QUANTUM GROUPS AND THE BAUM-CONNES CONJECTURE

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Abstract. In these lecture notes we explain the formulation and proof of the Baum-Connes conjecture for the dual of the quantum group $SU_q(2)$ of Woronowicz. Along the way we discuss some concepts and results of independent interest, like braided tensor products and the notion of torsion-free discrete quantum groups.

1. Introduction

Let G be a second countable locally compact group and let A be a separable G-C^*-algebra. The Baum-Connes conjecture with coefficients in A asserts that the assembly map

$$\mu_A : K_*^{\mathsf{top}}(G; A) \to K_*(G \ltimes_{\mathsf{r}} A)$$

is an isomorphism [3], [4]. Here $K_*(G \ltimes_{\mathsf{r}} A)$ is the K-theory of the reduced crossed product of A by G. The validity of this conjecture has applications in topology, geometry and representation theory. In particular, if G is discrete then the Baum-Connes conjecture with trivial coefficients $\mathbb{C}$ implies the Novikov conjecture on higher signatures and the Kadison-Kaplansky idempotent conjecture.

It is natural to ask what happens if one replaces the group G in this conjecture by a locally compact quantum group. Indeed, quantum groups give rise to interesting C^*-algebras, and an important aspect in the study of a C^*-algebra is the computation of its K-theory. Since the Baum-Connes conjecture provides powerful tools to calculate the K-theory of group C^*-algebras it should also be useful in the quantum case.

In these notes we discuss the formulation and proof of the Baum-Connes conjecture for a specific discrete quantum group, namely the dual of the quantum group $SU_q(2)$ of Woronowicz. Along the way we will review some results in connection with braided tensor products and the Drinfeld double. We shall also explain basic facts concerning torsion-free quantum groups. The material covered here is mostly taken from [19], [23].

In order to formulate the Baum-Connes problem for quantum groups we follow the approach proposed by Meyer and Nest using the language of triangulated categories and derived functors [16]. In this approach the left hand side of the assembly map is identified with the localisation $\mathbb{L}F$ of the functor $F(A) = K_*(G \ltimes_{\mathsf{r}} A)$ on the equivariant Kasparov category KK^G. The usual definition of the left hand side of the conjecture is based on the universal space for proper actions, a concept which does not translate to the quantum setting in an obvious way. Following [16], one has to specify instead an appropriate subcategory of the equivariant Kasparov category corresponding to compactly induced actions in the group case. This approach has been implemented in [17] for duals of compact groups.

Let us describe how the paper is organized. In section 2 we discuss some basic material on Hilbert modules and the definition of KK-theory. Section 3 contains

2000 *Mathematics Subject Classification.* 20G42, 46L80, 19K35.

the definition of coactions and equivariant KK-theory for quantum groups. Equivariant KK-theory is the most important tool in our considerations. In section 4 we define braided tensor products and their relation to the Drinfeld double $\mathsf{D}(G)$ of a quantum group G. Braided tensor products replace the usual tensor products in the context of quantum group actions. In section 5 we first review the definition of $SU_q(2)$ and the standard Podleś sphere $SU_q(2)/T$. Then we prove that, on the level of $KK^{\mathsf{D}(SU_q(2))}$, the Podleś sphere $C(SU_q(2)/T)$ is isomorphic to $\mathbb{C}^2$. This turns out to be the most important ingredient in the proof of the Baum-Connes conjecture below. In section 6 we discuss the notion of torsion-free quantum groups and formulate the Baum-Connes conjecture for these quantum groups. Finally, using the considerations from section 5 we prove in section 7 that the dual of $SU_q(2)$ satisfies the strong Baum-Connes conjecture. In two appendices we discuss the relation between Hilbert modules and continuous fields of Hilbert spaces and details on Yetter-Drinfeld structures. The first appendix on continuous fields is not needed in the main text and only included as additional information.
Let us make some remarks on notation. We write $\mathbb{L}(\mathcal{E})$ for the space of adjointable operators on a Hilbert A-module $\mathcal{E}$. Moreover $\mathbb{K}(\mathcal{E})$ denotes the space of compact operators. The closed linear span of a subset X of a Banach space is denoted by $[X]$. Depending on the context, the symbol $\otimes$ denotes either the tensor product of Hilbert spaces, the minimal tensor product of C^*-algebras, or the tensor product of von Neumann algebras. We write $\odot$ for algebraic tensor products. For operators on multiple tensor products we use the leg numbering notation.
It is a pleasure to thank E. Germain and R. Vergnioux for their kind hospitality in Caen. Moreover I thank J. Bichon and R. Vergnioux for discussions on the notion of torsion-freeness for quantum groups.

2. Hilbert modules and KK-theory

In this section we discuss Hilbert modules an explain the definition of KK-theory. We state most results without proof. For more details on Hilbert modules and KK-theory we refer to [12] and [5].
Let B be a C^*-algebra and let $\mathcal{E}$ be a right B-module. A B-valued inner product on $\mathcal{E}$ is a sesquilinear form $\mathcal{E} \times \mathcal{E} \to B, (\xi, \eta) \mapsto \langle \xi, \eta \rangle$, linear in the second and conjugate linear in the first variable, such that

$$\langle \xi, \eta b \rangle = \langle \xi, \eta \rangle b, \qquad \langle \xi, \eta \rangle = \langle \eta, \xi \rangle^*, \qquad \langle \xi, \xi \rangle \geq 0$$

for all $\xi, \eta \in \mathcal{E}$ and $b \in B$. Note that the last condition asserts positivity in the C^*-algebra B.

Definition 2.1. *Let B be a C^*-algebra and $\mathcal{E}$ a right B-module with a B-valued inner product $\langle \ , \ \rangle$. Then $\mathcal{E}$ is called a Hilbert B-module if*

$$||\xi|| = ||\langle \xi, \xi \rangle||^{1/2}$$

defines a norm on $\mathcal{E}$ for which $\mathcal{E}$ is complete. A Hilbert module is called full if the ideal $[\langle \mathcal{E}, \mathcal{E} \rangle] \subset B$ is equal to B.

Observe that $||\xi|| = 0$ holds iff $\langle \xi, \xi \rangle = 0$ in B. Hence for an element ξ in a Hilbert B-module $\mathcal{E}$ the relation $\langle \xi, \xi \rangle = 0$ holds if and only if $\xi = 0$.

Examples 2.2. *Let us consider some examples of Hilbert modules.*

a) $B = \mathbb{C}$. *It is easy to check that Hilbert $\mathbb{C}$-modules are the same thing as Hilbert spaces.*

b) $B = C_0(X)$ *for a locally compact Hausdorff space X. Hilbert $C_0(X)$-modules are the same thing as continuous fields of Hilbert spaces over X.*

c) $\mathcal{E} = B$ *with right multiplication and the scalar product* $\langle x, y\rangle = x^*y$ *is a Hilbert* B*-module for any* C^**-algebra* B*.*

For the definition of continuous fields of Hilbert spaces see [8] and appendix A. From example *b)* we see in particular that every hermitian vector bundle E over a locally compact space X yields a Hilbert $C_0(X)$-module by taking the global sections of E vanishing at infinity.
Many constructions for modules carry over to the setting of Hilbert modules. In particular, a submodule of a Hilbert B-module $\mathcal{E}$ is a closed subspace of $\mathcal{E}$ which is preserved by the B-module structure. If $\mathcal{E}$ and $\mathcal{F}$ are Hilbert B-modules, the direct sum $\mathcal{E} \oplus \mathcal{F}$ is a Hilbert B-module in the obvious way. One may also take infinite direct sums of Hilbert modules, in particular we may consider

$$\mathbb{H}_B = \bigoplus_{n=1}^{\infty} B = \left\{ (b_k) \in \prod_{n\in\mathbb{N}} B \,\middle|\, \sum_{k=1}^{\infty} b_k^* b_k \text{ converges in } B \right\}$$

with the inner product

$$\langle (a_k), (b_k)\rangle = \sum_{k=1}^{\infty} a_k^* b_k.$$

This Hilbert B-module is called the standard Hilbert B-module, and plays an important role according to the following theorem.

Theorem 2.3 (Kasparov's stabilization theorem)**.** *Let* B *be a* C^**-algebra and let* $\mathcal{E}$ *be a countably generated Hilbert* B*-module. Then* $\mathcal{E} \oplus \mathbb{H}_B \cong \mathbb{H}_B$ *as Hilbert* B*-modules.*

Here a Hilbert module $\mathcal{E}$ is called countably generated if there exists a countable set $X \subset \mathcal{E}$ such that the smallest Hilbert submodule of $\mathcal{E}$ containing X is $\mathcal{E}$ itself. The stabilization theorem asserts that every countably generated Hilbert module is a direct summand in the standard Hilbert module $\mathbb{H}_B$. We note that the C^*-algebra B itself does not need to be separable.
Next we introduce bounded operators on Hilbert modules.

Definition 2.4. *Let* $\mathcal{E}$ *be a Hilbert* B*-module. A bounded operator on* $\mathcal{E}$ *is a linear map* $T : \mathcal{E} \to \mathcal{E}$ *such that there exists a linear map* $T^* : \mathcal{E} \to \mathcal{E}$ *satisfying*

$$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$$

for all $\xi, \eta \in \mathcal{E}$*.*

It is easy to check that the map T^* is uniquely determined by T, and that T is automatically a B-module map. In fact, even linearity follows from the existence of T^*. A bounded operator is indeed bounded for the operator norm, this follows from the closed graph theorem. The algebra $\mathbb{L}(\mathcal{E})$ of all bounded operators on $\mathcal{E}$ is a C^*-algebra with the operator norm and the $*$-structure given by taking adjoints.

Example 2.5. *It is easy to find an example of Hilbert modules* $\mathcal{E}_B$ *and* $\mathcal{F}_B$ *and a bounded* B*-linear map* $\mathcal{E} \to \mathcal{F}$ *which is not adjointable, hence not a bounded operator in the sense of definition 2.4. Consider, for instance,* $B = C[0, 1]$ *and* $\mathcal{E} = C_0([0, 1)), \mathcal{F} = C[0, 1]$*. The inclusion map* $T : \mathcal{E} \to \mathcal{F}$ *is clearly* B*-linear and bounded. If* $T^* : \mathcal{F} \to \mathcal{E}$ *where an adjoint then we would have* $T^*(1)(x) = 1$ *for all* $x \in [0, 1)$*. This would contradict the fact that* $T^*(1)$ *has to be an element in* $C_0([0, 1))$*.*

Let $\mathcal{E}$ be a Hilbert module and let $\xi, \eta \in \mathcal{E}$. The rank-one operator $|\xi\rangle\langle\eta|$ is the linear map $\mathcal{E} \to \mathcal{E}$ defined by

$$|\xi\rangle\langle\eta|(\mu) = \xi\langle\eta, \mu\rangle$$

for $\mu \in \mathcal{E}$. One checks that $|\xi\rangle\langle\eta|$ is a bounded operator with adjoint

$$(|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|.$$

A finite sum of rank-one operators is called a finite rank operator. The finite rank operators form a two-sided ideal in $\mathbb{L}(\mathcal{E})$.

Definition 2.6. *Let $\mathcal{E}$ be a Hilbert B-module. A compact operator on $\mathcal{E}$ is a bounded operator which is the norm-limit of a sequence of finite rank operators.*

The set $\mathbb{K}(\mathcal{E})$ of all compact operators on $\mathcal{E}$ is a closed two-sided ideal in $\mathbb{L}(\mathcal{E})$, in particular it is itself a C^*-algebra.

Examples 2.7. *Let us consider some examples of bounded and compact operators.*

a) For $B = \mathbb{C}$ we reobtain the usual notions of bounded and compact operators on a Hilbert space.
b) Consider the Hilbert B-module $\mathcal{E} = B$. Then $\mathbb{K}(B) = B$ and $\mathbb{L}(B) = M(B)$, the multiplier algebra of B.
c) More generally, for any Hilbert module $\mathcal{E}$ one has $\mathbb{L}(\mathcal{E}) = M(\mathbb{K}(\mathcal{E}))$.
d) If $B = C_0(X)$ and $\mathcal{E} = (\mathcal{E}_x)_{x\in X}$ is a continuous field of Hilbert spaces, then

$$\mathbb{K}(\mathcal{E}) \cong \Gamma_0(\mathbb{K}(\mathcal{E}_x))$$

is the algebra of C_0-sections of the algebra bundle $\mathbb{K}(\mathcal{E})$, and

$$\mathbb{L}(\mathcal{E}) \cong \Gamma_b(\mathbb{L}(\mathcal{E}_x))$$

where on the right-hand side we consider strong-$*$-continuous bounded sections.

