hence the claim follows from Lemma 4.41. \square

4.5.5. *The quantum PRV determinant.* According to Theorem 2.154 there exists a $U_q(\mathfrak{g})$ -invariant linear subspace $\mathbb{H} \subset FU_q(\mathfrak{g})$ such that the multiplication map $\mathbb{H} \otimes ZU_q(\mathfrak{g}) \rightarrow FU_q(\mathfrak{g})$ is a $U_q(\mathfrak{g})$ -linear isomorphism. For $\mu \in \mathbf{P}^+$ let

$$\mathbb{H}^\mu \cong \bigoplus_{j=1}^m V(\mu)$$

be the μ -isotypical component of \mathbb{H} . It is also shown in Theorem 2.154 that the multiplicity m is given by $m = [\mathbb{H} : V(\mu)] = \dim(V(\mu)_0)$. Choose a basis v_1, \dots, v_m of the zero weight space of $V(\mu)$, and write $v_{i,j} \in \mathbb{H}^\mu$ for the vector v_j in the i -th copy of $V(\mu)$ with respect to the above identification. According to the remarks after Theorem 2.154, each of the copies of $V(\mu)$ in \mathbb{H}^μ can be assumed to be contained in a subspace of the form $U_q(\mathfrak{g}) \rightarrow K_{2\nu_i}$ for some $\nu_1, \dots, \nu_m \in \mathbf{P}^+$.

If we identify \mathbb{H}^μ with $\mathbb{H}^\mu \otimes 1 \subset FU_q(\mathfrak{g})$ via the above isomorphism, then the PRV determinant of $U_q(\mathfrak{g})$ associated with μ is defined as the determinant of the

matrix $\mathcal{P}_\mu = (\mathcal{P}(v_{ij}))$, where $\mathcal{P} : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{h})$ denotes the Harish-Chandra map. Note that this determinant, which we denote by $\det(\mathcal{P}_\mu) \in U_q(\mathfrak{h})$, is independent of the choice of basis v_1, \dots, v_m up to an invertible scalar in \mathbb{C} . Our aim in this subsection is to compute $\det(\mathcal{P}_\mu)$, at least up to multiplication by an invertible element of $U_q(\mathfrak{h})$. We start with a result on annihilators, see Lemma 7.1.10 in [41].

Lemma 4.43. *Let $H \subset FU_q(\mathfrak{g})$ be a submodule with respect to the adjoint action, and let $\lambda \in \mathfrak{h}_q^*$. Then $\chi_\lambda(\mathcal{P}(H)) = 0$ iff $H \subset \text{ann}_{FU_q(\mathfrak{g})}(V(\lambda))$.*

Proof. Recall that χ_λ denotes the character of $U_q(\mathfrak{h})$ given by $\chi_\lambda(K_\beta) = q^{(\lambda, \beta)}$. Let us write $N \subset V(\lambda)$ for the linear span of all weight spaces in $V(\lambda)$ apart from the highest weight space $V(\lambda)_\lambda = \mathbb{C}v_\lambda$.

We claim that $\chi_\lambda(\mathcal{P}(H)) = 0$ iff $H \cdot v_\lambda \subset N$. Indeed, if $H \cdot v_\lambda \subset N$ then the projection $\mathcal{P}(H)$ of H onto $U_q(\mathfrak{h}) = 1 \otimes U_q(\mathfrak{h}) \otimes 1 \subset U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}_+)$ must satisfy $\chi_\lambda(\mathcal{P}(H)) = 0$ since it acts on v_λ by the character χ_λ . Conversely, if $\chi_\lambda(\mathcal{P}(H)) = 0$ then only terms in $H \cdot v_\lambda$ which lower weights survive, which means $H \cdot v_\lambda \subset N$.

Assume $\chi_\lambda(\mathcal{P}(H)) = 0$ and let $X \in H$ be a weight vector of weight μ with respect to the adjoint action. Then

$$XF_j = q^{(\mu, \alpha_j)} K_j^{-1} X K_j F_j = q^{(\mu, \alpha_j)} (F_j X - (F_j \rightarrow X))$$

by definition of the adjoint action. We conclude $HU_q(\mathfrak{n}_-) = U_q(\mathfrak{n}_-)H$. Since $\chi_\lambda(\mathcal{P}(H)) = 0$ implies $H \cdot v_\lambda \subset N$ this implies

$$H \cdot V(\lambda) = HU_q(\mathfrak{n}_-) \cdot v_\lambda = U_q(\mathfrak{n}_-)H \cdot v_\lambda \subset N.$$

In addition, because H is ad -stable we have $U_q(\mathfrak{g})H = HU_q(\mathfrak{g})$ and $U_q(\mathfrak{g})H \cdot V(\lambda) = HU_q(\mathfrak{g}) \cdot V(\lambda) = H \cdot V(\lambda)$. That is, the vector space $H \cdot V(\lambda) \subset V(\lambda)$ is a $U_q(\mathfrak{g})$ -submodule. Since $H \cdot V(\lambda)$ is contained in N and $V(\lambda)$ is simple we conclude $H \cdot V(\lambda) = 0$. That is, we have $H \subset \text{ann}_{U_q(\mathfrak{g})}(V(\lambda))$, and therefore in particular $H \subset \text{ann}_{FU_q(\mathfrak{g})}(V(\lambda))$.

Conversely, if $H \subset \text{ann}_{FU_q(\mathfrak{g})}(V(\lambda))$ we obtain $H \cdot v_\lambda = 0 \subset N$, and our initial argument implies $\chi_\lambda(\mathcal{P}(H)) = 0$. \square

We note that Lemma 4.43 implies in particular that $\det(\mathcal{P}_\mu)$ is nonzero for any $\mu \in \mathbf{P}^+$. In fact, otherwise we would find a nonzero ad -stable subspace $H \subset \mathbb{H}^\mu \subset \mathbb{H}$ such that $\chi_\lambda(\mathcal{P}(H)) = 0$ for all $\lambda \in \mathfrak{h}_q^*$, and hence H would act by zero on all irreducible modules $V(\lambda)$. This is impossible due to Theorem 2.101.

For $\alpha \in \Delta^+$ and $n \in \mathbb{N}$ let us define

$$\Gamma_{n, \alpha} = \{\lambda \in \mathfrak{h}_q^* \mid q_\alpha^{(\lambda + \rho, \alpha^\vee)} = q_\alpha^n \text{ and } q_\beta^{(\lambda + \rho, \beta^\vee)} \notin \pm q_\beta^{\mathbb{Z}} \text{ for all } \beta \in \Delta^+, \beta \neq \alpha\}.$$

If $\lambda \in \Gamma_{n, \alpha}$ then this description implies in particular that $s_\alpha \cdot \lambda = s_{0, \alpha} \cdot \lambda$ is antidominant, see Definition 4.6. According to Theorem 4.15 it follows that $M(s_{0, \alpha} \cdot \lambda)$ is simple. Moreover the Jantzen sum formula from Theorem 4.24 shows $M(\lambda)^i = 0$ for $i > 1$ in this case and hence $M(\lambda)^1 = M(s_{0, \alpha} \cdot \lambda)$. In particular $M(\lambda)/M(s_{0, \alpha} \cdot \lambda)$ is simple.

Lemma 4.44. *Let $\lambda \in \Gamma_{n, \alpha}$. Then*

$$[F \text{End}(V(\lambda)) : V(\mu)] = \dim(V(\mu)_0) - \dim(V(\mu)_{n\alpha})$$

for all $\mu \in \mathbf{P}^+$.

Proof. Since $q_\beta^{(\lambda + \rho, \beta^\vee)} \notin \pm q_\beta^{-\mathbb{N}}$ for all $\beta \in \Delta^+$ by the definition of $\Gamma_{n, \alpha}$, it follows from Proposition 4.16 that $M(\lambda)$ is projective. Moreover $V(\lambda) = M(\lambda)/M(s_{0, \alpha} \cdot \lambda)$ is a proper quotient of $M(\lambda)$, so by Lemma 4.30 we have $d(V(\lambda)) < d(M(\lambda)) = d(M(s_{0, \alpha} \cdot \lambda))$. Therefore Proposition 4.32 b) shows that $F \text{Hom}(M(s_{0, \alpha} \cdot \lambda), V(\lambda)) = 0$, and hence the map $F \text{End}(V(\lambda), V(\lambda)) \rightarrow F \text{End}(M(\lambda), V(\lambda))$ is an isomorphism.

Since $M(\lambda)$ is projective the functor $F \operatorname{Hom}(M(\lambda), -)$ is exact, so applying this functor to the short exact sequence

$$0 \longrightarrow M(s_{0,\alpha}\lambda) \longrightarrow M(\lambda) \longrightarrow V(\lambda) \longrightarrow 0$$

gives

$$[F \operatorname{End}(V(\lambda), V(\lambda)) : V(\mu)] = [F \operatorname{Hom}(M(\lambda), M(\lambda)) : V(\mu)] - [F \operatorname{Hom}(M(\lambda), M(s_{0,\alpha}\lambda)) : V(\mu)].$$

Now Lemma 4.35 yields the claim, keeping in mind that $\dim(V(\mu)_{-n\alpha}) = \dim(V(\mu)_{n\alpha})$ since $\pm n\alpha$ are in the same orbit of the Weyl group action. \square

Lemma 4.45. *Let $\nu \in \mathbf{P}^+$. Then*

$$\mathcal{P}(U_q(\mathfrak{g}) \rightarrow K_{2\nu})K_{-2\nu} \subset \sum_{\gamma \in \nu + \mathbf{P}(V(-w_0\nu))} \mathbb{C}K_{-2\gamma} \subset \mathbb{C}[K_1^{-2}, \dots, K_N^{-2}],$$

where $\mathbf{P}(V(-w_0\nu))$ denotes the set of weights of $V(-w_0\nu) \cong V(\nu)^*$.

Proof. Let us first show $\mathcal{P}(X \rightarrow K_{2\nu})K_{-2\nu} \in \mathbb{C}[K_1^{-2}, \dots, K_N^{-2}]$ for any $X \in U_q(\mathfrak{g})$. For this it suffices to consider a monomial $X = E_{i_1} \cdots E_{i_r} K_\lambda F_{j_1} \cdots F_{j_s}$ with $1 \leq i_1, \dots, i_r \leq N, \lambda \in \mathbf{P}$ and $1 \leq j_1, \dots, j_s \leq N$. Using the commutation relations in $U_q(\mathfrak{g})$ one checks that $F_{j_1} \cdots F_{j_s} \rightarrow K_{2\nu}$ is a linear combination of monomials of the form $K_{2\lambda} F_{j_{\sigma(1)}} \cdots F_{j_{\sigma(s)}}$ where σ is a permutation of $1, \dots, s$. The same is true after applying the adjoint action of K_λ . In order for $\mathcal{P}(X \rightarrow K_{2\nu})$ to be nonzero it is therefore necessary that $E_{i_1} \cdots E_{i_r}$ contains exactly as many generators E_k as $F_{j_1} \cdots F_{j_s}$ contains generators F_k for all $1 \leq k \leq N$. Our assertion follows now from

$$E_i \rightarrow F_j = [E_i, F_j] K_i^{-1} = \frac{\delta_{ij}}{q_i - q_i^{-1}} (1 - K_i^{-2})$$

for $1 \leq i, j \leq N$ and the fact that the adjoint action is compatible with multiplication.

In order to prove the Lemma we recall that the isomorphism $J : FU_q(\mathfrak{g}) \cong \mathcal{O}(G_q)$ from Theorem 2.113 maps $U_q(\mathfrak{g}) \rightarrow K_{2\nu}$ onto $\operatorname{End}(V(-w_0\nu))^*$. Since J is compatible with the adjoint action on $FU_q(\mathfrak{g})$ and the coadjoint action on $\mathcal{O}(G_q)$ it follows that for $\mathcal{P}(X \rightarrow K_{2\nu})$ to be nonzero it is necessary that $E_{i_1} \cdots E_{i_r}$ is contained in $U_q(\mathfrak{n}_+)_\gamma$ for some $\gamma \in \mathbf{Q}^+$ such that $-\nu + \gamma \in \mathbf{P}(V(-w_0\nu))$. Combined with the above considerations this yields the claim. \square

As pointed out at the start of this subsection, the j -th copy of $V(\mu)$ in \mathbb{H}^μ can be assumed to be contained in a subspace of the form $U_q(\mathfrak{g}) \rightarrow K_{2\nu_j}$ for some $\nu_1, \dots, \nu_m \in \mathbf{P}^+$. Hence if we write $\nu = \nu_1 + \cdots + \nu_m$ then according to Lemma 4.45 it follows that $\det(\mathcal{P}_\mu)$ is contained in $\mathbb{C}[K_1^{-2}, \dots, K_N^{-2}]K_{2\nu} \subset U_q(\mathfrak{h})$. For technical reasons it will be convenient to remove the factor $K_{2\nu}$ from $\det(\mathcal{P}_\mu)$ in our considerations below, and work with $\operatorname{Det}(\mathcal{P}_\mu) = \det(\mathcal{P}_\mu)K_{-2\nu}$ instead. Thus we have $\operatorname{Det}(\mathcal{P}_\mu) \in \mathbb{C}[K_1^{-2}, \dots, K_N^{-2}] \subset \mathbb{C}[\mathbf{Q}]$ by construction.

Recall from Subsection 2.3.1 that we identify elements $\lambda \in \mathfrak{h}_q^*$ with the associated algebra characters $\chi_\lambda : U_q(\mathfrak{h}) \cong \mathbb{C}[\mathbf{P}] \rightarrow \mathbb{C}$. We may also restrict these to characters of $\mathbb{C}[\mathbf{Q}]$. For $\lambda \in \mathfrak{h}_q^*$ we denote by $\mathfrak{m}_\lambda \subset \mathbb{C}[K_1^{\pm 1}, \dots, K_N^{\pm 1}] = \mathbb{C}[\mathbf{Q}]$ the kernel of $\chi_\lambda : \mathbb{C}[\mathbf{Q}] \rightarrow \mathbb{C}$. Given $f \in \mathbb{C}[\mathbf{Q}]$ we shall say that λ is a zero of f of order $\geq d$ if $f \in \mathfrak{m}_\lambda^d$ for some $d \in \mathbb{N}$. We also say that λ is a zero of order d if $f \in \mathfrak{m}_\lambda^d$ and $f \notin \mathfrak{m}_\lambda^{d+1}$.

Lemma 4.46. *Let $\lambda \in \mathfrak{h}_q^*$ and $\mu \in \mathbf{P}^+$. Then λ is a zero of $\operatorname{Det}(\mathcal{P}_\mu)$ of order $\geq [\operatorname{ann}_{\mathbb{H}}(V(\lambda)) : V(\mu)]$.*

Proof. For each copy V of $V(\mu)$ in $\text{ann}_{\mathbb{H}}(V(\lambda))$ we have $\chi_\lambda(\mathcal{P}(V)) = 0$ by Lemma 4.43. In terms of the elements $v_{ij} \in \mathbb{H}^\mu$ defined at the start of this subsection this means that the rank of the matrix $\chi_\lambda(\mathcal{P}(v_{ij}))$ satisfies $\text{rk}(\chi_\lambda(\mathcal{P}(v_{ij}))) \leq m - [\text{ann}_{\mathbb{H}}(V(\lambda)) : V(\mu)]$. Hence the same is true for the matrix $\chi_\lambda(\mathcal{P}(u_{ij}))$ where $u_{ij} = v_{ij}K_{-2\nu_j} \in \mathbb{C}[K_1^{-2}, \dots, K_N^{-2}]$. Since $\text{Det}(\mathcal{P}_\mu) = \det(\mathcal{P}(u_{ij}))$ this yields the assertion. \square

Lemma 4.47. *Let $\lambda \in \Gamma_{n,\alpha}$ and $\mu \in \mathbf{P}^+$. Then λ is a zero of $\text{Det}(\mathcal{P}_\mu)$ of order $\geq \dim(V(\mu)_{n\alpha})$.*

Proof. According to our above observations we have $V(\lambda) = M(\lambda)/M(s_{0,\alpha}\cdot\lambda)$. Moreover the action of $FU_q(\mathfrak{g})$ on $V(\lambda) = M(\lambda)/M(s_{0,\alpha}\cdot\lambda)$ clearly defines an injection of $FU_q(\mathfrak{g})/\text{ann}_{FU_q(\mathfrak{g})}(V(\lambda))$ into $F\text{Hom}(V(\lambda), V(\lambda))$. Hence

$[\mathbb{H}/\text{ann}_{\mathbb{H}}(V(\lambda)) : V(\mu)] \leq \dim(V(\mu)_0) - \dim(V(\mu)_{n\alpha}) = [\mathbb{H} : V(\mu)] - \dim(V(\mu)_{n\alpha})$
by Lemma 4.44. We thus obtain

$$[\text{ann}_{\mathbb{H}}(V(\lambda)) : V(\mu)] = [\mathbb{H} : V(\mu)] - [\mathbb{H}/\text{ann}_{\mathbb{H}}(V(\lambda)) : V(\mu)] \geq \dim(V(\mu)_{n\alpha}).$$

Therefore the assertion follows from Lemma 4.46. \square

Lemma 4.48. *Let $\mu \in \mathbf{P}^+$. For each $\alpha \in \Delta^+$ and $n \in \mathbb{N}$, the element*

$$(K_\alpha - q_\alpha^{2n-2(\rho, \alpha^\vee)} K_{-\alpha})^{\dim(V(\mu)_{n\alpha})}$$

divides $\text{Det}(\mathcal{P}_\mu)$.

Proof. We shall show that $p^{\dim(V(\mu)_{n\alpha})}$ divides $\text{Det}(\mathcal{P}_\mu)$ where

$$p = 1 - q_\alpha^{2n-2(\rho, \alpha^\vee)} K_\alpha^{-2} \in \mathbb{C}[2\mathbf{Q}].$$

Note first that in $\mathbb{C}[\mathbf{Q}]$ we have the factorization $p = p_+ p_-$ where

$$p_\pm = 1 \pm q_\alpha^{n-(\rho, \alpha^\vee)} K_\alpha^{-1},$$

and for $\lambda \in \Gamma_{n,\alpha}$ the formula

$$1 \pm q_\alpha^{n-(\rho, \alpha^\vee)} K_\alpha^{-1} = 1 \pm q_\alpha^{(\lambda, \alpha^\vee)} K_\alpha^{-1} = 1 \pm q^{(\lambda, \alpha)} K_{-\alpha}$$

shows that $\chi_\lambda(p_-) = 0$ and $\chi_\lambda(p_+) \neq 0$.

We claim that p_\pm are irreducible elements in $\mathbb{C}[\mathbf{Q}]$ and that λ is a zero of p_- of order 1. This is clear if $\alpha = \alpha_i$ is a simple root. For general α we can use the Weyl group action on $\mathbb{C}[\mathbf{Q}]$ to transform p_\pm into a polynomial of the form $1 \pm q^{(\lambda, \alpha)} K_{\alpha_i}^{-1}$ for some simple root α_i , which easily yields the claim in this case as well.

Consider the linear subspace $N_\alpha = \{\gamma \in \mathfrak{h}^* \mid (\gamma, \alpha) = 0\}$ of \mathfrak{h}^* . The bilinear form (\cdot, \cdot) on \mathfrak{h}^* induces an orthogonal direct sum decomposition

$$\mathfrak{h}^* = N_\alpha \oplus N_\alpha^\perp,$$

such that $\eta \in \mathfrak{h}^*$ corresponds to $(\eta - \frac{1}{2}(\eta, \alpha^\vee)\alpha, \frac{1}{2}(\eta, \alpha^\vee)\alpha)$. We shall write $\pi : \mathfrak{h}^* \rightarrow N_\alpha$ for the canonical projection, given by

$$\pi(\eta) = \eta - \frac{1}{2}(\eta, \alpha^\vee)\alpha.$$

Note that when $\eta \in 2\mathbf{Q}$ we have $\frac{1}{2}(\eta, \alpha^\vee) \in \mathbb{Z}$, so that $\eta - \pi(\eta) \in \mathbb{Z}\alpha$ and $\pi(\eta) \in N_\alpha \cap \mathbf{Q}$. It follows that if we define $L_\alpha = \pi(2\mathbf{Q}) \subset N_\alpha \cap \mathbf{Q}$ then every element of $\mathbb{C}[2\mathbf{Q}]$ can be written in the form $\sum_{j=-l}^l g_j K_\alpha^j$ for some $l \geq 0$ and elements $g_j \in \mathbb{C}[L_\alpha]$.

In particular, we obtain such an expression for $\text{Det}(\mathcal{P}_\mu)$. It follows that we can write

$$\text{Det}(\mathcal{P}_\mu) = K_\alpha^l \sum_{j=d}^k f_j p_-^j,$$

where each f_j is contained in $\mathbb{C}[L_\alpha] \subset \mathbb{C}[\mathbf{Q}]$, and we have chosen $d \in \mathbb{N}_0$ to be the smallest integer with $f_d \neq 0$.

Consider the canonical embedding $\mathbb{C}[L_\alpha] \subset \mathbb{C}[\mathbf{Q}]$. Since the restriction of the bilinear form $(\ , \)$ to N_α is nondegenerate, the assignment $\eta \mapsto \chi_\eta$ induces a surjection $\chi : N_\alpha \rightarrow \text{Hom}(L_\alpha, \mathbb{C}^\times)$ such that the diagram

$$\begin{array}{ccc} \mathfrak{h}^* & \xrightarrow{\chi} & \text{Hom}(\mathbf{Q}, \mathbb{C}^\times) \\ \downarrow \pi & & \downarrow \\ N_\alpha & \xrightarrow{\chi} & \text{Hom}(L_\alpha, \mathbb{C}^\times) \end{array}$$

is commutative. Here the right hand vertical map is induced by restriction.

For $n \in \mathbb{N}$ let us define

$$Z_{n,\alpha} = \{ \lambda \in \mathfrak{h}^* \mid (\lambda + \rho, \alpha^\vee) = n \text{ and } (\lambda + \rho, \beta^\vee) \notin \mathbb{Z} + \frac{1}{2}i\hbar_\beta^{-1}\mathbb{Z} \text{ for all } \beta \in \Delta^+, \beta \neq \alpha \},$$

where we are using the notation $\hbar_\beta = d_\beta \hbar$. Note that under the surjection $\mathfrak{h}^* \rightarrow \mathfrak{h}_q^* \cong \text{Hom}(\mathbf{P}, \mathbb{C}^\times)$ the set $Z_{n,\alpha}$ gets mapped to $\Gamma_{n,\alpha}$. The set $Z_{n,\alpha}$ is open and dense in the affine subspace

$$\{ \lambda \in \mathfrak{h}^* \mid (\lambda + \rho, \alpha^\vee) = n \},$$

and it follows that $\pi(Z_{n,\alpha})$ is open and dense in N_α . Interpreting $\text{Hom}(L_\alpha, \mathbb{C}^\times)$ as the set of characters of $\mathbb{C}[L_\alpha]$ it follows easily that we find $\lambda \in Z_{n,\alpha}$ such that $\chi_\lambda(f_d)$ is nonzero. Since the image of λ in $\Gamma_{n,\alpha}$ is a zero of $\text{Det}(\mathcal{P}_\mu)$ of order $\geq \dim(V(\mu)_{n\alpha})$ by Lemma 4.47, we conclude that $d \geq \dim(V(\mu)_{n\alpha})$.

Finally, recall that $\text{Det}(\mathcal{P}_\mu)$ is contained in $\mathbb{C}[2\mathbf{Q}]$, and therefore invariant under the automorphism of $\mathbb{C}[\mathbf{Q}]$ which maps K_j to $-K_j$. It follows that p_+^d divides $\text{Det}(\mathcal{P}_\mu)$ as well. We have thus shown that the polynomial p^d divides $\text{Det}(\mathcal{P}_\mu)$, and this finishes the proof. \square

The following result is essentially the content of Theorem 8.2.10 in [41].

Theorem 4.49. *For $\mu \in \mathbf{P}^+$ we have*

$$\det(\mathcal{P}_\mu) = \prod_{n \in \mathbb{N}} \prod_{\alpha \in \Delta^+} (K_\alpha - q_\alpha^{2n-2(\rho, \alpha^\vee)} K_{-\alpha})^{\dim(V(\mu)_{n\alpha})},$$

up to multiplication by an invertible element of $U_q(\mathfrak{h})$.

Proof. Recall that we may choose the decomposition of \mathbb{H}^μ in such a way that the i -th copy of $V(\mu)$ is contained in $U_q(\mathfrak{g}) \rightarrow K_{2\nu_i}$ for some $\nu_i \in \mathbf{P}^+$ for $i = 1, \dots, m$, and we define $\nu = \nu_1 + \dots + \nu_m$. Clearly it suffices to prove the assertion for $\text{Det}(\mathcal{P}_\mu) = \det(\mathcal{P}_\mu)K_{-2\nu}$ instead of $\det(\mathcal{P}_\mu)$.

According to Lemma 4.48 we have

$$\text{Det}(\mathcal{P}_\mu) = f \prod_{n \in \mathbb{N}} \prod_{\alpha \in \Delta^+} (1 - q_\alpha^{2(n-(\rho, \alpha^\vee))} K_\alpha^{-2})^{\dim(V(\mu)_{n\alpha})}$$

for some element $f \in \mathbb{C}[K_1^{-2}, \dots, K_N^{-2}]$. Our strategy is to use degree considerations to show that f is actually an invertible scalar.

Let us first use the definition of deg to define a filtration on $\mathbb{C}[K_1^{-2}, \dots, K_N^{-2}]$ whereby, for a linear combination $X = \sum_j c_j K_{-\eta_j}$ with $\eta_j \in 2\mathbf{Q}^+$ and nonzero coefficients $c_j \in \mathbb{C}$, we set $\text{deg}(X) = \max_j \text{deg}(\eta_j)$. By Lemma 4.45 each element of $\mathcal{P}(U_q(\mathfrak{g}) \rightarrow K_{2\nu_j})K_{-2\nu_j}$ is a linear combination of monomials $K_{-2\gamma}$ with $\gamma \in \nu_j + \mathbf{P}(V(-w_0\nu_j))$. Since the action of $-w_0$ on \mathbf{P} preserves the set of positive roots

it is easy to see that $\deg(-2w_0\nu_j) = \deg(2\nu_j)$, so that each element $X \in \mathcal{P}(U_q(\mathfrak{g}) \rightarrow K_{2\nu_j})K_{-2\nu_j}$ satisfies

$$\deg(X) \leq \deg(2\nu_j) + \deg(-2w_0\nu_j) = 4 \deg(\nu_j).$$

It follows that $\deg(\text{Det}(\mathcal{P}_\mu)) \leq 4 \deg(\nu)$.

If we consider the Hilbert-Poincaré series

$$h_\mu^{\mathbb{H}}(z) = \sum_{k=0}^{\infty} [\mathbb{H}_k : V(\mu)] z^k$$

for \mathbb{H} from Definition 4.36, then by the definition of the ν_i we have

$$(\partial_z h_\mu^{\mathbb{H}})(1) = \sum_{k=0}^{\infty} k [\mathbb{H}_k : V(\mu)] = \deg(\nu_1) + \cdots + \deg(\nu_m) = \deg(\nu).$$

According to Proposition 4.42 we therefore obtain

$$\deg(\nu) = \frac{1}{2} \sum_{n \in \mathbb{N}} \sum_{\alpha \in \Delta^+} \dim(V(\mu)_{n\alpha}) \deg(\alpha).$$

This implies

$$\deg \left(\prod_{n \in \mathbb{N}} \prod_{\alpha \in \Delta^+} (1 - q_\alpha^{2(n - (\rho, \alpha^\vee))}) K_\alpha^{-2} \right)^{\dim(V(\mu)_{n\alpha})} = 4 \deg(\nu).$$

Combining our above considerations yields

$$4 \deg(\nu) \deg(f) = \deg(\text{Det}(\mathcal{P}_\mu)) \leq 4 \deg(\nu),$$

which implies $\deg(f) = 0$. We conclude that f is a scalar, which must be invertible since $\text{Det}(\mathcal{P}_\mu)$ is nonzero. This finishes the proof. \square

We note that Joseph establishes a slightly stronger version of the formula for the PRV determinant given in Theorem 4.49. Namely, it is shown in Theorem 8.2.10 of [41] that the formula holds even up to multiplication by a nonzero scalar of the ground field, and not only up to an invertible element of $U_q(\mathfrak{h})$. We shall not need this stronger assertion in the sequel.

In fact, our main application of Theorem 4.49 is the following key result on annihilators.

Proposition 4.50. *Let $\lambda \in \mathfrak{h}_q^*$. Then the Verma module $M(\lambda)$ is simple iff*

$$\text{ann}_{FU_q(\mathfrak{g})}(V(\lambda)) \cap \mathbb{H} = 0.$$

Proof. From our considerations in Lemma 4.17 we know that the canonical projection $M(\lambda) \rightarrow V(\lambda)$ has a nonzero kernel if and only if $\chi_\lambda(\det(\text{Sh}_\nu)) = 0$ for some Shapovalov determinant $\det(\text{Sh}_\nu)$ with $\nu \in \mathbf{Q}^+$.

We claim that $\text{ann}_{FU_q(\mathfrak{g})}(V(\lambda)) \cap \mathbb{H} \neq 0$ iff $\chi_\lambda(\det(\mathcal{P}_\mu)) = 0$ for some PRV determinant $\det(\mathcal{P}_\mu)$ with $\mu \in \mathbf{P}^+$. Indeed, consider the space $H = \text{ann}_{FU_q(\mathfrak{g})}(V(\lambda)) \cap \mathbb{H}$. Then H is invariant under the adjoint action and therefore $\chi_\lambda(\mathcal{P}(H)) = 0$ by Lemma 4.43. Hence if H is nonzero we have $\chi_\lambda(\det(\mathcal{P}_\mu)) = 0$ for some μ . Conversely, if $\chi_\lambda(\det(\mathcal{P}_\mu)) = 0$ there exist scalars c_1, \dots, c_m , not all zero, such that $\sum_i c_i \chi_\lambda(\mathcal{P}(v_{ij})) = 0$ for all j . Let us write V for the linear span of the vectors $\sum c_i v_{ij}$ for $j = 1, \dots, m$. Then V is the weight zero subspace of a uniquely determined $U_q(\mathfrak{g})$ -invariant subspace $L \subset \mathbb{H}^\mu$. Applying again Lemma 4.43 we have $L \subset H$, so H is nonzero.

According to Theorem 4.49, we find that $\chi_\lambda(\det(\mathcal{P}_\mu))$ is zero for some $\mu \in \mathbf{P}^+$ iff

$$q_\alpha^{2(\alpha^\vee, \lambda + \rho)} - q_\alpha^{2n} = 0$$

for some $\alpha \in \mathbf{\Delta}^+$ and $n \in \mathbb{N}$. By Theorem 4.22, this corresponds precisely to the condition for the vanishing of $\chi_\lambda(\det(\text{Sh}_\nu))$ for some $\nu \in \mathbf{Q}^+$. This yields the claim. \square

4.6. Annihilators of Verma modules. This section is devoted to the Verma module annihilator Theorem of Joseph and Letzter [44], see also Theorem 8.3.9 in [41]. We shall rely crucially on results obtained in Chapter 2 and the considerations in previous sections.

Let $\lambda \in \mathfrak{h}_q^*$ and consider the associated Verma module $M(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b})} \mathbb{C}_\lambda$, where we recall that \mathbb{C}_λ denotes the $U_q(\mathfrak{b})$ -module associated with the character χ_λ . The canonical left action of elements in $U_q(\mathfrak{g})$ on $M(\lambda)$ defines an algebra homomorphism $\phi_\lambda : U_q(\mathfrak{g}) \rightarrow \text{End}(M(\lambda))$.

The map ϕ_λ is $U_q(\mathfrak{g})$ -linear with respect to the adjoint action

$$\text{ad}(X)(Y) = X \rightarrow Y = X_{(1)}Y\hat{S}(X_{(2)})$$

on $U_q(\mathfrak{g})$ and the action given by

$$(X \cdot T)(m) = X_{(1)} \cdot T(\hat{S}(X_{(2)}) \cdot m)$$

on $\text{End}(M(\lambda))$. In particular, ϕ_λ induces an algebra homomorphism $FU_q(\mathfrak{g}) \rightarrow F\text{End}(M(\lambda))$, which we again denote by ϕ_λ .

We write $\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \subset FU_q(\mathfrak{g})$ for the annihilator of $M(\lambda)$ viewed as a $FU_q(\mathfrak{g})$ -module, that is, for the kernel of $\phi_\lambda : FU_q(\mathfrak{g}) \rightarrow F\text{End}(M(\lambda))$.

Recall that $ZU_q(\mathfrak{g}) \subset FU_q(\mathfrak{g})$ denotes the centre of $U_q(\mathfrak{g})$. In a similar way we write $\text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda)) \subset ZU_q(\mathfrak{g})$ for the annihilator of $M(\lambda)$ viewed as a $ZU_q(\mathfrak{g})$ -module.

Theorem 4.51 (Verma module Annihilator Theorem). *Let $\lambda \in \mathfrak{h}_q^*$. Then*

$$\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) = FU_q(\mathfrak{g}) \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda)),$$

and the linear map $\phi_\lambda : FU_q(\mathfrak{g})/\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \rightarrow F\text{End}(M(\lambda))$ is an isomorphism.

Proof. We first claim that

$$FU_q(\mathfrak{g}) = \mathbb{H} + FU_q(\mathfrak{g}) \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda)),$$

where \mathbb{H} is the subspace of $FU_q(\mathfrak{g})$ obtained in the Separation of Variables Theorem 2.154. Indeed, according to Theorem 2.154, we have $FU_q(\mathfrak{g}) = \mathbb{H} \otimes ZU_q(\mathfrak{g})$. Moreover, recall that the centre $ZU_q(\mathfrak{g})$ acts on $M(\lambda)$ by the central character ξ_λ . In particular, we have $Z - \xi_\lambda(Z)1 \in \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda))$ for all $Z \in ZU_q(\mathfrak{g})$. Hence we can write any element $h \otimes Z \in \mathbb{H} \otimes ZU_q(\mathfrak{g})$ in the form

$$\xi_\lambda(Z)h \otimes 1 + h \otimes (Z - \xi_\lambda(Z)1) \in \mathbb{H} + FU_q(\mathfrak{g}) \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda))$$

as desired.

Assume first that $M(\lambda)$ is simple. Then Proposition 4.50 implies

$$\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \cap \mathbb{H} = 0.$$

It follows that the restriction of the map $FU_q(\mathfrak{g}) \rightarrow FU_q(\mathfrak{g})/\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda))$ to \mathbb{H} is an injective map $\mathbb{H} \rightarrow FU_q(\mathfrak{g})/\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda))$. It is also surjective since $FU_q(\mathfrak{g}) = \mathbb{H} + FU_q(\mathfrak{g}) \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda))$ and $FU_q(\mathfrak{g}) \text{ann}_{ZU_q(\mathfrak{g})}(M(\lambda)) \subset \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda))$. Therefore, we have an isomorphism

$$FU_q(\mathfrak{g})/\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \cong \mathbb{H}.$$

Still assuming that $M(\lambda)$ is simple, Proposition 4.35 implies

$$[F\text{End}(M(\lambda)) : V(\nu)] = \dim(V(\nu)_0)$$

for any $\nu \in \mathbf{P}^+$, where $[N : V(\nu)]$ denotes the multiplicity of $V(\nu)$ inside N and $V(\nu)_0$ is the zero weight space of $V(\nu)$. From Separation of Variables in Theorem 2.154 we know that $[\mathbb{H} : V(\nu)] = \dim(V(\nu)_0)$ as well. By comparing the multiplicities of all isotypical components it follows that the inclusion map

$$\mathbb{H} \cong FU_q(\mathfrak{g}) / \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \rightarrow F \text{End}(M(\lambda))$$

is an isomorphism. This finishes the proof in the case that $M(\lambda)$ is simple.