Let $\mathcal{E}$ be a Hilbert A-module and $\mathcal{F}$ a Hilbert B-module. Then the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ is a right $A \odot B$-module and equipped with a $A \otimes B$-valued inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \xi_1, \xi_2\rangle\langle \eta_1, \eta_2\rangle.$$

The completion of $\mathcal{E} \odot \mathcal{F}$ with respect to the corresponding norm is a Hilbert $A\otimes B$-module, denoted by $\mathcal{E} \otimes \mathcal{F}$ and called the exterior tensor product of $\mathcal{E}$ and $\mathcal{F}$. One has natural $*$-homomorphisms $\mathbb{L}(\mathcal{E}) \to \mathbb{L}(\mathcal{E} \otimes \mathcal{F})$ and $\mathbb{L}(\mathcal{F}) \to \mathbb{L}(\mathcal{E} \otimes \mathcal{F})$ and an isomorphism $\mathbb{K}(\mathcal{E}) \otimes \mathbb{K}(\mathcal{F}) \cong \mathbb{K}(\mathcal{E} \otimes \mathcal{F})$.
Let $\mathcal{E}$ be a Hilbert A-module and $\mathcal{F}$ a Hilbert B-module. If $\phi : A \to \mathbb{L}(\mathcal{F})$ is a $*$-homomorphism we can define a Hilbert module $\mathcal{E} \otimes_A \mathcal{F}$, called the interior tensor product of $\mathcal{E}$ and $\mathcal{F}$ along ϕ, as follows. First we consider the algebraic tensor product $\mathcal{E} \odot_A \mathcal{F}$, which is defined as the quotient of the algebraic tensor product $\mathcal{E} \odot \mathcal{F}$ by the space of all elements

$$\xi a \otimes \eta - \xi \otimes \phi(a)(\eta)$$

for $\xi \in \mathcal{E}, a \in A$ and $\eta \in \mathcal{F}$. Then $\mathcal{E} \odot_A \mathcal{F}$ is a right B-module and carries a B-valued inner product given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \eta_1, \phi(\langle \xi_1, \xi_2\rangle)\eta_2\rangle.$$

One can show that this inner product is positive definite. By definition, the completion of $\mathcal{E} \odot_A \mathcal{F}$ with respect to the corresponding norm is the interior tensor product $\mathcal{E} \otimes_A \mathcal{F}$.
We are now ready to give the definition of Kasparov cycles.

Definition 2.8. *Let A and B be C^*-algebras. A Kasparov A-B-cycle $(\mathcal{E}, \phi, F)$ consists of*

a) a (countably generated) graded Hilbert B-module $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$
b) a graded $*$-homomorphism $\phi : A \to \mathbb{L}(\mathcal{E})$, that is, $\phi = \phi_0 \oplus \phi_1$ where $\phi_j : A \to \mathbb{L}(\mathcal{E}_j)$ for $j = 0, 1$

c) a bounded operator $F \in \mathbb{L}(\mathcal{E})$ which is odd, that is, of the form

$$F = \begin{pmatrix} 0 & F_{10} \\ F_{01} & 0 \end{pmatrix}.$$

These data are required to satisfy the condition that

$$\phi(a)(F^2 - 1), \qquad \phi(a)(F - F^*), \qquad [F, \phi(a)]$$

are in $\mathbb{K}(\mathcal{E})$ for all $a \in A$.

Two Kasparov A-B-cycles $(\mathcal{E}_0, \phi_0, F_0)$ and $(\mathcal{E}_1, \phi_1, F_1)$ are called unitarily equivalent if there is a unitary $U \in \mathbb{L}(\mathcal{E}_0, \mathcal{E}_1)$ of degree zero such that $U\phi_0(a) = \phi_1(a)U$ for all $a \in A$ and $F_1 U = U F_0$. We write $(\mathcal{E}_0, \phi_0, F_0) \cong (\mathcal{E}_1, \phi_1, F_1)$ in this case.
Let $\mathbb{E}(A, B)$ be the set of unitary equivalence classes of Kasparov A-B-cycles. This set is functorial for $*$-homomorphisms in both variables. If $f : B_1 \to B_2$ is a $*$-homomorphism and $(\mathcal{E}, \phi, F)$ is a Kasparov A-B_1-cycle, then

$$f_*(\mathcal{E}, \phi, F) = (\mathcal{E} \otimes_f B_2, \phi \otimes \mathrm{id}, F \otimes 1)$$

is the corresponding Kasparov A-B_2-cycle. A homotopy between Kasparov A-B-cycles $(\mathcal{E}_0, \phi_0, F_0)$ and $(\mathcal{E}_1, \phi_1, F_1)$ is a Kasparov A-$B[0,1]$-cycle $(\mathcal{E}, \phi, F)$ such that $(\mathrm{ev}_t)_*(\mathcal{E}, \phi, F) \cong (\mathcal{E}_t, \phi_t, F_t)$ for $t = 0, 1$. Here $B[0,1] = B \otimes C[0,1]$ and $\mathrm{ev}_t : B[0,1] \to B$ is evaluation at t.

Definition 2.9. *Let A and B be C^*-algebras. The Kasparov group $KK(A, B)$ is the set of homotopy classes of Kasparov A-B-cycles.*

The addition in $KK(A, B)$ is defined in the obvious way by taking direct sums of cycles.
Every $*$-homomorphism $\phi : A \to B$ defines an element $[\phi] \in KK(A, B)$ by taking $\mathcal{E} = \mathcal{E}_0 = B, \phi : A \to B = \mathbb{K}(\mathcal{E}) \subset \mathbb{L}(\mathcal{E})$ and $F = 0$.

Theorem 2.10. *There exists an associative bilinear product*

$$KK(A, B) \times KK(B, C) \to KK(A, C), \qquad (x, y) \mapsto x \otimes y$$

generalizing the composition of $*$-homomorphisms.

That is, $[\phi] \otimes [\psi] = [\psi \circ \phi]$ whenever $\phi : A \to B$ and $\psi : B \to C$ are $*$-homomorphisms. The element $\mathrm{id} = [\mathrm{id}]$ acts as a unit with respect to the product.
Using the Kasparov product, we may view KK as a category with objects all separable C^*-algebras and with the set of morphisms between A and B given by $KK(A, B)$.
The construction of the product and the proof of its associativity are difficult. We refer to [5] for the details. A simple special case of the Kasparov product arises if the operators in both cycles are zero. More precisely, if $(\mathcal{E}_1, \phi_1, 0)$ is a Kasparov A-B-module and $(\mathcal{E}_2, \phi_2, 0)$ is a Kasparov B-C-module then $(\mathcal{E}_1 \otimes_B \mathcal{E}_2, \phi_1 \otimes \mathrm{id}, 0)$ represents their Kasparov product.
This situation arises in the context of Morita equivalences. A Morita equivalence between C^*-algebras A and B is a full Hilbert B-module $\mathcal{E}$ together with an isomorphism $\phi : A \to \mathbb{K}(\mathcal{E})$ of C^*-algebras. Such a Hilbert module is called an A-B-imprimitivity bimodule. The triple $(\mathcal{E}, \phi, 0)$ defines an element in $KK(A, B)$, and this element is invertible with respect to Kasparov product. The inverse is implemented by the dual imprimitivity bimodule $\mathcal{E}^\vee = \mathbb{K}(\mathcal{E}, B)$. It follows that Morita equivalent C^*-algebras are isomorphic in KK.

3. Coactions, crossed products, and equivariant KK-theory

In this section we extend the definition of KK-theory to the equivariant setting. We work in the setting of Hopf-C^*-algebras.
Let us begin with the definition of a Hopf-C^*-algebra.

Definition 3.1. *A Hopf C^*-algebra is a C^*-algebra S together with an injective nondegenerate $*$-homomorphism $\Delta : S \to M(S \otimes S)$ such that the diagram*

$$\begin{tikzcd} S \arrow[r, "\Delta"] \arrow[d] & M(S \otimes S) \arrow[d, "\mathrm{id} \otimes \Delta"] \\ M(S \otimes S) \arrow[r, "\Delta \otimes \mathrm{id}"] & M(S \otimes S \otimes S) \end{tikzcd}$$

is commutative and $[\Delta(S)(1 \otimes S)] = S \otimes S = [(S \otimes 1)\Delta(S)]$.
A morphism between Hopf-C^*-algebras (S, Δ_S) and (T, Δ_T) is a nondegenerate $*$-homomorphism $\pi : S \to M(T)$ such that $\Delta_T \pi = (\pi \otimes \pi)\Delta_S$.

If S is a Hopf-C^*-algebra we write S^{cop} for the Hopf-C^*-algebra obtained by equipping S with the opposite comultiplication $\Delta^{\mathsf{cop}} = \sigma\Delta$.
All Hopf-C^*-algebras of interest to us are obtained from locally compact quantum groups, see [10], [11].

Definition 3.2. *A locally compact quantum group G is given by a von Neumann algebra $L^\infty(G)$ together with a normal unital $*$-homomorphism $\Delta : L^\infty(G) \to L^\infty(G) \otimes L^\infty(G)$ satisfying the coassociativity relation*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

and normal semifinite faithful weights ϕ and ψ on $L^\infty(G)$ which are left and right invariant, respectively.

Let G be a locally compact quantum group and let $\Lambda : \mathcal{N}_\phi \to \mathbb{H}_G$ be a GNS-construction for the weight ϕ where

$$\mathcal{N}_\phi = \{x \in M | \phi(x^*x) < \infty\}.$$

One obtains a unitary $W_G = W$ on $\mathbb{H}_G \otimes \mathbb{H}_G$ by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1))$$

for $x, y \in \mathcal{N}_\phi$. This unitary is multiplicative, which means that W satisfies the pentagonal equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Using W one obtains Hopf C^*-algebras

$$C_0^{\mathsf{r}}(G) = [(\mathrm{id} \otimes \mathbb{L}(\mathbb{H}_G)_*)(W)], \qquad \Delta(x) = W^*(1 \otimes x)W$$

and

$$C_{\mathsf{r}}^*(G) = [(\mathbb{L}(\mathbb{H}_G)_* \otimes \mathrm{id})(W)], \qquad \hat{\Delta}(y) = \Sigma W(y \otimes 1)W^*\Sigma = \hat{W}^*(1 \otimes y)\hat{W}$$

where $\Sigma \in \mathbb{L}(\mathbb{H}_G \otimes \mathbb{H}_G)$ is the flip map and $\hat{W} = \Sigma W^* \Sigma$. We note that $W \in M(C_0^{\mathsf{r}}(G) \otimes C_{\mathsf{r}}^*(G))$.
There are also Hopf-C^*-algebras $C_0^{\mathsf{f}}(G)$ and $C_{\mathsf{f}}^*(G)$ associated to G, together with canonical surjective morphisms $\hat{\pi} : C_{\mathsf{f}}^*(G) \to C_{\mathsf{r}}^*(G)$ and $\pi : C_0^{\mathsf{f}}(G) \to C_0^{\mathsf{r}}(G)$ of Hopf-C^*-algebras. One should view all these Hopf-C^*-algebras as different appearances of the quantum group G. The quantum group G is amenable if $\hat{\pi}$ is an isomorphism and coamenable if π is an isomorphism.
A quantum group G is called compact iff $C_0^{\mathsf{f}}(G)$ is unital. In this case we write $C^{\mathsf{f}}(G) = C_0^{\mathsf{f}}(G)$ and $C^{\mathsf{r}}(G) = C_0^{\mathsf{r}}(G)$. If G is in addition coamenable we write

$C(G) = C^{\mathrm{f}}(G) = C^{\mathrm{r}}(G)$. We will be interested in particular in the coamenable compact quantum group $SU_q(2)$.

Definition 3.3. *Let $q \in (0,1]$. The C^*-algebra $C(SU_q(2))$ is the universal C^*-algebra generated by elements α and γ satisfying the relations*

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

The comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

We will come back to $SU_q(2)$ in section 5.
Let us now introduce coactions on C^*-algebras.

Definition 3.4. *A (continuous, left) coaction of a Hopf C^*-algebra S on a C^*-algebra A is an injective nondegenerate $*$-homomorphism $\alpha : A \to M(S \otimes A)$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\alpha"] & M(S \otimes A) \arrow[d, "\Delta\otimes\mathrm{id}"] \\ M(S \otimes A) \arrow[r, "\mathrm{id}\otimes\alpha"] & M(S \otimes S \otimes A) \end{tikzcd}$$

is commutative and $[\alpha(A)(S \otimes 1)] = S \otimes A$.
An S-C^*-algebra (A, α) is a C^*-algebra A with a coaction α of S on A. If (A, α) and (B, β) are S-C^*-algebras, then a $*$-homomorphism $f : A \to M(B)$ is called S-colinear if $\beta f = (\mathrm{id} \otimes f)\alpha$.

A G-C^*-algebra is an S-C^*-algebra for $S = C_0^{\mathrm{r}}(G)$ for a locally compact quantum group G. In this case S-colinear $*$-homomorphisms will be called G-equivariant. We write G-Alg for the category of separable G-C^*-algebras and equivariant $*$-homomorphisms.

Examples 3.5. *Let us consider two examples of S-C^*-algebras.*

a) If $A = S$ then $\alpha = \Delta$ defines a coaction of S on A.
b) For any C^*-algebra B the trivial coaction $\beta : B \to M(S \otimes B), \beta(b) = 1 \otimes b$ turns B into an S-C^*-algebra.

Let $\mathcal{E}_B$ be a right Hilbert module. The multiplier module $M(\mathcal{E})$ of $\mathcal{E}$ is the right Hilbert-$M(B)$-module $M(\mathcal{E}) = \mathbb{L}(B, \mathcal{E})$. There is a natural embedding $\mathcal{E} \cong \mathbb{K}(B, \mathcal{E}) \to \mathbb{L}(B, \mathcal{E}) = M(\mathcal{E})$. If $\mathcal{E}_A$ and $\mathcal{F}_B$ are Hilbert modules, then a morphism from $\mathcal{E}$ to $\mathcal{F}$ is a linear map $\Phi : \mathcal{E} \to M(\mathcal{F})$ together with a $*$-homomorphism $\phi : A \to M(B)$ such that

$$\langle \Phi(\xi), \Phi(\eta) \rangle = \phi(\langle \xi, \eta \rangle)$$

for all $\xi, \eta \in \mathcal{E}$. In this case Φ is automatically norm-decreasing and satisfies $\Phi(\xi a) = \Phi(\xi)\phi(a)$ for all $\xi \in \mathcal{E}$ and $a \in A$. The morphism Φ is called nondegenerate if ϕ is nondegenerate and $[\Phi(\mathcal{E})B] = \mathcal{F}$.