Now consider an arbitrary $\lambda \in \mathfrak{h}_q^*$. Then according to Lemma 4.10 there exists $\lambda' \leq \lambda$ such that $M(\lambda') \subset M(\lambda)$ and $M(\lambda')$ is simple. Due to Proposition 4.50 we have $\text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \cap \mathbb{H} \subset \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda')) \cap \mathbb{H} = 0$. Hence we obtain

$$FU_q(\mathfrak{g}) / \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) \cong \mathbb{H}.$$

in the same way as above. Consider the commutative diagram

$$\begin{array}{ccc} F \text{Hom}(M(\lambda), M(\lambda')) & \longrightarrow & F \text{Hom}(M(\lambda), M(\lambda)) \\ \downarrow & & \downarrow \\ F \text{Hom}(M(\lambda'), M(\lambda')) & \longrightarrow & F \text{Hom}(M(\lambda'), M(\lambda)) \end{array}$$

induced by the inclusion $M(\lambda') \subset M(\lambda)$. Using Proposition 4.33 and exactness of the Hom-functor we see that the two vertical maps are injections and the bottom horizontal map is an isomorphism.

This gives us an injection $F \text{End}(M(\lambda)) \cong F \text{End}(M(\lambda'))$ and we obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{H} \cong FU_q(\mathfrak{g}) / \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda)) & \longrightarrow & F \text{End}(M(\lambda)) \\ \downarrow & & \downarrow \\ \mathbb{H} \cong FU_q(\mathfrak{g}) / \text{ann}_{FU_q(\mathfrak{g})}(M(\lambda')) & \longrightarrow & F \text{End}(M(\lambda')). \end{array}$$

The bottom horizontal map is an isomorphism since $M(\lambda')$ is simple. The left vertical map is also an isomorphism. The right vertical map is therefore surjective. Since we have shown it is injective, this map is in fact an isomorphism, and so is the top horizontal map. This finishes the proof. \square

5. REPRESENTATION THEORY OF COMPLEX SEMISIMPLE QUANTUM GROUPS

In this chapter, we discuss the representation theory of complex semisimple quantum groups. The appropriate notion of a G_q -representation here is that of a Harish-Chandra module for G_q , which means an essential $\mathfrak{D}(G_q)$ -module with K_q -types of finite multiplicity, see Section 5.2. In particular, the irreducible unitary representations of G_q belong to this class.

Our main focus will be the classification of irreducible Harish-Chandra modules. This is achieved by studying the (non-unitary) principal series representations of G_q and the intertwining operators between them. We shall begin, however, with some results on the Verma modules for the quantized universal enveloping algebra $U_q^{\mathbb{R}}(\mathfrak{g})$ associated to a complex semisimple quantum group G_q , see Section 5.1.

Although some of the constructions and results presented here work more generally, we shall assume throughout that $\mathbb{K} = \mathbb{C}$ and $1 \neq q = e^h$ is positive. We identify $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^{\vee}$ where $\hbar = \frac{h}{2\pi}$. Moreover we write

$$[z]_q = \frac{q^z - q^{-z}}{q - q^{-1}}$$

for the q -number associated with $z \in \mathbb{C}$, and similarly use the notation for q -binomial coefficients as in Section 2.1.

5.1. Verma modules for $U_q^{\mathbb{R}}(\mathfrak{g})$. In this section we discuss the theory of Verma modules for the quantized universal enveloping algebra

$$U_q^{\mathbb{R}}(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k}) \rtimes \mathfrak{C}^{\infty}(K_q)$$

of the complex quantum group G_q , which was introduced in Section 3.6.

5.1.1. Characters of $U_q^{\mathbb{R}}(\mathfrak{b})$. Characters of the algebra $U_q^{\mathbb{R}}(\mathfrak{b}) = U_q^{\mathbb{R}}(\mathfrak{t}) \rtimes \mathfrak{C}^{\infty}(K_q)$ are parametrized by pairs of weights (μ, λ) where $\mu, \lambda \in \mathfrak{h}_q^*$, as we now describe.

To $\mu \in \mathfrak{h}_q^*$ we associate the character χ_{μ} of $U_q^{\mathbb{R}}(\mathfrak{t}) = U_q(\mathfrak{h})$ given by $\chi_{\mu}(K_{\nu}) = q^{(\mu, \nu)}$ for all $\nu \in \mathbf{P}$, as in Subsection 2.3.1.

To $\lambda \in \mathfrak{h}_q^*$ we associate a character K_{λ} of $\mathfrak{C}^{\infty}(K_q)$ as follows. Briefly, if $\lambda = \sum_i a_i \alpha_i \in \mathfrak{h}^*$ with $a_i \in \mathbb{C}$ then we may define $K_{\lambda} = \prod_i K_i^{a_i} \in \mathcal{M}(\mathfrak{D}(K_q))$ by functional calculus. More explicitly, recalling that the multiplier algebra of $\mathfrak{D}(K_q)$ is

$$\mathcal{M}(\mathfrak{D}(K_q)) = \mathfrak{C}^{\infty}(\hat{K}_q) = \prod_{\gamma \in \mathbf{P}^+} \text{End}(V(\gamma)),$$

we define $K_{\lambda} \in \mathcal{M}(\mathfrak{D}(K_q))$ to be the element which acts on any vector $v \in V(\gamma)$ of weight ν by

$$K_{\lambda} \cdot v = q^{(\lambda, \nu)} v.$$

This is compatible with the existing notation for elements K_{λ} when $\lambda \in \mathbf{P}$, and depends only on the class of λ in $\mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^{\vee}$. The element K_{λ} determines a character of $\mathfrak{C}^{\infty}(K_q)$ by

$$f \mapsto (f, K_{\lambda}).$$

Note that we are using the reverse pairing $(f, K_{\lambda}) = (K_{-\lambda}, f)$ in the definition of this character, since we will want to interpret K_{λ} as an element of the function algebra $\mathfrak{C}^{\infty}(\hat{K}_q)$.

We remark that K_{λ} belongs to the subalgebra $\mathcal{M}(\mathfrak{D}(T)) \subset \mathcal{M}(\mathfrak{D}(K_q))$ where T is the maximal torus subgroup. It is in fact the pullback of a character K_{λ} of $\mathfrak{C}^{\infty}(T)$ by the quotient map $\mathfrak{C}^{\infty}(K_q) \rightarrow \mathfrak{C}^{\infty}(T)$. Moreover, every character of $\mathfrak{C}^{\infty}(K_q)$ is of this form, see [77]. In this way, the parameter space \mathfrak{h}_q^* is identified with the Chevalley complexification $T_{\mathbb{C}}$ of the torus T .

Combining the characters χ_μ and K_λ for any $\mu, \lambda \in \mathfrak{h}_q^*$ we obtain a character $\chi_{\mu, \lambda}$ of $U_q^{\mathbb{R}}(\mathfrak{b}) = U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(K_q)$ by setting

$$\chi_{\mu, \lambda}(X \bowtie f) = \chi_\mu(X)(f, K_\lambda).$$

Notice that $\chi_{\mu, \lambda}$ is typically not $*$ -preserving for the standard $*$ -structure on $U_q^{\mathbb{R}}(\mathfrak{b})$.

Let us conclude with some remarks about the restriction of these characters to the subalgebra $U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(PK_q)$, which corresponds to the Borel subgroup of the “connected component” G_q^0 of G_q that was discussed in Subsection 3.4.2. One sees that the character $\chi_{\mu, \lambda}$ restricts to the trivial character on $U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(PK_q)$ if and only if $(\mu, \lambda) = (0, i\hbar^{-1}\gamma)$ for some $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$. It follows that the characters of $U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(PK_q)$ are indexed by pairs $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}^*/i\hbar^{-1}\mathbf{P}^\vee$.

5.1.2. *Verma modules for $U_q^{\mathbb{R}}(\mathfrak{g})$.* The Verma module associated to $\chi_{\mu, \lambda}$ is defined by

$$M(\mu, \lambda) = U_q^{\mathbb{R}}(\mathfrak{g}) \otimes_{U_q^{\mathbb{R}}(\mathfrak{b})} \mathbb{C}_{\mu, \lambda},$$

where $\mathbb{C}_{\mu, \lambda} = \mathbb{C}$ is the representation space corresponding to the character $\chi_{\mu, \lambda}$ defined above. Following previous notation, we will denote by $v_{\mu, \lambda}$ the cyclic vector $1 \otimes 1 \in M(\mu, \lambda)$.

Recall from Lemma 3.27 that we have an embedding $U_q^{\mathbb{R}}(\mathfrak{g}) \subset U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ of algebras given explicitly by

$$\iota(X \bowtie f) = \hat{\Delta}(X)(l^-(f_{(1)}) \otimes l^+(f_{(2)})).$$

In this way we may view modules over $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ as modules over $U_q^{\mathbb{R}}(\mathfrak{g})$. However, it will be convenient in later sections to use a different embedding of $U_q^{\mathbb{R}}(\mathfrak{g})$ in $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ as follows.

Definition 5.1. We define the embedding $\iota' : U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ by

$$\iota'(X \bowtie f) = \hat{\Delta}(X)(l^+(f_{(1)}) \otimes l^-(f_{(2)}))$$

for $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$.

Note that ι and ι' differ only by the order of the l -functionals in the formula. Again, ι' is an algebra homomorphism.

We observe that for any $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$ we have

$$\iota'(X \bowtie f) = \sigma((\hat{S} \otimes \hat{S})(\iota((1 \bowtie S^{-1}(f))(\hat{S}^{-1}(X) \bowtie 1))))),$$

using Lemma 2.111. Thus we can also write

$$\iota' = \sigma \circ (\hat{S} \otimes \hat{S}) \circ \iota \circ \hat{S}_{G_q}^{-1},$$

where σ denotes the flip map.

In the following proposition, $N(\lambda)$ denotes the universal lowest weight module of $U_q(\mathfrak{g})$ with lowest weight $\lambda \in \mathfrak{h}_q^*$, namely

$$N(\lambda) = U_q(\mathfrak{g}) \otimes_{U_q(\mathfrak{b}_-)} \mathbb{C}_\lambda$$

where \mathbb{C}_λ denotes the one-dimensional $U_q(\mathfrak{b}_-)$ -module upon which $U_q(\mathfrak{n}_-)$ acts trivially and $U_q(\mathfrak{h})$ acts by the character χ_λ . We denote the cyclic vector $1 \otimes 1$ by v^λ .

Proposition 5.2. *Let $(\mu, \lambda) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ and let $l, r \in \mathfrak{h}_q^*$ be such that*

$$\mu = l - r, \quad \lambda = -l - r.$$

We have an isomorphism

$$M(\mu, \lambda) \cong M(l) \otimes N(-r)$$

of $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules which sends the cyclic vector $v_{\mu,\lambda}$ to $v_l \otimes v^{-r}$. Here the action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on $M(l) \otimes N(-r)$ is that induced by the embedding ι' above.

Proof. For all $\nu \in \mathbf{P}$, the action of $K_\nu \in U_q^{\mathbb{R}}(\mathfrak{k})$ on $v_l \otimes v^{-r}$ is given by

$$\begin{aligned} K_\nu \cdot (v_l \otimes v^{-r}) &= K_\nu \cdot v_l \otimes K_\nu \cdot v^{-r} \\ &= q^{(l,\nu)} v_l \otimes q^{(-r,\nu)} v^{-r} = \chi_\mu(K_\nu) v_l \otimes v^{-r}. \end{aligned}$$

We make a similar calculation for the action of a matrix coefficient $u_{ij}^\eta = \langle e_\eta^i | \bullet | e_j^\eta \rangle$ where $\eta \in \mathbf{P}^+$ and the e_j^η form a basis of weight vectors for $V(\eta)$, with dual basis vectors $e_\eta^i \in V(\eta)^*$. Writing ϵ_j for the weight of e_j^η we obtain

$$\begin{aligned} u_{ij}^\eta \cdot (v_l \otimes v^{-r}) &= \sum_k l^+(u_{ik}^\eta) \cdot v_l \otimes l^-(u_{kj}^\eta) \cdot v^{-r} \\ &= \delta_{ij} l^+(u_{ii}^\eta) \cdot v_l \otimes l^-(u_{ii}^\eta) \cdot v^{-r} \\ &= \delta_{ij} K_{\epsilon_i} \cdot v_l \otimes K_{-\epsilon_i} \cdot v^{-r} \\ &= \delta_{ij} q^{(l,\epsilon_i)} v_l \otimes q^{(r,\epsilon_i)} v^{-r} \\ &= \delta_{ij} q^{(-\lambda,\epsilon_i)} v_l \otimes v^{-r} \\ &= (u_{ij}^\eta, K_\lambda) v_l \otimes v^{-r}. \end{aligned}$$

Here, in the second step, we use that $l^\pm(u_{ij}^\eta)$ is contained in $U_q(\mathfrak{b}_\pm)_{\epsilon_j - \epsilon_i}$, and that the vectors v_{-l} and v_r are annihilated by $U_q(\mathfrak{n}_+)$ and $U_q(\mathfrak{n}_-)$, respectively. Note also that when $\epsilon_i = \epsilon_j$ we have $l^\pm(u_{ij}^\eta) = \delta_{ij} K_{\pm\epsilon_i}$.

By the definition of $M(\mu, \lambda)$, we thus obtain a $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear map $\gamma : M(\mu, \lambda) \rightarrow M(l) \otimes N(-r)$ such that $\gamma(v_{\mu,\lambda}) = v_l \otimes v^{-r}$.

For surjectivity, we will prove inductively that the subspaces $M(l)_{l-\nu} \otimes N(-r)$ are contained in $\text{im}(\gamma)$ for each $\nu \in \mathbf{Q}^+$. For the case $\nu = 0$, note that the action of $E_i \in U_q^{\mathbb{R}}(\mathfrak{k})$ on $v_l \otimes v^{-r}$ is given by

$$E_i \cdot (v_l \otimes v^{-r}) = v_l \otimes E_i \cdot v^{-r}.$$

It follows that $\mathbb{C}v_l \otimes N(-r) \subset \text{im}(\gamma)$. Now fix $\nu \in \mathbf{Q}^+$ and suppose that $M(l)_{l-\nu'} \otimes N(-r) \subset \text{im}(\gamma)$ for all $\nu' < \nu$. Note that $M(l)_{l-\nu} \otimes N(-r)$ is spanned by elements of the form $F_i \cdot m \otimes n$ with $m \in M(l)_{l-\nu+\alpha_i}$ and $n \in N(-r)$. Using the action of F_i on $n \otimes m$ we can write

$$(F_i \cdot m) \otimes n = F_i \cdot (m \otimes n) - K_i^{-1} \cdot m \otimes F_i \cdot n \in \text{im}(\gamma),$$

and surjectivity follows.

For injectivity, we note to begin with that $U_q^{\mathbb{R}}(\mathfrak{g})$ is a free right $U_q^{\mathbb{R}}(\mathfrak{b})$ -module generated by $U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+) \subset U_q(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k}) \subset U_q^{\mathbb{R}}(\mathfrak{g})$. That is, the map

$$U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+) \rightarrow M(\mu, \lambda); \quad Y \otimes X \mapsto YX \cdot v_{\mu,\lambda}$$

is an isomorphism. Consider an element

$$\sum_j Y_j \otimes X_j \in U_q(\mathfrak{n}_-) \otimes U_q(\mathfrak{n}_+)$$

where $Y_j \in U_q(\mathfrak{n}_-)_{-\nu'_j}$, $X_j \in U_q(\mathfrak{n}_+)_{\nu''_j}$ for some weights $\nu'_j, \nu''_j \in \mathbf{Q}^+$. We may arrange the sum so that ν''_0 is maximal among those ν''_j appearing nontrivially, and ν'_0 is maximal among the weights ν'_j such that $\nu''_j = \nu''_0$ and (ν'_j, ν''_j) appears nontrivially. By considering the actions of E_i and F_i , one sees that

$$\gamma(\sum_j Y_{\nu'_j} X_{\nu''_j} \cdot v_{\mu,\lambda}) = \sum_j Y_{\nu'_j} X_{\nu''_j} \cdot (v_l \otimes v^{-r})$$

contains a nonzero term in $M(l)_{l-\nu'_0} \otimes N(-r)_{-r+\nu''_0}$. This completes the proof. \square

In order to convert the lowest weight module $N(-r)$ into a highest weight module, we may twist the action of $U_q(\mathfrak{g})$ by the algebra automorphism

$$\theta = \tau \hat{S} = \hat{S}^{-1} \tau,$$

where τ is the algebra anti-automorphism defined in Lemma 2.16. We record the formulas

$$\theta(E_i) = -F_i, \quad \theta(F_i) = -E_i, \quad \theta(K_\nu) = K_{-\nu},$$

from which it follows that $N(-r)$ is isomorphic to $M(r)^\theta$, where the latter denotes the Verma module $M(r)$ endowed with the action obtained by composition with θ .

Let us further note that for any $U_q(\mathfrak{g})$ -module M , we have $M^{\hat{S}^2} \cong M$, where $M^{\hat{S}^2}$ denotes the same space with action twisted by the automorphism \hat{S}^2 . This is because the two modules are intertwined by the action of $K_{2\rho}$, see Lemma 2.7. Therefore, if we put $\theta' = \hat{S}\tau = \hat{S}^2\theta$ then we also have $N(-r) \cong M(r)^{\theta'}$.

We immediately obtain the following.

Corollary 5.3. *Let $(\mu, \lambda) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ and let $l, r \in \mathfrak{h}_q^*$ be such that*

$$\mu = l - r, \quad \lambda = -l - r.$$

We have an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules

$$M(\mu, \lambda) \cong M(l) \otimes M(r)$$

which sends the cyclic vector $v_{\mu, \lambda}$ to $v_l \otimes v_r$, and where the action of $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$ on $m \otimes n \in M(l) \otimes M(r)$ is given by

$$(X \bowtie f) \cdot (m \otimes n) = (\text{id} \otimes \theta)(\iota'(X \bowtie f)) \cdot (m \otimes n).$$

The same is true if θ is replaced by θ' .

We point out that for a given character $\chi_{\mu, \lambda}$ of $U_q^{\mathbb{R}}(\mathfrak{b})$ with $\mu, \lambda \in \mathfrak{h}_q^*$, there are 2^N choices for the associated parameters $(l, r) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$, where N is the rank of G . Specifically, if we lift μ, λ to \mathfrak{h}^* then we obtain the solutions for (l, r) given by

$$l = -\frac{1}{2}(\lambda - \mu) + \frac{1}{2}i\hbar^{-1}\alpha^\vee, \quad r = -\frac{1}{2}(\lambda + \mu) + \frac{1}{2}i\hbar^{-1}\alpha^\vee,$$

for $\alpha^\vee \in \mathbf{Q}^\vee/2\mathbf{Q}^\vee$.

In other words, if we let the extended Weyl group from Definition 2.125

$$\hat{W} = \mathbf{Y}_q \bowtie W = (\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee) \bowtie W$$

act diagonally upon the parameter space $\mathfrak{h}_q^* \times \mathfrak{h}_q^*$ by $\hat{w}(l, r) = (\hat{w}l, \hat{w}r)$ then the action of the translation subgroup $\mathbf{Y}_q = \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee$ does not change the isomorphism class of the associated $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules $M(l) \otimes M(r)$.

The following criterion allows us to characterize irreducibility of the Verma modules $M(\mu, \lambda)$ introduced above. We recall that in our conventions we use $\mathbb{N} = \{1, 2, \dots\}$.

Theorem 5.4. *The $U_q^{\mathbb{R}}(\mathfrak{g})$ -module $M(\mu, \lambda)$ is irreducible if and only if $(\mu, \lambda) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ satisfies*

$$q_\alpha^{(-\lambda+2\rho, \alpha^\vee)} \neq q_\alpha^{(\pm\mu, \alpha^\vee)+2m}$$

for all $\alpha \in \Delta^+$ and all $m \in \mathbb{N}$.

Proof. As in Proposition 5.3 we choose $l, r \in \mathfrak{h}_q^*$ such that $\mu = l - r, \lambda = -l - r$. Then $M(l) \otimes M(r)$ is irreducible as a $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ -module if and only if $M(l)$ and $M(r)$ are both irreducible as $U_q(\mathfrak{g})$ -modules. According to Theorem 4.15 and Definition 4.6, this is the case iff

$$q_\alpha^{2(l+\rho, \alpha^\vee)} \notin q_\alpha^{2\mathbb{N}} \quad \text{and} \quad q_\alpha^{2(r+\rho, \alpha^\vee)} \notin q_\alpha^{2\mathbb{N}}$$

for all $\alpha \in \Delta^+$. Using $2l = -\lambda + \mu$ and $2r = -\lambda - \mu$, this is equivalent to the condition

$$q_\alpha^{(-\lambda+2\rho, \alpha^\vee)} \neq q_\alpha^{(\pm\mu, \alpha^\vee)+2m}$$

for all $\alpha \in \Delta^+$, $m \in \mathbb{N}$.

It remains to check that $M(l) \otimes M(r)$ is irreducible as a $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ -module if and only if it is irreducible as a $U_q^{\mathbb{R}}(\mathfrak{g})$ -module with the action induced by $(\theta \otimes \text{id})\iota'$, compare the argument in the proof of Theorem 3.4 in [33]. For the nontrivial implication assume that $M(l) \otimes M(r)$ is irreducible as a $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ -module. According to Lemma 3.27, the image of ι' is $(1 \otimes \hat{S}(FU_q(\mathfrak{g})))\hat{\Delta}(U_q(\mathfrak{g}))$. Thus, Theorem 2.113 shows that the image of $(\text{id} \otimes \theta)\iota'$ contains $K_{\nu'} \otimes K_{2\nu'' - \nu'}$ for all $\nu' \in \mathbf{P}$ and $\nu'' \in \mathbf{P}^+$. It follows that any $U_q^{\mathbb{R}}(\mathfrak{g})$ -submodule $V \subset M(l) \otimes M(r)$ is the sum of its $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ -weight spaces.

Let $v = \sum_j x_j \otimes y_j \in V$ be a nonzero vector of weight (ϵ_l, ϵ_r) with $\epsilon = \epsilon_l + \epsilon_r$ maximal in the submodule V . We may assume that the y_j are linearly independent in $M(r)_{\epsilon_r}$. Since

$$E_i \cdot (x_j \otimes y_j) = E_i \cdot x_j \otimes K_i^{-1} \cdot y_j - x_j \otimes F_i \cdot y_j,$$

the maximality of ϵ forces $E_i \cdot x_j = 0$ for all i , and so each x_j is primitive. Since $M(l)$ is irreducible we conclude that each x_j is a scalar multiple of v_l , and thus $v = v_l \otimes y$ for some y . Likewise, using

$$F_i \cdot (v_l \otimes y) = F_i \cdot v_l \otimes y - K_i^{-1} \cdot v_l \otimes E_i \cdot y,$$

the maximality of ϵ shows that y is primitive, and hence is a nonzero multiple of v_r . By Corollary 5.3, $v_l \otimes v_r$ is a cyclic vector for the $U_q^{\mathbb{R}}(\mathfrak{g})$ -action, so $V = M(l) \otimes M(r)$. This yields the claim. \square

5.2. Representations of G_q .

5.2.1. *Harish-Chandra modules.* In this subsection we define the main notion of G_q -representation which we will be studying in the remainder of the chapter.

We recall that the convolution algebra of G_q is $\mathfrak{D}(G_q) = \mathfrak{D}(K_q) \bowtie \mathfrak{C}^\infty(K_q)$. Recall also that a $\mathfrak{D}(G_q)$ -module V is called essential if the multiplication map $\mathfrak{D}(G_q) \otimes_{\mathfrak{D}(G_q)} V \rightarrow V$ is an isomorphism. Using the central idempotents in $\mathfrak{D}(K_q)$ one sees that a $\mathfrak{D}(G_q)$ -module is essential if and only if it is an essential $\mathfrak{D}(K_q)$ -module under restriction.

Recall from Section 3.6 that the quantized universal enveloping algebra $U_q^{\mathbb{R}}(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k}) \bowtie \mathfrak{C}^\infty(K_q)$ of G_q sits inside the multiplier algebra of $\mathfrak{D}(G_q)$. Let us say that a $U_q^{\mathbb{R}}(\mathfrak{g})$ -module V is integrable if it is an integrable module for the action of $U_q^{\mathbb{R}}(\mathfrak{k}) \subset U_q^{\mathbb{R}}(\mathfrak{g})$. The following result is then essentially immediate from the definitions.

Lemma 5.5. *There is a canonical isomorphism between the category of essential $\mathfrak{D}(G_q)$ -modules and the category of integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules.*

Proof. Every essential $\mathfrak{D}(G_q)$ -module becomes an integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -module via the inclusion $U_q^{\mathbb{R}}(\mathfrak{g}) \subset \mathcal{M}(\mathfrak{D}(G_q))$.

Conversely, if V is an integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -module then the action of $U_q^{\mathbb{R}}(\mathfrak{k}) \subset U_q^{\mathbb{R}}(\mathfrak{g})$ corresponds uniquely to an essential $\mathfrak{D}(K_q)$ -module structure on V , and the latter combines with the action of $\mathfrak{C}^\infty(K_q) \subset U_q^{\mathbb{R}}(\mathfrak{g})$ to turn V into an essential $\mathfrak{D}(G_q)$ -module.

These procedures are inverse to each other and compatible with morphisms. \square

A third structure which is equivalent to an essential $\mathfrak{D}(G_q)$ -module is that of a $\mathfrak{C}^\infty(K_q)$ -Yetter Drinfeld module. We recall the general definition.

Definition 5.6. Let H be a Hopf algebra. A Yetter-Drinfeld module over H is a left H -module V which is at the same time a left H -comodule such that

$$(f \cdot v)_{(-1)} \otimes (f \cdot v)_{(0)} = f_{(1)}v_{(-1)}S(f_{(3)}) \otimes f_{(2)} \cdot v_{(0)}$$

for all $v \in V$ and $f \in H$. Here we write $\gamma(v) = v_{(-1)} \otimes v_{(0)}$ for the left coaction of H on V .

If V is a Yetter-Drinfeld module over $\mathfrak{C}^\infty(K_q)$ then we can convert the left $\mathfrak{C}^\infty(K_q)$ -coaction into an essential left action of $\mathfrak{D}(K_q)$ by the formula

$$x \cdot v = (\hat{S}(x), v_{(-1)})v_{(0)}, \quad x \in \mathfrak{D}(K_q).$$

A standard calculation shows that the resulting $\mathfrak{D}(K_q)$ -action and the given $\mathfrak{C}^\infty(K_q)$ -action satisfy the commutation relations in the Drinfeld double $\mathfrak{D}(G_q)$. It follows that the category of $\mathfrak{C}^\infty(K_q)$ -Yetter-Drinfeld modules is isomorphic to the category of essential $\mathfrak{D}(G_q)$ -modules.

Since any essential $\mathfrak{D}(G_q)$ -module is an essential $\mathfrak{D}(K_q)$ -module V by restriction, it decomposes as a direct sum

$$V = \bigoplus_{\gamma \in \mathbf{P}^+} V^\gamma,$$

where V^γ is a direct sum of copies of the simple module with highest weight γ . The subspace $V^\gamma \subset V$ is called the isotypical component corresponding to γ .

Definition 5.7. An essential $\mathfrak{D}(K_q)$ -module V is called admissible if each isotypical component V^γ is finite dimensional. An essential $\mathfrak{D}(G_q)$ -module which is admissible as a $\mathfrak{D}(K_q)$ -module will be called a Harish-Chandra module over G_q .

We remark that the term ‘‘Harish-Chandra module’’ has several different meanings in the classical literature, in particular with or without the imposition of admissibility. The imposition of admissibility in this definition avoids many technical annoyances, and as in the classical case, all irreducible unitary G_q -representations are admissible, see the following section.

5.2.2. Unitary G_q -representations. By definition, a unitary representation of G_q on a Hilbert space \mathcal{H} is a non-degenerate $*$ -homomorphism $C_r^*(G_q) \rightarrow \mathbb{L}(\mathcal{H})$.

In particular any unitary representation \mathcal{H} of G_q becomes a unitary representation of K_q by restriction. A vector $\xi \in \mathcal{H}$ is called K_q -finite if it is contained in a finite dimensional K_q -subrepresentation. The Harish-Chandra module $\text{HC}(\mathcal{H})$ associated with \mathcal{H} is the space of all K_q -finite vectors in \mathcal{H} . Explicitly, this is given by

$$\text{HC}(\mathcal{H}) = \mathfrak{D}(K_q) \cdot \mathcal{H} = \mathfrak{D}(G_q) \cdot \mathcal{H} \subset \mathcal{H}.$$

From this description it is clear that $\text{HC}(\mathcal{H})$ is dense in \mathcal{H} , and that $\text{HC}(\mathcal{H})$ naturally becomes an essential module over $\mathfrak{D}(G_q)$.

The goal of this subsection is to show that the $\mathfrak{D}(G_q)$ -module $\text{HC}(\mathcal{H})$ associated with an irreducible unitary representation \mathcal{H} of G_q is admissible, so that it is a Harish-Chandra module in the sense of Definition 5.7. The argument is the same as in the classical case, based on results of Godement.

Firstly, if A is any algebra then the n -commutator of n elements $a_1, \dots, a_n \in A$ is defined by

$$[a_1, \dots, a_n] = \sum_{\sigma \in S_n} \text{sign}(\sigma) a_{\sigma(1)} \cdots a_{\sigma(n)}.$$

Note that for $n = 2$ this reduces to the usual commutator $[a_1, a_2] = a_1a_2 - a_2a_1$. Let us say that A is n -abelian if all n -commutators $[a_1, \dots, a_n]$ for $a_1, \dots, a_n \in A$ vanish. Clearly, subalgebras and quotients of n -abelian algebras are again n -abelian.

By basic linear algebra, every finite dimensional algebra A is n -abelian for any $n > \dim(A)$. In particular, for every $k \in \mathbb{N}$ there exists a smallest number $r(k) \in \mathbb{N}$ such that $M_k(\mathbb{C})$ is $r(k)$ -abelian.

Lemma 5.8. *With the notation as above, we have $r(k + 1) > r(k)$ for all $k \in \mathbb{N}$.*

Proof. Put $n = r(k) - 1$. Then there exists $x_1, \dots, x_n \in M_k(\mathbb{C})$ such that $X = [x_1, \dots, x_n] \neq 0$. In particular, if we write $X = (X_{ij})$ then there exist indices $1 \leq i, j \leq k$ such that $X_{ij} \neq 0$. Set

$$y_l = \begin{pmatrix} x_l & 0 \\ 0 & 0 \end{pmatrix} \in M_{k+1}(\mathbb{C}) \quad \text{for } l = 1, \dots, n,$$

and $y_{n+1} = e_{j,k+1}$ the standard matrix unit with 1 in the $(j, k + 1)$ -position. Then $y_{n+1}y_l = 0$ for all $1 \leq l \leq n$ and thus

$$[y_1, \dots, y_{n+1}] = [y_1, \dots, y_n]y_{n+1} = [y_1, \dots, y_n]e_{j,k+1} \neq 0.$$

We conclude $r(k + 1) \geq r(k) + 1$ as claimed. □

We are now ready to prove admissibility of irreducible unitary representations of complex quantum groups, compare [2].

Theorem 5.9. *Let \mathcal{H} be an irreducible unitary representation of G_q . Then the associated $\mathfrak{D}(G_q)$ -module $\text{HC}(\mathcal{H})$ is admissible, and so a Harish-Chandra module. More precisely, for any $\mu \in \mathbf{P}^+$ the multiplicity of the K_q -type $\text{HC}(\mathcal{H})^\mu$ in $\text{HC}(\mathcal{H})$ is at most $\dim(V(\mu))$.*

Proof. According to Proposition 3.25, the embedding $\mathfrak{D}(G_q) \rightarrow \mathcal{M}(\mathfrak{D}(K_q) \otimes \mathfrak{D}(K_q))$ restricts to an embedding

$$p_\mu \mathfrak{D}(G_q) p_\mu \rightarrow \prod_{\eta, \nu \in \mathbf{P}^+} \text{End}(p_\mu \cdot (V(\eta) \otimes V(\nu))).$$

Let $v_\eta \in V(\eta)$ and $v^\nu \in V(\nu)$ be the highest weight vector and lowest weight vector, respectively. Then $v_\eta \otimes v^\nu$ is a cyclic vector for $V(\eta) \otimes V(\nu)$ as a $U_q(\mathfrak{g})$ -module, and therefore the linear map $\text{Hom}_{U_q(\mathfrak{g})}(V(\eta) \otimes V(\nu), V(\mu)) \rightarrow V(\mu)$ sending f to $f(v_\eta \otimes v^\nu)$ is injective. We conclude that the dimension of $p_\mu \cdot (V(\eta) \otimes V(\nu))$ is at most $d = \dim(V(\mu))^2$. Therefore the algebra $A = p_\mu \mathfrak{D}(G_q) p_\mu$ is $r(d)$ -abelian.

Now let $\pi : C_\#^*(G_q) \rightarrow \mathbb{L}(\mathcal{H})$ be an irreducible unitary representation. Then the von Neumann algebra $\pi(C_\#^*(G_q))''$ equals $\mathbb{L}(\mathcal{H})$ by irreducibility, and hence $\pi(p_\mu)\pi(C_\#^*(G_q))''\pi(p_\mu) = \mathbb{L}(\pi(p_\mu)\mathcal{H})$. If we set $A = p_\mu \mathfrak{D}(G_q) p_\mu$ this means that the strong closure $\pi(A)''$ of $\pi(A) \subset \mathbb{L}(p_\mu\mathcal{H})$ is equal to $\mathbb{L}(\pi(p_\mu)\mathcal{H})$. Since $\pi(A)$ is $r(d)$ -abelian, the same holds for its strong closure $\mathbb{L}(\pi(p_\mu)\mathcal{H})$. We conclude that $\pi(p_\mu) \in \mathbb{L}(\mathcal{H})$ is a finite rank projection of rank at most d . Hence the multiplicity of $V(\mu)$ in \mathcal{H} is at most $\dim(V(\mu))$ as claimed. □

Let π be an irreducible unitary representation of G_q on the Hilbert space \mathcal{H} . According to Theorem 5.9, the image of any element of $\mathfrak{D}(K_q) \subset \mathfrak{D}(G_q)$ is a finite-rank operator on \mathcal{H} . This implies that $\mathfrak{D}(G_q)$ acts by finite-rank operators. In particular, the associated representation $\pi : C_\#^*(G_q) \rightarrow \mathbb{L}(\mathcal{H})$ takes values in the algebra of compact operators on \mathcal{H} .

Let us say that a locally compact quantum group G is type I if the full group C^* -algebra $C_\#^*(G)$ is a C^* -algebra of type I . As an immediate consequence of the above observations we obtain the following result.

Corollary 5.10. *Complex semisimple quantum groups are type I .*

We remark that the corresponding result for classical semisimple groups is due to Harish-Chandra, see [31].

5.3. Action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on K_q -types. In this section we collect some facts on the structure of integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules, following Section 9.1 in [22].

Let V be an essential $\mathfrak{D}(G_q)$ -module. We may also view V as an integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -module, see Lemma 5.5. By definition, the module V decomposes as a direct sum

$$V = \bigoplus_{\gamma \in \mathbf{P}^+} V^\gamma$$

of its K_q -isotypical components.

For $\sigma \in \mathbf{P}^+$ we denote

$$I^\sigma = \ker(\pi_\sigma) = \text{ann}(V(\sigma)) \subset U_q^{\mathbb{R}}(\mathfrak{k}),$$

where $\pi_\sigma : U_q^{\mathbb{R}}(\mathfrak{k}) \rightarrow \text{End}(V(\sigma))$ is the homomorphism corresponding to the action on $V(\sigma)$. By construction we have $U_q^{\mathbb{R}}(\mathfrak{k})/I^\sigma \cong \text{End}(V(\sigma))$. Notice that

$$V^\tau = \{v \in V \mid X \cdot v = 0 \text{ for all } X \in I^\tau\}.$$

For $\sigma, \tau \in \mathbf{P}^+$ we define

$$U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma} = \{X \in U_q^{\mathbb{R}}(\mathfrak{g}) \mid I^\tau X \subset U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma\}.$$

By construction, we have $U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma} \cdot V^\sigma \subset V^\tau$ for any $U_q^{\mathbb{R}}(\mathfrak{g})$ -module V .

Lemma 5.11. *Let $\sigma \in \mathbf{P}^+$. With the notation as above, the following holds.*

- a) $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma \subset U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma}$ for all $\tau \in \mathbf{P}^+$.
- b) $U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ is an integrable $\mathfrak{D}(G_q)$ -module with respect to left multiplication. Its decomposition into K_q -types is

$$U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma = \bigoplus_{\tau \in \mathbf{P}^+} U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma.$$

- c) For all $\tau \in \mathbf{P}^+$, the space $U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma}$ consists precisely of all elements of $U_q^{\mathbb{R}}(\mathfrak{g})$ that map V^σ into V^τ for any $U_q^{\mathbb{R}}(\mathfrak{g})$ -module V .

Proof. a) is obvious.

b) Using $U_q^{\mathbb{R}}(\mathfrak{k})/I^\sigma \cong \text{End}(V(\sigma))$, one sees that the generator $1 \in U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ is K_q -finite. It follows that $U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ is integrable. The τ -isotypical subspace is the subspace annihilated by I^τ , that is,

$$(U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma)^\tau = \{X + U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma \mid I^\tau X \subset U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma\} = U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma.$$

This yields the claim.

- c) follows from the decomposition into K_q -types in part b). □

We will mostly be interested in $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma, \sigma}$ for any given $\sigma \in \mathbf{P}^+$. It is easy to check that this is an algebra. It contains $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ as an obvious two-sided ideal, and $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ acts as zero on every V^σ . We will typically factor this ideal out. Accordingly, for any $U_q^{\mathbb{R}}(\mathfrak{g})$ -module V we get a map $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma, \sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma \rightarrow \text{End}(V^\sigma)$.

Proposition 5.12. *If V is a simple integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -module then V^σ is a simple $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma, \sigma}$ -module or zero.*

Proof. Let $v \in V^\sigma$ be nonzero. By simplicity of V we have $U_q^{\mathbb{R}}(\mathfrak{g}) \cdot v = V$. Moreover, using the above observations and part b) of Lemma 5.11 we get $U_q^{\mathbb{R}}(\mathfrak{g}) \cdot v = \sum_{\tau \in \mathbf{P}^+} U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma} \cdot v$. Since $U_q^{\mathbb{R}}(\mathfrak{g})^{\tau, \sigma} \cdot v \subset V^\tau$, we have $V^\sigma = (U_q^{\mathbb{R}}(\mathfrak{g}) \cdot v)^\sigma = U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma, \sigma} \cdot v$. □

The converse is not true, that is, simplicity of V^σ as a $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ -module does not imply that V is simple. Nonetheless, we will show the existence of a unique simple $U_q^{\mathbb{R}}(\mathfrak{g})$ -module associated to each simple $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ -module.

We start with the following construction. Let V be a Harish-Chandra module and $\sigma \in \mathbf{P}^+$ such that $V^\sigma \neq 0$. Moreover assume that $W \subset V^\sigma$ is a $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ -submodule. Then we define

$$W^{\min} = U_q^{\mathbb{R}}(\mathfrak{g}) \cdot W, \quad W^{\max} = \{v \in V \mid U_q^{\mathbb{R}}(\mathfrak{g}) \cdot v \cap V^\sigma \subseteq W\}.$$

The following Proposition shows that W^{\min} and W^{\max} are minimal and maximal, respectively, among the $U_q^{\mathbb{R}}(\mathfrak{g})$ -submodules $V' \subset V$ satisfying $(V')^\sigma = W$.

Proposition 5.13. *We have $(W^{\min})^\sigma = (W^{\max})^\sigma = W$, and for any $U_q^{\mathbb{R}}(\mathfrak{g})$ -submodule V' of V such that $(V')^\sigma = W$, we have $W^{\min} \subset V' \subset W^{\max}$.*

Proof. From Lemma 5.11 c) we have that $(U_q^{\mathbb{R}}(\mathfrak{g}) \cdot W) \cap V^\sigma = U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} \cdot W = W$, and it follows that $(W^{\min})^\sigma = (W^{\max})^\sigma = W$. If V' is as given, then $W^{\min} = U_q^{\mathbb{R}}(\mathfrak{g}) \cdot W \subset U_q^{\mathbb{R}}(\mathfrak{g}) \cdot V' \subset W^{\max}$, as claimed. \square

Proposition 5.14. *For $\sigma \in \mathbf{P}^+$ define*

$$\begin{aligned} \mathcal{M}_\sigma &= \{\text{maximal left ideals of } U_q^{\mathbb{R}}(\mathfrak{g}) \text{ containing } U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma\}, \\ \mathcal{L}_\sigma &= \{\text{maximal left ideals of } U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} \text{ containing } U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma\}. \end{aligned}$$

Then there is a bijective correspondence $\phi : \mathcal{M}_\sigma \rightarrow \mathcal{L}_\sigma$ given by

$$\phi(M) = M \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}.$$

The inverse of ϕ is given by

$$\phi^{-1}(L) = \{x \in U_q^{\mathbb{R}}(\mathfrak{g}) \mid U_q^{\mathbb{R}}(\mathfrak{g})x \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} \subset L\}.$$

Proof. Notice that we have a natural bijection between left ideals of $U_q^{\mathbb{R}}(\mathfrak{g})$ containing $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ and $U_q^{\mathbb{R}}(\mathfrak{g})$ -submodules of $U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$. If M is a maximal left ideal of $U_q^{\mathbb{R}}(\mathfrak{g})$ then $1 \notin M$, and therefore $M \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} \neq U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$.

Similarly, we have a natural bijection between left ideals of $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ containing $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ and $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ -submodules of $(U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma)^\sigma$.

Now it suffices to apply Proposition 5.13 to $V = U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$. \square

Proposition 5.15. *Let $\sigma \in \mathbf{P}^+$. Consider*

$$\begin{aligned} \mathcal{S}_\sigma &= \{\text{simple integrable } U_q^{\mathbb{R}}(\mathfrak{g})\text{-modules } V \text{ with } V^\sigma \neq 0, \text{ modulo isomorphism}\}, \\ \mathcal{T}_\sigma &= \{\text{simple } U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma\text{-modules, modulo isomorphism}\}. \end{aligned}$$

There is a natural bijective correspondence $\psi : \mathcal{S}_\sigma \rightarrow \mathcal{T}_\sigma$ given by

$$\psi(V) = V^\sigma.$$

Proof. The map ψ is well-defined by Proposition 5.12.

For injectivity, suppose that V, V' are simple integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules with nontrivial σ -isotypical component, and that $f : V^\sigma \rightarrow (V')^\sigma$ is an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$ -modules. Let $v \in V^\sigma$ be nonzero and set

- $L = \text{annihilator of } v \text{ in } U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma},$
- $M = \text{annihilator of } v \text{ in } U_q^{\mathbb{R}}(\mathfrak{g}),$
- $L' = \text{annihilator of } f(v) \text{ in } U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma},$
- $M' = \text{annihilator of } f(v) \text{ in } U_q^{\mathbb{R}}(\mathfrak{g}).$

These are all maximal left ideals of $U_q^{\mathbb{R}}(\mathfrak{g})$ or $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ respectively, containing $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$. But

$$M \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} = L = L' = M' \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma},$$

so by the correspondence of Proposition 5.14 we deduce $M = M'$.

For surjectivity, let W be a simple $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ -module whose annihilator contains $U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma$. Moreover let $w \in W$ be nonzero. Put

- $L = \text{annihilator of } w \text{ in } U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$,
- $M = \phi^{-1}(L)$, the maximal left ideal of $U_q^{\mathbb{R}}(\mathfrak{g})$ such that $M \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} = L$, which is obtained from Proposition 5.14,
- $V = U_q^{\mathbb{R}}(\mathfrak{g})/M$.

Then V is simple and by Lemma 5.11 b),

$$\begin{aligned} V^\sigma &= (U_q^{\mathbb{R}}(\mathfrak{g})/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma)^\sigma / (M/U_q^{\mathbb{R}}(\mathfrak{g})I^\sigma)^\sigma \\ &= U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} / (M \cap U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}) \\ &= U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} / L = W. \end{aligned}$$

This finishes the proof. \square

5.4. Principal series representations. In this section we define principal series representations for the complex quantum group G_q . As in the classical case, the principal series is key to the analysis of the representation theory of complex quantum groups.

5.4.1. *The definition of principal series representations.* Recall from Subsection 5.1.1 that associated to any $\mu \in \mathfrak{h}_q^*$ we have a character of $U_q^{\mathbb{R}}(\mathfrak{t}) = U_q(\mathfrak{h})$ given by $\chi_\mu(K_\nu) = q^{(\mu,\nu)}$. In the case where $\mu \in \mathbf{P}$ is an integral weight, this character can be obtained as the pairing of $U_q^{\mathbb{R}}(\mathfrak{t})$ with an appropriately chosen matrix coefficient in $\mathfrak{C}^\infty(K_q)$. In other words, the character χ_μ can be viewed as an element of the function algebra $\mathfrak{C}^\infty(T)$, compare Section 3.7. We will denote this function by $e^\mu \in \mathfrak{C}^\infty(T)$, so that

$$(K_\nu, e^\mu) = \chi_\mu(K_\nu) = q^{(\mu,\nu)},$$

for all $\nu \in \mathbf{P}$.

In analogy with the classical case, there is an ‘‘associated line bundle’’ \mathcal{E}_μ over the quantum flag variety $\mathcal{X}_q = K_q/T$. This bundle is defined via its space of sections

$$\Gamma(\mathcal{E}_\mu) = \{\xi \in \mathfrak{C}^\infty(K_q) \mid (\text{id} \otimes \pi_T)\Delta(\xi) = \xi \otimes e^\mu\},$$

where $\pi_T : \mathfrak{C}^\infty(K_q) \rightarrow \mathfrak{C}^\infty(T)$ denotes the canonical projection map. Note that $\Gamma(\mathcal{E}_\mu) \subset \mathfrak{C}^\infty(K_q)$ is equal to the subspace of weight μ with respect to the left $U_q^{\mathbb{R}}(\mathfrak{k})$ -action

$$X \rightharpoonup \xi = \xi_{(1)}(X, \xi_{(2)}).$$

The space $\Gamma(\mathcal{E}_\mu)$ is a left $\mathfrak{C}^\infty(K_q)$ -comodule with coaction given by Δ . For $\lambda \in \mathfrak{h}_q^*$ we define the twisted left adjoint representation of $\mathfrak{C}^\infty(K_q)$ on $\Gamma(\mathcal{E}_\mu)$ by

$$f \cdot \xi = f_{(1)} \xi S(f_{(3)}) (K_{2\rho+\lambda}, f_{(2)}).$$

This combination of action and coaction make $\Gamma(\mathcal{E}_\mu)$ into a Yetter-Drinfeld module over $\mathfrak{C}^\infty(K_q)$, see Definition 5.6, since we have

$$\begin{aligned} \Delta(f \cdot \xi) &= f_{(1)} \xi_{(1)} S(f_{(5)}) \otimes f_{(2)} \xi_{(2)} S(f_{(4)}) (K_{2\rho+\lambda}, f_{(3)}) \\ &= f_{(1)} \xi_{(1)} S(f_{(3)}) \otimes f_{(2)} \cdot \xi_{(2)} \end{aligned}$$

for all $f \in \mathfrak{C}^\infty(K_q)$, $\xi \in \Gamma(\mathcal{E}_\mu)$.

We point out that the factor of 2ρ in the above formula is included merely to force a shift in the parameter λ . This shift is chosen such that for purely imaginary λ the resulting Yetter-Drinfeld module is always unitary, see Section 5.10 below.

Definition 5.16. We write $\Gamma(\mathcal{E}_{\mu,\lambda})$ for the space $\Gamma(\mathcal{E}_\mu)$ equipped with the action and coaction of $\mathfrak{C}^\infty(K_q)$ as above, and call it the *principal series Yetter-Drinfeld module*, or principal series representation, with parameter $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$.

Equivalently, $\Gamma(\mathcal{E}_{\mu,\lambda})$ can be viewed as an essential module over $\mathfrak{D}(G_q)$ by transforming the coaction into the associated left $\mathfrak{D}(K_q)$ -action

$$x \cdot \xi = (\hat{S}(x), \xi_{(1)})\xi_{(2)}$$

for $x \in \mathfrak{D}(K_q)$ and $\xi \in \Gamma(\mathcal{E}_\mu)$. Likewise, replacing $x \in \mathfrak{D}(K_q)$ by $X \in U_q^{\mathbb{R}}(\mathfrak{k})$ in this formula we obtain the realization of $\Gamma(\mathcal{E}_{\mu,\lambda})$ as an integrable $U_q^{\mathbb{R}}(\mathfrak{g})$ -module, see Lemma 5.5.

We record the following formula for the multiplicities of K_q -types in $\Gamma(\mathcal{E}_{\mu,\lambda})$.

Lemma 5.17. *The multiplicities of the K_q -types in $\Gamma(\mathcal{E}_{\mu,\lambda})$ are given by*

$$[\Gamma(\mathcal{E}_{\mu,\lambda}) : V(\nu)] = \dim((V(\nu)^*)_\mu) = \dim(V(\nu)_{-\mu}),$$

for all $\nu \in \mathbf{P}^+$.

Proof. Note that we have

$$\mathfrak{C}^\infty(K_q) \cong \bigoplus_{\nu \in \mathbf{P}^+} (\text{End } V(\nu))^* \cong \bigoplus_{\nu \in \mathbf{P}^+} V(\nu)^* \otimes V(\nu) \cong \bigoplus_{\nu \in \mathbf{P}^+} V(-w_0\nu) \otimes V(-w_0\nu)^*,$$

where the left and right regular $U_q^{\mathbb{R}}(\mathfrak{k})$ -actions correspond to the actions on the left and right tensor factors, respectively. Here, the dual modules are equipped with the contragredient action. Therefore, we have an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{k})$ -modules

$$\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \bigoplus_{\nu \in \mathbf{P}^+} V(\nu) \otimes (V(\nu)^*)_\mu.$$

This proves the first equality. The second equality follows from $\dim((V(\nu)^*)_\mu) = \dim((V(\nu)_{-\mu})^*) = \dim(V(\nu)_{-\mu})$. \square

Thus, in our conventions, the principal series representation $\Gamma_{\mu,\lambda}$ has a nontrivial K_q -isotypical component of highest weight ν if and only if $\nu \in (-\mu)^+ + \mathbf{Q}^+$, where $(-\mu)^+$ denotes the unique dominant integral weight which is conjugate to $-\mu$ under the Weyl group action. In particular, $\Gamma(\mathcal{E}_{\mu,\lambda})$ has a minimal K_q -type of highest weight $(-\mu)^+$ and this minimal K_q -type occurs with multiplicity one.

5.4.2. *Compact versus noncompact pictures.* Definition 5.16 is referred to as the “compact picture” of the principal series representation. One may simply accept this definition without motivation and check that it satisfies the Yetter-Drinfeld condition in Definition 5.6. But for more insight, we note that principal series representations also admit an interpretation as representations of G_q induced from characters of the parabolic quantum subgroup $B_q = T \rtimes \hat{K}_q$.

Let $\mathbb{C}_{\mu,\lambda}$ denote the one dimensional representation of B_q with character $\chi_{\mu,\lambda}$ as in the definition of the Verma module $M(\mu, \lambda)$ in Section 5.1.1. By definition, the (algebraic) unitarily induced representation of G_q is

$$\text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda}) = \{\xi \in \mathfrak{C}^\infty(G_q) \mid (\text{id} \otimes \pi_{B_q})\Delta_{G_q}(\xi) = \xi \otimes (e^\mu \otimes K_{2\rho+\lambda})\}.$$

Here $K_{2\rho+\lambda}$ is viewed as multiplier of $\mathfrak{D}(K_q)$ inside $\mathfrak{C}^\infty(G_q)$. Again, the shift by 2ρ is included to ensure a suitable parametrization with inner products later on.

Observe that the coproduct of $\mathfrak{C}_c^\infty(G_q)$ induces a left $\mathfrak{D}(G_q)$ -module structure on $\text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$ according to the formula

$$x \cdot \xi = (\hat{S}_{G_q}(x), \xi_{(1)})\xi_{(2)}$$

for $x \in \mathfrak{D}(G_q)$ and $\xi \in \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$.

On the other hand, recall that

$$\Gamma(\mathcal{E}_{\mu,\lambda}) = \{\xi \in \mathfrak{C}^\infty(K_q) \mid (\text{id} \otimes \pi_T)\Delta(\xi) = \xi \otimes e^\mu\}$$

admits a $\mathfrak{D}(G_q)$ -action corresponding to the Yetter-Drinfeld structure from Definition 5.16.

Lemma 5.18. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. The linear maps*

$$\text{ext} : \Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda}); \quad \text{ext}(\xi) = \xi \otimes K_{2\rho+\lambda}$$

and

$$\text{res} : \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu,\lambda}); \quad \text{res}(\sigma) = (\text{id} \otimes \hat{\epsilon})(\sigma)$$

are well-defined and are inverse isomorphisms of $\mathfrak{D}(K_q)$ -modules. Moreover, the Yetter-Drinfeld action of $\mathfrak{C}^\infty(K_q)$ on $\Gamma(\mathcal{E}_{\mu,\lambda})$ corresponds to the natural action of $\mathfrak{C}^\infty(K_q) \subset \mathcal{M}(\mathfrak{D}(G_q))$ on $\text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$.

Proof. Let us check that $\text{ext}(\xi)$ for $\xi \in \Gamma(\mathcal{E}_\mu)$ satisfies the correct invariance properties. Recall that we can write the multiplicative unitary W in the form

$$W = \sum_{\nu, i, j} u_{ij}^\nu \otimes \omega_{ij}^\nu$$

where (u_{ij}^ν) is a basis of matrix units for $\mathfrak{C}^\infty(K_q)$ and (ω_{ij}^ν) is its dual basis. Using the definition of Δ_{G_q} from Subsection 3.4.1, we compute

$$\begin{aligned} (\text{id} \otimes \pi_{B_q})\Delta_{G_q}(\text{ext}(\xi)) &= \sum_{\nu, \eta, i, j, r, s} \xi_{(1)} \otimes \omega_{ij}^\nu K_{2\rho+\lambda} \omega_{rs}^\eta \otimes \pi_{B_q}(u_{ij}^\nu \xi_{(2)}) S(u_{rs}^\eta) \otimes K_{2\rho+\lambda} \\ &= \sum_{\nu, i, j, s} \xi \otimes \omega_{ij}^\nu K_{2\rho+\lambda} \omega_{js}^\nu \otimes \pi_T(u_{ij}^\nu) e^\mu \pi_T(S(u_{js}^\nu)) \otimes K_{2\rho+\lambda} \\ &= \sum_{\nu, j} \xi \otimes \omega_{jj}^\nu K_{2\rho+\lambda} \omega_{jj}^\nu \otimes e^\mu \pi_T(u_{jj}^\nu) S(\pi_T(u_{jj}^\nu)) \otimes K_{2\rho+\lambda} \\ &= \sum_{\nu, j} \xi \otimes \omega_{jj}^\nu K_{2\rho+\lambda} \omega_{jj}^\nu \otimes e^\mu \otimes K_{2\rho+\lambda} \\ &= \xi \otimes K_{2\rho+\lambda} \otimes e^\mu \otimes K_{2\rho+\lambda} = \text{ext}(\xi) \otimes e^\mu \otimes K_{2\rho+\lambda} \end{aligned}$$

in $\mathcal{M}(\mathfrak{C}_c^\infty(G_q) \otimes \mathfrak{C}_c^\infty(B_q))$. Hence $\text{ext}(\xi)$ satisfies the invariance condition in the definition of $\text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$.

Similarly, for $\sigma \in \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$ the element $\text{res}(\sigma) = (\text{id} \otimes \hat{\epsilon})(\sigma)$ satisfies

$$\begin{aligned} (\text{id} \otimes \pi_T)\Delta(\text{res}(\sigma)) &= (\text{id} \otimes \hat{\epsilon} \otimes \pi_T \otimes \hat{\epsilon})\Delta_{G_q}(\sigma) \\ &= (\text{id} \otimes \hat{\epsilon} \otimes \text{id} \otimes \hat{\epsilon})(\text{id} \otimes \pi_{B_q})\Delta_{G_q}(\sigma) \\ &= (\text{id} \otimes \hat{\epsilon})(\sigma) \otimes e^\mu \hat{\epsilon}(K_{2\rho+\lambda}) \\ &= \text{res}(\sigma) \otimes e^\mu \end{aligned}$$

inside $\mathfrak{C}^\infty(K_q) \otimes \mathfrak{C}^\infty(T)$.

It is clear that $\text{res} \circ \text{ext}$ is the identity on $\Gamma(\mathcal{E}_{\mu,\lambda})$. For the reverse composition, we begin by observing that

$$(\text{id}_{\mathfrak{C}^\infty(K_q)} \otimes \hat{\epsilon} \otimes \epsilon \otimes \text{id}_{\mathfrak{D}(K_q)})\Delta_{G_q} = \text{id}_{\mathfrak{C}_c^\infty(G_q)}.$$

Then for $\sigma \in \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$ we obtain

$$\begin{aligned}\sigma &= (\text{id} \otimes \hat{\epsilon} \otimes \epsilon \otimes \text{id}) \Delta_{G_q}(\sigma) \\ &= (\text{id} \otimes \hat{\epsilon} \otimes \epsilon \otimes \text{id})(\text{id} \otimes \pi_{B_q}) \Delta_{G_q}(\sigma) \\ &= (\text{id} \otimes \hat{\epsilon} \otimes \epsilon \otimes \text{id})(\sigma \otimes (e^\mu \otimes K_{2\rho+\lambda})) \\ &= (\text{id} \otimes \hat{\epsilon})(\sigma) \otimes K_{2\rho+\lambda} \\ &= \text{ext} \circ \text{res}(\sigma).\end{aligned}$$

Therefore ext and res are isomorphisms.

Finally, we consider the actions. The action of $x = t \bowtie a \in \mathfrak{D}(G_q)$ on $\sigma \in \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$ is given by

$$\begin{aligned}(t \bowtie a) \cdot \sigma &= (\hat{S}_{G_q}(t \bowtie a), \sigma_{(1)}) \sigma_{(2)} \\ &= (t \bowtie a, S_{G_q}^{-1}(\sigma_{(1)})) \sigma_{(2)} \\ &= (t \bowtie a, (S^{-1} \otimes \hat{S}^{-1})(W \sigma_{(1)} W^{-1})) \sigma_{(2)} \\ &= (\hat{S}(t) \bowtie S(a), W \sigma_{(1)} W^{-1}) \sigma_{(2)},\end{aligned}$$

according to the formula before Proposition 3.19. To transfer this to the compact picture, we consider $\xi \in \Gamma(\mathcal{E}_{\mu,\lambda})$ and calculate

$$\begin{aligned}\text{res}((t \bowtie a) \cdot \text{ext}(\xi)) &= \sum_{\nu,\eta,i,j,r,s} (\hat{S}(t) \bowtie S(a), W(\xi_{(1)} \otimes \omega_{ij}^\nu K_{2\rho+\lambda} \omega_{rs}^\eta) W^{-1}) u_{ij}^\nu \xi_{(2)} S(u_{rs}^\eta) \\ &= \sum_{\substack{\nu,\eta,\alpha,\beta, \\ i,j,r,s,p,q,m,n}} (\hat{S}(t) \bowtie S(a), u_{pq}^\alpha \xi_{(1)} u_{mn}^\beta \otimes \omega_{pq}^\alpha \omega_{ij}^\nu K_{2\rho+\lambda} \omega_{rs}^\eta \hat{S}^{-1}(\omega_{mn}^\beta)) u_{ij}^\nu \xi_{(2)} S(u_{rs}^\eta) \\ &= \sum_{\nu,\eta,i,j,r,s} (\hat{S}(t_{(2)}), \xi_{(1)}) (\hat{S}(t_{(1)}) \omega_{ij}^\nu K_{2\rho+\lambda} \omega_{rs}^\eta \hat{S}^{-1}(\hat{S}(t_{(3)})), a) u_{ij}^\nu \xi_{(2)} S(u_{rs}^\eta) \\ &= (\hat{S}(t_{(1)}), a_{(1)})(t_{(3)}, a_{(5)}) a_{(2)}(t_{(2)} \cdot \xi) S(a_{(4)})(K_{2\rho+\lambda}, a_{(3)}) \\ &= (\hat{S}(t_{(1)}), a_{(1)})(t_{(3)}, a_{(3)}) a_{(2)} \cdot (t_{(2)} \cdot \xi) \\ &= t \cdot a \cdot \xi,\end{aligned}$$

where the final lines use the actions of $\mathfrak{D}(K_q)$ and $\mathfrak{C}^\infty(K_q)$ on $\Gamma(\mathcal{E}_{\mu,\lambda})$ given by

$$t \cdot \xi = (\hat{S}(t), \xi_{(1)}) \xi_{(2)}$$

for $t \in \mathfrak{D}(K_q)$ and

$$a \cdot \xi = a_{(1)} \xi S(a_{(3)})(K_{2\rho+\lambda}, a_{(2)}),$$

for $a \in \mathfrak{C}^\infty(K_q)$, respectively. \square

In our conventions, the principal series module $\Gamma(\mathcal{E}_{0,-2\rho})$ corresponds to the representation of G_q induced from the trivial representation of B_q when disregarding the ρ -shift. Geometrically, this is the algebra of functions on the flag variety G_q/B_q equipped with the regular representation. In particular, this algebra contains the constant function $1 = 1 \otimes 1 \in \mathfrak{C}^\infty(G_q)$, which is invariant under the $\mathfrak{D}(G_q)$ -action,

$$(x \bowtie f) \cdot 1 = (\hat{S}_{G_q}(x \bowtie f), 1) 1 = \hat{\epsilon}_{G_q}(x \bowtie f) 1.$$

That is, $\Gamma(\mathcal{E}_{0,-2\rho})$ contains the one-dimensional trivial representation of G_q as a submodule.

5.4.3. *The action of the centre on the principal series.* Recall from Subsection 3.4.2 that the quantum group G_q admits a finite group of one-dimensional unitary representations indexed by the centre $Z \cong \mathbf{P}^\vee/\mathbf{Q}^\vee$ of G . Explicitly, for each $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$ we have the unitary character $\hat{\epsilon}_\gamma : \mathfrak{D}(G_q) \rightarrow \mathbb{C}$ defined by

$$\hat{\epsilon}_\gamma(x \bowtie f) = \hat{\epsilon}(x)(K_{i\hbar^{-1}\gamma}, f),$$

see Definition 3.21. These characters induce an action of Z on the class of Harish-Chandra modules for G_q whereby $\gamma \in Z$ sends a Harish-Chandra module H to the module $H \otimes \mathbb{C}_{i\hbar^{-1}\gamma}$, with $\mathbb{C}_{i\hbar^{-1}\gamma}$ being the one-dimensional module with action $\hat{\epsilon}_\gamma$. On the principal series, this action is given as follows.

Lemma 5.19. *For any $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and $\gamma \in \mathbf{P}^\vee/\mathbf{Q}^\vee$, we have $\Gamma(\mathcal{E}_{\mu, \lambda}) \otimes \mathbb{C}_{i\hbar^{-1}\gamma} = \Gamma(\mathcal{E}_{\mu, \lambda + i\hbar^{-1}\gamma})$.*

Proof. Using the compact picture, $\Gamma(\mathcal{E}_{\mu, \lambda}) \otimes \mathbb{C}_{i\hbar^{-1}\gamma}$ and $\Gamma(\mathcal{E}_{\mu, \lambda + i\hbar^{-1}\gamma})$ are naturally identified as $\mathfrak{D}(K_q)$ -modules. It remains to compute the Yetter-Drinfeld action of $f \in \mathfrak{C}^\infty(K_q)$ on $\xi \otimes 1 \in \Gamma(\mathcal{E}_{\mu, \lambda}) \otimes \mathbb{C}_{i\hbar^{-1}\gamma}$. Using the fact that $K_{i\hbar^{-1}\gamma} \rightharpoonup f = f \leftarrow K_{i\hbar^{-1}\gamma}$ we obtain

$$\begin{aligned} f \cdot (\xi \otimes 1) &= f_{(1)} \xi S(f_{(3)}) (K_{2\rho + \lambda}, f_{(2)}) \otimes (K_{i\hbar^{-1}\gamma}, f_{(4)}) \\ &= f_{(1)} \xi S(f_{(4)}) (K_{2\rho + \lambda}, f_{(2)}) (K_{i\hbar^{-1}\gamma}, f_{(3)}) \otimes 1 \\ &= f_{(1)} \xi S(f_{(3)}) (K_{2\rho + \lambda + i\hbar^{-1}\gamma}, f_{(2)}) \otimes 1. \end{aligned}$$

This completes the proof. \square

This Z -action is responsible for the appearance of certain representations of G_q which do not have classical analogues. For instance, in the case of $SL_q(2, \mathbb{C})$, Podleś and Woronowicz observed the appearance of a nontrivial one-dimensional representation and of a pair of complementary series [63], as compared to the classical group $SL(2, \mathbb{C})$, which has only the trivial one-dimensional representation and only one complementary series. This is related to the action of $Z \cong \mathbb{Z}_2$.

In general, the various representations in a single Z -orbit all become isomorphic upon restriction to the ‘‘connected component’’ G_q^0 described in Subsection 3.4.2, since the characters $\hat{\epsilon}_\gamma$ are all trivial on $\mathfrak{D}(G_q^0)$. As a consequence, the natural parameter space for the principal series representations of G_q^0 is $\mathbf{P} \times \mathfrak{h}^*/i\hbar^{-1}\mathbf{P}^\vee$, compare the remarks at the end of Subsection 5.1.1.

5.4.4. *Duality for principal series modules.* One can introduce a general notion of the dual of a Harish-Chandra module.

Definition 5.20. Let V be a Harish-Chandra module over G_q , see Definition 5.7. We equip the dual space V^* with the $\mathfrak{D}(G_q)$ -action given by

$$(x \cdot \varphi)(v) = \varphi(\hat{S}_{G_q}(x) \cdot v).$$

for $x \in \mathfrak{D}(G_q)$, $\varphi \in V^*$, $v \in V$. The locally finite part $F(V^*) \subset V^*$ for the $\mathfrak{D}(K_q)$ -action is again a Harish-Chandra module over G_q , called the dual Harish-Chandra module.

Note that this defines an exact contravariant functor on the category of Harish-Chandra modules over G_q with $\mathfrak{D}(G_q)$ -linear morphisms. Note also that the double dual of V is isomorphic to V . Explicitly, the double dual of V is naturally identified with V as a vector space, equipped with the $\mathfrak{D}(G_q)$ -action given by precomposing the original action by $\hat{S}_{G_q}^2$. Using the relation

$$(K_{2\rho} \bowtie 1)(x \bowtie f)(K_{-2\rho} \bowtie 1) = \hat{S}^2(x) \bowtie S^2(f) = \hat{S}_{G_q}^2(x \bowtie f)$$

for all $x \bowtie f \in \mathfrak{D}(G_q)$, we see that the two actions are intertwined by the action of $K_{2\rho}$.

Lemma 5.21. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. The Haar integral ϕ on $L^2(K_q)$ induces a non-degenerate bilinear pairing*

$$\Gamma(\mathcal{E}_{\mu, \lambda}) \times \Gamma(\mathcal{E}_{-\mu, -\lambda}) \rightarrow \mathbb{C}; \quad (\xi, \eta) = \phi(\xi\eta)$$

which is $U_q^{\mathbb{R}}(\mathfrak{g})$ -invariant in the sense that for all $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$,

$$((X \bowtie f) \cdot \xi, \eta) = (\xi, \hat{S}_{G_q}(X \bowtie f) \cdot \eta).$$

Proof. Non-degeneracy and the fact that $(\hat{S}(X) \cdot \xi, \eta) = (\xi, X \cdot \eta)$ for $X \in U_q^{\mathbb{R}}(\mathfrak{k})$ follow from the basic properties of the Haar integral. For $f, g \in \mathfrak{C}^\infty(K_q)$ we have $S^2(f) = K_{2\rho} \rightharpoonup f \leftarrow K_{-2\rho}$, where we are using the notation

$$X \rightharpoonup f = (X, f_{(2)})f_{(1)} \quad f \leftarrow X = (X, f_{(1)})f_{(2)}.$$

We also have the modular property $\phi(fg) = \phi((K_{2\rho} \rightharpoonup g \leftarrow K_{2\rho})f)$, see Subsection 3.2.3 and Lemma 3.17. Therefore,

$$\begin{aligned} (f \cdot \xi, \eta) &= (K_\lambda, f_{(2)}) \phi(K_{2\rho} \rightharpoonup f_{(1)} \xi S(f_{(3)}) \eta) \\ &= (K_\lambda, f_{(2)}) \phi(\xi S(f_{(3)}) \eta f_{(1)} \leftarrow K_{-2\rho}) \\ &= (K_{-\lambda}, S(f_{(2)})) \phi(\xi S(f_{(3)}) \eta f_{(1)} \leftarrow K_{-2\rho}) \\ &= (K_{-\lambda}, S(f_{(2)})) \phi(\xi S(f_{(3)}) \eta K_{-2\rho} \rightharpoonup S^2(f_{(1)})) \\ &= (\xi, S(f) \cdot \eta). \end{aligned}$$

This completes the proof. \square

It follows that we have an isomorphism between $\Gamma(\mathcal{E}_{\mu, \lambda})$ and the dual Harish-Chandra module of $\Gamma(\mathcal{E}_{-\mu, -\lambda})$, in the sense of Definition 5.20.

5.4.5. *Relation between principal series modules and Verma modules.* Recall from Lemma 3.27 and Proposition 3.24 that we have an embedding $\iota : U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ of algebras and an embedding $\mathcal{O}(G_q) \bowtie \mathcal{O}(G_q) \rightarrow \mathfrak{C}^\infty(G_q)$ of multiplier Hopf $*$ -algebras. These embeddings are compatible with the pairings

$$\begin{aligned} U_q^{\mathbb{R}}(\mathfrak{g}) \times \mathfrak{C}^\infty(G_q) &\rightarrow \mathbb{C} \\ (X \bowtie c, a \otimes t) &= (X, a)(c, t) \end{aligned}$$

for $a \otimes t \in \mathfrak{C}^\infty(K_q) \otimes \mathcal{D}(K_q) \subset \mathfrak{C}^\infty(G_q)$ and $X \bowtie c \in U_q^{\mathbb{R}}(\mathfrak{k}) \bowtie \mathfrak{C}^\infty(K_q) = U_q^{\mathbb{R}}(\mathfrak{g})$, and

$$\begin{aligned} (U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})) \times (\mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)) &\rightarrow \mathbb{C}, \\ (X \otimes Y, f \bowtie g) &= (X, g)(Y, f) \end{aligned}$$

for $X \otimes Y \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ and $f \bowtie g \in \mathcal{O}(G_q) \bowtie \mathcal{O}(G_q)$, respectively, see the remark at the end of Section 3.6.

Lemma 5.22. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and consider the Yetter-Drinfeld module $\Gamma(\mathcal{E}_{\mu, \lambda}) \cong \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu, \lambda}) \subset \mathfrak{C}^\infty(G_q)$. The pairing $U_q^{\mathbb{R}}(\mathfrak{g}) \times \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu, \lambda}) \rightarrow \mathbb{C}$ descends to a well-defined bilinear pairing*

$$M(\mu, 2\rho + \lambda) \times \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu, \lambda}) \rightarrow \mathbb{C}.$$

This induces an injective $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear map

$$\Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow M(\mu, 2\rho + \lambda)^*$$

where $M(\mu, 2\rho + \lambda)^*$ is equipped with the action given by

$$(X \cdot T)(m) = T(\hat{S}_{G_q}(X) \cdot m)$$

for $X \in U_q^{\mathbb{R}}(\mathfrak{g})$, $T \in M(\mu, 2\rho + \lambda)^*$, $m \in M(\mu, 2\rho + \lambda)$.