Definition 3.6. *Let S be a Hopf-C^*-algebra and let (B, β) an S-C^*-algebra. A (continuous, left) coaction of S on a Hilbert module $\mathcal{E}_B$ is a nondegenerate morphism $\lambda : \mathcal{E} \to M(S \otimes \mathcal{E})$ such that the diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "\lambda"] \arrow[d, "\lambda"] & M(S \otimes \mathcal{E}) \arrow[d, "\Delta\otimes\mathrm{id}"] \\ M(S \otimes \mathcal{E}) \arrow[r, "\mathrm{id}\otimes\lambda"] & M(S \otimes S \otimes \mathcal{E}) \end{tikzcd}$$

is commutative and $[(S \otimes 1)\lambda(\mathcal{E})] = S \otimes \mathcal{E} = [\lambda(\mathcal{E})(S \otimes 1)]$.
An S-Hilbert B-module $(\mathcal{E}, \lambda$ is a Hilbert module $\mathcal{E}$ together with a coaction of S on

$\mathcal{E}$. A morphism $\Phi : \mathcal{E} \to M(\mathcal{F})$ of S-Hilbert B-modules with coactions $\lambda_{\mathcal{E}}$ and $\lambda_{\mathcal{F}}$, respectively, is called S-colinear if $\lambda_{\mathcal{F}}\Phi = (\mathrm{id} \otimes \Phi)\lambda_{\mathcal{E}}$.

If $\lambda : \mathcal{E} \to M(S \otimes \mathcal{E})$ is a coaction on the Hilbert-B-module $\mathcal{E}$ then the map λ is automatically isometric and hence injective.
If G is a locally compact quantum group and $S = C_0^{\mathrm{r}}(G)$, then S-Hilbert B-modules are also called G-Hilbert B-modules. Instead of S-colinear morphisms we also speak of equivariant morphisms between G-Hilbert B-modules.
Let S be a Hopf-C^*-algebra and $(\mathcal{E}_B, \lambda)$ an S-Hilbert B-module. We obtain a unitary operator $V_\lambda : \mathcal{E} \otimes_B (S \otimes B) \to S \otimes \mathcal{E}$ by

$$V_\lambda(\xi \otimes x) = \lambda(\xi)x$$

for $\xi \in \mathcal{E}$ and $x \in S \otimes B$. Here the tensor product over B is formed with respect to the coaction $\beta : B \to M(S \otimes B)$. This unitary satisfies the relation

$$(\mathrm{id} \otimes_{\mathbb{C}} V_\lambda)(V_\lambda \otimes_{(\mathrm{id} \otimes \beta)} \mathrm{id}) = V_\lambda \otimes_{(\Delta \otimes \mathrm{id})} \mathrm{id}$$

in $\mathbb{L}(\mathcal{E} \otimes_{(\Delta \otimes \mathrm{id})\beta} (S \otimes S \otimes B), S \otimes S \otimes \mathcal{E})$, compare [1]. Moreover, the equation

$$\mathsf{ad}_\lambda(T) = V_\lambda(T \otimes \mathrm{id})V_\lambda^*$$

determines a coaction $\mathsf{ad}_\lambda : \mathbb{K}(\mathcal{E}) \to M(S \otimes \mathbb{K}(\mathcal{E})) = \mathbb{L}(S \otimes \mathcal{E})$ which turns $\mathbb{K}(\mathcal{E})$ into an S-C^*-algebra.

Definition 3.7. *Let S be a Hopf-C^*-algebra and let A and B be S-C^*-algebras. An S-equivariant Kasparov A-B-module is a countably generated graded S-Hilbert B-module $\mathcal{E}$ together with an S-colinear graded $*$-homomorphism $\phi : A \to \mathbb{L}(\mathcal{E})$ and an odd operator $F \in \mathbb{L}(\mathcal{E})$ such that*

$$[F, \phi(a)], \qquad (F^2 - 1)\phi(a), \qquad (F - F^*)\phi(a)$$

are contained in $\mathbb{K}(\mathcal{E})$ for all $a \in A$ and F is almost invariant in the sense that

$$(\mathrm{id} \otimes \phi)(x)(1 \otimes F - \mathsf{ad}_\lambda(F)) \subset S \otimes \mathbb{K}(\mathcal{E})$$

for all $x \in S \otimes A$. Here $S \otimes \mathbb{K}(\mathcal{E}) = \mathbb{K}(S \otimes \mathcal{E})$ is viewed as a subset of $\mathbb{L}(S \otimes \mathcal{E})$ and ad_λ is the adjoint coaction associated to the given coaction $\lambda : \mathcal{E} \to M(S \otimes \mathcal{E})$ on $\mathcal{E}$.

As in the nonequivariant case one defines unitary equivalence and homotopy of S-equivariant Kasparov modules, taking into account compatibility with the S-coactions in the obvious way.

Definition 3.8. *Let S be Hopf-C^*-algebra and let A and B be S-C^*-algebras. The S-equivariant Kasparov group $KK^S(A, B)$ is the set of homotopy classes of S-equivariant Kasparov A-B-cycles.*

We will be interested in the case that $S = C_0^{\mathrm{r}}(G)$ for a locally compact quantum group G. In this case we write KK^G instead of KK^S.
Every G-equivariant $*$-homomorphism $\phi : A \to B$ between G-C^*-algebras defines an element $[\phi] \in KK^G(A, B)$ by taking $\mathcal{E} = \mathcal{E}_0 = B, \phi : A \to B = \mathbb{K}(\mathcal{E}) \subset \mathbb{L}(\mathcal{E})$ and $F = 0$. The construction of the Kasparov product carries over to the equivariant case.
A G-equivariant Morita equivalence between G-C^*-algebras A and B is given by an equivariant A-B-imprimitivity bimodule, that is, a full G-Hilbert B-module $\mathcal{E}$ together with an isomorphism $\phi : A \to \mathbb{K}(\mathcal{E})$ of G-C^*-algebras. As in the nonequivariant case, the triple $(\mathcal{E}, \phi, 0)$ gives an invertible element in $KK^G(A, B)$.
We shall now introduce some further concepts which play a role in the sequel.

Definition 3.9. *Let G be a locally compact quantum group and let (A,α) be a G-C^*-algebra. The reduced crossed product $C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} A$ is defined by*

$$C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} A = [(C^*_{\mathsf{r}}(G) \otimes 1)\alpha(A)] \subset \mathbb{L}(\mathbb{H}_G \otimes A) = M(\mathbb{K}_G \otimes A)$$

where $\mathbb{K}_G = \mathbb{K}(\mathbb{H}_G)$.

There is a nondegenerate $*$-homomorphism $j_A : A \to M(C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} A)$ induced by α. Similarly, we have a canonical nondegenerate $*$-homomorphism $g_A : C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \to M(C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} A)$.
The reduced crossed products admits a dual coaction of $C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$. The dual coaction leaves $j_A(A)$ invariant and acts by the (opposite) comultiplication on $g_A(C^*_{\mathsf{r}}(G)^{\mathsf{cop}})$.
If A is equipped with the trivial coaction then $C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} A = C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \otimes A$ is the minimal tensor product of $C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$ and A. Recall that the comultiplication $\Delta : C_0^{\mathsf{r}}(G) \to M(C_0^{\mathsf{r}}(G) \otimes C_0^{\mathsf{r}}(G))$ defines a coaction of $C_0^{\mathsf{r}}(G)$ on itself. A locally compact quantum group G is called regular if $C^*_{\mathsf{r}}(G)^{\mathsf{cop}} \ltimes_{\mathsf{r}} C_0^{\mathsf{r}}(G) \cong \mathbb{K}_G = \mathbb{K}(\mathbb{H}_G)$. All compact and all discrete quantum groups are regular, and we will only consider regular locally compact quantum groups in the sequel.

Theorem 3.10. *[Baaj-Skandalis duality] Let G be a regular locally compact quantum group and let $S = C_0^{\mathsf{r}}(G)$ and $\hat{S} = C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$. For all S-C^*-algebras A and B there is a canonical isomorphism*

$$J_S : KK^S(A,B) \to KK^{\hat{S}}(\hat{S} \ltimes_{\mathsf{r}} A, \hat{S} \ltimes_{\mathsf{r}} B)$$

which is multiplicative with respect to the composition product.

A proof of theorem 3.10 can be found in [1], [2].

Definition 3.11. *A unitary corepresentation of a Hopf-C^*-algebra S on a Hilbert B-module $\mathcal{E}$ is a unitary $X \in \mathbb{L}(S \otimes \mathcal{E})$ satisfying*

$$(\Delta \otimes \mathrm{id})(X) = X_{13}X_{23}.$$

Let B be a C^*-algebra equipped with the trivial coaction of the Hopf-C^*-algebra S and let $\lambda : \mathcal{E} \to M(S \otimes \mathcal{E})$ be a coaction on the Hilbert module $\mathcal{E}_B$. Then using the natural identification $\mathcal{E} \otimes_B (S \otimes B) \cong \mathcal{E} \otimes S \cong S \otimes \mathcal{E}$ the associated unitary V_λ determines a unitary corepresentation V_λ^* in $\mathbb{L}(S \otimes \mathcal{E})$. If G is a regular locally compact quantum group and $S = C_0^{\mathsf{r}}(G)$, then this defines a bijective correspondence between S-Hilbert B-modules and unitary corepresentations.

4. The Drinfeld double and braided tensor products

In this section we discuss the definition of the braided tensor product and the Drinfeld double.
To motivate the definition of braided tensor products let us first consider the problem that it is designed to solve. Although we are interested in G-C^*-algebras for a quantum group G, the problem is in fact purely algebraic.
If $\mathcal{C}$ is a monoidal category then an algebra in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with a morphism $\mu_A : A \otimes A \to A$ such that

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes \mu_A"] \arrow[d, "\mu_A \otimes \mathrm{id}"] & A \otimes A \arrow[d, "\mu_A"] \\ A \otimes A \arrow[r, "\mu_A"] & A \end{tikzcd}$$

is commutative.
Assume that A and B are algebras in $\mathcal{C}$. We may form the tensor product $A \otimes B$ as an object of $\mathcal{C}$, but in contrast to the situation for vector spaces it will be an algebra in $\mathcal{C}$

in general only when $\mathcal{C}$ is braided. If $\mathcal{C}$ is braided and $\gamma_{XY} : X \otimes Y \to Y \otimes X$ denotes the braiding, the multiplication $\mu_{A\otimes B}$ for $A \otimes B$ is defined as the composition

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes \gamma_{BA} \otimes \mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

We are interested in the case where $\mathcal{C}$ is, roughly speaking, the category of comodules for a Hopf-C^*-algebra associated to a quantum group. However, in order to explain the main ideas we shall first consider the algebraic situation of ordinary Hopf algebras.
The comodule category of a Hopf algebra H is braided if H is coquasitriangular.

Definition 4.1. *A Hopf algebra H is called coquasitriangular if there exists a linear form $r : H \otimes H \to \mathbb{C}$ such that r is invertible with respect to the convolution product and*

a) $\mu^{\mathsf{opp}} = r * \mu * r^{-1}$
a) $r(\mu \otimes \mathrm{id}) = r_{13}r_{23}$ *and* $r(\mathrm{id} \otimes \mu) = r_{13}r_{12}$

Here $\mu : H \otimes H \to H$ denotes the multiplication map.

If M and N are comodules over a coquasitriangular Hopf algebra H then

$$\gamma_{MN}(m \otimes n) = r(n_{(-1)} \otimes m_{(-1)})n_{(0)} \otimes m_{(0)}$$

is an H-colinear isomorphism which defines a braiding on the category of H-comodules.
It is easy to give examples of Hopf algebras which are not braided.

Example 4.2. *Consider a finite group G and the Hopf algebra $H = \mathbb{C}G$. A comodule for H is the same thing as a G-graded vector space V, that is,*

$$V = \bigoplus_{s \in G} V_s$$

If G is nonabelian then this category is not braided.
To see this let us write $\mathbb{C}_s$ for $\mathbb{C}$ with the coaction $\lambda_s : \mathbb{C}_s \to H \otimes \mathbb{C}_s$ given by $\lambda_s(1) = s \otimes 1$. We find

$$\mathbb{C}_s \otimes \mathbb{C}_t \cong \mathbb{C}_{st}$$

for all s, t so that $\mathbb{C}_s \otimes \mathbb{C}_t$ is not isomorphic to $\mathbb{C}_t \otimes \mathbb{C}_s$ unless s and t commute.

Hence a finite dimensional Hopf algebra H will usually fail to be coquasitriangular. However, there is a universal way to write H as a quotient of a coquasitriangular Hopf algebra. This is the Drinfeld double construction which we shall explain next.
Let H be a finite dimensional Hopf algebra and consider

$$\hat{w} = \sum_{j=1}^{n} S^{-1}(e^j) \otimes e_j \in (H^*)^{\mathsf{cop}} \otimes H$$

where $e_1, \dots, e_n$ is a basis of H with dual basis $e^1, \dots, e^n$ of $(H^*)^{\mathsf{cop}} = H^*$. We write Δ for the comultiplication in H and $\hat{\Delta}$ for the comultiplication in $(H^*)^{\mathsf{cop}}$.

Lemma 4.3. *The element $\hat{w}$ is a bicharacter of $(H^*)^{\mathsf{cop}} \otimes H$, that is, $\hat{w}$ is invertible and the formulas*

$$(\epsilon \otimes \mathrm{id})(\hat{w}) = 1, \qquad (\mathrm{id} \otimes \epsilon)(\hat{w}) = 1$$

and

$$(\hat{\Delta} \otimes \mathrm{id})(\hat{w}) = \hat{w}_{13}\hat{w}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\hat{w}) = \hat{w}_{13}\hat{w}_{12}$$

hold.