Proof. Let

$$\begin{aligned} X \bowtie c &\in U_q^{\mathbb{R}}(\mathfrak{k}) \bowtie \mathfrak{C}^\infty(K_q) = U_q^{\mathbb{R}}(\mathfrak{g}), \\ Y \bowtie d &\in U_q^{\mathbb{R}}(\mathfrak{t}) \bowtie \mathfrak{C}^\infty(K_q) = U_q^{\mathbb{R}}(\mathfrak{b}), \end{aligned}$$

and $a \in \Gamma(\mathcal{E}_\mu)$, so that

$$\text{ext}(a) = a \otimes K_{2\rho+\lambda} \in \text{ind}_{B_q^+}^{G_q}(\mathbb{C}_{\mu,\lambda}).$$

Note that since $Y \in U_q^{\mathbb{R}}(\mathfrak{t})$ we have

$$(XY, a) = (X \otimes Y, (\text{id} \otimes \pi_T)\Delta(a)) = (X, a)\chi_\mu(Y).$$

Using the fact that $U_q^{\mathbb{R}}(\mathfrak{t})$ is commutative and cocommutative, we obtain

$$\begin{aligned} &((X \bowtie c)(Y \bowtie d), a \otimes K_{2\rho+\lambda}) \\ &= \left((Y_{(1)}, c_{(1)}) (\hat{S}(Y_{(3)}), c_{(3)}) XY_{(2)} \bowtie c_{(2)}d, a \otimes K_{2\rho+\lambda} \right) \\ &= (Y_{(1)}, c_{(1)}) (\hat{S}(Y_{(3)}), c_{(3)}) (XY_{(2)}, a) (c_{(2)}d, K_{2\rho+\lambda}) \\ &= (Y_{(1)}, c_{(1)}) (\hat{S}(Y_{(3)}), c_{(3)}) (X, a)\chi_\mu(Y_{(2)}) (K_{-2\rho-\lambda}, c_{(2)}) (K_{-2\rho-\lambda}, d) \\ &= (Y_{(1)}K_{-2\rho-\lambda}\hat{S}(Y_{(2)}), c) (X, a)\chi_\mu(Y_{(3)}) (K_{-2\rho-\lambda}, d) \\ &= (X, a)\chi_\mu(Y) (c, K_{2\rho+\lambda}) (d, K_{2\rho+\lambda}) \\ &= \chi_{\mu, 2\rho+\lambda}(Y \bowtie d)(X \bowtie c, a \otimes K_{2\rho+\lambda}). \end{aligned}$$

This proves that the pairing is well-defined.

For the final statement we note that, with the action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on $\mathfrak{C}^\infty(G_q)$ given by

$$X \cdot f = (\hat{S}_{G_q}(X), f_{(1)})f_{(2)},$$

the pairing $U_q^{\mathbb{R}}(\mathfrak{g}) \times \mathfrak{C}^\infty(G_q) \rightarrow \mathbb{C}$ satisfies

$$(XY, f) = (Y, \hat{S}_{G_q}^{-1}(X) \cdot f).$$

Therefore, the map

$$\mathfrak{C}^\infty(G_q) \rightarrow U_q^{\mathbb{R}}(\mathfrak{g})^*; \quad f \mapsto (\bullet, f)$$

is $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear, and this restricts to the map stated in the lemma. Moreover, this map is injective by the nondegeneracy of the pairing. \square

The above lemma will allow us to identify the principal series module $\Gamma(\mathcal{E}_{\mu,\lambda})$ with the locally finite part of $(M(l) \otimes M(r))^*$ for well-chosen values of l and r in \mathfrak{h}_q^* . To do so, we need to clarify the relationship between $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules and $FU_q(\mathfrak{g})$ -bimodules with compatible adjoint action of $U_q(\mathfrak{g})$ as in Definition 4.26.

To begin with, suppose V is any $U_q^{\mathbb{R}}(\mathfrak{g})$ -module. The diagonal embedding $\iota : U_q^{\mathbb{R}}(\mathfrak{g}) \rightarrow U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ of Lemma 3.27 gives an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{g})$ onto the subalgebra $(FU_q(\mathfrak{g}) \otimes 1)\hat{\Delta}(U_q(\mathfrak{g})) = (1 \otimes \hat{S}^{-1}(FU_q(\mathfrak{g})))\hat{\Delta}(U_q(\mathfrak{g}))$. Therefore, we can define an $FU_q(\mathfrak{g})$ -bimodule structure on V by

$$Y \cdot v = \iota^{-1}(Y \otimes 1)v, \quad v \cdot Z = \iota^{-1}(1 \otimes \hat{S}^{-1}(Z))v.$$

Moreover, we have a $U_q(\mathfrak{g})$ -action on V given by the action of $U_q(\mathfrak{g}) = U_q^{\mathbb{R}}(\mathfrak{k}) \subset U_q^{\mathbb{R}}(\mathfrak{g})$. As usual, we refer to this $U_q(\mathfrak{g})$ -action as the adjoint action and denote it by $X \rightarrow v$ for $X \in U_q(\mathfrak{g})$ and $v \in V$. Note that for $X \in FU_q(\mathfrak{g})$ we have

$$\begin{aligned} (X_{(1)} \rightarrow v) \cdot X_{(2)} &= \iota^{-1}((1 \otimes \hat{S}^{-1}(X_{(2)}))\hat{\Delta}(X_{(1)}))v \\ &= \iota^{-1}(X \otimes 1)v \\ &= X \cdot v, \end{aligned}$$

so the actions indeed satisfy the compatibility condition of Definition 4.26.

Next, recall from Subsection 4.5.1 that we defined left, right and adjoint actions of $U_q(\mathfrak{g})$ on $(M \otimes N)^*$ by

$$\begin{aligned} (Y \cdot \varphi \cdot Z)(m \otimes n) &= \varphi(Z \cdot m \otimes \tau(Y) \cdot n), \\ (X \rightarrow \varphi)(m \otimes n) &= \varphi(\hat{S}(X_{(2)}) \cdot m \otimes \tau(X_{(1)}) \cdot n), \end{aligned}$$

for $X, Y, Z \in U_q(\mathfrak{g})$, where τ is the algebra antiautomorphism from Lemma 2.16. The locally finite part of $(M \otimes N)^*$ with respect to the adjoint action is denoted $F((M \otimes N)^*)$, and the above actions restrict to make $F((M \otimes N)^*)$ into an $FU_q(\mathfrak{g})$ -bimodule with compatible adjoint action of $U_q(\mathfrak{g})$, see Lemma 4.27 and the discussion preceding it.

These actions correspond to a $U_q^{\mathbb{R}}(\mathfrak{g})$ -module structure on $F((M \otimes N)^*)$. Explicitly, we can define an action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on $(M \otimes N)^*$ by first embedding $U_q^{\mathbb{R}}(\mathfrak{g})$ into $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ via the map ι and then letting $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ act on $(M \otimes N)^*$ by

$$\begin{aligned} ((Y \otimes Z)\varphi)(m \otimes n) &= (Y \cdot \varphi \cdot \hat{S}(Z))(m \otimes n) \\ &= \varphi(\hat{S}(Z) \cdot m \otimes \tau(Y) \cdot n) \end{aligned}$$

for $Y, Z \in U_q(\mathfrak{g})$. By the above discussion, this $U_q^{\mathbb{R}}(\mathfrak{g})$ -action restricts to the locally finite part $F((M \otimes N)^*)$.

With this in place, we have the following result.

Proposition 5.23. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and let $l, r \in \mathfrak{h}_q^*$ such that*

$$\mu = l - r, \quad \lambda + 2\rho = -l - r.$$

Then there exists an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules

$$\Gamma(\mathcal{E}_{\mu, \lambda}) \cong F((M(l) \otimes M(r))^*),$$

where $F((M(l) \otimes M(r))^)$ is equipped with the $U_q^{\mathbb{R}}(\mathfrak{g})$ -module structure defined above.*

Proof. By Corollary 5.3 we can identify the Verma module $M(\mu, \lambda + 2\rho)$ with $M(l) \otimes M(r)$ as a $U_q^{\mathbb{R}}(\mathfrak{g})$ -module if we equip $M(l) \otimes M(r)$ with the $U_q^{\mathbb{R}}(\mathfrak{g})$ -action given by

$$X \cdot (m \otimes n) = ((\text{id} \otimes \theta') \circ \iota'(X))(m \otimes n),$$

for $X \in U_q^{\mathbb{R}}(\mathfrak{g})$. Thus, from Lemma 5.22 we obtain an injective linear map $\gamma : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow (M(l) \otimes M(r))^*$.

By the $U_q^{\mathbb{R}}(\mathfrak{g})$ -linearity of the map in Lemma 5.22, we see that for any $X \in U_q^{\mathbb{R}}(\mathfrak{g})$ and $f \in \Gamma(\mathcal{E}_{\mu, \lambda})$ we have

$$\gamma(X \cdot f)(m \otimes n) = \gamma(f)(\hat{S}_{G_q}(X) \cdot (m \otimes n))$$

for all $m \otimes n \in M(l) \otimes M(r)$, where the action of $\hat{S}_{G_q}(X)$ on the right-hand side is given by

$$\begin{aligned} \hat{S}_{G_q}(X) \cdot (m \otimes n) &= (\text{id} \otimes \theta') \circ \sigma \circ (\hat{S} \otimes \hat{S}) \circ \iota(X)(m \otimes n) \\ &= ((\hat{S} \otimes \tau)(\sigma \circ \iota(X)))(m \otimes n), \end{aligned}$$

see Corollary 5.3. Thus, if we write $\iota(X) = \sum_j Y_j \otimes Z_j$ we get

$$\gamma(X \cdot f)(m \otimes n) = \sum_j \gamma(f)(\hat{S}(Z_j) \cdot m \otimes \tau(Y_j) \cdot n).$$

Comparing this with the formula for the $U_q^{\mathbb{R}}(\mathfrak{g})$ -action on $(M \otimes N)^*$ given just before the proposition, we see that γ is $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear.

Since the action of $U_q^{\mathbb{R}}(\mathfrak{g}) = U_q(\mathfrak{g})$ on $\Gamma(\mathcal{E}_{\mu, \lambda})$ is locally finite and γ is $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear, the image $\text{im}(\gamma)$ is contained in the locally finite part of $(M(l) \otimes M(r))^*$.

Moreover, γ is surjective onto each K_q -type, since Proposition 4.35 and Lemma 5.17 show that for every $\nu \in \mathbf{P}^+$,

$$[F((M(l) \otimes M(r))^*) : V(\nu)] = \dim(V(\nu)_{-\mu}) = [\Gamma(\mathcal{E}_{\mu,\lambda}) : V(\nu)].$$

This completes the proof. □

Using Lemma 4.27, we can also reformulate Proposition 5.23 in terms of the spaces $F \operatorname{Hom}(M(l), M(r)^\vee)$. Here we will state the result in terms of $FU_q(\mathfrak{g})$ -bimodules with compatible $U_q(\mathfrak{g})$ -action rather than $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules, using the discussion preceding Proposition 5.23.

Corollary 5.24. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and let $l, r \in \mathfrak{h}_q^*$ such that*

$$\mu = l - r, \quad \lambda + 2\rho = -l - r.$$

There is an isomorphism of $FU_q(\mathfrak{g})$ -bimodules

$$\Gamma(\mathcal{E}_{\mu,\lambda}) \cong F \operatorname{Hom}(M(l), M(r)^\vee)$$

which respects the adjoint actions of $U_q(\mathfrak{g})$. Here $F \operatorname{Hom}(M(l), M(r)^\vee)$ is equipped with the actions

$$\begin{aligned} (Y \cdot \varphi \cdot Z)(m)(n) &= \varphi(Z \cdot m)(\tau(Y) \cdot n), \\ (X \rightarrow \varphi)(m)(n) &= \varphi(\hat{S}(X_{(2)}) \cdot m)(\tau(X_{(1)}) \cdot n) \end{aligned}$$

for $Y, Z \in FU_q(\mathfrak{g})$, $X \in U_q(\mathfrak{g})$, $m \in M(l)$ and $n \in M(r)$.

In the sequel we will often identify $\Gamma(\mathcal{E}_{\mu,\lambda})$ and $F \operatorname{Hom}(M(l), M(r)^\vee)$, with the parameters matching as in Corollary 5.24.

As we noted in the remarks after Corollary 5.3, the parameters (l, r) and (l', r') correspond to the same pair $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ iff they differ by the diagonal action of the translation subgroup $\mathbf{Y}_q = \frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee/i\hbar^{-1}\mathbf{Q}^\vee$ of the extended Weyl group $\hat{W} = \mathbf{Y}_q \rtimes W$, that is, iff

$$(l', r') = (l + \frac{1}{2}i\hbar^{-1}\alpha^\vee, r + \frac{1}{2}i\hbar^{-1}\alpha^\vee)$$

for some $\alpha^\vee \in \mathbf{Q}^\vee$.

We also note that by applying Lemma 4.28, we have an isomorphism of $U_q(\mathfrak{g})$ -compatible $FU_q(\mathfrak{g})$ -bimodules

$$\Gamma(\mathcal{E}_{\mu,\lambda}) \cong F \operatorname{Hom}(M(r), M(l)^\vee)$$

where now $F \operatorname{Hom}(M(r), M(l)^\vee)$ should be equipped with the actions

$$\begin{aligned} (Y \cdot \varphi \cdot Z)(m)(n) &= \varphi(\tau(Y) \cdot m)(Z \cdot n), \\ (X \rightarrow \varphi)(m)(n) &= \varphi(\tau(X_{(1)}) \cdot m)(\hat{S}(X_{(2)}) \cdot n) \end{aligned}$$

for $Y, Z \in FU_q(\mathfrak{g})$ and $X \in U_q(\mathfrak{g})$.

5.5. An equivalence of categories. In this section we discuss the relation between certain categories of $FU_q(\mathfrak{g})$ -bimodules, which are related to the Harish-Chandra modules of Section 5.2, and subcategories of category \mathcal{O} , following Chapter 8.4 in [41]. The results are due to Joseph and Letzter [44]. The corresponding theory in the classical setting was developed independently by Bernstein-Gelfand in [13], Enright [27], and Joseph [40].

We first introduce certain subcategories of category \mathcal{O} .

Definition 5.25. For $l \in \mathfrak{h}_q^*$ let \mathcal{O}_l be the full subcategory of \mathcal{O} consisting of all modules with weight spaces associated to weights in $l + \mathbf{P} \subset \mathfrak{h}_q^*$.

Comparing with the set-up in [41], note that for all modules N in \mathcal{O} the annihilator $\text{ann}_{ZU_q(\mathfrak{g})}(N)$ has finite codimension in $ZU_q(\mathfrak{g})$. This is due to the fact that we require modules in \mathcal{O} to be finitely generated.

Next we introduce a category of $U_q(\mathfrak{g})$ -compatible $FU_q(\mathfrak{g})$ -bimodules as in Definition 4.26, but with some additional structural hypotheses.

Definition 5.26. Let H be a $FU_q(\mathfrak{g})$ -bimodule with a compatible locally finite action of $U_q(\mathfrak{g})$, meaning that

$$(X_{(1)} \rightarrow v) \cdot X_{(2)} = X \cdot v$$

for all $X \in FU_q(\mathfrak{g})$ and $v \in V$. We say that H is a Harish-Chandra bimodule if the following conditions are satisfied.

- a) All isotypical components for the adjoint action of $U_q(\mathfrak{g})$ are finite dimensional and integrable.
- b) The annihilators of both the left and right actions of $ZU_q(\mathfrak{g})$ have finite codimension.
- c) H is finitely generated as a right $FU_q(\mathfrak{g})$ -module.

We write HC for the category of Harish-Chandra bimodules with morphisms being $FU_q(\mathfrak{g})$ -bimodule maps which are also $U_q(\mathfrak{g})$ -linear.

Let us observe that every irreducible Harish-Chandra module V naturally defines a Harish-Chandra bimodule. Indeed, according to Definition 5.7 Harish-Chandra modules are admissible as $\mathfrak{D}(K_q)$ -modules, so that the isotypical components of V are finite dimensional and integrable by assumption in this case. Irreducibility implies that the annihilators for the action of left and right action of $ZU_q(\mathfrak{g})$ have finite codimension because the centre must act by fixed left/right central characters. If V is simple then every finite dimensional $\mathfrak{D}(K_q)$ -submodule W of V generates V as a $U_q^{\mathbb{R}}(\mathfrak{g})$ -module, and the compatibility condition in Definition 4.26 implies that W generates V even as a right $FU_q(\mathfrak{g})$ -module.

Recall that $\xi_l : ZU_q(\mathfrak{g}) \rightarrow \mathbb{C}$ is the central character corresponding to $l \in \mathfrak{h}_q^*$ as in Definition 2.123, and note that $\ker(\xi_l)$ annihilates $M(l)$.

Definition 5.27. For $l \in \mathfrak{h}_q^*$ let HC_l be the full subcategory of HC consisting of all objects for which the annihilator of the right action of $ZU_q(\mathfrak{g})$ contains $\ker(\xi_l)$.

We shall now relate Verma modules and compatible bimodules in the Harish-Chandra category, at least with suitable extra conditions on both sides.

Proposition 5.28. For $l \in \mathfrak{h}_q^*$ we obtain a covariant functor $\mathcal{F}_l : \mathcal{O}_l \rightarrow \text{HC}_l$ by setting

$$\mathcal{F}_l(M) = F \text{Hom}(M(l), M)$$

for $M \in \mathcal{O}_l$, equipped with the compatible bimodule structure

$$(Y \cdot \varphi \cdot Z)(m) = Y \cdot \varphi(Z \cdot m)$$

$$(X \rightarrow \varphi)(m) = X_{(1)} \cdot \varphi(\hat{S}(X_{(2)}) \cdot m)$$

for $Y, Z \in FU_q(\mathfrak{g})$, $X \in U_q(\mathfrak{g})$, $\varphi \in F \text{Hom}(M(l), M)$ and $m \in M(l)$.

Proof. Let us check that $\mathcal{F}_l(M)$ is indeed contained in HC_l . Firstly, the adjoint action of $U_q(\mathfrak{g})$ is clearly locally finite on the right hand side. Moreover, since the weights of M belong to $l + \mathbf{P}$, one easily checks that the adjoint action has weights in \mathbf{P} , so is integrable. The annihilator of the left action of $ZU_q(\mathfrak{g})$ has finite codimension because $M \in \mathcal{O}$, and the annihilator of the right action of $ZU_q(\mathfrak{g})$ has finite codimension because it contains $\ker(\xi_l)$.

For the finite multiplicity requirement, note that modules in category \mathcal{O} have finite length by Theorem 4.3, and since every simple module embeds in a dual

Verma module it is therefore enough to consider the case that $M = M(r)^\vee$ is a dual Verma module. In this case it follows from Lemma 4.27 and Proposition 4.35 a).

Hence $\mathcal{F}_l(M)$ is indeed contained in HC_l . Moreover \mathcal{F}_l clearly maps $U_q(\mathfrak{g})$ -linear maps to $U_q(\mathfrak{g})$ -linear $FU_q(\mathfrak{g})$ -bimodule maps. \square

Note that when $M = M(r)^\vee$ for $r \in \mathfrak{h}_q^*$, the bimodule structure in Proposition 5.28 is exactly the same as that from Corollary 5.24. In other words, with $\mu = l - r$ and $\lambda + 2\rho = -l - r$ we have

$$F_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu,\lambda}).$$

Let F be the functor on $U_q(\mathfrak{g})$ -modules that assigns to M its locally finite part FM . This is *not* an exact functor. If $0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$ is exact then clearly $0 \rightarrow FK \rightarrow FE \rightarrow FQ \rightarrow 0$ is exact at FK and FE . Surjectivity may fail however. Consider for instance $E = M(\mu), Q = V(\mu), K = I(\mu)$ for $\mu \in \mathbf{P}^+$. Then $FE = 0$ and $FQ = Q$.

Despite this fact we have the following result.

Lemma 5.29. *Let $l \in \mathfrak{h}_q^*$ be dominant. Then $\mathcal{F}_l : \mathcal{O}_l \rightarrow \text{HC}_l$ is an exact functor.*

Proof. Recall from Definition 4.6 that $l \in \mathfrak{h}_q^*$ is called dominant if $q_\alpha^{(l+\rho, \alpha^\vee)} \notin \pm q_\alpha^{-\mathbb{N}}$ for all $\alpha \in \mathbf{\Delta}^+$, where $\mathbb{N} = \{1, 2, \dots\}$. According to Proposition 4.16, dominance of l implies that $M(l)$ is projective. Hence the invariant part of the induced sequence $0 \rightarrow \text{Hom}(M(l), K) \rightarrow \text{Hom}(M(l), E) \rightarrow \text{Hom}(M(l), Q) \rightarrow 0$ is exact. Tensoring with a simple module $V(\mu)$ for $\mu \in \mathbf{P}^+$ we see that this applies to all μ -isotypical components. Indeed, the μ -isotypical component of $\text{Hom}(M(l), N)$ is the invariant part of

$$\text{Hom}(M(l), N) \otimes V(\mu)^* \cong \text{Hom}(M(l), N \otimes V(\mu)^*),$$

so exactness follows again from projectivity of $M(l)$. \square

Our next aim is to define a functor in the reverse direction. For this we need some auxiliary considerations.

Firstly, assume that H is any $FU_q(\mathfrak{g})$ -bimodule with a compatible action of $U_q(\mathfrak{g})$, and let V be a finite dimensional integrable $U_q(\mathfrak{g})$ -module. Consider the right $FU_q(\mathfrak{g})$ -module structure on the second factor of $V \otimes H$ and the left $U_q(\mathfrak{g})$ -module structure given by the diagonal action

$$X \rightarrow (v \otimes h) = (X_{(1)} \cdot v) \otimes (X_{(2)} \rightarrow h)$$

for $X \in U_q(\mathfrak{g})$. Let us define a left $FU_q(\mathfrak{g})$ -module structure on $V \otimes H$ by setting

$$Y \cdot (v \otimes h) = (Y_{(1)} \cdot v) \otimes Y_{(2)} \cdot h,$$

for $Y \in FU_q(\mathfrak{g})$. This makes sense since $FU_q(\mathfrak{g})$ is a left coideal, see Lemma 2.112. Then we compute

$$\begin{aligned} (Y_{(1)} \rightarrow (v \otimes h)) \cdot Y_{(2)} &= (Y_{(1)} \cdot v) \otimes (Y_{(2)} \rightarrow h) \cdot Y_{(3)} \\ &= (Y_{(1)} \cdot v) \otimes (Y_{(2)} \cdot h) \\ &= Y \cdot (v \otimes h), \end{aligned}$$

so that the resulting $FU_q(\mathfrak{g})$ -bimodule $V \otimes H$ is compatible with the $U_q(\mathfrak{g})$ -action.

Proposition 5.30. *Let V be a finite dimensional integrable $U_q(\mathfrak{g})$ -module and assume that $H \in \text{HC}$. Then the compatible $FU_q(\mathfrak{g})$ -bimodule $V \otimes H$ constructed above is again contained in HC . For $H \in \text{HC}_l$ we have $V \otimes H \in \text{HC}_l$.*

Proof. We have to verify that the conditions in Definition 5.26 are satisfied for $V \otimes H$. For condition *a*) notice that V is locally finite and integrable for the left $U_q(\mathfrak{g})$ -module structure, and that local finiteness and integrability are preserved under tensor products.

The key point to check is condition *b*). For this we may assume that $V = V(\mu)$ is simple. The annihilator of $ZU_q(\mathfrak{g})$ for the left action on H being of finite codimension implies that we have

$$\text{ann}_{ZU_q(\mathfrak{g})}(H) \supseteq (\ker(\xi_{l_1}) \cdots \ker(\xi_{l_m}))^r$$

for some weights $l_1, \dots, l_m \in \mathfrak{h}_q^*$ and $r \in \mathbb{N}$. Note that, thanks to the decomposition series of $V \otimes M(l_j)$ from Lemma 4.34, we have

$$\begin{aligned} \text{ann}_{ZU_q(\mathfrak{g})}(V(\mu) \otimes M(l_j)) &\supseteq \prod_{\nu \in \mathbf{P}(V(\mu))} (\text{ann}_{ZU_q(\mathfrak{g})} M(l_j + \nu))^s \\ &= \prod_{\nu \in \mathbf{P}(V(\mu))} (\ker(\xi_{l_j + \nu}))^s \end{aligned}$$

for sufficiently large s . It follows that

$$\text{ann}_{ZU_q(\mathfrak{g})}(V(\mu) \otimes H) \supseteq \prod_j \prod_{\nu \in \mathbf{P}(V(\mu))} \ker(\xi_{l_j + \nu})^{s'}$$

for some s' and hence $\text{ann}_{ZU_q(\mathfrak{g})}(V(\mu) \otimes H)$ has finite codimension.

Finite codimension for the annihilator of the right action of $ZU_q(\mathfrak{g})$ is obvious since $H \in \text{HC}$.

Condition *c*) is evidently satisfied because the right action of $FU_q(\mathfrak{g})$ on $V \otimes H$ only sees the factor H . We conclude that $V \otimes H$ is contained in HC .

The final assertion regarding HC_l is again immediate because the right action of $ZU_q(\mathfrak{g})$ depends only on the second tensor factor of $V \otimes H$. \square

Let $H \in \text{HC}$ be given. Since H is finitely generated as a right $FU_q(\mathfrak{g})$ -module we find a finite dimensional integrable $U_q(\mathfrak{g})$ -submodule $V \subset H$ such that the canonical map $V \otimes FU_q(\mathfrak{g}) \rightarrow H$ is surjective.

If $H \in \text{HC}_l$ then the right annihilator of H contains $\ker(\xi_l) \subset ZU_q(\mathfrak{g})$, and hence Theorem 4.51 implies that the the projection $V \otimes FU_q(\mathfrak{g}) \rightarrow H$ factorizes through $V \otimes F\text{End}(M(l))$.

Lemma 5.31. *Let $l \in \mathfrak{h}_q^*$ and let $H \in \text{HC}_l$. The left action of $FU_q(\mathfrak{g})$ on the first leg of $H \otimes_{FU_q(\mathfrak{g})} M(l)$ extends to a left action of $U_q(\mathfrak{g})$ via the formula*

$$X \cdot (h \otimes m) = X_{(1)} \rightarrow h \otimes X_{(2)} \cdot m,$$

and we obtain a functor $\mathcal{T}_l : \text{HC}_l \rightarrow \mathcal{O}_l$ by setting

$$\mathcal{T}_l(H) = H \otimes_{FU_q(\mathfrak{g})} M(l)$$

for $H \in \text{HC}_l$.

Proof. Note that when $X \in FU_q(\mathfrak{g})$, we have

$$(X \cdot h) \otimes m = (X_{(1)} \rightarrow h) \cdot X_{(2)} \otimes m = X_{(1)} \rightarrow h \otimes X_{(2)} \cdot m,$$

for all $h \otimes m \in H \otimes_{FU_q(\mathfrak{g})} M(l)$. We need to show that the expression on the right-hand side remains well-defined on the balanced tensor product when $X \in U_q(\mathfrak{g})$.

Since H is a Harish-Chandra bimodule, we have a direct sum decomposition $H = \bigoplus_{\mu \in \mathbf{P}} H_\mu$ into weight spaces for the adjoint action. If $h \in H_\mu$, $m \in M(l)_\lambda$ then the above calculation shows $(K_\nu \cdot h) \otimes m = q^{(\mu + \lambda, \nu)} h \otimes m$ for every $\nu \in 2\mathbf{P}^+$.

It follows that these elements $K_\nu \in FU_q(\mathfrak{g})$ are simultaneously diagonalizable on $H \otimes_{FU_q(\mathfrak{g})} M(l)$ with eigenspaces

$$(H \otimes_{FU_q(\mathfrak{g})} M(l))_\gamma = \bigoplus_{\mu+\lambda=\gamma} H_\mu \otimes M(l)_\lambda.$$

Now it makes sense to define the action of any $K_\nu \in U_q(\mathfrak{h})$ by $K_\nu \cdot v = q^{(\gamma, \nu)} v$ whenever $v \in (H \otimes_{FU_q(\mathfrak{g})} M(l))_\gamma$, and this agrees with the formula stated in the lemma. By Lemma 2.115, this action is well-defined for all $X \in U_q(\mathfrak{g})$.

The resulting $U_q(\mathfrak{g})$ -module $\mathcal{T}_l(H)$ is clearly a weight module with weights contained in $l + \mathbf{P}$. According to our considerations above there exists some finite dimensional integrable $U_q(\mathfrak{g})$ -module V such that H can be written as a quotient of $V \otimes_{FU_q(\mathfrak{g})} M(l)$. Hence $\mathcal{T}_l(H) = H \otimes_{FU_q(\mathfrak{g})} M(l)$ is a quotient of $V \otimes M(l)$, and the latter is in category \mathcal{O}_l . It follows that $\mathcal{T}_l(H)$ is contained in \mathcal{O}_l .

It is clear that \mathcal{T}_l maps morphisms in HC_l to morphisms in \mathcal{O}_l . \square

We obtain a Frobenius reciprocity relation for the functors \mathcal{F}_l and \mathcal{T}_l .

Proposition 5.32. *For $H \in \text{HC}_l$ and $N \in \mathcal{O}_l$ there is a natural isomorphism*

$$\text{Hom}_{U_q(\mathfrak{g})}(\mathcal{T}_l(H), N) \cong \text{Hom}_{\text{HC}}(H, \mathcal{F}_l(N)),$$

that is, the functor \mathcal{T}_l is left adjoint to \mathcal{F}_l .

Proof. The standard Hom-tensor adjunction yields an isomorphism

$$\phi : \text{Hom}_{FU_q(\mathfrak{g})-FU_q(\mathfrak{g})}(H, \text{Hom}(M(l), N)) \rightarrow \text{Hom}_{FU_q(\mathfrak{g})}(H \otimes_{FU_q(\mathfrak{g})} M(l), N),$$

sending $f \in \text{Hom}_{FU_q(\mathfrak{g})-FU_q(\mathfrak{g})}(H, \text{Hom}(M(l), N))$ to $\phi(f)$ given by $\phi(f)(h \otimes m) = f(h)(m)$. The inverse isomorphism is given by $\psi(g)(h)(m) = g(h \otimes m)$.

Let us show that ϕ maps morphisms in HC to $U_q(\mathfrak{g})$ -linear maps and ψ maps $U_q(\mathfrak{g})$ -linear maps to morphisms in HC . We do this by verifying that the unit and counit of the above adjunction have these properties. For $H \in \text{HC}_l$ the unit $u : H \rightarrow \mathcal{F}_l \mathcal{T}_l(H) = F \text{Hom}(M(l), H \otimes_{FU_q(\mathfrak{g})} M(l))$ is given by $u(h)(m) = h \otimes m$. We compute

$$\begin{aligned} u(X \rightarrow h)(m) &= (X \rightarrow h) \otimes m \\ &= (X_{(1)} \rightarrow h) \otimes (X_{(2)} \hat{S}(X_{(3)}))m \\ &= X_{(1)} \cdot (h \otimes \hat{S}(X_{(2)}) \cdot m) \\ &= X_{(1)} \cdot u(h)(\hat{S}(X_{(2)}) \cdot m) = (X \rightarrow u(h))(m). \end{aligned}$$

The counit $e : \mathcal{T}_l \mathcal{F}_l(M) = F \text{Hom}(M(l), M) \otimes_{FU_q(\mathfrak{g})} M(l) \rightarrow M$ is given by $e(T \otimes m) = T(m)$. For any $\nu \in \mathbf{P}$ we have

$$\begin{aligned} e(K_\nu \cdot (T \otimes m)) &= q^{(\nu, \mu+\lambda)} e(T \otimes m) \\ &= q^{(\nu, \mu+\lambda)} T(m) \\ &= (K_\nu \rightarrow T)(K_\nu \cdot m) \\ &= K_\nu \cdot T(K_\nu^{-1} \cdot (K_\nu \cdot m)) \\ &= K_\nu \cdot T(m) \\ &= K_\nu \cdot e(T \otimes m) \end{aligned}$$

if T has weight μ and m has weight λ . In light of Lemma 2.115, this suffices to conclude that e is $U_q(\mathfrak{g})$ -linear.

We conclude that the map ϕ constructed above restricts to an isomorphism

$$\text{Hom}_{\text{HC}}(H, F \text{Hom}(M(l), N)) \rightarrow \text{Hom}_{U_q(\mathfrak{g})}(H \otimes_{FU_q(\mathfrak{g})} M(l), N)$$

as desired. \square

Proposition 5.33. *Assume $l \in \mathfrak{h}_q^*$ is dominant. Then the canonical map $H \rightarrow \mathcal{F}_l \mathcal{T}_l(H)$ induced from the adjunction of \mathcal{F}_l and \mathcal{T}_l is an isomorphism for $H \in \text{HC}_l$.*

Proof. First, take $H = F \text{End}(M(l))$. By the Verma module annihilator theorem 4.51 we have $F \text{End}(M(l)) = FU_q(\mathfrak{g}) / \text{ann}_{FU_q(\mathfrak{g})}(M(l))$ and so the canonical multiplication map $F \text{End}(M(l)) \otimes_{FU_q(\mathfrak{g})} M(l) \rightarrow M(l)$ is an isomorphism. It follows that

$$\mathcal{T}_l(H) = F \text{End}(M(l)) \otimes_{FU_q(\mathfrak{g})} M(l) \cong M(l).$$

Hence we get $\mathcal{F}_l \mathcal{T}_l(H) = \mathcal{F}_l(M(l)) = F \text{End}(M(l)) = H$ in this case.

Next assume that $H = V \otimes F \text{End}(M(l))$ for some finite dimensional integrable $U_q(\mathfrak{g})$ -module V . Then by our previous considerations

$$\mathcal{T}_l(H) \cong (V \otimes F \text{End}(M(l))) \otimes_{FU_q(\mathfrak{g})} M(l) = V \otimes \mathcal{T}_l(F \text{End}(M(l))) = V \otimes M(l),$$

and hence

$$\begin{aligned} \mathcal{F}_l \mathcal{T}_l(H) &\cong \mathcal{F}_l(V \otimes M(l)) \\ &= F \text{Hom}(M(l), V \otimes M(l)) \\ &\cong V \otimes F \text{End}(M(l)). \end{aligned}$$

This means $\mathcal{F}_l \mathcal{T}_l(H) \cong H$ in this case as well.

Finally, suppose that $H \in \text{HC}_l$ is arbitrary. According to the discussion preceding Lemma 5.31, there exists a finite dimensional integrable $U_q(\mathfrak{g})$ -module V_1 and a surjective homomorphism $V_1 \otimes F \text{End}(M(l)) \rightarrow H$. Moreover, as $FU_q(\mathfrak{g})$ is Noetherian by Theorem 2.137, the kernel of the projection map is again in HC_l . Hence we obtain a short exact sequence of the form

$$V_2 \otimes F \text{End}(M(l)) \longrightarrow V_1 \otimes F \text{End}(M(l)) \longrightarrow H \longrightarrow 0$$

Since \mathcal{F}_l is exact by Lemma 5.29 and \mathcal{T}_l is right exact we obtain a commutative diagram

$$\begin{array}{ccccccc} V_2 \otimes F \text{End}(M(l)) & \longrightarrow & V_1 \otimes F \text{End}(M(l)) & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \\ V_2 \otimes F \text{End}(M(l)) & \longrightarrow & V_1 \otimes F \text{End}(M(l)) & \longrightarrow & \mathcal{F}_l \mathcal{T}_l(H) & \longrightarrow & 0 \end{array}$$

with exact rows. The 5-lemma shows that the right hand vertical arrow is an isomorphism. \square

Lemma 5.34. *Assume that $l \in \mathfrak{h}_q^*$ is dominant. If $V \in \mathcal{O}_l$ is simple then $\mathcal{F}_l(V) = 0$ or $\mathcal{F}_l(V)$ is simple. All simple objects in HC_l are obtained in this way.*

Proof. Assume $\mathcal{F}_l(V) \neq 0$ and let $H \subset \mathcal{F}_l(V)$ be a nonzero submodule. If $i : H \rightarrow \mathcal{F}_l(V)$ denotes the embedding map then the map $j : \mathcal{T}_l(H) \rightarrow V$ corresponding to i is nonzero according to Proposition 5.32. Since V is simple it follows that j is a surjection. Since \mathcal{F}_l is exact by Lemma 5.29 we conclude that $\mathcal{F}_l(j) : \mathcal{F}_l \mathcal{T}_l(H) \rightarrow \mathcal{F}_l(V)$ is a surjection. Composing the latter with the isomorphism $H \rightarrow \mathcal{F}_l \mathcal{T}_l(H)$ we reobtain our original map i . Hence i is surjective, which means that $H = \mathcal{F}_l(V)$. Hence $\mathcal{F}_l(V)$ is simple.

Now let $H \in \text{HC}_l$ be an arbitrary simple object. Then we have $H \cong \mathcal{F}_l \mathcal{T}_l(H)$ by Proposition 5.33, so that $\mathcal{T}_l(H)$ is nonzero. Since every object in \mathcal{O}_l has finite length we find a simple quotient V of $\mathcal{T}_l(H)$. Then $\mathcal{F}_l(V)$ is simple, and the quotient map $\mathcal{T}_l(H) \rightarrow V$ corresponds to a nonzero homomorphism $H \rightarrow \mathcal{F}_l(V)$. Since both H and $\mathcal{F}_l(V)$ are simple this means $H \cong \mathcal{F}_l(V)$. \square

In fact, in Proposition 5.38 below we will see that with suitable hypotheses on l , the bimodule $\mathcal{F}_l(V)$ is never zero for V simple, so that \mathcal{F}_l maps simple objects to simple objects.

As a final ingredient we shall discuss under what conditions we have $\mathcal{F}_l(V) \neq 0$ for all simple V . For this we need to introduce the concept of a regular weight and discuss the translation principle.

Let $\lambda \in \mathfrak{h}_q^*$. Recall from Section 4.1 that $\hat{W}_{[\lambda]} = \{\hat{w} \in \hat{W} \mid \hat{w}.\lambda - \lambda \in \mathbf{Q} \subset \mathfrak{h}_q^*/\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee\}$. We adopt the following definition from Section 8.4.9 in [41].

Definition 5.35. A weight $\lambda \in \mathfrak{h}_q^*$ is called regular if the only element $\hat{w} \in \hat{W}_{[\lambda]}$ with $\hat{w}.\lambda = \lambda$ is $\hat{w} = e$.

Observe that any element $\hat{w} \in \hat{W}$ which satisfies $\hat{w}.\lambda = \lambda$ is automatically in $\hat{W}_{[\lambda]}$, so the definition is equivalent to demanding that e is the only element of \hat{W} which fixes λ . We conclude that the group \hat{W} acts freely on the set of all regular elements in \mathfrak{h}_q^* .

Lemma 5.36. Let $\mu \in \mathbf{P}^+$. If $l \in \mathfrak{h}_q^*$ is dominant and regular then

$$\hat{W}.(l + \mu) \cap (l + \mathbf{P}(V(\mu))) = \{l + \mu\},$$

where $\mathbf{P}(V(\mu)) \subset \mathbf{P}$ denotes the set of all weights of $V(\mu)$.

Proof. It is clear that $l + \mu$ is contained in $\hat{W}.(l + \mu) \cap (l + \mathbf{P}(V(\mu)))$.

Conversely, assume $\hat{w} \in \hat{W}$ satisfies $\hat{w}.(l + \mu) \in (l + \mathbf{P}(V(\mu)))$. Then $\hat{w} \in \hat{W}_{[l]}$ since

$$\hat{w}(l + \mu + \rho) \in l + \mu + \rho + \mathbf{Q}$$

and $\hat{W}_{[l+\mu+\rho]} = \hat{W}_{[l]}$. Since l is dominant it is maximal in its $\hat{W}_{[l]}$ -orbit according to Proposition 4.8, which means that we have $\hat{w}^{-1}.l = l - \gamma$ for some $\gamma \in \mathbf{Q}^+$, or equivalently $\hat{w}(l - \gamma + \rho) = l + \rho$. From the latter relation we conclude

$$\hat{w}.(l + \mu) = \hat{w}(l + \mu + \rho) - \rho = l + \hat{w}(\mu + \gamma),$$

which implies in particular $\hat{w}(\mu + \gamma) \in \mathbf{P}(V(\mu))$. Moreover, since $\gamma \in \mathbf{Q}^+$ and $\mu \in \mathbf{P}^+$ we have $(\mu, \gamma) \geq 0$, and we get

$$(\hat{w}(\mu + \gamma), \hat{w}(\mu + \gamma)) = (\mu + \gamma, \mu + \gamma) = (\mu, \mu) + 2(\mu, \gamma) + (\gamma, \gamma) \geq (\mu, \mu)$$

with a strict inequality iff $\gamma \neq 0$. However, for $\nu \in \mathbf{P}(V(\mu))$ we have $(\nu, \nu) \leq (\mu, \mu)$, see Lemma 1.7 in [26]. Hence $\gamma = 0$ and we conclude $\hat{w}^{-1}.l = l$. This implies $\hat{w} = e$ by regularity of l . \square

Let $M \in \mathcal{O}$. For a central character $\chi : ZU_q(\mathfrak{g}) \rightarrow \mathbb{C}$, let us define the χ -primary component of M by

$$M^\chi = \{m \in M \mid \text{for all } Z \in ZU_q(\mathfrak{g}) \text{ we have } (Z - \chi(Z))^n \cdot m = 0 \text{ for some } n \in \mathbb{N}\},$$

compare Section 1.12 in [37]. Then $M^\chi \subset M$ is a $U_q(\mathfrak{g})$ -submodule, and as in the classical case one checks that M decomposes into a finite direct sum of its primary components.

Using this fact we obtain a direct sum decomposition of \mathcal{O} into the full subcategories \mathcal{O}^χ of modules for which $M = M^\chi$. We also refer to \mathcal{O}^χ as the primary component corresponding to χ .

Note that according to Proposition 2.129 every central character is of the form $\chi = \xi_\lambda$ for some $\lambda \in \mathfrak{h}_q^*$ in our setting.

Proposition 5.37. Let $\mu \in \mathbf{P}^+$ and let $l \in \mathfrak{h}_q^*$ be dominant and regular. Then for all $\hat{w} \in \hat{W}$ the Verma module $M(\hat{w}.(l + \mu))$ is a direct summand of $V(\mu) \otimes M(\hat{w}.l)$.

Proof. Consider the primary component of $M = V(\mu) \otimes M(\hat{w}.l)$ corresponding to $\xi_{l+\mu} : ZU_q(\mathfrak{g}) \rightarrow \mathbb{C}$. All Verma modules in the primary component $\mathcal{O}^{\xi_{l+\mu}}$ have central character $\xi_{l+\mu}$, and hence highest weights in $\hat{W}.(l + \mu)$. By Lemma 4.34, M admits a Verma flag with highest weights of the form $\hat{w}.l + \eta$ for $\eta \in \mathbf{P}(V(\mu))$. Hence the Verma modules occurring as subquotients of M for this flag are of the form $M(\nu)$ for $\nu \in (\hat{W}.(l + \mu)) \cap (\hat{w}.l + \mathbf{P}(V(\mu)))$. Since $\mathbf{P}(V(\mu))$ is stable under the (unshifted) action of W we obtain

$$\hat{w}.l + \mathbf{P}(V(\mu)) = \hat{w}(l + \rho) - \rho + \mathbf{P}(V(\mu)) = \hat{w}(l + \mathbf{P}(V(\mu)) + \rho) - \rho = \hat{w}.(l + \mathbf{P}(V(\mu)))$$

Since clearly $\hat{W}.(l + \mu) = \hat{w}.\hat{W}.(l + \mu)$ we deduce

$$(\hat{W}.(l + \mu)) \cap (\hat{w}.l + \mathbf{P}(V(\mu))) = \hat{w}.(\hat{W}.(l + \mu) \cap (l + \mathbf{P}(V(\mu)))) = \hat{w}.(l + \mu)$$

by Lemma 5.36. Hence the only possibility is $\nu = \hat{w}.(l + \mu)$. The result follows. \square

We shall now show that \mathcal{F}_l maps simple objects to simple objects in the case that l is regular.

Proposition 5.38. *Assume $l \in \mathfrak{h}_q^*$ is dominant and regular. Then $\mathcal{F}_l(V)$ is simple for each simple module $V \in \mathcal{O}_l$.*

Proof. We have to show that $F \operatorname{Hom}(M(l), V)$ is nonzero if V is a simple module. For this we may assume $V = V(l + \lambda)$ for some $\lambda \in \mathbf{P}$. According to Proposition 5.37 we have that $M(l + \mu)$ is a direct summand of $V(\mu) \otimes M(l)$ for any $\mu \in \mathbf{P}^+$. Since $F \operatorname{Hom}(M(l), V) \otimes V(\mu)^* \cong F \operatorname{Hom}(V(\mu) \otimes M(l), V)$ it therefore suffices to show that $F \operatorname{Hom}(M(l + \mu), V(l + \lambda))$ is nonzero for some μ .

Choose μ such that $\nu = \mu - \lambda \in \mathbf{P}^+$, and let $V(-\nu)$ denote the simple module of lowest weight $-\nu$. Then Proposition 4.35 and Lemma 4.27 show that

$$[F \operatorname{Hom}(M(l + \mu), M(l + \lambda)^\vee) : V(-\nu)] = \dim(V(-\nu)_{-\nu}) = 1$$

and

$$[F \operatorname{Hom}(M(l + \mu), M(l + \lambda - \gamma)^\vee) : V(-\nu)] = \dim(V(-\nu)_{-\nu - \gamma}) = 0$$

for all $\gamma \in \mathbf{P}^+ \setminus \{0\}$. Since $V(l + \lambda) \subset M(l + \lambda)^\vee$ and the corresponding quotient has a filtration with subquotients isomorphic to $V(l + \lambda - \gamma)$ for $\gamma \in \mathbf{P}^+ \setminus \{0\}$ this yields the claim. \square

Let us now summarize the results obtained so far.

Theorem 5.39. *Let $l \in \mathfrak{h}_q^*$ be dominant.*

- a) *The functor $\mathcal{T}_l : \operatorname{HC}_l \rightarrow \mathcal{O}_l$ embeds HC_l as a full subcategory into \mathcal{O}_l .*
- b) *If $l \in \mathfrak{h}_q^*$ is regular then $\mathcal{F}_l : \mathcal{O}_l \rightarrow \operatorname{HC}_l$ is an equivalence of \mathbb{C} -linear categories.*

Proof. a) It follows from Propositions 5.32 and 5.33 that we have

$$\operatorname{Hom}_{\operatorname{HC}}(H, K) \cong \operatorname{Hom}_{\operatorname{HC}}(H, \mathcal{F}_l \mathcal{T}_l(K)) \cong \operatorname{Hom}_{U_q(\mathfrak{g})}(\mathcal{T}_l(H), \mathcal{T}_l(K))$$

for all $H, K \in \operatorname{HC}_l$, hence \mathcal{T}_l is fully faithful. This means precisely that \mathcal{T}_l embeds HC_l as a full subcategory into \mathcal{O}_l .

b) Assume first that $H \in \operatorname{HC}_l$ is simple. If $\mathcal{T}_l(H)$ is non-simple there is a composition series $0 \subset V_0 \subset \cdots \subset V_n = \mathcal{T}_l(H)$ in \mathcal{O}_l with simple subquotients for some $n > 0$. Since \mathcal{F}_l is exact by Lemma 5.29 and maps simple modules to simple modules by Proposition 5.38, this induces a composition series $0 \subset \mathcal{F}_l(V_0) \subset \cdots \subset \mathcal{F}_l(V_n) = \mathcal{F}_l \mathcal{T}_l(H) \cong H$ with simple subquotients. Now our assumption that H is simple implies $n = 0$, which is a contradiction. Hence $\mathcal{T}_l(H)$ is simple.

We conclude that \mathcal{T}_l maps simple objects in HC_l to simple objects in \mathcal{O}_l . Taking into account Proposition 5.38 it follows that \mathcal{F}_l and \mathcal{T}_l induce inverse equivalences on simple objects.

Let $0 \rightarrow K \rightarrow E \rightarrow Q \rightarrow 0$ be an extension in \mathcal{O}_l , and assume that the counit of the adjunction from Proposition 5.32 induces isomorphisms $\mathcal{T}_l \mathcal{F}_l(K) \cong K$ and $\mathcal{T}_l \mathcal{F}_l(Q) \cong Q$. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{T}_l \mathcal{F}_l(K) & \longrightarrow & \mathcal{T}_l \mathcal{F}_l(E) & \longrightarrow & \mathcal{T}_l \mathcal{F}_l(Q) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong \\ 0 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q \longrightarrow 0 \end{array}$$

The bottom row is exact by assumption, so the map $K \cong \mathcal{T}_l \mathcal{F}_l(K) \rightarrow \mathcal{T}_l \mathcal{F}_l(E)$ must be injective. Since \mathcal{F}_l is exact and \mathcal{T}_l is right exact, it follows that the upper row of the diagram is exact as well. This implies $\mathcal{T}_l \mathcal{F}_l(E) \cong E$ as desired.

The full subcategory of \mathcal{O}_l on which the counit $e : \mathcal{T}_l \mathcal{F}_l \rightarrow \text{id}$ of the adjunction is an isomorphism contains all simple objects, and every object in \mathcal{O}_l has finite length. We conclude $\mathcal{T}_l \mathcal{F}_l \cong \text{id}$, and this finishes the proof. \square

5.6. Irreducible Harish-Chandra modules. We shall now derive consequences for the category of Harish-Chandra modules. Some of these results have been sketched in [3].

Recall from the comments following Lemma 5.17 that the principal series representation $\Gamma(\mathcal{E}_{\mu,\lambda})$ has a minimal K_q -type which occurs with multiplicity one.

Definition 5.40. Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. We denote by $V_{\mu,\lambda}$ the unique irreducible subquotient of $\Gamma(\mathcal{E}_{\mu,\lambda})$ with the same minimal K_q -type.

To observe that $V_{\mu,\lambda}$ is indeed well-defined, consider the submodule of $\Gamma(\mathcal{E}_{\mu,\lambda})$ generated by any nonzero vector in the minimal K_q -type. This module has a unique maximal submodule, namely the sum of all submodules not containing the minimal K_q -type, with corresponding quotient $V_{\mu,\lambda}$.

Throughout this section, given $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$, we shall use (l, r) to denote a pair of weights in $\mathfrak{h}_q^* \times \mathfrak{h}_q^*$ such that

$$\mu = l - r, \quad \lambda + 2\rho = -l - r,$$

compare Corollary 5.24. We recall that the pair (l, r) is well-defined only up to addition by an element of the form $(\frac{1}{2}i\hbar^{-1}\alpha^\vee, \frac{1}{2}i\hbar^{-1}\alpha^\vee)$ for some $\alpha^\vee \in \mathbf{Q}^\vee$.

We record the following equivalent formulations of a common integrality condition on the parameters (μ, λ) .

Lemma 5.41. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and let $l, r \in \mathfrak{h}_q^*$ be such that $\mu = l - r$ and $\lambda + 2\rho = -l - r$. For any positive root $\alpha \in \mathbf{\Delta}^+$, the following are equivalent:*

- a) $q_\alpha^{(\lambda, \alpha^\vee)} \in q_\alpha^{-|(\mu, \alpha^\vee)| - 2\mathbb{N}}$.
- b) Both $q_\alpha^{(l+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$.

Likewise, the following are equivalent:

- c) $q_\alpha^{(\lambda, \alpha^\vee)} \in q_\alpha^{(|(\mu, \alpha^\vee)| + 2\mathbb{N}}$.
- d) Both $q_\alpha^{(l+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}}$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}}$.

Proof. Let us put $\hbar_\alpha = \hbar d_\alpha$. Then condition a) is equivalent to

$$(\lambda, \alpha^\vee) \in -|(\mu, \alpha^\vee)| - 2\mathbb{N} \bmod i\hbar_\alpha^{-1}\mathbb{Z}.$$

If we define $l' = l + \rho$, $r' = r + \rho$, so that we have

$$\mu = l' - r', \quad \lambda = -l' - r',$$

then the above condition is equivalent to having both

$$\begin{aligned} -(l' + r', \alpha^\vee) &\in -(l' - r', \alpha^\vee) - 2\mathbb{N} \bmod i\hbar_\alpha^{-1}\mathbb{Z}, \\ -(l' + r', \alpha^\vee) &\in -(r' - l', \alpha^\vee) - 2\mathbb{N} \bmod i\hbar_\alpha^{-1}\mathbb{Z}. \end{aligned}$$

These conditions simplify to

$$\begin{aligned} (l', \alpha^\vee) &\in \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}, \\ (r', \alpha^\vee) &\in \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}, \end{aligned}$$

which is equivalent to b).

The equivalence of c) and d) follows by replacing (μ, λ) by $(-\mu, -\lambda)$. \square

We now begin the study of the simple modules $V_{\mu, \lambda}$.

Lemma 5.42. *If there is no $\alpha \in \Delta^+$ such that $q_\alpha^{(l+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{Z}}$ then $\Gamma(\mathcal{E}_{\mu, \lambda})$ is simple, and we have*

$$V_{\mu, \lambda} \cong \mathcal{F}_l(V(r)) = \mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu, \lambda}).$$

Proof. Since $r - l$ is a weight, the hypothesis implies $q_\alpha^{(r+\rho, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{Z}}$ for any $\alpha \in \Delta^+$. In particular, r is antidominant, so $M(r)$ is simple by Theorem 4.15. We get $\mathcal{F}_l(V(r)) = \mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu, \lambda})$, which is nonzero, and hence irreducible by Lemma 5.34. By definition, we have $V_{\mu, \lambda} = \Gamma(\mathcal{E}_{\mu, \lambda})$ in this situation. \square

Our next result gives a sufficient condition for the simple modules $V_{\mu, \lambda}$ and $V_{\mu', \lambda'}$ to be isomorphic. We will shortly show that this condition is also necessary, see Theorem 5.47.

Theorem 5.43. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. Then $V_{\mu, \lambda} \cong V_{w\mu, w\lambda}$ for all $w \in W$.*

Proof. Fix $w \in W$ and let us write $(\mu', \lambda') = (w\mu, w\lambda)$. We begin by proving that $V_{\mu', \lambda'} \cong V_{\lambda, \mu}$ in the case where μ is fixed and λ belongs to the dense subset

$$Z = \{\lambda \in \mathfrak{h}_q^* \mid q_\alpha^{(\lambda - \mu, \alpha^\vee)} \notin q_\alpha^{2\mathbb{Z}} \text{ for all } \alpha \in \Delta^+\} \subset \mathfrak{h}_q^*.$$

Using the relation $2l = \mu - \lambda - 2\rho$ we get $q_\alpha^{(l, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{Z}}$ for $\lambda \in Z$, and hence $V_{\mu, \lambda} = \Gamma(\mathcal{E}_{\mu, \lambda})$ by Lemma 5.42. The condition $q_\alpha^{(l, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{Z}}$ implies that $\Delta_{[l]} = \emptyset$, in the notation of Subsection 4.1.3, and thus also $\Delta_{[r]} = \emptyset$ since r and l differ by an integral weight. Note also that l and r are both dominant and antidominant in this case.

The parameters associated to (μ', λ') can be chosen as $l' = w.l$, $r' = w.r$, which are again both dominant and antidominant. By Theorem 2.128 we have $\xi_{l'} = \xi_l$, and so $\text{HC}_{l'} = \text{HC}_l$. Therefore, Lemma 5.34 implies that $\Gamma(\mathcal{E}_{\mu', \lambda'})$ is isomorphic to $\mathcal{F}_{l'}(V)$ for some simple module $V \in \mathcal{O}_l$. Specifically, we have $V \cong V(s)$ for some $s \in l + \mathbf{P}$. From the above condition on l we see that s is again dominant and antidominant. In particular, according to Theorem 4.15, we have $V \cong M(s)^\vee$ and thus $\mathcal{F}_{l'}(M(r')^\vee) \cong \mathcal{F}_l(M(s)^\vee)$.

The minimal K_q -type of $\mathcal{F}_{l'}(M(r')^\vee)$ is conjugate to $-\mu' = r' - l'$ under the Weyl group action and hence to $-\mu = r - l$. Likewise, the minimal K_q -type of $\mathcal{F}_l(M(s)^\vee)$ is conjugate to $s - l$. It follows that $r - l \in \mathbf{P}$ and $s - l \in \mathbf{P}$ are in the same orbit of the Weyl group action, and hence $s - r \in \mathbf{Q}$. Moreover, since $\xi_s = \xi_{r'}$, we see that s and r' must be \hat{W} -linked, and hence s and r are \hat{W} -linked as well. This implies that $s = \hat{v}.r$ for some $\hat{v} \in \hat{W}_{[r]}$, in the notation of Subsection 4.1.3. But we observed above that $\Delta_{[r]} = \emptyset$, so $\hat{W}_{[r]}$ is trivial, according to Proposition 4.5. Therefore $s = r$, and we conclude that $V_{\mu, \lambda} = \Gamma(\mathcal{E}_{\mu, \lambda}) \cong \Gamma(\mathcal{E}_{w\mu, w\lambda}) = V_{w\mu, w\lambda}$ for all $\lambda \in Z$.

Now we extend this to all $\lambda \in \mathfrak{h}_q^*$ as follows. Recall from Subsection 5.4.1 that the principal series modules $\Gamma(\mathcal{E}_{\mu,\lambda})$ for fixed μ all have the same underlying $\mathfrak{D}(K_q)$ -module $\Gamma(\mathcal{E}_\mu)$. Let us write σ for the minimal K_q -type in $\Gamma(\mathcal{E}_\mu)$ and denote the subspaces of K_q -type σ by $E \subset \Gamma(\mathcal{E}_\mu)$ and $E' \subset \Gamma(\mathcal{E}_{w\mu})$, respectively. Since the minimal K_q -type occurs with multiplicity one, there is a unique K_q -invariant isomorphism $\varphi : E \rightarrow E'$ up to scalar.

Recall from Section 5.3 that under any of the representations $\pi_{\mu,\lambda}$, the subalgebra $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma} \subset U_q^{\mathbb{R}}(\mathfrak{g})$ preserves E . By the above discussion, when $\lambda \in Z$, there is a bijective intertwiner $\Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{w\mu,w\lambda})$, and after rescaling we may assume that it agrees with φ on E . In particular, for any $X \in U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ we have $\varphi \circ \pi_{\mu,\lambda}(X)|_E = \pi_{w\mu,w\lambda}(X)|_{E'} \circ \varphi$. From the definition of the principal series representations $\pi_{\mu,\lambda}$ in Subsection 5.4.1, it is easy to check that the functions

$$\begin{aligned} \mathfrak{h}_q^* &\rightarrow \text{End}(E); & \lambda &\mapsto \pi_{\mu,\lambda}(X), \\ \mathfrak{h}_q^* &\rightarrow \text{End}(E'); & \lambda &\mapsto \pi_{w\mu,w\lambda}(X) \end{aligned}$$

are algebraic. Since Z is dense in \mathfrak{h}_q^* we deduce that $\varphi \circ \pi_{\mu,\lambda}(X)|_E = \pi_{w\mu,w\lambda}(X)|_{E'} \circ \varphi$ for all $\lambda \in \mathfrak{h}_q^*$. Therefore, E and E' are isomorphic as $U_q^{\mathbb{R}}(\mathfrak{g})^{\sigma,\sigma}$ -modules for all $\lambda \in \mathfrak{h}_q^*$, and so Proposition 5.15 shows that $V_{\mu,\lambda} \cong V_{w\mu,w\lambda}$ as $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules. \square

To obtain more precise information on the structure of $\Gamma(\mathcal{E}_{\mu,\lambda})$, we will make use of the following technical lemma. We use the notation $\|\mu\|^2 = (\mu, \mu)$, when $\mu \in \mathbf{P}$. Note that when $\mu, \mu' \in \mathbf{P}^+$ we have $\mu \leq \mu'$ implies $\|\mu\| \leq \|\mu'\|$.

Lemma 5.44. *Let $\alpha \in \mathbf{\Delta}^+$ and assume that $l, r \in \mathfrak{h}^*$ are such that $l - r \in \mathbf{P}$ and that $(l + \rho, \alpha^\vee) \in \mathbb{N}_0$ and $(r + \rho, \alpha^\vee) \in \mathbb{N}_0$. Then*

$$\|l - r\| \leq \|l - s_\alpha.r\|.$$

Moreover equality holds iff the shifted action of s_α stabilizes l or r .

Proof. Note first that $s_\alpha.r = r - (r + \rho, \alpha^\vee)\alpha$ and hence $l - s_\alpha.r = l - r + (r + \rho, \alpha^\vee)\alpha$. We compute

$$\begin{aligned} \|l - s_\alpha.r\|^2 - \|l - r\|^2 &= (l - s_\alpha.r, l - s_\alpha.r) - (l - r, l - r) \\ &= 2(l - r, (r + \rho, \alpha^\vee)\alpha) + (r + \rho, \alpha^\vee)^2(\alpha, \alpha) \\ &= 2(l - r, (r + \rho, \alpha^\vee)\alpha) + 2(r + \rho, \alpha^\vee)(r + \rho, \alpha) \\ &= 2(r + \rho + l - r, (r + \rho, \alpha^\vee)\alpha) \\ &= 2(l + \rho, \alpha)(r + \rho, \alpha^\vee). \end{aligned}$$

This yields the claim. \square

We now consider the modules $V_{\mu,\lambda}$ when the associated parameter l is dominant.

Lemma 5.45. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ such that the associated parameter $l \in \mathfrak{h}_q^*$ is dominant.*

a) *If there is no $\alpha \in \mathbf{\Delta}^+$ such that both $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r+\rho, \alpha)} \in \pm q_\alpha^{\mathbb{N}}$ then*

$$V_{\mu,\lambda} \cong \mathcal{F}_l(V(r)) \subset \mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu,\lambda}).$$

Otherwise $\mathcal{F}_l(V(r)) = 0$.

b) *The simple module $V_{\mu,\lambda}$ is a submodule of $\Gamma(\mathcal{E}_{\mu,\lambda})$. Explicitly, there exists $\hat{u} \in \hat{W}$ with $\hat{u}.l = l$ such that the pair $(l, \hat{u}.r)$ verifies the conditions in a) above, and we have*

$$V_{\mu,\lambda} \cong \mathcal{F}_l(V(\hat{u}.r)) \subset \mathcal{F}_l(M(\hat{u}.r)^\vee) \cong \mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu,\lambda}).$$

Proof. a) If l fulfils the hypothesis of Lemma 5.42 we are done, so we assume henceforth that this is not the case. Given that l is assumed dominant, this means that $q_\alpha^{(l+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}_0}$ for some $\alpha \in \mathbf{\Delta}^+$.

Let $I(r)$ denote the kernel of the surjection $M(r) \rightarrow V(r)$. By Lemma 4.2, we have a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = I(r),$$

where for each $1 \leq i \leq n$ the quotient M_i/M_{i-1} is a highest weight module. By the BGG Theorem 4.25, if $t \in \mathfrak{h}_q^*$ is a highest weight for any M_i/M_{i-1} with $1 \leq i \leq n$ then t is strongly linked to r , meaning that $t = s_{k_1, \alpha_1} \cdots s_{k_m, \alpha_m} \cdot r$ for some nonempty chain of affine reflections s_{α_j, k_j} with $\alpha_j \in \mathbf{\Delta}^+$ and $k_j \in \mathbb{Z}_2$, satisfying

$$r > s_{k_m, \alpha_m} \cdot r > s_{k_{m-1}, \alpha_{m-1}} s_{k_m, \alpha_m} \cdot r > \cdots > s_{k_1, \alpha_1} \cdots s_{k_m, \alpha_m} \cdot r = t.$$

Note that this implies

$$q_{\alpha_j}^{(s_{\alpha_{j+1}} \cdots s_{\alpha_m} \cdot r + \rho, \alpha_j^\vee)} \in \pm q_{\alpha_j}^{\mathbb{N}}$$

for $1 \leq j \leq m$. Moreover, since $r - s_{k_{j+1}, \alpha_{j+1}} \cdots s_{k_m, \alpha_m} \cdot r \in \mathbf{Q}$ and $l - r \in \mathbf{P}$ we have $q_{\alpha_j}^{(l+\rho, \alpha_j^\vee)} \in \pm q_{\alpha_j}^{\mathbb{Z}}$ for each α_j appearing in this chain, and hence $q_{\alpha_j}^{(l+\rho, \alpha_j^\vee)} \in \pm q_{\alpha_j}^{\mathbb{N}_0}$ by dominance.

Therefore, for each $1 \leq i \leq n$ we have a surjective morphism $M(t_i) \rightarrow M_i/M_{i-1}$ for some t_i strongly linked to r as above, and therefore an injective morphism

$$\mathcal{F}_l((M_i/M_{i+1})^\vee) \rightarrow \mathcal{F}_l(M(t_i)^\vee).$$

If there is no $\alpha \in \mathbf{\Delta}^+$ such that $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$, then an inductive application of Lemma 5.44 along the chain of reflections s_{α_j} shows that $\|\operatorname{Re}(l) - \operatorname{Re}(t_i)\| > \|\operatorname{Re}(l) - \operatorname{Re}(r)\|$, where the real part of an element in \mathfrak{h}_q^* refers to its component in the \mathbb{R} -span of $\mathbf{\Delta}$, see the remarks before Proposition 4.5. Note that we have a strict inequality here because the strict inequalities in the strongly linking chain imply that the reflection s_{α_j} does not fix $s_{\alpha_{j+1}} \cdots s_{\alpha_m} \cdot \operatorname{Re}(r)$. According to Proposition 4.35 a) and Proposition 5.23, the minimal K_q -type in $\mathcal{F}_l(M(t_i)^\vee)$ is strictly larger than that of $\mathcal{F}_l((M(r)^\vee)^\vee)$, and therefore $\mathcal{F}_l((M_i/M_{i-1})^\vee)$ does not contain the minimal K_q -type of $\mathcal{F}_l(M(r)^\vee)$. By induction, using the short exact sequences

$$0 \rightarrow \mathcal{F}_l((M_i/M_{i-1})^\vee) \rightarrow \mathcal{F}_l(M_i^\vee) \rightarrow \mathcal{F}_l(M_{i-1}^\vee) \rightarrow 0,$$

we deduce that $\mathcal{F}_l(M_i^\vee)$ does not contain the minimal K_q -type of $\mathcal{F}_l(M(r)^\vee)$ for any $1 \leq i \leq n$.

Therefore, in the exact sequence

$$0 \rightarrow \mathcal{F}_l(V(r)) \rightarrow \mathcal{F}_l(M(r)^\vee) \rightarrow \mathcal{F}_l(I(r)^\vee) \rightarrow 0,$$

the inclusion on the left is necessarily an isomorphism on the minimal K_q -type. It follows that $\mathcal{F}_l(V(r))$ is nonzero, and hence it is an irreducible Harish-Chandra module by Lemma 5.34. Moreover, since $\mathcal{F}_l(V(r))$ contains the minimal K_q -type of $\mathcal{F}_l(M(r)^\vee)$ we have $\mathcal{F}_l(V(r)) \cong V_{\mu, \lambda}$ in this case.

Conversely, assume that there is an $\alpha \in \mathbf{\Delta}^+$ satisfying both $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$. Then $M(s_{k, \alpha} \cdot r)$ is a submodule of $M(r)$ for suitable k , by Theorem 4.14, and hence $V(r)$ is a quotient of $M(r)/M(s_{k, \alpha} \cdot r)$. Equivalently, $V(r)$ is contained in the kernel of the natural projection $M(r)^\vee \rightarrow M(s_{k, \alpha} \cdot r)^\vee$. Applying the exact functor \mathcal{F}_l shows that $\mathcal{F}_l(V(r))$ is contained in the kernel of the surjective map $\mathcal{F}_l(M(r)^\vee) \rightarrow \mathcal{F}_l(M(s_{k, \alpha} \cdot r)^\vee)$. Note that $s_{k, \alpha}$ fixes l up to a translation by an element of $\frac{1}{2}i\hbar^{-1}\mathbf{Q}^\vee$, and since $s_{k, \alpha} \cdot r$ and r differ by an element of \mathbf{P} the same must be true of $s_{k, \alpha} \cdot l$ and l , whence $s_{k, \alpha} \cdot l = l$. It follows that both $\mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu, \lambda})$ and $\mathcal{F}_l(M(s_{k, \alpha} \cdot r)^\vee) \cong \Gamma(\mathcal{E}_{s_{\alpha\mu}, s_{\alpha\lambda}})$ have the same K_q -type multiplicities, so that

the surjection $\mathcal{F}_l(M(r)^\vee) \rightarrow \mathcal{F}_l(M(s_{k,\alpha}.r)^\vee)$ is in fact an isomorphism. It follows that $\mathcal{F}_l(V(r))$ is zero.

b) Under the conditions stated in a) we obtain the assertion with $\hat{u} = 1$, so it remains to consider the case where both $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^\mathbb{N}$ for some $\alpha \in \Delta^+$. Let $\beta_1, \dots, \beta_m \in \Delta^+$ be the positive roots such that $q_{\beta_i}^{(l+\rho, \beta_i^\vee)} = \pm 1$. Consider the subgroup U of \hat{W} generated by the affine reflections corresponding to β_1, \dots, β_m and pick $\hat{u} \in U$ such that $r' = \hat{u}.r$ is minimal. We claim that then $q_{\beta_i}^{(r'+\rho, \beta_i^\vee)} \notin \pm q_{\beta_i}^\mathbb{N}$ for all $1 \leq i \leq m$. Indeed, if $q_{\beta_i}^{(r'+\rho, \beta_i^\vee)} \in \pm q_{\beta_i}^\mathbb{N}$ we would get $s_{k, \beta_i}.r' < r'$ for suitable k , contradicting our choice of r' . Thus, by our definition of r' there is no $\alpha \in \Delta^+$ such that $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r'+\rho, \alpha^\vee)} \in \pm q_\alpha^\mathbb{N}$.

We are thus back in a situation as in a). Using Theorem 5.43, we obtain an embedding

$$V_{\mu, \lambda} \cong V_{u\mu, u\lambda} \cong \mathcal{F}_l(\hat{u}.r) \subseteq \mathcal{F}_l(M(\hat{u}.r)^\vee),$$

where u denotes the image of \hat{u} under the canonical projection $\hat{W} \rightarrow W$. We may assume that \hat{u} is of the form $\hat{u} = s_{k_1, \beta_{i_1}} \cdots s_{k_p, \beta_{i_p}} \in U$ such that

$$s_{k_1, \beta_{i_1}} \cdots s_{k_p, \beta_{i_p}}.r < s_{k_2, \beta_{i_2}} \cdots s_{k_p, \beta_{i_p}}.r < \cdots < s_{k_p, \beta_{i_p}}.r < r,$$

and we have $s_{k_j, \beta_{i_j}} \cdots s_{k_p, \beta_{i_p}}.l = l$ for all $1 \leq j \leq p$. In this situation, as in a), we see that the corresponding quotient maps

$$\mathcal{F}_l(M(r)^\vee) \rightarrow \mathcal{F}_l(M(s_{k_p, \beta_{i_p}}.r)^\vee) \rightarrow \cdots \rightarrow \mathcal{F}_l(M(\hat{u}.r)^\vee)$$

are all isomorphisms. Hence we get an inclusion $V_{\mu, \lambda} \subset \mathcal{F}_l(M(\hat{u}.r)^\vee) \cong \mathcal{F}_l(M(r)^\vee) \cong \Gamma(\mathcal{E}_{\mu, \lambda})$ as claimed. \square

Lemma 5.46. *Assume $l \in \mathfrak{h}_q^*$ is dominant, and let $r, r' \in l + \mathbf{P}$ with $\mathcal{F}_l(V(r)) \neq 0$. Then $\mathcal{F}_l(V(r)) \cong \mathcal{F}_l(V(r'))$ iff $r = r'$.*

Proof. Assume $r \neq r'$ and $\mathcal{F}_l(V(r)) \cong \mathcal{F}_l(V(r'))$. Let us write $H = \mathcal{F}_l(V(r))$. Since $H \neq 0$, it is simple by Lemma 5.34. The adjointness relation from Proposition 5.32 implies that there are nonzero $U_q(\mathfrak{g})$ -linear maps $\mathcal{T}_l(H) \rightarrow V(r), \mathcal{T}_l(H) \rightarrow V(r')$. These are necessarily surjective by the simplicity of $V(r), V(r')$. Consider the direct sum $\mathcal{T}_l(H) \rightarrow V(r) \oplus V(r')$ of the maps thus obtained. This map is again surjective because $V(r)$ is not isomorphic to $V(r')$. By exactness of \mathcal{F}_l and Proposition 5.33, we obtain a surjection $H \cong \mathcal{F}_l \mathcal{T}_l(H) \rightarrow \mathcal{F}_l(V(r)) \oplus \mathcal{F}_l(V(r'))$. This contradicts the simplicity of H . Hence $\mathcal{F}_l(V(r)) \not\cong \mathcal{F}_l(V(r'))$. \square

Theorem 5.47. *Two simple Harish-Chandra modules of the form $V_{\mu, \lambda}$ and $V_{\mu', \lambda'}$ for $(\mu, \lambda), (\mu', \lambda') \in \mathbf{P} \times \mathfrak{h}_q^*$ are isomorphic iff there exists $w \in W$ such that $(\mu', \lambda') = (w\mu, w\lambda)$.*

Proof. Sufficiency was proven in Theorem 5.43, so it remains to check that the condition is necessary. Suppose then that $V_{\mu, \lambda} \cong V_{\mu', \lambda'}$, and let (l, r) and (l', r') be the associated parameters in $\mathfrak{h}_q^* \times \mathfrak{h}_q^*$. By Theorem 2.128, l and l' are \hat{W} -linked. Therefore, after applying the isomorphisms of Theorem 5.43, we may assume that $l = l'$ and moreover that l is dominant.