Proof. Let us write

$$\langle x, f\rangle = x(f) = \langle f, x\rangle$$

for the canonical evaluation of $x \in (H^*)^{\mathsf{cop}} = H^*$ and $f \in H$. Note that S^{-1} is the antipode of $(H^*)^{\mathsf{cop}}$ in the above formula for $\hat{w}$, and that $\langle S^{-1}(x), f\rangle = \langle x, S(f)\rangle$ for all $x \in (H^*)^{\mathsf{cop}}$ and $f \in H$. For $f, g \in H$ and $x \in H^*$ we compute

$$\begin{aligned}
\langle(\hat{\Delta} \otimes \mathrm{id})(\hat{w}), f \otimes g \otimes x\rangle &= \sum_{j=1}^{n} \langle \hat{\Delta}(S^{-1}(e^j)) \otimes e_j, f \otimes g \otimes x\rangle \\
&= \sum_{j=1}^{n} \langle S^{-1}(e^j), gf\rangle\langle e_j, x\rangle \\
&= x(S(gf)) \\
&= \sum_{j,k=1}^{n} \langle e^j, S(f)\rangle\langle e^k, S(g)\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j), f\rangle\langle S^{-1}(e^k), g\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j) \otimes S^{-1}(e^k) \otimes e_j e_k, f \otimes g \otimes x\rangle \\
&= \langle \hat{w}_{13}\hat{w}_{23}, f \otimes g \otimes x\rangle
\end{aligned}$$

The verification of the remaining relations can be found in appendix B. □

Using the bicharacter $\hat{w}$ we can define a new Hopf algebra as follows.

Definition 4.4. *Let H be a finite dimensional Hopf algebra. The Drinfeld codouble of H is*

$$\mathsf{D}_H = H \otimes (H^*)^{\mathsf{cop}}$$

with the tensor product algebra structure, the counit $\epsilon_{\mathsf{D}}(f \otimes x) = \epsilon(f)\epsilon(x)$, the antipode $S_{\mathsf{D}} = w^{-1}(S \otimes S)w$, the comultiplication

$$\begin{aligned}
\Delta_{\mathsf{D}}(f \otimes x) &= f_{(1)} \otimes \hat{w}^{-1}(x_{(1)} \otimes f_{(2)})\hat{w} \otimes x_{(2)} \\
&= (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathsf{ad}_w \otimes \mathrm{id})(\Delta \otimes \hat{\Delta})(f \otimes x)
\end{aligned}$$

where $\hat{w} \in (H^*)^{\mathsf{cop}} \otimes H$ as above, $w = (\hat{w}^{-1})_{21} \in H \otimes (H^*)^{\mathsf{cop}}$ and ad_w denotes conjugation by w.

Using lemma 4.3 one checks that D_H becomes a Hopf algebra such that the projection maps $\pi : \mathsf{D}_H \to H, \pi(f \otimes x) = f\epsilon(x)$ and $\hat{\pi} : \mathsf{D}_H \to (H^*)^{\mathsf{cop}}, \hat{\pi}(f \otimes x) = \epsilon(f)x$ are Hopf algebra homomorphisms. The Hopf algebra D_H is coquasitriangular with universal r-form

$$r(f \otimes x \otimes g \otimes y) = \epsilon(f)g(x)\epsilon(y).$$

If M and N are left D_H-comodules, this can be used to define a D_H-colinear isomorphism $\gamma_{MN} : M \otimes N \to N \otimes N$ given by

$$\gamma_{MN}(m \otimes n) = r(n_{(-1)} \otimes m_{(-1)})n_{(0)} \otimes m_{(0)}$$

providing a braiding for the category of D_H-comodules.

We want to describe comodules for D_H in terms of H. For this we need the following definition.

Definition 4.5. *Let H be a Hopf algebra. An H-Yetter-Drinfeld module M is a vector space which is both a left H-module via $H \otimes M \to M, f \otimes m \mapsto f \cdot m$ and a left H-comodule via $M \to H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$ such that*
$$(f \cdot m)_{(-1)} \otimes (f \cdot m)_{(0)} = f_{(1)} m_{(-1)} S(f_{(3)}) \otimes f_{(2)} \cdot m_{(0)}$$
for all $f \in H$ and $m \in M$.

With this definition we can formulate the following basic result.

Proposition 4.6. *Let H be a finite dimensional Hopf algebra. Then there is a bijective correspondence between H-Yetter-Drinfeld modules and left D_H-comodules.*

The correspondence is as follows. If $\lambda : M \to \mathsf{D}_H \otimes M$ is a coaction, then using the morphisms π and $\hat{\pi}$ we obtain coactions $M \to H \otimes M$ and $M \to (H^*)^{\mathsf{cop}} \otimes M$. The latter corresponds to a left H-module structure.
Using proposition 4.6 we can give a basic example of a D_H-comodule. Consider $A = H$ with the regular coaction $\Delta : A \to H \otimes A$ and the left action
$$f \cdot a = f_{(1)} a S(f_{(2)})$$
for $f \in H$ and $a \in A$. Then A is an H-Yetter-Drinfeld module, in fact even a D_H-comodule algebra.
Now we shall go back to C^*-algebras. If G is a locally compact quantum group then the Drinfeld double $\mathsf{D}(G)$ is given by $C_0^{\mathsf{r}}(\mathsf{D}(G)) = C_0^{\mathsf{r}}(G) \otimes C_{\mathsf{r}}^*(G)$ with comultiplication
$$\Delta_{\mathsf{D}(G)} = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \mathsf{ad}(W) \otimes \mathrm{id})(\Delta \otimes \hat{\Delta}).$$
Here $\mathsf{ad}(W)$ denotes the adjoint action of the left regular multiplicative unitary $W \in M(C_0^{\mathsf{r}}(G) \otimes C_{\mathsf{r}}^*(G))$ and σ is the flip map. To relate this with the algebraic setting we note that $\hat{w}$ corresponds to the multiplicative unitary $\hat{W} = \Sigma W^* \Sigma$. If G is regular then $\mathsf{D}(G)$ is again regular.
The analogue of definition 4.5 for actions on C^*-algebras is as follows.

Definition 4.7. *Let G be a locally compact quantum group and let $S = C_0^{\mathsf{r}}(G)$ and $\hat{S} = C_{\mathsf{r}}^*(G)$ be the associated reduced Hopf-C^*-algebras. A G-Yetter-Drinfeld C^*-algebra is a C^*-algebra A equipped with continuous coactions α of S and λ of $\hat{S}$ such that the diagram*
$$\begin{array}{ccccc} A & \xrightarrow{\lambda} & M(\hat{S} \otimes A) & \xrightarrow{\mathrm{id} \otimes \alpha} & M(\hat{S} \otimes S \otimes A) \\ \downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle \sigma \otimes \mathrm{id}} \\ M(S \otimes A) & \xrightarrow{\mathrm{id} \otimes \lambda} & M(S \otimes \hat{S} \otimes A) & \xrightarrow{\mathsf{ad}(W) \otimes \mathrm{id}} & M(S \otimes \hat{S} \otimes A) \end{array}$$
is commutative. Here $\mathsf{ad}(W)(x) = W x W^*$ denotes the adjoint action of the fundamental unitary $W \in M(S \otimes \hat{S})$.

We could also consider actions on Hilbert modules, this would actually be even closer to the case of comodules in the algebraic setting treated above. For simplicity however, we restrict to actions on C^*-algebras here.
Definition 4.7 is made such that the following analogue of proposition 4.6 holds.

Proposition 4.8. *Let G be a locally compact quantum group and let $\mathsf{D}(G)$ be its Drinfeld double. Then a G-Yetter-Drinfeld C^*-algebra is the same thing as a $\mathsf{D}(G)$-C^*-algebra.*

The proof of proposition 4.8 can be found in [19].
We will now define a suitable tensor product for G-Yetter-Drinfeld C^*-algebras A and B. In fact, for the construction we do not need a G-YD-algebra structure on

B. More precisely, let A be a G-YD-algebra and let B be a G-C^*-algebra. The C^*-algebra B acts on the Hilbert module $\mathbb{H}_G \otimes B$ by β where we view $C_0^r(G) \subset \mathbb{L}(\mathbb{H}_G)$. Similarly, the C^*-algebra A acts on $\mathbb{H}_G \otimes A$ by λ where we view $C_r^*(G) \subset \mathbb{L}(\mathbb{H})$. From this we obtain two $*$-homomorphisms $\iota_A = \lambda_{12} : A \to \mathbb{L}(\mathbb{H} \otimes A \otimes B)$ and $\iota_B = \beta_{13} : B \to \mathbb{L}(\mathbb{H} \otimes A \otimes B)$.

Definition 4.9. *Let G be a locally compact quantum group, let A be a G-YD-algebra and B a G-C^*-algebra. With the notation as above, the braided tensor product $A \boxtimes_G B$ is the C^*-subalgebra of $\mathbb{L}(\mathbb{H} \otimes A \otimes B)$ generated by all elements $\iota_A(a)\iota_B(b)$ for $a \in A$ and $b \in B$.*

We will also write $A \boxtimes B$ instead of $A \boxtimes_G B$ if the quantum group G is clear from the context. The braided tensor product $A \boxtimes B$ is in fact equal to the closed linear span $[\iota_A(A)\iota_B(B)]$. This follows from proposition 8.3 in [22]. In particular, we have natural nondegenerate $*$-homomorphisms $\iota_A : A \to M(A \boxtimes B)$ and $\iota_B : B \to M(A \boxtimes B)$.
The braided tensor product $A \boxtimes B$ becomes a G-C^*-algebra in a canonical way such that the $*$-homomorphisms ι_A and ι_B are G-equivariant. If B is a G-YD-algebra then $A \boxtimes B$ is a G-YD-algebra such that ι_A and ι_B are $\mathsf{D}(G)$-equivariant. It is this symmetric situation which is the exact analogue of the braided tensor product in the algebraic setting. For the proofs of the above assertions and more details we refer to [19].

5. The quantum group $SU_q(2)$ and the Podleś sphere

In this section we discuss some constructions and results related to $SU_q(2)$ and the standard Podleś sphere. Background material on compact quantum groups and q-deformations can be found in [9]. Our main aim is to compute the equivariant KK-theory of the Podleś sphere. Throughout we consider $q \in (0, 1]$.
Let us first recall the definition of the quantum group $SU_q(2)$. In definition 3.3 we introduced the C^*-algebra $C(SU_q(2))$ as the universal C^*-algebra generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma, \quad \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that the fundamental matrix

$$u = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is unitary.
Recall also the comultiplication $\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$ given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

Using the above matrix notation, we can write the definition of Δ in a concise way as

$$\Delta \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \otimes \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}.$$

It is important that $SU_q(2)$ can be described equivalently on the level of Hopf-$*$-algebras. The algebra $\mathbb{C}[SU_q(2)]$ of polynomial functions on $SU_q(2)$ is the $*$-subalgebra of $C(SU_q(2))$ generated by α and γ. It is a Hopf-$*$-algebra with comultiplication $\Delta : \mathbb{C}[SU_q(2)] \to \mathbb{C}[SU_q(2)] \otimes \mathbb{C}[SU_q(2)]$ given by the same formula as above. Using again matrix notation, the counit $\epsilon : \mathbb{C}[SU_q(2)] \to \mathbb{C}$ is defined by

$$\epsilon \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the antipode $S : \mathbb{C}[SU_q(2)] \to \mathbb{C}[SU_q(2)]$ by

$$S \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} \alpha^* & \gamma^* \\ -q\gamma & \alpha \end{pmatrix}.$$

Recall that the antipode in a Hopf algebra is always an algebra anti-homorphism, so that it is uniquely determined by describing it on algebra generators.
We use the Sweedler notation $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for the comultiplication and write

$$f \rightharpoonup x = x_{(1)} f(x_{(2)}), \qquad x \leftharpoonup f = f(x_{(1)}) x_{(2)}$$

for elements $x \in \mathbb{C}[SU_q(2)]$ and linear functionals $f : \mathbb{C}[SU_q(2)] \to \mathbb{C}$. The antipode satisfies $S(S(x^*)^*) = x$ for all $x \in \mathbb{C}[SU_q(2)]$. In particular the map S is invertible, and the inverse of S can be written as

$$S^{-1}(x) = f_1 \rightharpoonup S(x) \leftharpoonup f_{-1}$$

where $f_{\pm 1} : \mathbb{C}[SU_q(2)] \to \mathbb{C}$ are the modular characters given by

$$f_{\pm 1} \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} q^{\mp 1} & 0 \\ 0 & q^{\pm 1} \end{pmatrix}.$$