By Theorem 5.45 b), we have $V_{\mu, \lambda} \cong \mathcal{F}_l(V(\hat{u}.r))$ and $V_{\mu', \lambda'} \cong \mathcal{F}_l(V(\hat{v}.r'))$ for some $\hat{u}, \hat{v} \in \hat{W}$ such that $\hat{u}.l = \hat{v}.l = l$. By Lemma 5.46, this implies $\hat{u}.r = \hat{v}.r'$. Therefore, $(l', r') = (l, \hat{v}^{-1}\hat{u}.r) = (\hat{v}^{-1}\hat{u}.l, \hat{v}^{-1}\hat{u}.r)$ and hence $(\mu', \lambda') = (v^{-1}u\mu, v^{-1}u\lambda)$, where u and v are the images in W of the canonical quotient map $\hat{W} \rightarrow W$. This completes the proof. \square

Theorem 5.48. *Every simple Harish-Chandra module is of the form $V_{\mu, \lambda}$ for some $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$.*

Proof. Let V be a simple Harish-Chandra module. Then the elements of the algebra $1 \otimes ZU_q(\mathfrak{g}) \subset FU_q(\mathfrak{g}) \otimes FU_q(\mathfrak{g})$ commute with the action of $U_q^{\mathbb{R}}(\mathfrak{g})$, in particular with the action of $U_q^{\mathbb{R}}(\mathfrak{k})$. Hence they preserve the minimal K_q -type of V . According to Theorem 5.12 and Schur’s Lemma they act by scalars. Hence, the action of $1 \otimes ZU_q(\mathfrak{g})$ is determined by a central character $\xi_l : ZU_q(\mathfrak{g}) \rightarrow \mathbb{C}$ for some $l \in \mathfrak{h}_q^*$, see Proposition 2.129. Since the central characters are invariant under the shifted \hat{W} -action by Theorem 2.128 we may assume without loss of generality that l is dominant. It follows that V is contained in HC_l , and according to Lemma 5.34 V is isomorphic to $\mathcal{F}_l(V(r))$ for some simple module $V(r) \in \mathcal{O}_l$. Since $V \neq 0$, Lemma 5.45 a) shows that $V \cong \mathcal{F}_l(V(r)) \cong V_{\mu, \lambda}$ where $\mu = l - r$ and $\lambda = -l - r - 2\rho$. \square

Theorem 5.48 implies in particular that every irreducible Harish-Chandra module has a unique minimal K_q -type, which occurs with multiplicity one.

Theorem 5.49. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. Then the principal series module $\Gamma(\mathcal{E}_{\mu, \lambda})$ is an irreducible Yetter-Drinfeld module iff $q_\alpha^{(\lambda, \alpha^\vee)} \neq q_\alpha^{\pm(|(\mu, \alpha^\vee)|+2k)}$ for all $k \in \mathbb{N}$ and all $\alpha \in \mathbf{\Delta}^+$.*

Proof. The assumption on (μ, λ) is equivalent to saying that for every $\alpha \in \mathbf{\Delta}^+$ we do not have nonzero integers $m, n \in \mathbb{Z}$ of the same sign such that $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm q_\alpha^m$ and $q_\alpha^{(r+\rho, \alpha^\vee)} = \pm q_\alpha^n$.

We begin with the case where l is dominant. Under the above condition, this means that for each $\alpha \in \mathbf{\Delta}^+$, either $q_\alpha^{(l+\rho, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{Z}}$, in which case $q_\alpha^{(r+\rho, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{Z}}$, or $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm q_\alpha^m$ for some $m \in \mathbb{N}$, in which case $q_\alpha^{(r+\rho, \alpha^\vee)} \notin \pm q_\alpha^{\mathbb{N}}$. In either case, r is antidominant, so that $M(r)$ is irreducible by Theorem 4.15 and therefore $\Gamma(\mathcal{E}_{\mu, \lambda}) \cong \mathcal{F}_l(M(r)^\vee)$ is irreducible by Lemma 5.34.

Conversely, if the above condition on l and r does not hold, with l dominant, then there is some $\alpha \in \mathbf{\Delta}^+$ such that $q_\alpha^{(l+\rho, \alpha^\vee)}$ and $q_\alpha^{(r+\rho, \alpha^\vee)}$ both belong to $\pm q_\alpha^{\mathbb{N}}$. In this case, we have an embedding $M(s_{k, \alpha} \cdot r) \subset M(r)$ for some $k \in \mathbb{Z}_2$ by Theorem 4.14, and hence a surjective morphism

$$\Gamma(\mathcal{E}_{\mu, \lambda}) \cong \mathcal{F}_l(M(r)^\vee) \rightarrow \mathcal{F}_l(M(s_{k, \alpha} \cdot r)^\vee) \cong \Gamma(\mathcal{E}_{\mu', \lambda'}),$$

where $\mu' = l - s_{k, \alpha} \cdot r$ and $\lambda = -l - s_{k, \alpha} \cdot r - 2\rho$. Moreover, since $(\text{Re}(l) + \rho, \alpha^\vee) \neq 0$, Lemma 5.44 shows that $\|\text{Re}(l) - \text{Re}(r)\| < \|\text{Re}(l) - s_\alpha \cdot \text{Re}(r)\|$, so that the minimal K_q -type of $\mathcal{F}_l(M(s_{k, \alpha} \cdot r)^\vee)$ is strictly larger than that of $\mathcal{F}_l(M(r)^\vee)$. It follows that $\Gamma(\mathcal{E}_{\mu, \lambda})$ is not irreducible.

Consider now an arbitrary weight (μ, λ) and let $\hat{w} \in \hat{W}$ be such that $\hat{w} \cdot l$ is dominant. Due to Theorem 5.47 we know that $V_{w\mu, w\lambda}$ has the same K_q -type multiplicities as $V_{\mu, \lambda}$, where $w \in W$ denotes the image of \hat{w} under the canonical projection of \hat{W} onto W . By Lemma 5.17, we also know that the principal series modules $\Gamma(\mathcal{E}_{\mu, \lambda})$ and $\Gamma(\mathcal{E}_{w\mu, w\lambda})$ have the same K_q -type multiplicities. Since the condition $q_\alpha^{(\lambda, \alpha^\vee)} \neq q_\alpha^{\pm(|(\mu, \alpha^\vee)|+2k)}$ for all $k \in \mathbb{N}$ and all $\alpha \in \mathbf{\Delta}^+$ is stable under the action of W , the result follows. \square

5.7. The principal series for $SL_q(2, \mathbb{C})$. In this section we will determine the structure of the principal series representations for $SL_q(2, \mathbb{C})$ as well as all intertwiners between them. As a result we obtain a complete classification of the irreducible Harish-Chandra modules for $SL_q(2, \mathbb{C})$. These results were first proven by Pusz-Woronowicz [65]. Here we obtain them as a consequence of the relationship between the categories HC_l and \mathcal{O}_l described in the previous sections.

Recall that when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ we identify \mathfrak{h}^* with \mathbb{C} via the correspondence which sends $\lambda \in \mathfrak{h}^*$ to $\frac{1}{2}(\lambda, \alpha^\vee)$. With this convention, $\mathbf{P} = \frac{1}{2}\mathbb{Z}$, $\mathbf{Q}^\vee = \mathbf{Q} = \mathbb{Z}$ and

$\mathfrak{h}_q^* = \mathbb{C}/i\hbar^{-1}\mathbb{Z}$. The unique simple root is $\alpha = 1$ and we have $\rho = \frac{1}{2}$. It is important to note that the invariant bilinear form on \mathfrak{h}^* is given by $(\lambda_1, \lambda_2) = 2\lambda_1\lambda_2$.

As in the previous section, for parameters $(\mu, \lambda) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}/i\hbar^{-1}\mathbb{Z}$, we let $(l, r) \in \mathbb{C}/i\hbar^{-1}\mathbb{Z} \times \mathbb{C}/i\hbar^{-1}\mathbb{Z}$ denote a pair such that

$$\mu = l - r, \quad \lambda + 1 = -l - r.$$

There are always two such choices for (l, r) which differ by $(\frac{1}{2}i\hbar^{-1}, \frac{1}{2}i\hbar^{-1})$. A weight $l \in \mathbb{C}/i\hbar^{-1}\mathbb{Z}$ is dominant in the sense of Definition 4.6 iff

$$l + \frac{1}{2} \notin -\frac{1}{2}\mathbb{N} \bmod \frac{1}{4}i\hbar^{-1}\mathbb{Z},$$

and antidominant iff

$$l + \frac{1}{2} \notin \frac{1}{2}\mathbb{N} \bmod \frac{1}{4}i\hbar^{-1}\mathbb{Z},$$

where as usual $\mathbb{N} = \{1, 2, 3, \dots\}$.

Theorem 5.50. *Let $(\mu, \lambda) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}/i\hbar^{-1}\mathbb{Z}$. The principal series representation $\Gamma(\mathcal{E}_{\mu, \lambda})$ for $SL_q(2, \mathbb{C})$ is irreducible iff $\lambda \notin \pm(|\mu| + \mathbb{N}) \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$. Moreover, when $\lambda \in \pm(|\mu| + \mathbb{N}) \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ the simple module $V_{\mu, \lambda}$ is finite dimensional and $\Gamma(\mathcal{E}_{\mu, \lambda})$ is a non-split extension of simple modules as follows:*

$$\begin{aligned} 0 \rightarrow V_{\mu, \lambda} \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{\lambda, \mu}) \rightarrow 0 & \quad \text{if } \lambda \in -|\mu| - \mathbb{N}, \\ 0 \rightarrow V_{\mu, \lambda} \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{\lambda + \frac{1}{2}i\hbar^{-1}, \mu + \frac{1}{2}i\hbar^{-1}}) \rightarrow 0 & \quad \text{if } \lambda \in -|\mu| - \mathbb{N} + \frac{1}{2}i\hbar^{-1}\mathbb{Z}, \\ 0 \rightarrow \Gamma(\mathcal{E}_{\lambda, \mu}) \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow V_{\mu, \lambda} \rightarrow 0 & \quad \text{if } \lambda \in |\mu| + \mathbb{N}, \\ 0 \rightarrow \Gamma(\mathcal{E}_{\lambda + \frac{1}{2}i\hbar^{-1}, \mu + \frac{1}{2}i\hbar^{-1}}) \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow V_{\mu, \lambda} \rightarrow 0 & \quad \text{if } \lambda \in |\mu| + \mathbb{N} + \frac{1}{2}i\hbar^{-1}\mathbb{Z}. \end{aligned}$$

Proof. The statement about irreducibility follows directly from Theorem 5.49.

If $\lambda \in -|\mu| - \mathbb{N}$ then $l + \frac{1}{2}, r + \frac{1}{2} \in \frac{1}{2}\mathbb{N}$, so that l is dominant and regular and r is not antidominant. We have a short exact sequence in category \mathcal{O} ,

$$0 \rightarrow V(r) \rightarrow M(r)^\vee \rightarrow M(-r - 1)^\vee \rightarrow 0.$$

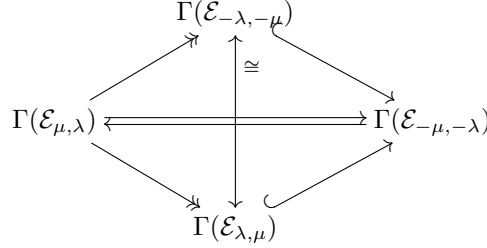
Note that $\mathcal{F}_l(M(-r - 1)^\vee) \cong \Gamma(\mathcal{E}_{-\lambda, -\mu})$ is irreducible, and so is isomorphic to $\Gamma(\mathcal{E}_{\lambda, \mu})$ by Theorem 5.47. Therefore, applying the functor \mathcal{F}_l to the above short exact sequence gives the first of the stated extensions. It is non-split by Theorem 5.39. To see that $V_{\mu, \lambda}$ is finite dimensional, it suffices to compare the K_q -type multiplicities in the two principal series representations in the extension. Specifically, Lemma 5.17 shows that $\Gamma(\mathcal{E}_{\mu, \lambda})$ contains the K_q -types with highest weights $\nu \in |\mu| + \mathbb{N}$, each with multiplicity one.

The case $\lambda \in -|\mu| - \mathbb{N} + \frac{1}{2}i\hbar^{-1}$ is obtained similarly, and the final two cases follow by duality using Lemma 5.21. \square

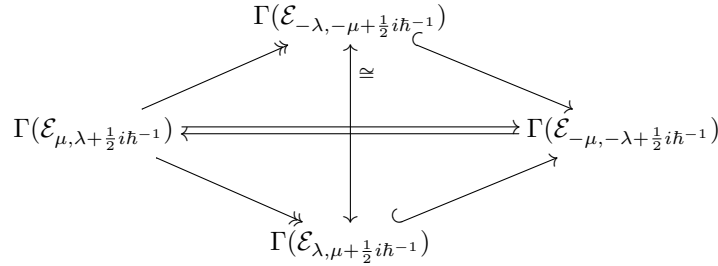
Theorem 5.51. *The following is an exhaustive list of non-zero intertwining operators between principal series representations of $SL_q(2, \mathbb{C})$.*

1. **Trivial intertwiners:** For every $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$, the only self-intertwiners $\Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda})$ are the scalar multiples of the identity.
2. **Standard intertwiners:** For every $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ there is a unique intertwiner, up to scalars, $\Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ as follows:
 - a) If $\lambda \in -|\mu| - \mathbb{N} \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ the intertwiner is Fredholm with kernel and cokernel isomorphic to $V_{\mu, \lambda}$.
 - b) If $\lambda \in |\mu| + \mathbb{N} \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ the intertwiner is finite-rank and factors through $V_{\mu, \lambda}$.
 - c) Otherwise, the intertwiner is bijective.

3. **Zelobenko intertwiners:** With $\lambda \in -|\mu| - \mathbb{N}$, there are additional intertwiners as follows:



and



where the right- and left-pointing horizontal arrows are the standard intertwiners from 2a) and b) respectively, the vertical arrows are the standard intertwiners from 2c), and the diagonal arrows are obtained using the extensions in Theorem 5.50. These diagrams commute, up to scalar multiple, except for the finite rank intertwiners from right to left which have composition zero with any other.

Proof. By Theorem 5.50, $\Gamma(\mathcal{E}_{\mu, \lambda})$ is either simple or a non-split extension of two non-isomorphic simple modules. In either case, $\Gamma(\mathcal{E}_{\mu, \lambda})$ does not admit any nontrivial self-intertwiners.

Inspecting the extensions in Theorem 5.50, we see that the non-simple principal series representations are all mutually non-isomorphic. It follows that there exists a unique intertwiner, up to scalars, between two principal series modules $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$ and $\Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ if and only if $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$ admits a simple quotient which is isomorphic to a simple submodule of $\Gamma(\mathcal{E}_{\mu_2, \lambda_2})$. The simple Harish-Chandra modules are all of the form $V_{\mu, \lambda}$ for some $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$, and we have $V_{\mu, \lambda} = \Gamma(\mathcal{E}_{\mu, \lambda})$ if $\lambda \notin \pm(|\mu| + \mathbb{N}) \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$. Also, by Theorem 5.47 we have $V_{\mu, \lambda} \cong V_{\mu', \lambda'}$ if and only if $(\mu', \lambda') = \pm(\mu, \lambda)$. Therefore, to complete the proof, we can make a case-by-case examination using the structure of the principal series modules given in Theorem 5.50.

Here are the details. First, consider the case where both $\lambda_1 \notin \pm(|\mu_1| + \mathbb{N}) \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$ and $\lambda_2 \notin \pm(|\mu_2| + \mathbb{N}) \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$. Then $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$ and $\Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ are both simple, so the only nontrivial intertwiners in this situation are the bijective intertwiners $\Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ from case 2c).

Next, we consider the cases where either $\lambda_1 \in \pm(|\mu_1| + \mathbb{N})$ or $\lambda_2 \in \pm(|\mu_2| + \mathbb{N})$ on the nose. We will obtain precisely the intertwiners in the first diagram from part 3.

- If $\lambda_1 \in -|\mu_1| - \mathbb{N}$ then the image of T must be isomorphic to $\Gamma(\mathcal{E}_{\lambda_1, \mu_1}) = V_{\lambda_1, \mu_1}$. The only principal series modules which contain $\Gamma(\mathcal{E}_{\lambda_1, \mu_1})$ as a submodule are $\Gamma(\mathcal{E}_{\lambda_1, \mu_1})$ itself, $\Gamma(\mathcal{E}_{-\lambda_1, -\mu_1})$ which is simple and isomorphic to $\Gamma(\mathcal{E}_{\lambda_1, \mu_1})$, and $\Gamma(\mathcal{E}_{-\mu_1, -\lambda_1})$ which contains $\Gamma(\mathcal{E}_{-\lambda_1, -\mu_1})$ as a proper submodule according to Theorem 5.50. These correspond to the three intertwiners issuing from the left-hand module in the first diagram.

- If $\lambda_1 \in |\mu_1| + \mathbb{N}$ then the image of T must be isomorphic to V_{μ_1, λ_1} and the only principal series module containing this as a submodule is $\Gamma(\mathcal{E}_{-\mu_1, -\lambda_1})$. This corresponds to the horizontal arrow from right to left.
- If $\lambda_2 \in -|\mu_2| - \mathbb{N}$ then the image of T must be isomorphic to V_{μ_2, λ_2} and the only principal series admitting this as a quotient module is $\Gamma(\mathcal{E}_{-\mu_2, -\lambda_2})$. This corresponds again to the horizontal arrow from right to left.
- If $\lambda_2 \in |\mu_2| + \mathbb{N}$ then the image of T must be isomorphic to $\Gamma(\mathcal{E}_{\lambda_2, \mu_2})$ and the only principal series modules admitting this as a quotient module are $\Gamma(\mathcal{E}_{\lambda_2, \mu_2}) \cong \Gamma(\mathcal{E}_{-\lambda_2, -\mu_2})$ and the non-simple module $\Gamma(\mathcal{E}_{-\mu_2, -\lambda_2})$. These correspond to the three intertwiners mapping to the right-hand module.

Finally, the cases where $\lambda_1 \in \pm(|\mu_1| + \mathbb{N}) + \frac{1}{2}i\hbar^{-1}$ or $\lambda_2 \in \pm(|\mu_2| + \mathbb{N}) + \frac{1}{2}i\hbar^{-1}$ are treated similarly, and we obtain the intertwiners in the second diagram from part 3. \square

We note that the two diagrams in part 3 of Theorem 5.51 are related by the action of the group $Z = \mathbf{P}^\vee/\mathbf{Q}^\vee \cong \mathbb{Z}_2$ which is described in Subsection 5.4.3. More precisely, the representations in the second diagram become isomorphic to the corresponding representations in the first diagram upon tensoring by the one-dimensional module $\mathbb{C}_{\frac{1}{2}i\hbar^{-1}}$, see Lemma 5.19. Therefore, the corresponding principal series representations in these diagrams become identical upon restriction to the “connected component” $SL_q(2, \mathbb{C})^0$. Moreover, under the canonical identification of the vector spaces $\Gamma(\mathcal{E}_{\mu, \lambda}) = \Gamma(\mathcal{E}_{\mu, \lambda + \frac{1}{2}i\hbar^{-1}})$ for each pair (μ, λ) , the corresponding intertwiners in the two diagrams are given by exactly the same linear operators.

Finally, let us remark that the Żelobenko intertwiners can be interpreted geometrically. For instance, from the case $(\mu, \lambda) = (0, 1)$ we obtain a system of $\mathfrak{D}(G_q)$ -linear operators

$$\mathbb{C} \longrightarrow \Gamma(\mathcal{E}_{0,-1}) \begin{array}{l} \nearrow \Gamma(\mathcal{E}_{1,0}) \\ \searrow \Gamma(\mathcal{E}_{-1,0}) \end{array} \begin{array}{l} \longrightarrow \Gamma(\mathcal{E}_{0,1}) \\ \longrightarrow \Gamma(\mathcal{E}_{0,1}) \end{array} \longrightarrow \mathbb{C}$$

where \mathbb{C} carries the trivial representation. In Subsection 5.8.3, we will show that the diagonal intertwiners in this diagram are given by the right regular action of the elements E or F in $U_q^{\mathbb{R}}(\mathfrak{k})$.

The principal series representation $\Gamma(\mathcal{E}_{0,-1})$ is the quantum analogue of the space of polynomial functions on the flag variety G/B . In this vein, the above diagram can be viewed as a quantum analogue of the $(\partial, \bar{\partial})$ -complex:

$$\mathbb{C} \longrightarrow \Omega^{0,0}(G/B) \begin{array}{l} \xrightarrow{\bar{\partial}} \Omega^{0,1}(G/B) \\ \xrightarrow{\partial} \Omega^{1,0}(G/B) \end{array} \begin{array}{l} \xrightarrow{\partial} \Omega^{1,1}(G/B) \\ \xrightarrow{\bar{\partial}} \Omega^{1,0}(G/B) \end{array} \longrightarrow \mathbb{C}.$$

Similarly, from the other values of (μ, λ) appearing in part 3 of Theorem 5.51 we obtain quantum analogues of the $(\partial, \bar{\partial})$ -complex twisted by a G -equivariant vector bundle.

5.8. Intertwining operators in higher rank.

5.8.1. *Intertwiners in the compact picture.* Fix $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. Let $f = \langle v' | \bullet | v \rangle \in \mathfrak{C}^\infty(K_q)$ and $\xi = \langle w' | \bullet | w \rangle \in \Gamma(\mathcal{E}_{\mu, \lambda})$ be matrix coefficients of finite dimensional $\mathfrak{D}(K_q)$ -modules V and W , respectively. That is, we have $v \in V, v' \in V^*, w \in W, w' \in W^*$, and moreover w has weight μ . Let e_1, \dots, e_n be a weight basis for V , with e_j of weight ϵ_j for each $1 \leq j \leq n$, and let e^1, \dots, e^n be the dual basis of V^* .

We shall write cV for the dual space V^* equipped with the precontragredient representation of $U_q^{\mathbb{R}}(\mathfrak{k})$, namely

$$(X \cdot v')(v) = v'(\hat{S}^{-1}(X) \cdot v)$$

for $X \in U_q^{\mathbb{R}}(\mathfrak{k})$, $v' \in {}^cV$ and $v \in V$. This is relevant for the antipode in $\mathfrak{C}^\infty(K_q)$, since

$$S(\langle v' | \bullet | v \rangle) = \langle v | \bullet | v' \rangle,$$

where the latter is a matrix coefficient for the pre-contragredient representation cV .

Lemma 5.52. *With the above notation, the Yetter-Drinfeld action of $f = \langle v' | \bullet | v \rangle \in \mathfrak{C}^\infty(K_q)$ on $\xi = \langle w' | \bullet | w \rangle \in \Gamma(\mathcal{E}_{\mu, \lambda})$ becomes*

$$\begin{aligned} f \cdot \xi &= \sum_j (K_{2\rho+\lambda}, \langle e^j | \bullet | e_j \rangle) \langle v \otimes w' \otimes v' | \bullet | e^j \otimes w \otimes e_j \rangle \\ &= \sum_j q^{(\lambda+2\rho, \epsilon_j)} \langle v \otimes w' \otimes v' | \bullet | e^j \otimes w \otimes e_j \rangle, \end{aligned}$$

where the latter is viewed as a matrix coefficient for ${}^cV \otimes W \otimes V$.

Proof. This follows immediately from the definitions. Indeed, we compute

$$\begin{aligned} (x, f \cdot \xi) &= (x, f_{(1)} \xi S(f_{(3)}))(K_{2\rho+\lambda}, f_{(2)}) \\ &= (x_{(3)}, f_{(1)})(x_{(2)}, \xi)(x_{(1)}, S(f_{(3)}))(K_{2\rho+\lambda}, f_{(2)}) \\ &= \sum_{i,j} (K_{2\rho+\lambda}, \langle e^i | \bullet | e_j \rangle)(x_{(1)}, \langle v | \bullet | e^j \rangle)(x_{(2)}, \langle w' | \bullet | w \rangle)(x_{(3)}, \langle v' | \bullet | e_i \rangle) \\ &= \sum_j (x, (K_{2\rho+\lambda}, \langle e^j | \bullet | e_j \rangle) \langle v \otimes w' \otimes v' | \bullet | e^j \otimes w \otimes e_j \rangle) \end{aligned}$$

for all $x \in \mathfrak{D}(K_q)$, as desired. □

Let $T \in \mathcal{M}(\mathfrak{D}(K_q))$ be an element of weight $\beta \in \mathbf{P}$ for the adjoint action, meaning

$$K_\lambda T K_{-\lambda} = q^{(\lambda, \beta)} T$$

for all $\lambda \in \mathbf{P}$. Then the right regular action of T on $\mathfrak{C}^\infty(K_q)$,

$$T \rightarrow \xi = (T, \xi_{(2)}) \xi_{(1)},$$

restricts to a morphism of $\mathfrak{D}(K_q)$ -modules $\Gamma(\mathcal{E}_\mu) \rightarrow \Gamma(\mathcal{E}_{\mu+\beta})$ for any $\mu \in \mathbf{P}$. We will denote this morphism simply by T .

In general, any $\mathfrak{D}(K_q)$ -linear map $f : \Gamma(\mathcal{E}_\mu) \rightarrow \Gamma(\mathcal{E}_{\mu+\beta})$ is of this form for some (in fact, many) $T \in \mathcal{M}(\mathfrak{D}(K_q))$. To confirm this, use the Peter-Weyl decomposition of $\mathfrak{C}^\infty(K_q)$ to write f in the form

$$f = \bigoplus_{\sigma \in \mathbf{P}^+} \text{id} \otimes T_\sigma : \bigoplus_{\sigma \in \mathbf{P}^+} V(\sigma)^* \otimes V(\sigma)_\mu \rightarrow \bigoplus_{\sigma \in \mathbf{P}^+} V(\sigma)^* \otimes V(\sigma)_{\mu+\beta}$$

for some family of linear maps $T_\sigma : V(\sigma)_\mu \rightarrow V(\sigma)_{\mu+\beta}$. Extend these maps to $T_\sigma : V(\sigma) \rightarrow V(\sigma)$, for instance by zero on all other weight spaces, to obtain $T = \bigoplus_\sigma T_\sigma \in \mathcal{M}(\mathfrak{D}(K_q))$ which acts as f .

Given this characterization of the $\mathfrak{D}(K_q)$ -linear maps between principal series representations, Lemma 5.52 immediately yields the following characterization of intertwiners of principal series representations.

Lemma 5.53. *Fix $(\mu_1, \lambda_1), (\mu_2, \lambda_2) \in \mathbf{P} \times \mathfrak{h}_q^*$ and let $T \in \mathcal{M}(\mathfrak{D}(K_q))$ have weight $\mu_2 - \mu_1$ for the adjoint action. Then the following conditions are equivalent.*

- a) $T : \Gamma(\mathcal{E}_{\mu_1, \lambda_1}) \rightarrow \Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ is an intertwiner of $\mathfrak{D}(G_q)$ -modules.

b) For any finite dimensional $\mathfrak{D}(K_q)$ -modules V and W , for any $w \in W_{\mu_1}$ we have

$$\sum_j q^{(\lambda_1+2\rho, \epsilon_j)} T \cdot (e^j \otimes w \otimes e_j) = \sum_j q^{(\lambda_2+2\rho, \epsilon_j)} e^j \otimes T \cdot w \otimes e_j.$$

where e_1, \dots, e_m is a weight basis for V with weights $\epsilon_1, \dots, \epsilon_m$, respectively, and e^1, \dots, e^m is the dual basis of cV .

5.8.2. *Intertwiners associated to simple roots.* Fix a simple root α . Consider the Hopf $*$ -algebra $U_{q_\alpha}^{\mathbb{R}}(\mathfrak{su}(2))$ generated by elements E, F and $K = K_\varpi$ as usual, where ϖ is the fundamental weight of $\mathfrak{sl}(2, \mathbb{C})$. We write $U_{q_\alpha}^{\mathbb{R}}(\mathfrak{psu}(2))$ for the $*$ -Hopf subalgebra generated by E, F and $K_\alpha = K^2$.

There is a unique Hopf $*$ -algebra morphism

$$\iota_\alpha : U_{q_\alpha}^{\mathbb{R}}(\mathfrak{psu}(2)) \rightarrow U_q^{\mathbb{R}}(\mathfrak{k})$$

sending E, F, K^2 to $E_\alpha, F_\alpha, K_\alpha$, respectively. In this way, every $U_q^{\mathbb{R}}(\mathfrak{k})$ -module V restricts to a $U_{q_\alpha}^{\mathbb{R}}(\mathfrak{psu}(2))$ -module. If V is integrable then the restriction extends uniquely to an integrable module over $U_{q_\alpha}^{\mathbb{R}}(\mathfrak{su}(2))$, where K acts by the positive square root of K^2 . This gives a well-defined restriction functor from integrable $U_q^{\mathbb{R}}(\mathfrak{k})$ -modules to integrable $U_{q_\alpha}^{\mathbb{R}}(\mathfrak{su}(2))$ -modules and thus a map

$$\iota_\alpha : \mathfrak{D}(SU_{q_\alpha}(2)) \rightarrow \mathcal{M}(\mathfrak{D}(K_q)).$$

We will write S_α for the quantum subgroup of K_q which is obtained in this way.

Let V be an integrable $U_q^{\mathbb{R}}(\mathfrak{k})$ -module. If $v \in V$ is a weight vector of weight $\mu \in \mathbf{P}$, then

$$\iota_\alpha(K) \cdot v = q^{\frac{1}{2}(\alpha, \mu)} v = q_\alpha^{\frac{1}{2}(\alpha^\vee, \mu)} v.$$

In other words, upon restricting V to a $\mathfrak{D}(S_\alpha)$ -module, v has weight $\frac{1}{2}(\alpha^\vee, \mu) \in \frac{1}{2}\mathbb{Z}$. Motivated by this, we define

$$\mu_\alpha = \frac{1}{2}(\alpha^\vee, \mu),$$

and call it the restriction of μ with respect to α . Note in particular that $\rho_\alpha = \frac{1}{2}$.

More generally, we write $\lambda_\alpha = \frac{1}{2}(\lambda, \alpha^\vee) \in \mathbb{C}$ for any $\lambda \in \mathfrak{h}^*$. This yields the orthogonal decomposition

$$\lambda = \lambda' + \lambda_\alpha \alpha,$$

where $\lambda' = \lambda - \lambda_\alpha \alpha \in \alpha^\perp$.

We also wish to define λ_α when $\lambda \in \mathfrak{h}_q^* = \mathfrak{h}^*/i\hbar^{-1}\mathbf{Q}^\vee$. We have

$$\frac{1}{2}(i\hbar^{-1}\mathbf{Q}^\vee, \alpha^\vee) = \frac{1}{2}i\hbar^{-1}d_\alpha^{-1}(\mathbf{Q}^\vee, \alpha) \subseteq \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z},$$

where we are using the notation $\hbar_\alpha = d_\alpha \hbar$. Therefore, we obtain a map

$$\mathfrak{h}_q^* \rightarrow \mathbb{C}/\frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}; \quad \lambda \mapsto \lambda_\alpha.$$

We point out that that the factor of $\frac{1}{2}$ in the above quotient means that the restriction of parameters

$$\begin{aligned} \mathbf{P} \times \mathfrak{h}_q^* &\rightarrow \frac{1}{2}\mathbb{Z} \times \mathbb{C}/\frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z} \\ (\mu, \lambda) &\mapsto (\mu_\alpha, \lambda_\alpha) \end{aligned}$$

sends principal series parameters for G_q to principal series parameters for the complex quantum group $SL_{q_\alpha}(2, \mathbb{C})^0$ rather than $SL_{q_\alpha}(2, \mathbb{C})$, see the remarks at the end of Subsection 5.1.1. This point will not play a significant role in what follows, thanks to the observation after Theorem 5.51 that the intertwining operators corresponding to different lifts of λ_α to $\mathbb{C}/i\hbar_\alpha^{-1}\mathbb{Z}$ coincide.

Recall from Subsection 5.8.1 that the intertwiners of $SL_{q_\alpha}(2, \mathbb{C})$ -principal series representations are always given by the right regular action of some element $T \in \mathcal{M}(\mathfrak{D}(SU_{q_\alpha}(2)))$. Since α is a simple root, we can map T to an element $T_\alpha =$

$\iota_\alpha(T) \in \mathcal{M}(\mathfrak{D}(K_q))$ as described above. We will show that the right regular action of T_α is itself an intertwiner of appropriate G_q -principal series representations.

Lemma 5.54. *Let α be a simple root, and suppose the right regular action of $T \in \mathcal{M}(\mathfrak{D}(SU_{q_\alpha}(2)))$ defines an intertwiner of $SL_{q_\alpha}(2, \mathbb{C})$ -principal series representations*

$$T : \Gamma(\mathcal{E}_{\mu_1, \lambda_1}) \rightarrow \Gamma(\mathcal{E}_{\mu_2, \lambda_2})$$

for some $(\mu_1, \lambda_1), (\mu_2, \lambda_2) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}/i\hbar_\alpha^{-1}\mathbb{Z}$. Then the right regular action of the corresponding element $T_\alpha \in \mathcal{M}(\mathfrak{D}(K_q))$ defines an intertwiner of G_q -principal series representations

$$T_\alpha : \Gamma(\mathcal{E}_{\mu' + \mu_1 \alpha, \lambda' + \lambda_1 \alpha}) \rightarrow \Gamma(\mathcal{E}_{\mu' + \mu_2 \alpha, \lambda' + \lambda_2 \alpha}),$$

for any s_α -fixed points $\mu' \in \mathfrak{h}^*$ and $\lambda' \in \mathfrak{h}_q^*$ such that $\mu' + \mu_i \alpha \in \mathbf{P}$ for $i = 1, 2$. Moreover, the intertwiner T_α is injective or surjective if and only if $T : \Gamma(\mathcal{E}_{\mu_1, \lambda_1}) \rightarrow \Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ is injective or surjective, respectively. Likewise T_α is injective on the minimal K_q -type if and only if $T : \Gamma(\mathcal{E}_{\mu_1, \lambda_1}) \rightarrow \Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ is injective on the minimal $SU_{q_\alpha}(2)$ -type.

Proof. We will appeal to Lemma 5.53. Let V and W be finite dimensional essential $\mathfrak{D}(K_q)$ -modules and let $w \in W_{\mu' + \mu_1 \alpha}$. We need to fix a weight basis for V ; let us do so by first decomposing V into $\mathfrak{D}(S_\alpha)$ -submodules, $V = \bigoplus_i V_i$, and then fixing a weight basis $(e_{i,j})_{j=1}^{N_i}$ for each V_i . Let $\epsilon_{i,j}$ denote the weight of $e_{i,j}$. Note that the weights in any fixed V_i differ by multiples of α , so that we can write

$$\epsilon_{i,j} = \epsilon'_i + k_{i,j} \alpha,$$

where $k_{i,j} = \frac{1}{2}(\epsilon_{i,j}, \alpha^\vee) \in \frac{1}{2}\mathbb{Z}$ is the weight of $e_{i,j}$ in V_i as a $\mathfrak{D}(S_\alpha)$ -module, and the orthogonal component ϵ'_i does not depend on j . The analogous orthogonal decomposition of ρ is

$$\rho = \rho' + \rho_\alpha \alpha = \rho' + \frac{1}{2}\alpha,$$

where $\rho' = \rho - \frac{1}{2}\alpha$. Note that $\frac{1}{2}\alpha$ is itself the half-sum of positive roots associated to the quantum subgroup S_α .