We note that $f_{-1}(x) = f_1(S(x))$ for all $x \in \mathbb{C}[SU_q(2)]$. These maps are actually members of a canonical family $(f_z)_{z \in \mathbb{C}}$ of characters. The character f_1 describes the modular properties of the Haar state ϕ of $C(SU_q(2))$ in the sense that

$$\phi(xy) = \phi(y(f_1 \rightharpoonup x \leftharpoonup f_1))$$

for all $x, y \in \mathbb{C}[SU_q(2)]$.
We write $L^2(SU_q(2))$ for the Hilbert space obtained using the inner product

$$\langle x, y \rangle = \phi(x^* y)$$

on $C(SU_q(2))$. It is a $SU_q(2)$-Hilbert space with the coaction $\lambda : L^2(SU_q(2)) \to C(SU_q(2)) \otimes L^2(SU_q(2))$ given by

$$\lambda(\xi) = W^*(1 \otimes \xi).$$

This is the left regular representation of $SU_q(2)$. According to the Peter-Weyl theorem, the Hilbert space $L^2(SU_q(2))$ has an orthonormal basis given by the decomposition of the left regular representation into isotypical components. Explicitly, we have basis vectors $e^{(l)}_{i,j}$ where $l \in \frac{1}{2}\mathbb{N}$ and $-l \leq i, j \leq l$ for all l.
The classical torus $T = S^1$ is a closed quantum subgroup of $SU_q(2)$. Explicitly, the inclusion $T \subset SU_q(2)$ is determined by the $*$-homomorphism $\pi : \mathbb{C}[SU_q(2)] \to \mathbb{C}[T] = \mathbb{C}[z, z^{-1}]$ given by

$$\pi \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}.$$

By definition, the standard Podleś sphere $SU_q(2)/T$ is the space of coinvariants with respect to the restricted action, that is

$$C(SU_q(2)/T) = \{x \in C(SU_q(2)) | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\},$$

see [20]. Inside $C(SU_q(2)/T)$ we have the dense $*$-subalgebra

$$\mathbb{C}[SU_q(2)/T] = \{x \in \mathbb{C}[SU_q(2)] | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes 1\}$$

corresponding to polynomial functions. In the classical case $q = 1$, these algebras define the continuous and polynomial functions on the 2-sphere $SU(2)/T = S^2$, respectively.
Generalizing these constructions, we define for $k \in \mathbb{Z}$ the space

$$\Gamma(E_k) = \{x \in \mathbb{C}[SU_q(2)] | (\mathrm{id} \otimes \pi)\Delta(x) = x \otimes z^k\} \subset \mathbb{C}[SU_q(2)]$$

and let $C(E_k)$ and $L^2(E_k)$ be the closures of $\Gamma(E_k)$ in $C(SU_q(2))$ and $L^2(SU_q(2))$, respectively. The space $\Gamma(E_k)$ is a $\mathbb{C}[SU_q(2)/T]$-bimodule in a natural way which is finitely generated and projective both as a left and right $\mathbb{C}[SU_q(2)/T]$-module. The latter follows from the fact that $\mathbb{C}[SU_q(2)/T] \subset \mathbb{C}[SU_q(2)]$ is a faithfully flat Hopf-Galois extension, compare [21]. The space $C(E_k)$ is naturally a $SU_q(2)$-equivariant Hilbert $C(SU_q(2)/T)$-module with coaction given by comultiplication. The space $L^2(E_k)$ is naturally a $SU_q(2)$-Hilbert space. These structures are induced from $C(SU_q(2))$ and $L^2(SU_q(2))$ by restriction.
In the classical case $q = 1$, the above constructions correspond to looking at induced vector bundles over the homogeneous space $SU(2)/T \cong S^2$. More precisely, if $\mathbb{C}_k$ is the irreducible representation of T weight $k \in \mathbb{Z}$, then

$$\Gamma(E_k) = \Gamma(SU(2) \times_T \mathbb{C}_k)$$

is the space of polynomial sections of the vector bundle

$$SU(2) \times_T \mathbb{C}_k = (SU(2) \times \mathbb{C})/\sim$$

over $SU(2)/T$ where

$$(gt, \lambda) \sim (g, t^k \lambda)$$

for all $g \in SU(2), \lambda \in \mathbb{C}$ and $t \in T$. Similarly, $C(E_k)$ and $L^2(E_k)$ are the spaces of continuous sections and L^2-sections, respectively, in this case.
Recall from section 4 the definition of the Drinfeld double $\mathsf{D}(G)$ of a locally compact quantum group G. The C^*-algebra $C(SU_q(2)/T)$ is a $\mathsf{D}(SU_q(2))$-C^*-algebra with the coaction $\lambda : C(SU_q(2)/T) \to M(C^*(SU_q(2)) \otimes C(SU_q(2)/T))$ given by

$$\lambda(g) = \hat{W}^*(1 \otimes g)\hat{W}$$

where $\hat{W} = \Sigma W^* \Sigma$ as usual. This coaction is determined on the algebraic level by the adjoint action

$$h \cdot g = h_{(1)} g S(h_{(2)})$$

of $\mathbb{C}[SU_q(2)]$ on $\mathbb{C}[SU_q(2)/T]$. To see that this action preserves $\mathbb{C}[SU_q(2)/T]$ we compute

$$\begin{aligned}
(\mathrm{id} \otimes \pi)\Delta(h \cdot g) &= (\mathrm{id} \otimes \pi)\Delta(h_{(1)} g S(h_{(2)})) \\
&= (\mathrm{id} \otimes \pi)(h_{(1)} g_{(1)} S(h_{(4)}) \otimes h_{(2)} g_{(2)} S(h_{(3)})) \\
&= h_{(1)} g_{(1)} S(h_{(4)}) \otimes \pi(h_{(2)}) \pi(g_{(2)}) \pi(S(h_{(3)})) \\
&= h_{(1)} g_{(1)} S(h_{(4)}) \otimes \pi(h_{(2)} S(h_{(3)})) \\
&= h_{(1)} g_{(1)} S(h_{(2)}) \otimes 1 = (h \cdot g) \otimes 1,
\end{aligned}$$

so that $h \cdot g$ is indeed contained in $\mathbb{C}[SU_q(2)/T]$ for all $h \in \mathbb{C}[SU_q(2)]$. The same construction turns the spaces $C(E_k)$ for $k \in \mathbb{Z}$ into $\mathsf{D}(SU_q(2))$-equivariant Hilbert $C(SU_q(2)/T)$-modules for every $k \in \mathbb{Z}$.
In the case of Hilbert spaces we need to twist the action as follows.

Lemma 5.1. *For every $k \in \mathbb{Z}$ the formula*

$$\omega(h)(\xi) = h_{(1)} \xi \, f_1 \rightharpoonup S(h_{(2)})$$

determines a $*$-homomorphism $\omega : C(SU_q(2)) \to \mathbb{L}(L^2(E_k))$ which turns $L^2(E_k)$ into a $\mathsf{D}(SU_q(2))$-Hilbert space.

The proof of lemma 5.1 is a straightforward computation. The twist by the modular character f_1 is needed because the Haar integral ϕ fails to be a trace unless $q = 1$.
Our aim is to describe the Podleś sphere as an element in the equivariant KK-category $KK^{\mathsf{D}(SU_q(2))}$. It is easy to show that the C^*-algebra $C(SU_q(2)/T)$ of

the Podleś sphere for $q \neq 1$ is isomorphic to $\mathbb{K}^+$, the compact operators $\mathbb{K}$ on a separable Hilbert space with a unit adjoined. Hence one obtains an extension

$$0 \longrightarrow \mathbb{K} \longrightarrow \mathbb{K}^+ \longrightarrow \mathbb{C} \longrightarrow 0$$

of C^*-algebras. Consider the $*$-homomorphism $\rho : \mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}^+ \to \mathbb{K}^+$ given by $\rho(1,0) = 1 - e_{11}, \rho(0,1) = e_{11}$. Since $K(\mathbb{K}) = \mathbb{Z} = K(\mathbb{C})$ we see that ρ induces an isomorphism on the level of K-theory and KK-theory. We thus obtain the following statement.

Lemma 5.2. *The map ρ induces an isomorphism $C(SU_q(2)/T) \cong \mathbb{C} \oplus \mathbb{C}$ in KK.*

However, the isomorphism in lemma 5.2 does not respect the $\mathsf{D}(SU_q(2))$-actions, in fact not even the $SU_q(2)$-actions. Here $\mathbb{C} \oplus \mathbb{C}$ is of course equipped with the trivial action. To improve on this we need more refined arguments.
Firstly, we recall the definition of the equivariant Fredholm module corresponding to the Dirac operator on the standard Podleś sphere $SU_q(2)/T$, compare [6], [19]. The underlying graded $SU_q(2)$-Hilbert space is

$$\mathcal{H} = L^2(E_1) \oplus L^2(E_{-1})$$

as defined above. The representation ϕ of $C(SU_q(2)/T)$ is given by left multiplication. We obtain a G-equivariant self-adjoint unitary operator F on $\mathcal{H}$ by

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

by identifying the basis vectors $e^{(l)}_{i,1/2}$ and $e^{(l)}_{i,-1/2}$ in even and odd degrees.

Proposition 5.3. *With the notation as above, $D = (\mathcal{H}, \phi, F)$ is a $SU_q(2)$-equivariant Fredholm module defining an element in $KK^{SU_q(2)}(C(SU_q(2)/T), \mathbb{C})$.*

Our considerations above show that the Drinfeld double $\mathsf{D}(SU_q(2))$ acts on the ingredients of the above cycle. The following proposition shows that this additional symmetry is compatible with the structure of D.

Proposition 5.4. *Let $q \in (0, 1]$. The Fredholm module D defined above induces an element $[D]$ in $KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), \mathbb{C})$.*

Proof. We note that for $q = 1$ all the actions defined above are trivial. Hence only the case $q < 1$ has to be treated. We have to verify that F commutes with the action of $\mathsf{D}(SU_q(2))$ up to compact operators. Since F is $SU_q(2)$-equivariant this amounts to showing

$$(C^*(SU_q(2)) \otimes 1)(1 \otimes F - \mathsf{ad}_\lambda(F)) \subset C^*(SU_q(2)) \otimes \mathbb{K}(\mathcal{H})$$

where $\lambda : \mathcal{H} \to M(C^*(SU_q(2)) \otimes \mathcal{H})$ is the coaction on $\mathcal{H}$ obtained from lemma 5.1. It suffices to check that F commutes with the corresponding action $\omega : C(SU_q(2)) \to \mathbb{L}(\mathcal{H})$ up to compact operators. This is a straightforward explicit calculation, in fact, F commutes exactly with this action. □

Recall that $C(E_k)$ for $k \in \mathbb{Z}$ is a $\mathsf{D}(SU_q(2))$-equivariant Hilbert $C(SU_q(2)/T)$-module in a natural way. Left multiplication yields a $\mathsf{D}(SU_q(2))$-equivariant $*$-homomorphism $\psi : C(SU_q(2)/T) \to \mathbb{K}(C(E_k))$, and it is straightforward to check that $(C(E_k), \psi, 0)$ defines a class $[[E_k]]$ in $KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), C(SU_q(2)/T))$. Moreover

$$[[E_m]] \otimes_{C(SU_q(2)/T)} [[E_n]] = [[E_{m+n}]]$$

for all $m, n \in \mathbb{Z}$.
We define $[D_k] \in KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), \mathbb{C})$ by

$$[D_k] = [[E_k]] \otimes_{C(SU_q(2)/T)} [D]$$

where $[D] \in KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), \mathbb{C})$ is the element obtained in proposition 5.4. Remark that $[D_0] = [D]$ since $[[E_0]] = 1$.
The unit map $u : \mathbb{C} \to C(SU_q(2)/T)$ given by $u(1) = 1$ induces a class $[u]$ in $KK^{\mathsf{D}(SU_q(2))}(\mathbb{C}, C(SU_q(2)/T))$. We define $[E_k]$ in $KK^{\mathsf{D}(SU_q(2))}(\mathbb{C}, C(SU_q(2)/T))$ by

$$[E_k] = [u] \otimes_{C(SU_q(2)/T)} [[E_k]].$$

Using this notation we define $\alpha_q \in KK^{\mathsf{D}(SU_q(2))}(C(SU_q(2)/T), \mathbb{C} \oplus \mathbb{C})$ and $\beta_q \in KK^{\mathsf{D}(SU_q(2))}(\mathbb{C} \oplus \mathbb{C}, C(SU_q(2)/T))$ by

$$\alpha_q = [D_0] \oplus [D_{-1}], \qquad \beta_q = (-[\mathcal{E}_1]) \oplus [\mathcal{E}_0],$$

respectively.

Theorem 5.5. *Let $q \in (0, 1]$. The standard Podleś sphere $C(SU_q(2)/T)$ is isomorphic to $\mathbb{C} \oplus \mathbb{C}$ in $KK^{\mathsf{D}(SU_q(2))}$.*

Proof. We claim that β_q and α_q define inverse isomorphisms. The crucial part is the relation

$$\beta_q \circ \alpha_q = \mathrm{id}$$

in $KK^{\mathsf{D}(SU_q(2))}(\mathbb{C} \oplus \mathbb{C}, \mathbb{C} \oplus \mathbb{C})$. In order to prove it we have to compute the Kasparov products $[\mathcal{E}_0] \circ [D]$ and $[\mathcal{E}_{\pm 1}] \circ [D]$.
The class $[\mathcal{E}_0] \circ [D]$ is obtained from the $\mathsf{D}(SU_q(2))$-equivariant Fredholm module D by forgetting the left action of $C(SU_q(2)/T)$. The operator F intertwines the representations of $C(SU_q(2))$ on $L^2(E_1)$ and $L^2(E_{-1})$ induced from the $\mathsf{D}(SU_q(2))$-Hilbert space structure. It follows that the resulting $\mathsf{D}(SU_q(2))$-equivariant Kasparov $\mathbb{C}$-$\mathbb{C}$-module is degenerate, and hence $[\mathcal{E}_0] \circ [D] = 0$ in $KK^{\mathsf{D}(SU_q(2))}(\mathbb{C}, \mathbb{C})$.
For $[\mathcal{E}_{\pm 1}] \circ [D]$ the situation is much more complicated. For instance, it is easy to check $[\mathcal{E}_{-1}] \circ [D] = 1$ in $KK^G(\mathbb{C}^2, \mathbb{C}^2)$. The entire difficulty lies in constructing a $\mathsf{D}(SU_q(2))$-equivariant homotopy to obtain the same relation on the level of $KK^{\mathsf{D}(SU_q(2))}$. For the details we refer to [23]. □
For $q = 1$ the difficulties with the $\mathsf{D}(SU_q(2))$-actions disappear since in this case the discrete part of the Drinfeld double acts trivially. Thus there is an important difference between the classical and the deformed situations.

6. Torsion-free discrete quantum groups and the Baum-Connes conjecture

In this section we formulate the Baum-Connes conjecture for torsion-free discrete quantum groups following Meyer [14]. This involves some general concepts from homological algebra in triangulated categories. For more detailed information we refer to [15], [14], [16].
Our first task is, however, to explain what it means for a discrete quantum group to be torsion-free. Recall that a discrete group is called torsion-free if it does not contain nontrivial elements of finite order. For discrete quantum groups the following definition was proposed by Meyer [14].

Definition 6.1. *Let G be discrete quantum group and set $\hat{S} = C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$. Then G is called (quantum) torsion-free iff every finite dimensional $\hat{S}$-C^*-algebra is equivariantly isomorphic to a direct sum of $\hat{S}$-C^*-algebras that are equivariantly Morita equivalent to $\mathbb{C}$.*

In order to explain this definition let us first review the following basic construction. Assume that $\mathcal{H}$ is a finite dimensional Hilbert space and that $\lambda : \mathcal{H} \to \hat{S} \otimes \mathcal{H}$ is a coaction. As above we write here $\hat{S} = C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$ for the unital Hopf-C^*-algebra associated to a discrete quantum group G. Since $\hat{S}$ is unital, there are no multipliers involved in the definition of the coaction, see definition 3.6. As indicated at the end of section 3, the map λ is uniquely determined by a unitary element $U_\lambda \in \mathbb{L}(\hat{S} \otimes \mathcal{H}) \cong \mathbb{L}(\mathcal{H} \otimes \hat{S})$ satisfying

$$\lambda(\xi) = U_\lambda^*(1 \otimes \xi).$$

The operator U_λ is a unitary corepresentation, that is, the relation

$$(\hat{\Delta}^{\mathsf{cop}} \otimes \mathrm{id})(U_\lambda) = (U_\lambda)_{13}(U_\lambda)_{23}$$

holds in $\hat{S} \otimes \hat{S} \otimes \mathcal{H}$. Moreover we obtain a coaction $\mathsf{ad}_\lambda : \mathbb{K}(\mathcal{H}) \to \hat{S} \otimes \mathbb{K}(\mathcal{H})$ by the formula

$$\mathsf{ad}_\lambda(T) = U_\lambda^*(\mathrm{id} \otimes T)U_\lambda$$

which turns $\mathbb{K}(\mathcal{H})$ into a $\hat{S}$-C^*-algebra.
Roughly speaking, torsion-freeness of G amounts to saying that every coaction of $\hat{S}$ on a finite dimensional C^*-algebra is obtained using this construction. More precisely, G is torsion-free iff for every finite dimensional $\hat{S}$-C^*-algebra A there are finite dimensional Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_l$ and unitary corepresentations $U_j \in \mathbb{L}(\hat{S} \otimes \mathcal{H}_j)$ such that

$$A \cong \mathbb{K}(\mathcal{H}_1) \oplus \cdots \oplus \mathbb{K}(\mathcal{H}_l)$$

as $\hat{S}$-C^*-algebras. Here each matrix block $\mathbb{K}(\mathcal{H}_j)$ is equipped with the adjoint action implemented by U_j.
An $\hat{S}$-C^*-algebra A with coaction $\alpha : A \to \hat{S} \otimes A$ is called ergodic iff its fixed point subalgebra

$$A_e = \{a \in A | \alpha(a) = 1 \otimes a\}$$

is equal to $\mathbb{C}$. In order to study the property described in definition 6.1 the following terminology is useful.

Definition 6.2. *Let G be a discrete quantum group and $\hat{S} = C^*_{\mathsf{r}}(G)^{\mathsf{cop}}$.*

a) G has no permutation torsion if there are no nonsimple finite dimensional C^*-algebras with ergodic coactions of $\hat{S}$.
b) G has no projective torsion if for every finite dimensional simple $\hat{S}$-C^*-algebra A there exists a finite dimensional unitary corepresentation $\mathcal{H}$ of $\hat{S}$ such that $A \cong \mathbb{K}(\mathcal{H})$ as $\hat{S}$-C^*-algebras.

A basic source of permutation torsion in a discrete quantum group arises from nontrivial finite quantum subgroups and their permutation actions. Similarly, projective torsion arises from projective representations of the dual compact quantum group.

Proposition 6.3. *A discrete quantum group G is quantum torsion-free iff it has no permutation torsion and no projective torsion.*

Proof. Assume first that G is quantum torsion-free. Then clearly G has no projective torsion. Now let A be a nonsimple finite dimensional ergodic $\hat{S}$-C^*-algebra. According to torsion-freeness there are corepresentations $\mathcal{H}_1 \dots, \mathcal{H}_l$ such that

$$A \cong \mathbb{K}(\mathcal{H}_1) \oplus \cdots \oplus \mathbb{K}(\mathcal{H}_l)$$

as $\hat{S}$-C^*-algebras. Since A is nonsimple we must have $l > 1$. But for $l > 1$ the right hand side is not an ergodic $\hat{S}$-C^*-algebra. Hence such an $\hat{S}$-C^*-algebra cannot exist, and G has no permutation torsion.
Conversely, assume that G has neither permutation torsion nor projective torsion and let A be a finite dimensional $\hat{S}$-C^*-algebra. We may assume that A is not simple and not ergodic. Let

$$A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$$

and $P_1, \dots, P_m$ be mutually orthogonal mimimal projections in the invariant subalgebra $A_e \subset A$ such that $P_1 + \cdots + P_m = 1$. Write

$$P_j = (p_1^j, \dots, p_l^j)$$

for all j and consider $P_j A P_j$. Since P_j is invariant, $P_j A P_j$ is a $\hat{S}$-C^*-algebra, and since P_j is minimal in A_e it is ergodic. Since G has no permutation torsion we conclude that $P_j A P_j$ is simple, and hence

$$P_j A P_j = \mathbb{K}(\mathcal{H}_j)$$

as $\hat{S}$-C^*-algebras because G has no projective torsion. In particular, $p_j^k = 0$ except for one k. Accordingly, every P_j is supported on a single matrix block, and using $P_1 + \cdots + P_m = 1$ it follows that each central projection in A must be invariant. Hence we may reduce to the case that A is a single matrix block, and since A has no projective torsion we obtain the assertion. □

Let us now check that definition 6.1 is compatible with the classical definition of torsion-freeness in the case of discrete groups.

Proposition 6.4. *A discrete group G is torsion-free in the quantum sense above iff it is torsion-free in the usual sense.*

Proof. Assume first that G is not torsion-free in the classical sense. Then there exists an element $t \in G$ different from the identity e and $n > 1$ such that $t^n = e$. Consider the finite subgroup $F \subset G$ generated by t. Then the C^*-algebra $A = C^*(F)$ is finite dimensional, and we obtain a coaction $A \to C^*_{\mathsf{r}}(G) \otimes A$ from the restriction of the comultiplication to A. Since F is abelian we have $A \cong \mathbb{C}^n$. If G were torsion-free in the quantum sense we would find one-dimensional corepresentations $\mathcal{H}_1, \dots, \mathcal{H}_n$ of $C^*_{\mathsf{r}}(G)$ such that $A \cong \mathbb{K}(\mathcal{H}_1) \oplus \cdots \oplus \mathbb{K}(\mathcal{H}_n)$ as $C^*_{\mathsf{r}}(G)$-C^*-algebras. This would imply that the coaction on A is trivial. We obtain a contradition since $\Delta(t) = t \otimes t$ and t is different from the identity element by assumption. Note that the above coaction on A is ergodic, so that G has permutation torsion.

Conversely let us assume that G is torsion-free in the classical sense. In order to show that G is quantum torsion-free it suffices according to proposition 6.3 to show that G has no permutation torsion and no projective torsion. Let A be a finite dimensional C^*-algebra and let $\alpha : A \to C^*_{\mathsf{r}}(G) \otimes A$ be a coaction. Since A is finite dimensional, we may view α as an algebraic coaction $\alpha : A \to \mathbb{C}G \odot A$. This corresponds to a G-grading on A. That is,

$$A = \bigoplus_{s \in G} A_s$$

where $A_s A_t \subset A_{st}$ and $(A_s)^* = A_{s^{-1}}$ for all $s \in G$.

Let us first show that $A_e = \mathbb{C}$ implies $A = \mathbb{C}$. If A is differnt from $A_e = \mathbb{C}$ we may pick an element $t \in G$ such that A_t is nonzero. If $x \in A_t$ is nonzero then $x^* x \in A_e$ is nonzero and hence invertible. It follows that x itself is invertible. Since t has infinite order and $x^n \in A_{t^n}$ is nonzero for all n we would obtain that A is infinite dimensional. We conclude that $A = A_e = \mathbb{C}$ as claimed.

It follows in particular that G has no permutation torsion. Hence it remains to show that G has no projective torsion. We shall prove by induction on the dimension of the simple C^*-algebra A that the coaction α has the required form. For $\dim(A) = 1$ the coaction is necessarily trivial, and therefore the claim is true in this case. Now assume that the assertion is proved for all dimensions $\leq n$ and let A be of dimension $n+1$. By our above reasoning, A_e contains a nontrivial invariant projection p. Let us write

$$A = M_n(\mathbb{C})$$

and cut down A with p to obtain a coaction of $C^*_\mathsf{r}(G)$ on pAp. Since pAp is simple we know by induction hypothesis that the $C^*_\mathsf{r}(G)$-C^*-algebra pAp is of the form

$$pAp \cong \mathbb{K}(\mathcal{H})$$

for some corepresentation $\mathcal{H}$. Since every corepresentation of $C^*_\mathsf{r}(G)$ is a direct sum of one-dimensional corepresentations the algebra $\mathbb{K}(\mathcal{H})$ contains an invariant minimal projection. It follows that A itself contains an invariant minimal projection e. Pick an invariant state ϕ on A such that $\phi(e)$ is nonzero. We write $\mathcal{K}$ for linear subspace Ae inside the corresponding GNS-construction. It carries naturally a coaction of $\hat{S}$. The GNS-representation restricts to a $*$-homomorphism $\mu : A \to \mathbb{K}(\mathcal{K})$ given by $\mu(a)(x) = ax$. By construction, the map μ is covariant. Since it is nonzero it is an isomorphism of $\hat{S}$-C^*-algebras by dimension reasons. Hence G has no projective torsion. □

Definition 6.1 is motivated from the study of torsion phenomena that occur for coactions of compact groups [17].

Lemma 6.5. *The dual of a compact Lie group G has no permutation torsion iff G is connected.*

Proof. If G is connected, then every action of G on

$$A = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C})$$

must preserve the individual matrix blocks. Indeed, if $(h_t)_{t\in[0,1]}$ is a path from e to s in G then every central projection p_j must obviously stay fixed under each h_t. Therefore $s \cdot x = s \cdot (p_j x) = p_j(s \cdot x)$ for all $x \in M_{n_j}(\mathbb{C})$, so that $s \cdot x \in M_{n_j}(\mathbb{C})$ as well.

Conversely assume that G is not connected. Then the quotient $F = G/G_0$ of G by its connected component is a nontrivial finite group, and since F is finite it has permutation torsion. Hence G has permutation torsion as well. □

The dual of a compact connected Lie group may still have projective torsion.

Example 6.6. *The dual $\hat{G}$ of $G = SO(3)$ is not torsion-free. To see this consider the vector representation of $SU(2)$ on $\mathbb{C}^2$. The corresponding conjugation action on $M_2(\mathbb{C})$ is trivial on the center $Z(SU(2))$ of $SU(2)$ since*

$$Z(SU(2)) = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z}_2$$

Hence it descends to an action of $SO(3) \cong SU(2)/Z(SU(2))$. This action is not $SO(3)$-equivariantly Morita equivalent to a trivial action since the spinor space $\mathbb{C}^2$ is only a projective representation of $SO(3)$.

In fact, the dual $\hat{G}$ of a compact Lie group G has no projective torsion iff there are no nontrivial projective representations of G. If G is connected this happens iff the fundamental group $\pi_1(G)$ of G is torsion-free.

We finally mention the following result in the case of $SU_q(2)$.

Proposition 6.7. *Let $q \in (0, 1]$. Then the discrete quantum group dual to $SU_q(2)$ is torsion-free.*

For the proof of proposition 6.7 we refer to [23]. In the case $q < 1$ it is based on a computation using the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$. For $q = 1$ one may do a similar computation using the ordinary enveloping algebra $U(\mathfrak{sl}_2)$, or instead refer to example a) above, the proof of which is contained in [17].

We shall now review some basics on triangulated categories. Let $\mathcal{T}$ be an additive category and let $\Sigma : \mathcal{T} \to \mathcal{T}$ be an additive automorphism. A diagram

$$\Sigma(B) \xrightarrow{h} E \xrightarrow{g} A \xrightarrow{f} B$$

in $\mathcal{T}$ is called a triangle if $f \circ g = 0$, $g \circ h = 0$ and $h \circ \Sigma(f) = 0$.
Let us now state the definition of a triangulated category, see [18].

Definition 6.8. *A triangulated category is an additive category $\mathcal{T}$ together with an additive automorphism $\mathcal{T}$, called the translation functor, and a class of exact triangles such that the following conditions are satisfied.*

(TR0) *Any triangle which is isomorphic to an exact triangle is exact. Every triangle of the form*

$$\Sigma(B) \longrightarrow 0 \longrightarrow B \xrightarrow{\mathrm{id}} B$$

is exact.