Now we calculate, as in Lemma 5.53,

$$\begin{aligned} & \sum_{i,j} q^{(\lambda' + \lambda_1 \alpha + 2\rho, \epsilon_{i,j})} T_\alpha(e^{i,j} \otimes w \otimes e_{i,j}) \\ &= \sum_i q^{(\lambda' + 2\rho', \epsilon'_i)} \left(\sum_j q^{(\lambda_1 \alpha + \alpha, k_{i,j} \alpha)} T_\alpha(e^{i,j} \otimes w \otimes e_{i,j}) \right) \\ &= \sum_i q^{(\lambda' + 2\rho', \epsilon'_i)} \left(\sum_j q_\alpha^{2(\lambda_1 + 1)k_{i,j}} T_\alpha(e^{i,j} \otimes w \otimes e_{i,j}) \right). \end{aligned}$$

Recall that for the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with the identification $\mathfrak{h}^* \cong \mathbb{C}$, the bilinear form on \mathfrak{h}^* is given by $(\nu_1, \nu_2) = 2\nu_1 \nu_2$ and the half-sum of positive roots is $\frac{1}{2}$. Therefore, since T defines an $SL_{q_\alpha}(2, \mathbb{C})$ -intertwiner from $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$ to $\Gamma(\mathcal{E}_{\mu_2, \lambda_2})$, Lemma 5.53 for the group $SL_{q_\alpha}(2, \mathbb{C})$ shows that the above sum equals

$$\begin{aligned} & \sum_i q^{(\lambda' + 2\rho', \epsilon'_i)} \left(\sum_j q_\alpha^{2(\lambda_2 + 1)k_{i,j}} e^{i,j} \otimes T_\alpha(w) \otimes e_{i,j} \right) \\ &= \sum_{i,j} q^{(\lambda' + \lambda_2 \alpha + 2\rho, \epsilon_{i,j})} e^{i,j} \otimes T_\alpha(w) \otimes e_{i,j}. \end{aligned}$$

Thus T_α satisfies the condition of Lemma 5.53 for G_q , and so $T_\alpha : \Gamma(\mathcal{E}_{s_\alpha \mu, s_\alpha \lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda})$ is an intertwiner of G_q -representations.

For the statement concerning injectivity, note that $\Gamma(\mathcal{E}_{\mu'+\mu_1\alpha, \lambda'+\lambda_1\alpha})$ is spanned by matrix coefficients of the form $\xi = \langle v' | \bullet | v \rangle$ with $v' \in V(\nu)^*$ and $v \in V(\nu)_{\mu'+\mu_1\alpha}$ where $\nu \in \mathbf{P}^+$, and where we may moreover assume that v belongs to some simple $\mathfrak{D}(S_\alpha)$ -submodule of $V(\nu)$. Note that v has weight μ_1 for the $\mathfrak{D}(S_\alpha)$ -action, and so the highest weight of its $\mathfrak{D}(S_\alpha)$ -submodule is equal to $|\mu_1| + n$ for some $n \in \mathbb{N}_0$. Moreover, every such highest weight in $|\mu_1| + \mathbf{N}_0$ can occur for ν sufficiently large. Since $T_\alpha(\langle v' | \bullet | v \rangle) = \langle v' | \bullet | \iota_\alpha(T) \cdot v \rangle$, it follows that T_α annihilates ξ if and only if the action of T annihilates the μ_1 -weight space of $V(|\mu_1| + n)$. Therefore T_α is injective if and only if T acts injectively on the μ_1 -weight space of every simple $\mathfrak{D}(SU_{q_\alpha}(2))$ -module, and hence if and only if $T : \Gamma(\mathcal{E}_{\mu_1, \lambda_1}) \rightarrow \Gamma(\mathcal{E}_{\mu_2, \lambda_2})$ is injective.

The statement about surjectivity is proven similarly.

Finally, suppose $\xi = \langle v' | \bullet | v \rangle$ belongs to the minimal K_q -type in $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$. Then v is an extremal weight vector in $V(\mu' + \mu_1\alpha)$, meaning that its weight lies in the Weyl orbit of the highest weight $\mu' + \mu_1\alpha$. In particular, v is annihilated by either E_α or F_α , so that it is an extremal weight vector in the $\mathfrak{D}(S_\alpha)$ -submodule it generates, which must therefore have highest weight $|\mu_1|$. Since $|\mu_1|$ is the minimal S_α -type of $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$, it follows that T_α annihilates ξ if and only if T annihilates the minimal S_α -type of $\Gamma(\mathcal{E}_{\mu_1, \lambda_1})$. \square

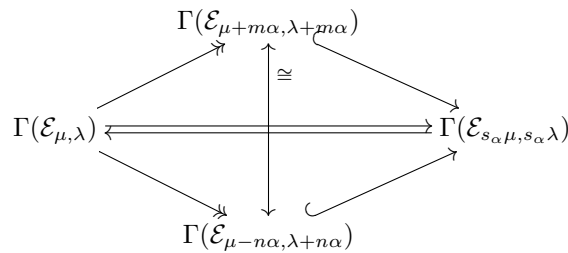
Theorem 5.55. *Fix a simple root α . For any $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ there exists a nonzero intertwiner*

$$T_\alpha : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha\mu, s_\alpha\lambda}),$$

where $T \in \mathfrak{D}(SU_{q_\alpha}(2))$ implements the intertwiner $T : \Gamma(\mathcal{E}_{\mu_\alpha, \lambda_\alpha}) \rightarrow \Gamma(\mathcal{E}_{-\mu_\alpha, -\lambda_\alpha})$ of $SL_{q_\alpha}(2, \mathbb{C})^0$ -principal series. We can identify the following particular cases:

- a) If $\lambda_\alpha \in -|\mu_\alpha| - \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$, then the intertwiner is zero on the minimal K_q -type of $\Gamma(\mathcal{E}_{\mu, \lambda})$.
- b) If $\lambda_\alpha \in |\mu_\alpha| + \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$, then the intertwiner is nonzero on the minimal K_q -type, but is not bijective.
- c) If $\pm\lambda_\alpha \notin |\mu_\alpha| + \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$, the intertwiner is bijective.

Moreover, if $\lambda_\alpha \in -(|\mu_\alpha| + \mathbb{N}) \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$, then we have additional \check{Z} elobenko intertwiners as follows:



where $m = 2\text{Re}(r_\alpha) + 1 = -\mu_\alpha - \text{Re}(\lambda_\alpha)$, $n = 2\text{Re}(l_\alpha) + 1 = \mu_\alpha - \text{Re}(\lambda_\alpha)$, and where the horizontal and vertical arrows are those from a), b) and c) above.

Proof. Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. As usual, we write $\mu' = \mu - \mu_\alpha\alpha$ for the component of μ orthogonal to α , so that

$$\mu = \mu' + \mu_\alpha\alpha, \quad s_\alpha\mu = \mu' - \mu_\alpha\alpha.$$

With λ we must be a little careful since in general λ_α is only well-defined modulo $\frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$ rather than $i\hbar_\alpha^{-1}\mathbb{Z}$. Let $\tilde{\lambda} \in \mathfrak{h}^*$ be any lift of $\lambda \in \mathfrak{h}_q^*$ and put $\tilde{\lambda}' = \tilde{\lambda} - \tilde{\lambda}_\alpha\alpha$. If we write λ' for the projection of $\tilde{\lambda}'$ in \mathfrak{h}_q^* , then

$$\lambda = \lambda' + \tilde{\lambda}_\alpha\alpha, \quad s_\alpha\lambda = \lambda' - \tilde{\lambda}_\alpha\alpha.$$

Hence applying Lemma 5.54 to the standard $SL_{q_\alpha}(2, \mathbb{C})$ -intertwiner $T : \Gamma(\mathcal{E}_{\mu_\alpha, \tilde{\lambda}_\alpha}) \rightarrow \Gamma(\mathcal{E}_{-\mu_\alpha, -\tilde{\lambda}_\alpha})$ from Theorem 5.51 (2) yields the intertwiner

$$T_\alpha : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha \mu, s_\alpha \lambda}).$$

We remark that a different choice of lift $\tilde{\lambda}$ would only alter $\tilde{\lambda}_\alpha$ by an integer multiple of $\frac{1}{2}i\hbar_\alpha^{-1}$, and so would result in the same intertwiner $T_\alpha : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha \mu, s_\alpha \lambda})$ by the remark after Theorem 5.51.

Using Lemma 5.54, the statements about bijectivity and action on minimal K_q -types follow from the analogous statements in Theorem 5.51.

The construction of the Želobenko intertwiners runs similarly. Suppose that $\lambda_\alpha \in -(|\mu_\alpha| + \mathbb{N}) \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$. As above, we lift $\lambda \in \mathfrak{h}_q^*$ to $\tilde{\lambda} \in \mathfrak{h}^*$ and write

$$\tilde{\lambda} = \tilde{\lambda}' + \tilde{\lambda}_\alpha \alpha,$$

where $\tilde{\lambda}'$ is fixed by s_α and $\tilde{\lambda}_\alpha \in -(|\mu_\alpha| + \mathbb{N}) \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$. This means we can apply Lemma 5.54 to one of the two diagrams of Želobenko intertwiners in Theorem 5.51. However, it will simplify the exposition if we arrange to have $\tilde{\lambda}_\alpha \in -(|\mu_\alpha| + \mathbb{N})$ on the nose. To this end, suppose that the imaginary part of $\tilde{\lambda}_\alpha$ is equal to $\frac{k}{2}i\hbar_\alpha^{-1}$ for some $k \in \mathbb{Z}$. Note that $\frac{1}{2}i\hbar^{-1}\alpha^\vee$ is fixed by s_α modulo $i\hbar^{-1}\mathbf{Q}^\vee$, so that if we decompose $\tilde{\lambda}$ as

$$\tilde{\lambda} = (\tilde{\lambda}' + \frac{k}{2}i\hbar^{-1}\alpha^\vee) + (\tilde{\lambda}_\alpha - \frac{k}{2}i\hbar_\alpha^{-1})\alpha,$$

then the first term is still fixed by s_α when understood as an element of \mathfrak{h}_q^* . Thus, without loss of generality, we may assume that $\tilde{\lambda}_\alpha \in -(|\mu_\alpha| + \mathbb{N})$ on the nose.

In this situation, if we define $m = -\mu_\alpha - \tilde{\lambda}_\alpha$ and $n = \mu_\alpha - \tilde{\lambda}_\alpha$, then the various principal series parameters in the diagram in the statement decompose as

$$\begin{aligned} \mu &= \mu' + \mu_\alpha \alpha, & \lambda &= \tilde{\lambda}' + \tilde{\lambda}_\alpha \alpha, \\ \mu + m\alpha &= \mu' - \tilde{\lambda}_\alpha \alpha, & \lambda + m\alpha &= \tilde{\lambda}' - \mu_\alpha \alpha, \\ \mu - n\alpha &= \mu' + \tilde{\lambda}_\alpha \alpha, & \lambda + n\alpha &= \tilde{\lambda}' + \mu_\alpha \alpha, \\ s_\alpha \mu &= \mu' - \mu_\alpha \alpha, & s_\alpha \lambda &= \tilde{\lambda}' - \tilde{\lambda}_\alpha \alpha. \end{aligned}$$

Applying Lemma 5.54 to the intertwiners in part 3a) of Theorem 5.51, we obtain the diagram of Želobenko intertwiners as claimed. \square

5.8.3. Explicit formulas for intertwiners. In this subsection, we will give explicit formulas for some of the intertwining operators between principal series representations. As usual, if $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ are parameters for the principal series, we will use $(l, r) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ to denote a pair such that

$$\mu = l - r, \quad \lambda + 2\rho = -l - r.$$

We begin with the Želobenko intertwiners. Recall from Theorem 4.14 that if $r' \in \mathfrak{h}_q^*$ is strongly linked to r , then we have an inclusion $M(r') \subset M(r)$. Recall also the algebra involution $\theta = \tau \hat{S}$ of $U_q(\mathfrak{g})$, which is given on generators by

$$\theta(E_i) = -F_i, \quad \theta(F_i) = -E_i, \quad \theta(K_\nu) = K_{-\nu},$$

see the remarks before Corollary 5.3.

Theorem 5.56. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ with associated parameters $(l, r) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$.*

a) *Suppose that $r' \in \mathfrak{h}_q^*$ is strongly linked to r and fix $Y \in U_q(\mathfrak{n}_-)$ such that $Y \cdot v_r \in M(r)$ is a primitive vector of weight r' . Then the right regular action of $\theta(Y)$ on $\Gamma(\mathcal{E}_\mu)$ defines an intertwiner*

$$\theta(Y) \rightarrow \bullet : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu', \lambda'})$$

where $\mu' = l - r'$, $\lambda' + 2\rho = -l - r'$.

- b) Suppose that $l' \in \mathfrak{h}_q^*$ is strongly linked to l and fix $Y \in U_q(\mathfrak{n}_-)$ such that $Y \cdot v_l \in M(l)$ is a primitive vector of weight l' . Then the right regular action of Y on $\Gamma(\mathcal{E}_\mu)$ defines an intertwiner

$$Y \rightarrow \bullet : \Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu',\lambda'})$$

where $\mu' = l' - r$, $\lambda' + 2\rho = -l' - r$.

Proof. a) It will be convenient to work with the involution $\theta' = \hat{S}\tau$ in place of $\theta = \tau\hat{S}$ in the calculations to follow. Note that Y is of weight $r' - r$ for the adjoint action, and hence

$$\theta'(Y) = \hat{S}^2(\theta(Y)) = K_{2\rho}\theta(Y)K_{-2\rho} = q^{(2\rho, r-r')}\theta(Y),$$

so that $\theta'(Y)$ differs from $\theta(Y)$ only by a scalar multiple. Therefore, it is equivalent to prove the claim with θ' in place of θ . Also, since $\mu' - \mu = r - r'$, the right regular action

$$\theta'(Y) \rightarrow \xi = (\theta'(Y), \xi_{(2)})\xi_{(1)}$$

does indeed map $\Gamma(\mathcal{E}_\mu)$ to $\Gamma(\mathcal{E}_{\mu'})$.

From Corollary 5.3 we have an isomorphism of $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules $M(\mu, \lambda + 2\rho) \cong M(l) \otimes M(r)$ which sends the cyclic vector $v_{\mu,\lambda}$ to $v_l \otimes v_r$. This requires a particular choice of action of $U_q^{\mathbb{R}}(\mathfrak{g})$ on $M(l) \otimes M(r)$, and in particular for $X \in U_q(\mathfrak{g})$, we have

$$(X \bowtie 1) \cdot (m \otimes n) = X_{(1)} \cdot m \otimes \theta'(X_{(2)}) \cdot n.$$

Note that $\theta'(Y) \in U_q(\mathfrak{n}_+)$, and so $\hat{\Delta}(\theta'(Y)) \in U_q(\mathfrak{n}_+) \otimes U_q(\mathfrak{b}_+)$. Since the action of $U_q(\mathfrak{n}_+)$ on v_l factors through $\hat{\epsilon}$, we get

$$\begin{aligned} (\theta'(Y) \bowtie 1) \cdot (v_l \otimes v_r) &= \theta'(Y)_{(1)} \cdot v_l \otimes \theta'(\theta'(Y)_{(2)}) \cdot v_r \\ &= v_l \otimes \theta'(\theta'(Y)) \cdot v_r \\ &= v_l \otimes Y \cdot v_r. \end{aligned}$$

Therefore the inclusion $M(l) \otimes M(r') \rightarrow M(l) \otimes M(r)$ gives rise to an inclusion of $U_q^{\mathbb{R}}(\mathfrak{g})$ -modules $j : M(\mu', \lambda' + 2\rho) \rightarrow M(\mu, \lambda + 2\rho)$ such that $j(v_{\mu', \lambda' + 2\rho}) = (\theta'(Y) \bowtie 1) \cdot v_{\mu, \lambda + 2\rho}$.

Let $\text{ext} : \Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda})$ denote the extension from the compact to the induced picture, see Lemma 5.18, so that $\text{ext}(\xi) = \xi \otimes K_{2\rho+\lambda}$, for all $\xi \in \Gamma(\mathcal{E}_{\mu,\lambda})$. For any $X \bowtie f \in U_q^{\mathbb{R}}(\mathfrak{g})$, the pairing $M(\mu, \lambda + 2\rho) \times \text{ind}_{B_q}^{G_q}(\mathbb{C}_{\mu,\lambda}) \rightarrow \mathbb{C}$ from Lemma 5.22 satisfies

$$\begin{aligned} (j((X \bowtie f) \cdot v_{\mu', \lambda' + 2\rho}), \text{ext}(\xi)) &= (f, K_{\lambda' + 2\rho})(j((X \bowtie 1) \cdot v_{\mu', \lambda' + 2\rho}), \xi \otimes K_{\lambda + 2\rho}) \\ &= (f, K_{\lambda' + 2\rho})((X\theta'(Y) \bowtie 1) \cdot v_{\mu, \lambda + 2\rho}, \xi \otimes K_{\lambda + 2\rho}) \\ &= (f, K_{\lambda' + 2\rho})(X, \xi_{(1)})(\theta'(Y), \xi_{(2)}) \\ &= ((X \bowtie f) \cdot v_{\mu', \lambda' + 2\rho}, \text{ext}(\theta'(Y) \rightarrow \xi)), \end{aligned}$$

for $\xi \in \Gamma(\mathcal{E}_{\mu,\lambda})$. Thus j is dual to the right regular action $\theta'(Y) \rightarrow \bullet : \Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu',\lambda'})$ under this pairing. By Lemma 5.22, we conclude that the right regular action of $\theta'(Y)$ is $U_q^{\mathbb{R}}(\mathfrak{g})$ -linear.

- b) The inclusion $M(l') \otimes M(r) \rightarrow M(l) \otimes M(r)$ corresponds to a map

$$j : M(\mu', \lambda' + 2\rho) \rightarrow M(\mu, \lambda + 2\rho)$$

which sends $v_{\mu', \lambda' + 2\rho}$ to $(Y \bowtie 1) \cdot v_{\mu, \lambda + 2\rho}$, and this map is dual to the right regular action of Y from $\Gamma(\mathcal{E}_{\mu,\lambda})$ to $\Gamma(\mathcal{E}_{\mu',\lambda'})$ under the pairing from Lemma 5.22. The rest of the argument is analogous to part a). \square

In particular, let us consider the Želobenko intertwiners corresponding to simple roots, that is, the diagonal arrows in the diagram from Theorem 5.55 part (3). Suppose that $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ is such that $\lambda_\alpha \in -|\mu_\alpha| - \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$. This is equivalent to the condition that l_α and r_α belong to $\frac{1}{2}\mathbb{N}_0 \bmod \frac{1}{4}i\hbar_\alpha^{-1}\mathbb{Z}$. Therefore, putting $l' = l - 2\text{Re}(l_\alpha)\alpha$ and $r' = r - 2\text{Re}(r_\alpha)\alpha$ we have inclusions of Verma modules

$$M(l') \subset M(l), \quad M(r') \subset M(r).$$

Explicitly, $F_\alpha^{2\text{Re}(l_\alpha)} \cdot v_l$ is a primitive vector of weight l' in $M(l)$ and $F_\alpha^{2\text{Re}(r_\alpha)} \cdot v_r$ is a primitive vector of weight r' in $M(r)$. Therefore, an application of Theorem 5.56 gives the diagram of intertwiners

$$\begin{array}{ccc}
 & \Gamma(\mathcal{E}_{\mu+m\alpha, \lambda+m\alpha}) & \\
 E_\alpha^m \nearrow & \uparrow \cong & \searrow F_\alpha^n \\
 \Gamma(\mathcal{E}_{\mu, \lambda}) & \longleftrightarrow & \Gamma(\mathcal{E}_{s_\alpha\mu, s_\alpha\lambda}) \\
 F_\alpha^n \searrow & \downarrow & \nearrow E_\alpha^m \\
 & \Gamma(\mathcal{E}_{\mu-n\alpha, \lambda+n\alpha}) &
 \end{array} \tag{5.1}$$

Next, we give an explicit formula for the standard intertwiners $T_\alpha : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha\mu, s_\alpha\lambda})$ corresponding to a simple root, as in Theorem 5.55 (2).

We begin with the case of $G_q = SL_q(2, \mathbb{C})$. Let $(\mu, \lambda) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}/i\hbar^{-1}\mathbb{Z}$. Recall that we have

$$\Gamma(\mathcal{E}_{\mu, \lambda}) \cong \bigoplus_{n \in |\mu| + \mathbb{N}_0} V(n)^* \otimes V(n)_\mu.$$

and therefore any $\mathfrak{D}(K_q)$ -linear map $T : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ is specified uniquely by a sequence of linear maps

$$T_n : V(n)_\mu \rightarrow V(n)_{-\mu}$$

for all $n \in |\mu| + \mathbb{N}_0$. Since the weight spaces of simple $\mathfrak{D}(SU_q(2))$ -modules are all one dimensional, the map T_n is specified by a scalar once we have fixed a weight basis of $V(n)$.

For this, we will use the orthonormal weight basis with respect to the invariant inner product on $V(n)$. In comparison to the weight basis $v_n, v_{n-1}, \dots, v_{-n}$ from Theorem 2.38, the orthonormal basis is given by

$$e_m^n = q^{\frac{1}{2}(n-m)(n+m-1)} \left(\frac{[n+m]_q!}{[2n]_q! [n-m]_q!} \right)^{\frac{1}{2}} v_m.$$

where $m = -n, -n+1, \dots, n$.

To see that it is indeed orthonormal for the invariant inner product, one can first confirm that the action of $U_q(\mathfrak{sl}(2))$ on this basis is given by

$$\begin{aligned}
 K \cdot e_m^n &= q^m e_m^n, & K_\alpha \cdot e_m^n &= q^{2m} e_m^n, \\
 E \cdot e_m^n &= q^m [n-m]_q^{\frac{1}{2}} [n+m+1]_q^{\frac{1}{2}} e_{m+1}^n, \\
 F \cdot e_m^n &= q^{-(m-1)} [n+m]_q^{\frac{1}{2}} [n-m+1]_q^{\frac{1}{2}} e_{m-1}^n,
 \end{aligned}$$

and then check these formulas are compatible with the involution on $U_q^{\mathbb{R}}(\mathfrak{su}(2))$ when they are declared to be orthonormal, compare Section 3.2.1 in [48]. We point out that one must be careful with statements about unitary modules in [48], since there are different versions of the enveloping algebra in play, and their preferred real forms do not always agree under the Hopf algebra morphisms they use.

We can then specify a $\mathfrak{D}(K_q)$ -linear operator $T : \Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu,-\lambda})$ by the scalars $(T_n)_{n \in |\mu| + \mathbb{N}_0}$ such that $T(e_\mu^n) = T_n e_{-\mu}^n$. Our reason for working in this basis is that we will be using the Clebsch-Gordan formulas from [48]. Looking at Lemma 5.53, we will need to consider the action of T on tensor products of the form

$$e_{\pm \frac{1}{2}}^{\frac{1}{2}\vee} \otimes e_\mu^n \otimes e_{\pm \frac{1}{2}}^{\frac{1}{2}}$$

for $n \in |\mu| + \mathbb{N}_0$. There is a double multiplicity of the representation $V(n)$ in the tensor product $V(\frac{1}{2}) \otimes V(n) \otimes V(\frac{1}{2})$, and to deal with this we will write

$$V(\frac{1}{2}) \otimes (V(n) \otimes V(\frac{1}{2})) \cong V(n+1) \oplus V(n^+) \oplus V(n^-) \oplus V(n-1),$$

where $V(n^\pm)$ signifies the copy of $V(n)$ contained in $V(\frac{1}{2}) \otimes V(n \pm \frac{1}{2})$, respectively.

The Clebsch-Gordan coefficients for tensor products with the fundamental representation are as follows:

$$\begin{aligned} e_\mu^n \otimes e_{\frac{1}{2}}^{\frac{1}{2}} &= q^{\frac{1}{2}(n-\mu)} \frac{[n+\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu+\frac{1}{2}}^{n+\frac{1}{2}} - q^{-\frac{1}{2}(n+\mu+1)} \frac{[n-\mu]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu+\frac{1}{2}}^{n-\frac{1}{2}}, \\ e_\mu^n \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} &= q^{-\frac{1}{2}(n+\mu)} \frac{[n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu-\frac{1}{2}}^{n+\frac{1}{2}} + q^{\frac{1}{2}(n-\mu+1)} \frac{[n+\mu]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu-\frac{1}{2}}^{n-\frac{1}{2}}, \\ e_{\frac{1}{2}}^{\frac{1}{2}} \otimes e_\mu^n &= q^{-\frac{1}{2}(n-\mu)} \frac{[n+\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu+\frac{1}{2}}^{n+\frac{1}{2}} + q^{\frac{1}{2}(n+\mu+1)} \frac{[n-\mu]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu+\frac{1}{2}}^{n-\frac{1}{2}}, \\ e_{-\frac{1}{2}}^{\frac{1}{2}} \otimes e_\mu^n &= q^{\frac{1}{2}(n+\mu)} \frac{[n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu-\frac{1}{2}}^{n+\frac{1}{2}} - q^{-\frac{1}{2}(n-\mu+1)} \frac{[n+\mu]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} e_{\mu-\frac{1}{2}}^{n-\frac{1}{2}}. \end{aligned}$$

For this, see e.g. Equations (3.68)–(3.69) in [48], but note the typographical error in Equation (3.68), where the factor $[l \pm j + \frac{1}{2}]^{\frac{1}{2}}$ should be $[l \pm j + 1]^{\frac{1}{2}}$.

For $SU_q(2)$, the precontragredient representation ${}^cV(n)$ is isomorphic to $V(n)$. In particular, for the fundamental representation the following map is an isomorphism:

$$e_{\frac{1}{2}}^{\frac{1}{2}\vee} \mapsto iq^{-\frac{1}{2}} e_{-\frac{1}{2}}^{\frac{1}{2}}, \quad e_{-\frac{1}{2}}^{\frac{1}{2}\vee} \mapsto -iq^{\frac{1}{2}} e_{\frac{1}{2}}^{\frac{1}{2}}.$$

Combining this with the Clebsch-Gordan formulae above gives

$$\begin{aligned} e_{\frac{1}{2}}^{\frac{1}{2}\vee} \otimes (e_\mu^n \otimes e_{\frac{1}{2}}^{\frac{1}{2}}) &= iq^{-\frac{1}{2}} \left(e_{-\frac{1}{2}}^{\frac{1}{2}} \otimes (e_\mu^n \otimes e_{\frac{1}{2}}^{\frac{1}{2}}) \right) \\ &= iq^{\frac{1}{2}(n-\mu-1)} \frac{[n+\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} \left(e_{-\frac{1}{2}}^{\frac{1}{2}} \otimes e_{\mu+\frac{1}{2}}^{n+\frac{1}{2}} \right) \\ &\quad - iq^{-\frac{1}{2}(n+\mu+2)} \frac{[n+\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} \left(e_{-\frac{1}{2}}^{\frac{1}{2}} \otimes e_{\mu+\frac{1}{2}}^{n-\frac{1}{2}} \right) \\ &= iq^n \frac{[n+\mu+1]^{\frac{1}{2}} [n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+1} - iq^{-1} \frac{[n+\mu+1]}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+} \\ &\quad - iq^{-1} \frac{[n-\mu]}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-} + iq^{-n-1} \frac{[n+\mu]^{\frac{1}{2}} [n-\mu]^{\frac{1}{2}}}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-1}. \\ e_{-\frac{1}{2}}^{\frac{1}{2}\vee} \otimes (e_\mu^n \otimes e_{-\frac{1}{2}}^{\frac{1}{2}}) &= -iq^{\frac{1}{2}} \left(e_{\frac{1}{2}}^{\frac{1}{2}} \otimes (e_\mu^n \otimes e_{-\frac{1}{2}}^{\frac{1}{2}}) \right) \\ &= -iq^{\frac{1}{2}(-n-\mu+1)} \frac{[n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} \left(e_{\frac{1}{2}}^{\frac{1}{2}} \otimes e_{\mu-\frac{1}{2}}^{n+\frac{1}{2}} \right) \\ &\quad - iq^{\frac{1}{2}(n-\mu+2)} \frac{[n+\mu]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}}} \left(e_{\frac{1}{2}}^{\frac{1}{2}} \otimes e_{\mu-\frac{1}{2}}^{n-\frac{1}{2}} \right) \\ &= -iq^{-n} \frac{[n+\mu+1]^{\frac{1}{2}} [n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+1} - iq \frac{[n-\mu+1]}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+} \\ &\quad - iq \frac{[n+\mu]}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-} - iq^{n+1} \frac{[n+\mu]^{\frac{1}{2}} [n-\mu]^{\frac{1}{2}}}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-1}. \end{aligned}$$

This allows us to write the sums that appear in Lemma 5.53 as

$$\begin{aligned}
& \sum_{j=\pm\frac{1}{2}} q^{(\lambda+2\rho, \epsilon_j)} e_j^{\frac{1}{2}\vee} \otimes e_\mu^n \otimes e_j^{\frac{1}{2}} \\
&= q^{\lambda+1} e_{\frac{1}{2}}^{\frac{1}{2}\vee} \otimes e_\mu^n \otimes e_{\frac{1}{2}}^{\frac{1}{2}} + q^{-\lambda-1} e_{-\frac{1}{2}}^{\frac{1}{2}\vee} \otimes e_\mu^n \otimes e_{-\frac{1}{2}}^{\frac{1}{2}} \\
&= i(q^{n+\lambda+1} - q^{-n-\lambda-1}) \frac{[n+\mu+1]^{\frac{1}{2}} [n-\mu+1]^{\frac{1}{2}}}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+1} \\
&\quad - i \frac{q^\lambda [n+\mu+1] + q^{-\lambda} [n-\mu+1]}{[2n+1]^{\frac{1}{2}} [2n+2]^{\frac{1}{2}}} e_\mu^{n+} \\
&\quad - i \frac{q^\lambda [n-\mu] + q^{-\lambda} [n+\mu]}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-} \\
&\quad + i(q^{-n+\lambda} - q^{n-\lambda}) \frac{[n+\mu]^{\frac{1}{2}} [n-\mu]^{\frac{1}{2}}}{[2n]^{\frac{1}{2}} [2n+1]^{\frac{1}{2}}} e_\mu^{n-1} \\
&= A e_\mu^{n+1} + B e_\mu^{n+} + C e_\mu^{n-} + D e_\mu^{n-1},
\end{aligned}$$

with coefficients

$$\begin{aligned}
A &= A(\mu, \lambda, n) = i(q - q^{-1}) \frac{[n + \lambda + 1][n + \mu + 1]^{\frac{1}{2}} [n - \mu + 1]^{\frac{1}{2}}}{[2n + 1]^{\frac{1}{2}} [2n + 2]^{\frac{1}{2}}}, \\
B &= B(\mu, \lambda, n) = -i \frac{q^\lambda [n + \mu + 1] + q^{-\lambda} [n - \mu + 1]}{[2n + 1]^{\frac{1}{2}} [2n + 2]^{\frac{1}{2}}}, \\
C &= C(\mu, \lambda, n) = -i \frac{q^\lambda [n - \mu] + q^{-\lambda} [n + \mu]}{[2n]^{\frac{1}{2}} [2n + 1]^{\frac{1}{2}}}, \\
D &= D(\mu, \lambda, n) = -i(q - q^{-1}) \frac{[n - \lambda][n + \mu]^{\frac{1}{2}} [n - \mu]^{\frac{1}{2}}}{[2n]^{\frac{1}{2}} [2n + 1]^{\frac{1}{2}}}.
\end{aligned}$$

Therefore, the intertwiner condition in Lemma 5.53 reduces to the following four conditions on the scalars T_n for $n \in |\mu_1| + \mathbb{N}_0$:

$$\begin{aligned}
T_{n+1}A(\mu_1, \lambda_1, n) &= T_n A(\mu_2, \lambda_2, n), \\
T_n B(\mu_1, \lambda_1, n) &= T_n B(\mu_2, \lambda_2, n), \\
T_n C(\mu_1, \lambda_1, n) &= T_n C(\mu_2, \lambda_2, n), \\
T_{n-1}D(\mu_1, \lambda_1, n) &= T_n D(\mu_2, \lambda_2, n).
\end{aligned}$$

Note that when $n = |\mu_1|$ we have $D(\mu_1, \lambda_1, n) = 0$, so that the last equation should be interpreted as saying $T_n D(\mu_2, \lambda_2, n) = 0$ in this case.

The following theorem gives explicit formulas for the standard intertwiners from part 2 of Theorem 5.51.

Theorem 5.57. *Let $(\mu, \lambda) \in \frac{1}{2}\mathbb{Z} \times \mathbb{C}/i\hbar^{-1}\mathbb{Z}$. Up to scalar, the unique intertwiner of $SL_q(2, \mathbb{C})$ -principal series $T : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ acts on matrix coefficients as*

$$T(\langle v' | \bullet | e_\mu^n \rangle) = T_n \langle v' | \bullet | e_{-\mu}^n \rangle$$

where the coefficients $(T_n)_{n \in |\mu| + \mathbb{N}_0}$ are as follows:

a) If $\lambda \in -|\mu| - \mathbb{N} \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$, we have

$$T_n = \begin{cases} 0, & |\mu| \leq n < -\operatorname{Re}(\lambda), \\ \prod_{k=-\operatorname{Re}(\lambda)+1}^n \frac{[k-\lambda]_q}{[k+\lambda]_q}, & n \geq -\operatorname{Re}(\lambda). \end{cases}$$

b) Otherwise, we have

$$T_n = \prod_{k=|\mu|+1}^n \frac{[k-\lambda]_q}{[k+\lambda]_q}.$$

Proof. We know that such intertwiners exist from Theorem 5.51 (2), so it only remains to deduce the explicit formulas for the coefficients T_n from the recurrence relations stated before the theorem. With $(\mu_1, \lambda_1) = (\mu, \lambda) = (-\mu_2, -\lambda_2)$, the last equation before the theorem reduces to

$$T_n[n + \lambda]_q = T_{n-1}[n - \lambda]_q$$

for all $n \in |\mu| + \mathbb{N}$. Note that the q -numbers $[n \pm \lambda]_q$ are all nonzero except for at most one value as follows:

- if $\lambda \in -|\mu| - \mathbb{N} \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$, then $[n + \lambda]_q = 0$ for $n = -\text{Re}(\lambda)$;
- if $\lambda \in |\mu| + \mathbb{N} \bmod \frac{1}{2}i\hbar^{-1}\mathbb{Z}$, then $[n - \lambda]_q = 0$ for $n = \text{Re}(\lambda)$;
- otherwise, they are all nonzero.

In every case, we observe that the recurrence relation will have a unique solution up to an overall scalar multiple, and a direct check shows that the values of T_n stated in the Theorem are indeed solutions. This completes the proof. \square

We note that the intertwiners $T : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ given in part b) of Theorem 5.57 form a meromorphic family of operators $\Gamma(\mathcal{E}_{\mu}) \rightarrow \Gamma(\mathcal{E}_{-\mu})$ as a function of λ , with poles at the points where $\lambda \in -|\mu| - \mathbb{N} + \frac{1}{2}i\hbar^{-1}\mathbb{Z}$. We also point out, when $\lambda \in |\mu| + \mathbb{N} + \frac{1}{2}i\hbar^{-1}\mathbb{Z}$, the coefficients T_n in b) are zero for all $n \geq \text{Re}(\lambda)$, so that the intertwiner T is indeed finite rank in this case.

Using Theorem 5.55, the results of Theorem 5.57 immediately give an explicit formula for the intertwiners in higher rank associated to a simple reflection. To specify this formula, we will make use of elements $\text{Ph}(E_\alpha)$ and $\text{Ph}(F_\alpha)$ in $\mathcal{M}(\mathfrak{D}(K_q))$ which are defined as the operator phases in the polar decomposition of E_α and F_α , respectively. Explicitly, if $V \cong V(n)$ is a $\mathfrak{D}(SU_{q_\alpha}(2))$ -submodule of highest weight $n \in \frac{1}{2}\mathbb{N}_0$ inside a $\mathfrak{D}(K_q)$ -module, $\text{Ph}(E_\alpha)$ and $\text{Ph}(F_\alpha)$ act as

$$\text{Ph}(E_\alpha) \cdot e_m^n = e_{m+1}^n, \quad \text{Ph}(F_\alpha) \cdot e_m^n = e_{m-1}^n,$$

where e_m^n, \dots, e_{-m}^n denotes an orthonormal weight basis of V isomorphic to that described above for $V(n)$. In order to avoid having to deal with different cases, we will abuse notation and write, for any $k \in \mathbb{Z}$

$$\text{Ph}(F_\alpha)^k = \begin{cases} \text{Ph}(F_\alpha)^k & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \text{Ph}(E_\alpha)^{-k} & \text{if } k < 0. \end{cases}$$

We can now give formulas for the standard intertwiners corresponding to simple reflections from Theorem 5.55 a), b) and c).