(TR1) *For any morphism $f : A \to B$ in $\mathcal{T}$ there exists an exact triangle*

$$\Sigma(B) \longrightarrow C \longrightarrow A \xrightarrow{f} B$$

(TR2) *A triangle*

$$\Sigma(B) \xrightarrow{h} C \xrightarrow{g} A \xrightarrow{f} B$$

is exact iff

$$\Sigma(A) \xrightarrow{-\Sigma(f)} \Sigma(B) \xrightarrow{-h} C \xrightarrow{-g} A$$

is exact.

(TR3) *If*

$$\begin{tikzcd}
\Sigma(B) \arrow[r] \arrow[d, "\Sigma(\beta)"] & C \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\
\Sigma(B') \arrow[r] & C' \arrow[r] & A' \arrow[r] & B'
\end{tikzcd}$$

is a commutative diagram in $\mathcal{T}$ such that both rows are exact triangles, then there exist a morphism $\gamma : C \to C'$ such that the completed diagram is still commutative.

(TR4) *(Octahedral axiom) If $f : B \to D$ and $g : A \to B$ are morphisms in $\mathcal{T}$ there exists a commutative diagram*

$$\begin{tikzcd}
\Sigma^2(D) \arrow[r] \arrow[d] & \Sigma(C_f) \arrow[r] \arrow[d] & \Sigma(B) \arrow[r, "\Sigma(f)"] \arrow[d] & \Sigma(D) \arrow[d] \\
0 \arrow[r] \arrow[d] & C_g \arrow[r, equal] \arrow[d] & C_g \arrow[r] \arrow[d] & 0 \arrow[d, "\beta"] \\
\Sigma(D) \arrow[r] \arrow[d] & C_{fg} \arrow[r] \arrow[d] & A \arrow[r, "fg"] \arrow[d, "g"] & D \arrow[d, equal] \\
\Sigma(D) \arrow[r] & C_f \arrow[r] & B \arrow[r, "f"] & D
\end{tikzcd}$$

whose rows and columns are exact triangles. Moreover, the morphisms $\Sigma(B) \to \Sigma(D) \to C_{fg}$ and $\Sigma(B) \to C_g \to C_{fg}$ in this diagram coincide.

We also need the notion of a localizing subcategory.

Definition 6.9. *Let $\mathcal{T}$ be a triangulated category with countable direct sums. A triangulated subcategory of $\mathcal{T}$ is a full subcategory $\mathcal{S} \subset \mathcal{T}$ that is closed under Σ and has the property that if*

$$\Sigma B \longrightarrow C \longrightarrow A \longrightarrow B$$

is an exact triangle with $A, B \in \mathcal{S}$, then $C \in \mathcal{S}$ as well.
A triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is called localising if it is closed under countable direct sums.

Every localising subcategory $\mathcal{S} \subset \mathcal{T}$ is closed under retracts. That is, if $X = X_1 \oplus X_2$ is a direct sum in $\mathcal{T}$ and $X \in \mathcal{S}$, then also $X_1, X_2 \in \mathcal{S}$.
Let G be a discrete quantum group. The equivariant Kasparov category KK^G has as objects all separable G-C^*-algebras, and $KK^G(A, B)$ as the set of morphisms between two objects A and B. Composition of morphisms is given by the Kasparov product.
The category KK^G is triangulated with translation automorphism $\Sigma : KK^G \to KK^G$ given by the suspension

$$\Sigma A = C_0(\mathbb{R}, A) = C_0(\mathbb{R}) \otimes A$$

of a G-C^*-algebra A. Here $C_0(\mathbb{R})$ carries the trivial G-action.
Every G-equivariant $*$-homomorphism $f : A \to B$ induces a diagram of the form

$$\Sigma B \longrightarrow C_f \longrightarrow A \xrightarrow{f} B$$

where C_f denotes the mapping cone of f. By definition

$$C_f = \{(a, b) \in A \oplus C_0((0, 1], B) | f(a) = b(1)\} \subset A \oplus C_0((0, 1], B),$$

and the maps $\Sigma B \to C_f$ and $C_f \to A$ are given by inclusion in the second variable and projection onto the first variable, respectively. Diagrams of the form above are called mapping cone triangles.
By definition, an exact triangle is a diagram in KK^G of the form $\Sigma Q \to K \to E \to Q$ which is isomorphic to a mapping cone triangle.
For the formulation of the Baum-Connes conjecture for a locally compact group G we have to consider G-C^*-algebras that are induced from compact subroups. The easiest situation arises when there are no nontrivial compact subgroups. For a discrete group G this means of course just that G is torsion-free.
Since torsion for quantum groups is not well-understood we consider in the sequel only the case that G is a torsion-free discrete quantum group. Associated with the inclusion of the trivial quantum subgroup E in G we have the obvious restriction functor $\mathrm{res}^G_E : KK^G \to KK^E = KK$ and an induction functor $\mathrm{ind}^G_E : KK \to KK^G$. Explicitly, $\mathrm{ind}^G_E(A) = C_0(G) \otimes A$ for $A \in KK$ with action of G given by comultiplication on the copy of $C_0(G)$.
We consider the following full subcategories of KK^G,

$$\mathcal{CC}_G = \{A \in KK^G | \, \mathrm{res}^G_E(A) = 0 \in KK\}$$
$$\mathcal{CI}_G = \{\mathrm{ind}^G_E(A) | A \in KK\},$$

and refer to their objects as compactly contractible and compactly induced G-C^*-algebras, respectively. If there is no risk of confusion we will write $\mathcal{CC}$ and $\mathcal{CI}$ instead of $\mathcal{CC}_G$ and $\mathcal{CI}_G$. The subcategory $\mathcal{CC}$ is localising, and we denote by $\langle \mathcal{CI} \rangle$ the localising subcategory generated by $\mathcal{CI}$.
The following result follows from theorem 3.21 in [14].

Theorem 6.10. *Let G be a torsion-free quantum group. Then the pair of localising subcategories $(\langle \mathcal{CI} \rangle, \mathcal{CC})$ in KK^G is complementary. That is, $KK^G(P, N) = 0$ for all $P \in \langle \mathcal{CI} \rangle$ and $N \in \mathcal{CC}$, and every object $A \in KK^G$ fits into an exact triangle*

$$\Sigma N \longrightarrow \tilde{A} \longrightarrow A \longrightarrow N$$

with $\tilde{A} \in \langle \mathcal{CI} \rangle$ and $N \in \mathcal{CC}$.

A triangle as in theorem 6.10 is uniquely determined up to isomorphism and depends functorially on A. The morphism $\tilde{A} \to A$ is called a Dirac element for A. The localisation $\mathbb{L}F$ of a functor F on KK^G at $\mathcal{CC}$ is given by

$$\mathbb{L}F(A) = F(\tilde{A})$$

where $\tilde{A} \to A$ is a Dirac element for A. By construction, there is an obvious map $\mathbb{L}F(A) \to F(A)$ for every $A \in KK^G$.
In the sequel we write $G \ltimes_{\mathsf{r}} A$ for the reduced crossed products of A by G.

Definition 6.11. *Let G be a torsion-free discrete quantum group and consider the functor $F(A) = K_*(G \ltimes_{\mathsf{r}} A)$ on KK^G. The Baum-Connes assembly map for G with coefficients in A is the map*

$$\mu_A : \mathbb{L}F(A) \to F(A).$$

We say that G satisfies the Baum-Connes conjecture with coefficients in A if μ_A is an isomorphism. We say that G satisfies the strong Baum-Connes conjecture if $\langle \mathcal{CI} \rangle = KK^G$.

The strong Baum-Connes conjecture implies the Baum-Connes conjecture with arbitrary coefficients. Indeed, for $A \in \langle \mathcal{CI} \rangle$ the assembly map μ_A is an isomorphism since $\mathrm{id} : A \to A$ is a Dirac morphism for A.
By the work of Meyer and Nest [16], the above terminology is consistent with the classical definitions in the case that G is a torsion-free discrete group. The strong Baum-Connes conjecture amounts to the assertion that G has a γ-element and $\gamma = 1$ in this case.

7. The Baum-Connes conjecture for the dual of $SU_q(2)$

In this section we show that the dual of $SU_q(2)$ satisfies the strong Baum-Connes conjecture. We work within the general setup explained in the previous section, taking into account proposition 6.7 which asserts that the dual of $SU_q(2)$ is torsion-free. Let us remark that the strong Baum-Connes conjecture for the dual of the classical group $SU(2)$ is a special case of the results in [17].
As a preparation we need some results on induction and restriction. Let G be a locally compact quantum group and let H be a closed quantum subgroup. If B is a G-C^*-algebra we may restrict the coaction to obtain an H-C^*-algebra $\mathrm{res}^G_H(B)$. In this way we obtain a triangulated functor $\mathrm{res}^G_H : G\text{-}\mathsf{Alg} \to H\text{-}\mathsf{Alg}$.

Examples 7.1. *Instead of giving the precise definition of the restriction functor in general, we only give two examples that are relevant for our purposes.*

a) In the previous section we considered the trivial (quantum) subgroup $E = \{e\}$ of a discrete quantum group G. The restriction functor G-$\mathsf{Alg} \to E$-Alg is nothing but the forgetful functor to C^*-algebras.

b) Below we will need restriction in the case that $G = SU_q(2)$ and $H = T \subset G$ is the maximal torus. Restriction of coactions is obtained using the quotient map $C(G) \to C(T)$.

Conversely, let G be a strongly regular quantum group and let $H \subset G$ be a closed quantum subgroup. Given an H-C^*-algebra B, there exists an induced G-C^*-algebra $\mathrm{ind}^G_H(B)$, see [22]. If B is an H-YD-algebra then the induced C^*-algebra $\mathrm{ind}^G_H(B)$ is a G-YD-algebra in a natural way. These constructions yield triangulated functors $\mathrm{ind}^G_H : H\text{-}\mathsf{Alg} \to G\text{-}\mathsf{Alg}$ and $\mathrm{ind}^G_H : \mathsf{D}(H)\text{-}\mathsf{Alg} \to \mathsf{D}(G)\text{-}\mathsf{Alg}$.

Examples 7.2. *Let us give two examples of induced C^*-algebras.*

a) If $E \subset G$ is the trivial quantum subgroup then $\mathrm{ind}^G_E(B) = C^{\mathsf{r}}_0(G) \otimes B$ where the coaction is given by comultiplication on the left tensor factor.

b) If $G = SU_q(2)$ and $H = T \subset G$ is the maximal torus, we have $\operatorname{ind}_T^G(\mathbb{C}) = C(G/T)$, the Podleś sphere.

The following theorem is a fundamental result relating induction with braided tensor products.

Theorem 7.3. *Let G be a strongly regular quantum group and let $H \subset G$ be a closed quantum subgroup. Moreover let A be an H-YD-algebra and let B be a G-algebra. Then there is a natural G-equivariant isomorphism*

$$\operatorname{ind}_H^G(A \boxtimes_H \operatorname{res}_H^G(B)) \cong \operatorname{ind}_H^G(A) \boxtimes_G B.$$

The proof of theorem 7.3 can be found in [19].
Now we are ready to prove the Baum-Connes conjecture for the dual of $SU_q(2)$.

Theorem 7.4. *Let $q \in (0, 1]$. The dual discrete quantum group of $SU_q(2)$ satisfies the strong Baum-Connes conjecture.*

Proof. In the sequel we write $G = SU_q(2)$. According to Baaj-Skandalis duality it suffices to prove $KK^G = \langle \mathcal{T} \rangle$ where $\langle \mathcal{T} \rangle$ denotes the localizing subcategory of KK^G generated by all trivial G-C^*-algebras.
Let A be a G-C^*-algebra. Theorem 5.5 implies that A is a retract of $C(G/T) \boxtimes_G A$ in KK^G, and according to theorem 7.3 we have a G-equivariant isomorphism

$$C(G/T) \boxtimes_G A = \operatorname{ind}_T^G(\mathbb{C}) \boxtimes_G A \cong \operatorname{ind}_T^G \operatorname{res}_T^G(A).$$

Since $\hat{T} = \mathbb{Z}$ is a torsion-free discrete abelian group the strong Baum-Connes conjecture holds for $\hat{T}$. That is, we have

$$KK^{\mathbb{Z}} = \langle \mathcal{CI} \rangle$$

where $\mathcal{CI} = \mathcal{CI}_{\mathbb{Z}}$ denotes the full subcategory in $KK^{\mathbb{Z}}$ of compactly induced $\mathbb{Z}$-C^*-algebras. Equivalently, we have

$$KK^T = \langle \mathcal{T} \rangle$$

where $\mathcal{T} \subset KK^T$ is the full subcategory of trivial T-C^*-algebras. In particular we obtain

$$\operatorname{res}_T^G(A) \in \langle \mathcal{T} \rangle \subset KK^T.$$

Due to theorem 5.5 we know that

$$\operatorname{ind}_T^G(B) = C(G/T) \otimes B \cong (\mathbb{C} \oplus \mathbb{C}) \otimes B$$

is contained in $\langle \mathcal{T} \rangle$ inside KK^G for any trivial T-C^*-algebra B. Since the induction functor $\operatorname{ind}_T^G : KK^T \to KK^G$ is triangulated it therefore maps $\langle \mathcal{T} \rangle$ to $\langle \mathcal{T} \rangle$. This yields

$$\operatorname{ind}_T^G \operatorname{res}_T^G(A) \in \langle \mathcal{T} \rangle$$

in KK^G. Combining the above considerations shows $A \in \langle \mathcal{T} \rangle$, and we conclude $KK^G = \langle \mathcal{T} \rangle$ as desired. □

Appendix A: Continuous fields of Hilbert spaces and Hilbert modules

In this appendix we show that continuous field of Hilbert spaces over a locally compact space X are in bijective correspondence with Hilbert $C_0(X)$-modules. We refer to [7], [8] for more information.
Let us first recall the definition of a continuous field of Banach spaces. If X is a topological space then a field of Banach spaces over X is simply a family $(B_x)_{x \in X}$ of Banach spaces indexed by X. A section of such a field is an element

$$\xi = (\xi(x))_{x \in X} \in \prod_{x \in X} B_x,$$

it is called bounded if

$$||\xi|| = \sup_{x \in X} ||\xi(x)|| < \infty.$$

The set of all bounded sections forms a Banach space, it is simply the l^∞-direct product of the Banach spaces B_x. In particular, the topology of X does not play any role in the considerations so far.