Corollary 5.58. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and fix a simple root α . The intertwiner $T : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha \mu, s_\alpha \lambda})$ acts on matrix coefficients by*

$$T(\langle v' | \bullet | v \rangle) = T_n \langle v' | \bullet | \text{Ph}(F_\alpha)^{2\mu_\alpha} v \rangle$$

whenever v belongs to a $\mathfrak{D}(SU_{q_\alpha}(2))$ -submodule of highest weight n , and where the coefficients $T_n \in \mathbb{C}$ are as follows:

a) If $\lambda_\alpha \in -|\mu_\alpha| - \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$,

$$T_n = \begin{cases} 0, & |\mu_\alpha| \leq n < -\text{Re}(\lambda_\alpha), \\ \prod_{k=-\text{Re}(\lambda_\alpha)+1}^n \frac{[k - \lambda_\alpha]_{q_\alpha}}{[k + \lambda_\alpha]_{q_\alpha}}, & n \geq -\text{Re}(\lambda_\alpha). \end{cases}$$

b) Otherwise,

$$T_n = \prod_{k=|\mu_\alpha|+1}^n \frac{[k - \lambda_\alpha]_{q_\alpha}}{[k + \lambda_\alpha]_{q_\alpha}}.$$

5.9. Submodules and quotient modules of the principal series. In this section we will elaborate on the structure of the principal series representation $\Gamma(\mathcal{E}_{\mu,\lambda})$. We closely follow the proofs of the corresponding classical results in [26].

For $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ and $\alpha \in \Delta^+$ any positive root, we continue to use the notation

$$\mu_\alpha = \frac{1}{2}(\mu, \alpha^\vee) \in \frac{1}{2}\mathbb{Z}, \quad \lambda_\alpha = \frac{1}{2}(\lambda, \alpha^\vee) \in \mathfrak{h}^*/\frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z},$$

as in the previous section.

Recall that for any $w \in W$ we define

$$S(w) = \{\alpha \in \Delta^+ \mid w\alpha \in \Delta^-\} = \Delta^+ \cap w^{-1}\Delta^-,$$

and note that $l(w) = |S(w)|$.

Proposition 5.59. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$ with associated parameters $(l, r) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ as usual. Let $w \in W$ and suppose that for all $\alpha \in S(w)$ we have*

$$\pm\lambda_\alpha \notin |\mu_\alpha| + \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}.$$

Then $\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \Gamma(\mathcal{E}_{w\mu,w\lambda})$

Proof. We proceed by induction on the length of w . If $w = 1$, the result is obvious. So suppose that $l(w) \geq 1$, and write $w = s_\alpha v$ for some simple root α and some $v \in W$ with $l(w) = l(v) + 1$. In this case, we have $S(w) = S(v) \cup \{v^{-1}\alpha\}$. In particular, by the inductive hypothesis we have $\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \Gamma(\mathcal{E}_{v\mu,v\lambda})$.

Further, since $v^{-1}\alpha \in S(w)$, the hypothesis gives that

$$\pm\frac{1}{2}(\lambda, v^{-1}\alpha^\vee) \notin |\frac{1}{2}(\mu, v^{-1}\alpha^\vee)| + \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z},$$

and hence $\pm(v\lambda)_\alpha \notin |(v\mu)_\alpha| + \mathbb{N} \bmod \frac{1}{2}i\hbar_\alpha^{-1}\mathbb{Z}$. Therefore, by Theorem 5.55, we have a bijective intertwiner from $\Gamma(\mathcal{E}_{v\mu,v\lambda})$ to $\Gamma(\mathcal{E}_{w\mu,w\lambda})$. The result follows. \square

The next two results are quantum analogues of Theorems I.4.3 and I.4.2 in [26].

Theorem 5.60. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. Every nonzero submodule of $\Gamma(\mathcal{E}_{\mu,\lambda})$ contains the minimal K_q -type if and only if $q_\alpha^{(\lambda, \alpha^\vee)} \notin q_\alpha^{|\mu, \alpha^\vee| + 2\mathbb{N}}$ for every positive root $\alpha \in \Delta^+$. In this case, the submodule generated by the minimal K_q -type is equal to $V_{\mu,\lambda}$, and this is the only simple submodule of $\Gamma(\mathcal{E}_{\mu,\lambda})$.*

Proof. We fix a pair $(l, r) \in \mathfrak{h}_q^* \times \mathfrak{h}_q^*$ with $\mu = l - r$ and $\lambda + 2\rho = -l - r$ as usual. Suppose that $q_\alpha^{(\lambda, \alpha^\vee)} \notin q_\alpha^{|\mu, \alpha^\vee| + 2\mathbb{N}}$ for every positive root $\alpha \in \Delta^+$. By Lemma 5.41, this is equivalent to saying that for every $\alpha \in \Delta^+$, the quantities $q_\alpha^{(l+\rho, \alpha^\vee)}$ and $q_\alpha^{(r+\rho, \alpha^\vee)}$ do not both lie in $\pm q_\alpha^{-\mathbb{N}}$.

Recall that we write $\text{Re}(l)$ for the component of l in the \mathbb{R} -span of Δ , see the remarks before Proposition 4.5. To begin with, let us suppose that the real part of $l + \rho$ lies in the closure of the dominant Weyl chamber, meaning that

$$(\text{Re}(l) + \rho, \alpha^\vee) \geq 0$$

for all $\alpha \in \Delta^+$. This means in particular that l is dominant. According to Lemma 5.45 b), there exists $\hat{u} \in \hat{W}$ such that, if we put $r' = \hat{u}.r$, we have $\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \mathcal{F}_l(M(r')^\vee)$ and there is no $\alpha \in \Delta^+$ with both $q_\alpha^{(l+\rho, \alpha^\vee)} = \pm 1$ and $q_\alpha^{(r'+\rho, \alpha^\vee)} \in \pm q_\alpha^{\mathbb{N}}$.

Let H be a submodule of $\mathcal{F}_l(M(r')^\vee)$. By Proposition 5.32, we have a nonzero $U_q(\mathfrak{g})$ -linear map $\varphi : \mathcal{T}_l(H) \rightarrow M(r')^\vee$. Every nontrivial quotient module of $M(r')^\vee$ projects to $V(r')$, and consequently every nontrivial submodule of $M(r')^\vee$ contains $V(r')$. In particular, $V(r') \subset \text{im}(\varphi)$.

Using Lemma 5.45 b) and applying the exact functor \mathcal{F}_l , we obtain an inclusion

$$V_{\mu,\lambda} \cong \mathcal{F}_l(V(r')) \subset \mathcal{F}_l(\text{im}(\varphi)) \cong \text{im}(\mathcal{F}_l(\varphi)).$$

By Proposition 5.33, the inclusion of H into $\mathcal{F}_l(M(r')^\vee)$ factorizes as

$$H \cong \mathcal{F}_l(T_l(H)) \xrightarrow{\mathcal{F}_l(\varphi)} \mathcal{F}_l(M(r')^\vee),$$

so that $\text{im}(\mathcal{F}_l(\varphi)) = H$ and so the inclusion above shows that $V_{\mu,\lambda}$ is isomorphic to a submodule of H . By considering minimal K_q -types, it follows that $V_{\mu,\lambda}$ is indeed a submodule of H as claimed.

Now consider the case of arbitrary l . We can find $w \in W$ such that the real part of $w(l + \rho)$ belongs to the closure of the dominant Weyl chamber. For any $\alpha \in S(w)$ we have $w\alpha \in \Delta^-$, so that

$$(\text{Re}(l) + \rho, \alpha^\vee) = (w(\text{Re}(l) + \rho), w\alpha^\vee) \leq 0.$$

In particular, according to Lemma 5.41, we do not have $q_\alpha^{(\lambda, \alpha^\vee)} \in q_\alpha^{-|(\mu, \alpha^\vee)| - 2\mathbb{N}}$, and by hypothesis we do not have $q_\alpha^{(\lambda, \alpha^\vee)} \in q_\alpha^{|\mu, \alpha^\vee| + 2\mathbb{N}}$. Therefore, Proposition 5.59 shows that $\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \Gamma(\mathcal{E}_{w\mu, w\lambda})$. By the previous case, every nonzero submodule of $\Gamma(\mathcal{E}_{w\mu, w\lambda})$ contains $V_{w\mu, w\lambda}$, and the result follows.

In this situation, the intersection of all nonzero submodules of $\Gamma(\mathcal{E}_{\mu,\lambda})$ contains the minimal K_q -type, and in particular is nonzero. It is necessarily simple, and so must be equal to $V_{\mu,\lambda}$. Moreover, there can be no other simple submodule, since otherwise the above intersection would be zero.

Conversely, suppose that there exists $\alpha \in \Delta^+$ with $q_\alpha^{(\lambda, \alpha^\vee)} \in q_\alpha^{|\mu, \alpha^\vee| + 2\mathbb{N}}$, and so $q_\alpha^{(l+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}}$ and $q_\alpha^{(r+\rho, \alpha^\vee)} \in \pm q_\alpha^{-\mathbb{N}}$, by Lemma 5.41. In this case, r is strongly linked to $r' = s_{k,\alpha}.r$ for some $k \in \mathbb{Z}_2$, see Definition 4.13 and the remark which follows it. By Theorem 5.56 we have a nonzero intertwiner $T : \Gamma(\mathcal{E}_{\mu', \lambda'}) \rightarrow \Gamma(\mathcal{E}_{\mu, \lambda})$ where $\mu' = l - r'$ and $\lambda' + 2\rho = -l - r'$.

We have $(\text{Re}(l) + \rho, \alpha^\vee) \in -\mathbb{N}$ and $(\text{Re}(r) + \rho, \alpha^\vee) \in -\mathbb{N}$, so $(s_\alpha.\text{Re}(l) + \rho, \alpha^\vee) \in \mathbb{N}$ and $(s_\alpha.\text{Re}(r) + \rho, \alpha^\vee) \in \mathbb{N}$. Thus, neither $\text{Re}(l)$ nor $\text{Re}(r)$ is fixed by the shifted action of s_α , and an application of Lemma 5.44 shows that $\|\mu'\| > \|\mu\|$. Therefore, $\text{im}(T)$, is a submodule of $\Gamma(\mathcal{E}_{\mu,\lambda})$ which does not contain the minimal K_q -type. This completes the proof. \square

Corollary 5.61. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. The principal series representation $\Gamma(\mathcal{E}_{\mu,\lambda})$ is generated by its subspace of minimal K_q -type if and only if $q_\alpha^{(\lambda, \alpha^\vee)} \notin q_\alpha^{-|(\mu, \alpha^\vee)| - 2\mathbb{N}}$ for every positive root $\alpha \in \Delta^+$.*

Proof. First suppose that $q_\alpha^{(\lambda, \alpha^\vee)} \notin q_\alpha^{-|(\mu, \alpha^\vee)| - 2\mathbb{N}}$ for every positive root $\alpha \in \Delta^+$. Let $V \subset \Gamma(\mathcal{E}_{\mu,\lambda})$ be the submodule generated by the minimal K_q -type. Using the invariant pairing from Lemma 5.21, we can define a submodule of $\Gamma(\mathcal{E}_{-\mu, -\lambda})$ by

$$V^\perp = \{\eta \in \Gamma(\mathcal{E}_{-\mu, -\lambda}) \mid \phi(\xi\eta) = 0 \text{ for all } \xi \in V\}.$$

The pairing decomposes as a sum of nondegenerate pairings between the subspaces of each K_q -type. Since V contains the minimal K_q -type, V^\perp does not contain the minimal K_q -type of $\Gamma(\mathcal{E}_{-\mu, -\lambda})$. Note, however, that $\Gamma(\mathcal{E}_{-\mu, -\lambda})$ fulfils the conditions of Theorem 5.60, so we must have $V^\perp = \{0\}$, and hence $V = \Gamma(\mathcal{E}_{\mu,\lambda})$.

Conversely, suppose that $\Gamma(\mathcal{E}_{\mu,\lambda})$ is generated by its subspace of minimal K_q -type. Let $U \subset \Gamma(\mathcal{E}_{-\mu, -\lambda})$ be a submodule which does not contain the minimal K_q -type. Then the module

$$U^\perp = \{\xi \in \Gamma(\mathcal{E}_{\mu,\lambda}) \mid \phi(\xi\eta) = 0 \text{ for all } \eta \in \Gamma(\mathcal{E}_{-\mu, -\lambda})\} \subset \Gamma(\mathcal{E}_{\mu,\lambda})$$

contains the minimal K_q -type, and hence $U^\perp = \Gamma(\mathcal{E}_{\mu,\lambda})$ by hypothesis. Therefore $U = 0$, and we deduce that every nonzero submodule of $\Gamma(\mathcal{E}_{-\mu, -\lambda})$ contains the minimal K_q -type. Theorem 5.60 implies that $q_\alpha^{(-\lambda, \alpha^\vee)} \notin q_\alpha^{|(-\mu, \alpha^\vee)| + 2\mathbb{N}}$ for every positive root $\alpha \in \Delta^+$, and the result follows. \square

5.10. Unitary representations. In this section we briefly comment on the question of unitarizability of Harish-Chandra modules. For more information we refer to the work of Arano [2], [3].

Let us write $\mathfrak{a}^* \subset \mathfrak{h}^*$ for the \mathbb{R} -span of the roots Δ , and $\mathfrak{t}_q^* = i\mathfrak{a}^*/i\hbar^{-1}\mathbf{Q}^\vee$. Recall that for $\lambda \in \mathfrak{h}_q^*$ we have a decomposition $\lambda = \operatorname{Re}(\lambda) + i\operatorname{Im}(\lambda)$ where $\operatorname{Re}(\lambda) \in \mathfrak{a}^*$ and $i\operatorname{Im}(\lambda) \in \mathfrak{t}_q^*$. Accordingly we write $\bar{\lambda} = \operatorname{Re}(\lambda) - i\operatorname{Im}(\lambda)$. Note that the characters $K_\lambda \in \mathcal{M}(\mathfrak{D}(K_q))$ with $\lambda \in \mathfrak{h}_q^*$, as defined in Subsection 5.1.1, satisfy $K_\lambda^* = K_{\bar{\lambda}}$.

Proposition 5.62. *Let $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{h}_q^*$. The standard inner product on $\mathfrak{C}^\infty(K_q)$ defined by $\langle \xi, \eta \rangle = \phi(\xi^* \eta)$, where ϕ is the Haar functional, restricts to a non-degenerate sesquilinear pairing between $\Gamma(\mathcal{E}_{\mu, -\bar{\lambda}})$ and $\Gamma(\mathcal{E}_{\mu, \lambda})$ which is G_q -invariant in the sense that*

$$\langle \xi, \pi_{\mu, \lambda}(x \bowtie f) \eta \rangle = \langle \pi_{\mu, -\bar{\lambda}}((x \bowtie f)^*) \xi, \eta \rangle$$

for all $x \bowtie f \in \mathfrak{D}(G_q)$, $\xi \in \Gamma(\mathcal{E}_{\mu, -\bar{\lambda}})$, $\eta \in \Gamma(\mathcal{E}_{\mu, \lambda})$.

Proof. Consider the bijective conjugate-linear map $*$: $\Gamma(\mathcal{E}_{\mu, -\bar{\lambda}}) \rightarrow \Gamma(\mathcal{E}_{-\mu, -\lambda})$ which sends ξ to ξ^* . We claim that for any $x \bowtie f \in \mathfrak{D}(G_q)$, we have

$$* \circ \pi_{\mu, -\bar{\lambda}}((x \bowtie f)^*) = \pi_{-\mu, -\lambda}(\hat{S}_{G_q}^{-1}(x \bowtie f)) \circ *.$$

It suffices to prove this for elements of the form $x \bowtie 1$ and $1 \bowtie f$ with $x \in \mathfrak{D}(K_q)$ and $f \in \mathfrak{C}^\infty(K_q)$. Let $\xi \in \Gamma(\mathcal{E}_{-\mu, -\lambda})$. Using the compatibility of the $*$ -structures, as recorded after Definition 3.15, we calculate

$$\begin{aligned} \pi_{-\mu, -\lambda}(\hat{S}_{G_q}^{-1}(x \bowtie 1))(\xi^*) &= (x, \xi_{(1)}^*) \xi_{(2)}^* \\ &= \overline{(\hat{S}(x^*), \xi_{(1)})} \xi_{(2)}^* \\ &= (\pi_{\mu, -\bar{\lambda}}(x^* \bowtie 1) \xi)^*, \end{aligned}$$

and similarly,

$$\begin{aligned} \pi_{-\mu, -\lambda}(\hat{S}_{G_q}^{-1}(1 \bowtie f))(\xi^*) &= S^{-1}(f_{(3)}) \xi^* f_{(1)} (K_{2\rho-\lambda}, S^{-1}(f_{(2)})) \\ &= S^{-1}(f_{(3)}) \xi^* f_{(1)} (\hat{S}(K_{2\rho-\bar{\lambda}}^*), f_{(2)}) \\ &= (f_{(1)}^* \xi S(f_{(3)}^*))^* \overline{(K_{2\rho-\bar{\lambda}}, f_{(2)}^*)} \\ &= (\pi_{\mu, -\bar{\lambda}}(1 \bowtie f^*) \xi)^*, \end{aligned}$$

which proves the claim.

Therefore, using the invariance property of the non-degenerate bilinear pairing $(,) : \Gamma(\mathcal{E}_{-\mu, -\lambda}) \times \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \mathbb{C}$ from Lemma 5.21 we have

$$\begin{aligned} \langle \xi, \pi_{\mu, \lambda}(x \bowtie f) \eta \rangle &= (\xi^*, \pi_{\mu, \lambda}(x \bowtie f) \eta) \\ &= (\pi_{-\mu, -\lambda}(\hat{S}_{G_q}^{-1}(x \bowtie f))(\xi^*), \eta) \\ &= ((\pi_{\mu, -\bar{\lambda}}((x \bowtie f)^*) \xi)^*, \eta) \\ &= \langle \pi_{\mu, -\bar{\lambda}}((x \bowtie f)^*) \xi, \eta \rangle, \end{aligned}$$

for any $\xi \in \Gamma(\mathcal{E}_{\mu, -\bar{\lambda}})$ and $\eta \in \Gamma(\mathcal{E}_{\mu, \lambda})$ and $x \bowtie f \in \mathfrak{D}(G_q)$, as claimed. \square

A nondegenerate $\mathfrak{D}(G_q)$ -module V is *unitarizable* if it admits a positive definite Hermitian form $(,)$ which is invariant in the sense that

$$(x \cdot v, w) = (v, x^* \cdot w)$$

for all $v, w \in V$ and $x \in \mathfrak{D}(G_q)$. By Proposition 5.62, the invariant sesquilinear forms $(,)$ on $\Gamma(\mathcal{E}_{\mu, \lambda})$ are in one-to-one correspondence with the intertwining operators $T : \Gamma(\mathcal{E}_{\mu, \lambda}) \rightarrow \Gamma(\mathcal{E}_{\mu, -\bar{\lambda}})$ via the formula

$$(\eta, \xi) = \langle T(\eta), \xi \rangle, \quad \xi, \eta \in \Gamma(\mathcal{E}_{\mu, \lambda}).$$

Moreover, if $\Gamma(\mathcal{E}_{\mu,\lambda})$ admits a nonzero invariant sesquilinear form (\cdot, \cdot) then we can arrange it to be Hermitian. Indeed, by polarization, we must have $(\xi, \xi) \neq 0$ for some ξ , and after multiplying the form by some scalar we can ensure this is strictly positive. Then the Hermitian form defined by

$$\langle \eta, \xi \rangle = \frac{1}{2} \left((\eta, \xi) + \overline{(\xi, \eta)} \right)$$

is invariant and nonzero since $\langle \xi, \xi \rangle > 0$.

On the other hand, there is no guarantee that this Hermitian form will be positive definite. Two particular classes of unitarizable principal series are well-known. We use the notation $\mathfrak{a}^* \subset \mathfrak{h}^*$ for the \mathbb{R} -span of the roots Δ and put

$$\mathfrak{t}_q^* = i\mathfrak{a}^*/i\hbar^{-1}\mathbf{Q}^\vee \subset \mathfrak{h}_q^*.$$

Thus, \mathfrak{t}_q^* is a compact torus of dimension N .

Theorem 5.63. *Let G_q be a complex semisimple quantum group.*

- a) (Unitary principal series) *The principal series representation $\Gamma(\mathcal{E}_{\mu,\lambda})$ equipped with the standard inner product is unitary if and only if $\lambda \in \mathfrak{t}_q^*$.*
- b) (Complementary series) *Let α be a simple root. There is an invariant inner product on the principal series representations $\Gamma(\mathcal{E}_{\mu,\lambda'+t\alpha})$ where $\mu \in \mathbf{P}, \lambda' \in \mathfrak{t}_q^*$ are both s_α -fixed and $-1 < t < 1$.*

All of these representations are irreducible, and they are unitarily equivalent if and only if their parameters lie in the same orbit of the Weyl group action on $\mathbf{P} \times \mathfrak{h}_q^$.*

Proof. We have the following facts.

- a) The standard inner product is G_q -invariant if and only if $(\mu, \lambda) = (\mu, -\bar{\lambda})$, which is equivalent to $\lambda \in \mathfrak{t}_q^*$.
- b) Let $\mu, \lambda = \lambda' + t\alpha$ be as stated. Note that

$$-\bar{\lambda} = \lambda' - t\alpha = s_\alpha(\lambda),$$

so the standard intertwiner $\Gamma(\mathcal{E}_{\mu,\lambda}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha\mu, s_\alpha\lambda}) = \Gamma(\mathcal{E}_{\mu, -\bar{\lambda}})$ from Theorem 5.55 combined with the sesquilinear pairing from Proposition 5.62 yields an invariant Hermitian form on $\Gamma(\mathcal{E}_{\mu,\lambda})$. Moreover, since $\lambda_\alpha = t$ and $\mu_\alpha = 0$, this intertwiner is bijective for all $-1 < t < 1$. At $t = 0$, the Hermitian form is positive definite by part a). Therefore, by a standard continuity argument, they are positive definite for all $-1 < t < 1$.

The irreducibility of these representations follows from Theorem 5.49. The intertwiners associated to the simple reflections are therefore bijective and automatically unitary. \square

This is far from an exhaustive list of irreducible unitary representations of G_q . In particular, some subquotients of generalized principal series are unitarizable. Arano [3] has used continuity arguments in q to compare the unitary dual of G_q with the classical unitary dual of G . This yields a complete classification for $SL_q(n, \mathbb{C})$, and an almost complete classification for general G_q .

The case of $SL_q(2, \mathbb{C})$ was already completed by Pusz [66]; see also [65]. We state their result without proof.

Theorem 5.64. *Up to unitary equivalence, the irreducible unitary representations of $SL_q(2, \mathbb{C})$ are the Hilbert space completions of the following Harish-Chandra modules.*

- a) *The unitary principal series $\Gamma(\mathcal{E}_{\mu,\lambda})$ with $\mu \in \frac{1}{2}\mathbb{Z}, \lambda \in i\mathbb{R}/i\hbar^{-1}\mathbb{Z}$, modulo the unitary equivalences $\Gamma(\mathcal{E}_{\mu,\lambda}) \cong \Gamma(\mathcal{E}_{-\mu, -\lambda})$,*
- b) *The two complementary series $\Gamma(\mathcal{E}_{0,t})$ and $\Gamma(\mathcal{E}_{\frac{1}{2}i\hbar^{-1}, t})$ with $-1 < t < 1$, modulo the unitary equivalences $\Gamma(\mathcal{E}_{0,t}) \cong \Gamma(\mathcal{E}_{0, -t})$ and $\Gamma(\mathcal{E}_{\frac{1}{2}i\hbar^{-1}, t}) \cong \Gamma(\mathcal{E}_{\frac{1}{2}i\hbar^{-1}, -t})$,*

c) *The trivial representation and the unitary character χ_z where z is the nontrivial element of the centre of $SU(2)$.*

As a final remark, we recall that Corollary 5.58 gives explicit formulas for the intertwiners between unitary principal series representations. In particular, we can obtain a particularly simple formula for the intertwiners between the base of principal series representations. This fact was needed in [75].

Corollary 5.65. *For any $\mu \in \mathbf{P}$ and any simple root α , the operator*

$$\text{Ph}(F_\alpha)^{2\mu_\alpha} : \Gamma(\mathcal{E}_{\mu,0}) \rightarrow \Gamma(\mathcal{E}_{s_\alpha\mu,0}),$$

defined by the right regular action, is a unitary intertwiner.

We recall that before Corollary 5.58 we introduced the notation $\text{Ph}(F_\alpha)^k \in \mathcal{M}(\mathfrak{D}(K_q))$ for the operator phase of F_α^k if $k \geq 0$ or E_α^{-k} if $k < 0$.

REFERENCES

- [1] Henning Haahr Andersen and Volodymyr Mazorchuk. Category \mathcal{O} for quantum groups. *J. Eur. Math. Soc. (JEMS)*, 17(2):405–431, 2015.
- [2] Yuki Arano. Unitary spherical representations of Drinfeld doubles. *preprint arXiv:1410.6238, to appear in J. Reine Angew. Math.*, 2014.
- [3] Yuki Arano. Comparison of unitary duals of Drinfeld doubles and complex semisimple lie groups. *preprint arXiv:1605.00866*, 2016.
- [4] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.
- [5] Saad Baaj and Georges Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^* -algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.
- [6] Saad Baaj and Stefaan Vaes. Double crossed products of locally compact quantum groups. *J. Inst. Math. Jussieu*, 4(1):135–173, 2005.
- [7] Teodor Banica. Fusion rules for representations of compact quantum groups. *Exposition. Math.*, 17(4):313–337, 1999.
- [8] P. Baumann. Another proof of Joseph and Letzter’s separation of variables theorem for quantum groups. *Transform. Groups*, 5(1):3–20, 2000.
- [9] Pierre Baumann. On the center of quantized enveloping algebras. *J. Algebra*, 203(1):244–260, 1998.
- [10] Pierre Baumann. Canonical bases and the conjugating representation of a semisimple group. *Pacific J. Math.*, 206(1):25–37, 2002.
- [11] E. Bédos and L. Tuset. Amenability and co-amenability for locally compact quantum groups. *Internat. J. Math.*, 14(8):865–884, 2003.
- [12] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.
- [13] J. N. Bernstein and S. I. Gel’fand. Tensor products of finite- and infinite-dimensional representations of semisimple Lie algebras. *Compositio Math.*, 41(2):245–285, 1980.
- [14] Walter Borho and Hanspeter Kraft. Über die Gelfand-Kirillov-Dimension. *Math. Ann.*, 220(1):1–24, 1976.
- [15] Ken A. Brown and Ken R. Goodearl. *Lectures on algebraic quantum groups*. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel, 2002.
- [16] P. Caldero. Algèbres enveloppantes quantifiées: action adjointe et représentations. *PhD thesis*, 1993.
- [17] Vyjayanthi Chari and Andrew Pressley. *A guide to quantum groups*. Cambridge University Press, Cambridge, 1995. Corrected reprint of the 1994 original.
- [18] C. De Concini and C. Procesi. Quantum groups. In *D-modules, representation theory, and quantum groups (Venice, 1992)*, volume 1565 of *Lecture Notes in Math.*, pages 31–140. Springer, Berlin, 1993.
- [19] Corrado De Concini and Victor G. Kac. Representations of quantum groups at roots of 1. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progr. Math.*, pages 471–506. Birkhäuser Boston, Boston, MA, 1990.
- [20] Corrado De Concini and Victor G. Kac. Representations of quantum groups at roots of 1. In *Modern quantum field theory (Bombay, 1990)*, pages 333–335. World Sci. Publ., River Edge, NJ, 1991.
- [21] L. Delvaux and A. Van Daele. The Drinfel’d double of multiplier Hopf algebras. *J. Algebra*, 272(1):273–291, 2004.
- [22] Jacques Dixmier. *Enveloping algebras*, volume 11 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996. Revised reprint of the 1977 translation.
- [23] Bernhard Drabant and Alfons Van Daele. Pairing and quantum double of multiplier Hopf algebras. *Algebr. Represent. Theory*, 4(2):109–132, 2001.
- [24] Bernhard Drabant, Alfons Van Daele, and Yinhua Zhang. Actions of multiplier Hopf algebras. *Comm. Algebra*, 27(9):4117–4172, 1999.
- [25] V. G. Drinfeld. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820, Providence, RI, 1987. Amer. Math. Soc.
- [26] Michel Duflo. Représentations irréductibles des groupes semi-simples complexes. In *Analyse harmonique sur les groupes de Lie (Sém., Nancy-Strasbourg, 1973–75)*, pages 26–88. Lecture Notes in Math., Vol. 497. Springer, Berlin, 1975.
- [27] Thomas J. Enright. *Lectures on representations of complex semisimple Lie groups*, volume 66 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin-New York, 1981. Lecture notes by Vyjayanthi Sundar.

- [28] Daniel R. Farkas and Gail Letzter. Quantized representation theory following Joseph. In *Studies in Lie theory*, volume 243 of *Progr. Math.*, pages 9–17. Birkhäuser Boston, Boston, MA, 2006.
- [29] D. Gaitsgory. Existence and uniqueness of the R -matrix in quantum groups. *J. Algebra*, 176(2):653–666, 1995.
- [30] Fabio Gavarini. The quantum duality principle. *Ann. Inst. Fourier (Grenoble)*, 52(3):809–834, 2002.
- [31] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953.
- [32] I. Heckenberger and H. Yamane. Drinfel’d doubles and Shapovalov determinants. *Rev. Un. Mat. Argentina*, 51(2):107–146, 2010.
- [33] Timothy J. Hodges. Double quantum groups and Iwasawa decomposition. *J. Algebra*, 192(1):303–325, 1997.
- [34] Jin Hong and Seok-Jin Kang. *Introduction to quantum groups and crystal bases*, volume 42 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [35] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1978. Second printing, revised.
- [36] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
- [37] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG category O* , volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [38] Nathan Jacobson. *Lie algebras*. Dover Publications, Inc., New York, 1979. Republication of the 1962 original.
- [39] Jens Carsten Jantzen. *Lectures on quantum groups*, volume 6 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.
- [40] A. Joseph. Dixmier’s problem for Verma and principal series submodules. *J. London Math. Soc. (2)*, 20(2):193–204, 1979.
- [41] Anthony Joseph. *Quantum groups and their primitive ideals*, volume 29 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1995.
- [42] Anthony Joseph and Gail Letzter. Local finiteness of the adjoint action for quantized enveloping algebras. *J. Algebra*, 153(2):289–318, 1992.
- [43] Anthony Joseph and Gail Letzter. Separation of variables for quantized enveloping algebras. *Amer. J. Math.*, 116(1):127–177, 1994.
- [44] Anthony Joseph and Gail Letzter. Verma module annihilators for quantized enveloping algebras. *Ann. Sci. École Norm. Sup. (4)*, 28(4):493–526, 1995.
- [45] M. Kashiwara. On crystal bases of the Q -analogue of universal enveloping algebras. *Duke Math. J.*, 63(2):465–516, 1991.
- [46] Masaki Kashiwara. Crystalizing the q -analogue of universal enveloping algebras. *Comm. Math. Phys.*, 133(2):249–260, 1990.
- [47] Christian Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [48] Anatoli Klimyk and Konrad Schmüdgen. *Quantum groups and their representations*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.
- [49] Leonid I. Korogodski and Yan S. Soibelman. *Algebras of functions on quantum groups. Part I*, volume 56 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- [50] Ulrich Krähmer. FRT-duals as quantum enveloping algebras. *J. Algebra*, 264(1):68–81, 2003.
- [51] J. Kustermans and A. van Daele. C^* -algebraic quantum groups arising from algebraic quantum groups. *Internat. J. Math.*, 8(8):1067–1139, 1997.
- [52] Johan Kustermans. Locally compact quantum groups in the universal setting. *Internat. J. Math.*, 12(3):289–338, 2001.
- [53] Johan Kustermans. The analytic structure of algebraic quantum groups. *J. Algebra*, 259(2):415–450, 2003.
- [54] Johan Kustermans and Stefaan Vaes. Locally compact quantum groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.
- [55] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J. Amer. Math. Soc.*, 3(2):447–498, 1990.
- [56] George Lusztig. Quantum groups at roots of 1. *Geom. Dedicata*, 35(1-3):89–113, 1990.
- [57] George Lusztig. *Introduction to quantum groups*. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2010. Reprint of the 1994 edition.

- [58] Ann Maes and Alfons Van Daele. Notes on compact quantum groups. *Nieuw Arch. Wisk. (4)*, 16(1-2):73–112, 1998.
- [59] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition, 2001. With the cooperation of L. W. Small.
- [60] Susan Montgomery. *Hopf algebras and their actions on rings*, volume 82 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993.
- [61] Gerard J. Murphy. *C^* -algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.
- [62] K. R. Parthasarathy, R. Ranga Rao, and V. S. Varadarajan. Representations of complex semi-simple Lie groups and Lie algebras. *Ann. of Math. (2)*, 85:383–429, 1967.
- [63] P. Podleś and S. L. Woronowicz. Quantum deformation of Lorentz group. *Comm. Math. Phys.*, 130(2):381–431, 1990.
- [64] W. Pusz and S. L. Woronowicz. Unitary representations of quantum Lorentz group. In *Non-compact Lie groups and some of their applications (San Antonio, TX, 1993)*, volume 429 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 453–472. Kluwer Acad. Publ., Dordrecht, 1994.
- [65] W. Pusz and S. L. Woronowicz. Representations of quantum Lorentz group on Gelfand spaces. *Rev. Math. Phys.*, 12(12):1551–1625, 2000.
- [66] Wiesław Pusz. Irreducible unitary representations of quantum Lorentz group. *Comm. Math. Phys.*, 152(3):591–626, 1993.
- [67] Marc Rosso. Représentations des groupes quantiques. *Astérisque*, (201-203):Exp. No. 744, 443–483 (1992), 1991. Séminaire Bourbaki, Vol. 1990/91.
- [68] Stephen F. Sawin. Quantum groups at roots of unity and modularity. *J. Knot Theory Ramifications*, 15(10):1245–1277, 2006.
- [69] M. E. Sweedler. *Hopf algebras*. Benjamin, New York, 1969.
- [70] Toshiyuki Tanisaki. Harish-Chandra isomorphisms for quantum algebras. *Comm. Math. Phys.*, 127(3):555–571, 1990.
- [71] Toshiyuki Tanisaki. Killing forms, Harish-Chandra isomorphisms, and universal R -matrices for quantum algebras. In *Infinite analysis, Part A, B (Kyoto, 1991)*, volume 16 of *Adv. Ser. Math. Phys.*, pages 941–961. World Sci. Publ., River Edge, NJ, 1992.
- [72] Toshiyuki Tanisaki. Invariance of the drinfeld pairing of a quantum group. *arXiv:1503.04573*, 2015.
- [73] A. Van Daele. Multiplier Hopf algebras. *Trans. Amer. Math. Soc.*, 342(2):917–932, 1994.
- [74] A. Van Daele. An algebraic framework for group duality. *Adv. Math.*, 140(2):323–366, 1998.
- [75] Christian Voigt and Robert Yuncken. Equivariant Fredholm modules for the full quantum flag manifold of $SU_q(3)$. *Doc. Math.*, 20:433–490, 2015.
- [76] S. L. Woronowicz. Compact quantum groups. In *Symétries quantiques (Les Houches, 1995)*, pages 845–884. North-Holland, Amsterdam, 1998.
- [77] Milen Yakimov. On the spectra of quantum groups. *Mem. Amer. Math. Soc.*, 229(1078):vi+91, 2014.

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF GLASGOW, 15 UNIVERSITY GARDENS, GLASGOW G12 8QW, UK

E-mail address: christian.voigt@glasgow.ac.uk

LABORATOIRE DE MATHÉMATIQUES, UNIVERSITÉ BLAISE PASCAL, COMPLEXE UNIVERSITAIRE DES CÉZEAUX, 63177 AUBIÈRE CEDEX, FRANCE

E-mail address: Robert.Yuncken@math.univ-bpclermont.fr