Definition 7.5. *Let X be a locally compact space. A continuous field of Banach spaces over X is a family $(B_x)_{x \in X}$ of Hilbert spaces together with a linear subspace*

$$F \subset \prod_{x \in X} B_x$$

such that

a) For every $\xi \in F$ the function $x \mapsto ||\xi(x)||$ is continuous.
b) We have

$$B_x = \{\xi(x) | \xi \in F\}$$

for every $x \in X$.
c) If $\eta \in \prod_{x \in X} B_x$ is a section such that for every $x \in X$ and every $\epsilon > 0$ there exists a neighborhood U of x and an element $\xi \in F$ such that

$$||\eta(y) - \xi(y)|| < \epsilon$$

for all $y \in U$, then $\eta \in F$.

The above definition works in fact for arbitrary topological spaces. We note that the norms of the Banach spaces B_x enter the definition via condition a). Replacing the norm of B_x by an equivalent norm is ususally not compatible with the continuous field structure. In other words, we always consider Banach spaces with a fixed norm.
By slight abuse of notation, we will write $(B_x)_{x \in X}$ or F for a continuous field of Banach spaces $((B_x)_{x \in X}, F)$. The elements of F are also called the continuous sections of the field. Of course, both the family of Banach spaces and the space of continuous sections are needed to describe a continuous field.
We note that condition b) in definition 7.5 can be relaxed. More precisely, it suffices to require that the set $\{\xi(x) | \xi \in F\}$ is dense in B_x for every $x \in X$, see proposition 10.1.10 in [7] or proposition 3 in [8]. We shall give the proof of another basic property of continuous fields, compare proposition 10.1.9 in [7].

Lemma 7.6. *Let X be a locally compact space and let F be a continuous field of Banach spaces over X. Then the space F is a (right) $C(X)$-module with the module structure*

$$(\xi f)(x) = \xi(x) f(x)$$

for $\xi \in F$ and $f \in C(X)$.

Proof. We only have to show that ξf is contained in F for $\xi \in F$ and $f \in C(X)$. Fix $x \in X$ and let U be a neighborhood of x such that $||f(x) - f(y)|| < \epsilon$ for all $y \in U$. Then

$$||\xi(y) f(y) - \xi(y) f(x)|| \leq \epsilon ||\xi(y)||$$

for all $y \in U$. By condition a) we may assume that the norms $||\xi(y)||$ are bounded on U. Now condition c) implies that ξf is contained in F. $\square$
Observe that $C(X)$ is not a C^*-algebra unless X is compact. We note that lemma 7.6 remains valid over arbitrary topological spaces X.
It is now easy to define continuous fields of Hilbert spaces.

Definition 7.7. *Let X be a locally compact space. A continuous field of Hilbert spaces over X is a continuous field of Banach spaces $(\mathcal{H}_x)_{x\in X}$ over X such that $\mathcal{H}_x$ is a Hilbert space for each $x \in X$.*
A unitary isomorphism of continuous fields $(\mathcal{H}_x)_{x\in X}$ and $(\mathcal{K}_x)_{x\in X}$ of Hilbert spaces over X is a family $(U_x)_{x\in X}$ of unitaries $U_x : \mathcal{H}_x \to \mathcal{K}_x$ such that $x \mapsto U_x\sigma(x)$ is a continuous section of $(\mathcal{K}_x)_{x\in X}$ for every continuous section σ of $(\mathcal{H}_x)_{x\in X}$.

The scalar products in a continuous field of Hilbert spaces $(\mathcal{H}_x)_{x\in X}$ are automatically compatible with the continuous field structure. More precisely, the polarization identity

$$4\langle\xi,\eta\rangle = ||\xi+\eta||^2 - ||\xi-\eta||^2 - i||\xi+i\eta||^2 + i||\xi-i\eta||^2$$

shows that the functions $x \mapsto \langle\xi(x),\eta(x)\rangle$ are continuous for all $\xi,\eta \in F$. Recall that our scalar products are always linear in the second variable.
Let X be a locally compact space and let F be a continuous field of Hilbert spaces over X. We associate to F the right $C_0(X)$-module $HM(F)$ of all $\xi \in F$ such that $x \mapsto ||\xi(x)||$ is in $C_0(X)$. According to the previous remark, the sesquilinear form

$$\langle\xi,\eta\rangle(x) = \langle\xi(x),\eta(x)\rangle$$

takes values in $C_0(X)$.

Lemma 7.8. *Let X be a locally ompact space and let F be a continuous field of Hilbert spaces over X. The space $HM(F)$ with the sesquilinear form defined above is a Hilbert $C_0(X)$-module.*

Proof. It is clear that the above sesquilinear form defines a $C_0(X)$-valued inner product on $HM(F)$ and that $\langle\xi,\xi\rangle = 0$ iff $\xi = 0$. It remains to check that $HM(F)$ is complete with respect to the norm $||\xi|| = ||\langle\xi,\xi\rangle||^{1/2}$. Assume that $(\xi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to this norm. Then clearly ξ_n converges pointwise to a section ξ of $\prod_{x\in X}\mathcal{H}_x$. More precisely, we have $\xi_n \to \xi$ in the Banach space of all bounded elements in $\prod_{x\in X}\mathcal{H}_x$ with respect to the l^∞-norm. Hence we find for every $\epsilon > 0$ an $n \in \mathbb{N}$ such that $||\xi(x) - \xi_n(x)|| < \epsilon$ for all $x \in X$. According to condition c) in definition 7.5 we conclude that ξ is contained in F. Using the triangle inequality

$$||\xi(x)|| \leq ||\xi(x) - \xi_n(x)|| + ||\xi_n(x)||$$

we see that we have in fact $\xi \in HM(F)$. □
Conversely, let X be a locally compact space and let $\mathcal{E}$ be a Hilbert $C_0(X)$-module. For $x \in X$ we consider the Hilbert space $\mathcal{E}_x = \mathcal{E} \otimes_{\mathrm{ev}_x} \mathbb{C}$, where $\mathrm{ev}_x : C_0(X) \to \mathbb{C}$ denotes evaluation at x. Recall that $\mathcal{E} \otimes_{\mathrm{ev}_x} \mathbb{C}$ is the completion of $\mathcal{E} \odot_{C_0(X)} \mathbb{C}$ with respect to the scalar product

$$\langle\xi\otimes 1, \eta\otimes 1\rangle = \mathrm{ev}_x(\langle\xi,\eta\rangle) = \langle\xi(x),\eta(x)\rangle.$$

Let us write $I_x \subset C_0(X)$ for the kernel of ev_x. Clearly I_x is a C^*-algebra, and $[\mathcal{E}I_x] \subset \mathcal{E}$ is a nondegenerate right I_x-module. By the Cohen factorization theorem, every element in $\mathcal{E}I_x$ can be written as ξf for some $\xi \in [\mathcal{E}I_x]$ and $f \in I_x$. Hence we obtain $[\mathcal{E}I_x] = \mathcal{E}I_x$, in other words, the linear subspace $\mathcal{E}I_x \subset \mathcal{E}$ is closed.

Lemma 7.9. *With the notation as above, we have a canonical isomorphism*

$$\mathcal{E}_x \cong \mathcal{E}/(\mathcal{E}I_x)$$

of Hilbert spaces.

Proof. Let us write $||\ ||_q$ for the quotient norm on the Banach space $\mathcal{E}/(\mathcal{E}I_x)$ and $||\ ||_x$ for the seminorm $\xi \mapsto ||\xi(x)||$. It is clear that $||\ ||_x \leq ||\ ||_q$, and to see that we have in fact equality let $\xi \in \mathcal{E}$ and $\epsilon > 0$. Then we find a neighborhood U of x such that $||\xi(y)|| \leq ||\xi(x)|| + \epsilon$ for all $y \in U$. We choose a compact neighborhood $K \subset U$

of x, and using the Urysohn theorem we find a continuous function $\chi : X \to [0,1]$ such that $\chi(y) = 1$ for $y \in K$ and $supp(\chi) \subset U$. Let us set $\eta = \xi\chi - \xi$. Then we have $\eta(y) = 0$ for all $y \in K$, and using again Urysohn we find a continuous function μ on X such that $\mu(x) = 0$ and $\mu = 1$ on $X \setminus K$. Hence $\eta = \eta\mu \in \mathcal{E}I_x$. We now obtain

$$||\xi||_q \leq ||\xi + \eta|| = ||\xi\chi|| \leq ||\xi(x)|| + \epsilon = ||\xi||_x + \epsilon,$$

and this yields the claim. In particular $|| \, ||_x$ is a norm on $\mathcal{E}/\mathcal{E}I_x$.
Consider the canonical linear map $\pi : \mathcal{E}/(\mathcal{E}I_x) \to \mathcal{E} \odot_{C_0(X)} \mathbb{C} \subset \mathcal{E}_x$ given by $\pi(\xi) = \xi \otimes 1$. According to our previous remark this map is an isometry. Since $\mathcal{E}/\mathcal{E}I_x$ is complete and the range of π is clearly $\mathcal{E} \odot_{C_0(X)} \mathbb{C}$, we conclude

$$\mathcal{E}/(\mathcal{E}I_x) \cong \mathcal{E} \odot_{C_0(X)} \mathbb{C} = \mathcal{E}_x$$

as desired. □

We remark that the assumption that X is locally compact is used in the proof of lemma 7.9.
We have now constructed a field $(\mathcal{E}_x)_{x\in X}$ of Hilbert spaces out of the Hilbert module $\mathcal{E}$. Moreover the canonical map $\mathcal{E} \to \prod_{x\in X} \mathcal{E}_x$ is injective and provides a linear subspace satisfying conditions a) and b) in definition 7.5. According to proposition 10.2.3 in [7] there exists a unique continuous field of Hilbert spaces $CF(\mathcal{E}) \subset \prod_{x\in X} \mathcal{E}_x$ containing $\mathcal{E}$. The elements in $CF(\mathcal{E})$ are precisely the sections η such that for every $x \in X$ and $\epsilon > 0$ there exists a neighborhood U of x and $\xi \in \mathcal{E}$ such that $||\xi(y) - \eta(y)|| < \epsilon$ for all $y \in U$.

Proposition 7.10. *Let X be a locally compact space.*

a) If $\mathcal{E}$ is a Hilbert $C_0(X)$-module then there is a canonical unitary isomorphism

$$HM(CF(\mathcal{E})) \cong \mathcal{E}$$

of Hilbert $C_0(X)$-modules.

b) If $F \subset \prod_{x\in X} \mathcal{H}_x$ is a continuous field of Hilbert spaces over X then there is a canonical unitary isomorphism

$$CF(HM(F)) \cong F$$

of continuous fields.

Hence there is a bijective correspondence between isomorphism classes of Hilbert $C_0(X)$-modules and continuous fields of Hilbert spaces over X. Under this correspondence, full Hilbert modules are identified with continuous fields $\prod_{x\in X} \mathcal{H}_x$ with all fibres $\mathcal{H}_x$ being nonzero.

Proof. a) By construction we may view $\mathcal{E} \subset HM(CF(\mathcal{E}))$, and the scalar product of $HM(CF(\mathcal{E}))$ restricted to $\mathcal{E}$ agrees with the given scalar product. Assume $\xi \in HM(CF(\mathcal{E}))$ and let $\epsilon > 0$. Then there is a compact subset $K \subset X$ such that $||\xi(x)|| < \epsilon$ for $x \in X \setminus K$. For every $x \in K$ we find an open neighborhood $U_x \subset X$ and an element $\eta_x \in \mathcal{E}$ such that $||\eta_x(y) - \xi(y)|| < \epsilon$ for all $y \in U_x$. Since K is compact there is a finite covering $K \subset U_{x_1} \cup \cdots \cup U_{x_n}$, and we let $\chi_1, \dots, \chi_n$ be a subordinate partition of unity. That is, $\chi_k : X \to [0,1]$ is continuous, the support $supp(\chi_k)$ is compact and contained in U_{x_k} for all k and

$$\sum_{j=1}^{n} \chi_j(x) = \begin{cases} 1 & x \in K \\ \leq 1 & x \in X \setminus K. \end{cases}$$

We define $\eta \in \mathcal{E}$ by

$$\eta(x) = \sum_{j=1}^{n} \chi_j(x)\eta_{x_j}(x)$$

and obtain

$$||\xi(x)-\eta(x)|| = \Big\|\sum_{j=1}^n \chi_j(x)\xi(x)-\chi_j(x)\eta_{x_j}(x)\Big\| \leq \sum_{j=1}^n \chi_j(x)||\xi(x)-\eta_{x_j}(x)|| \leq \epsilon$$

for all $x \in X$. Since ϵ was arbitrary it follows that we find a sequence η_n in $\mathcal{E}$ converging to ξ with respect to $||\ ||$ in $HM(CF(\mathcal{E}))$. This implies $\xi \in \mathcal{E}$ because $\mathcal{E}$ is closed in $HM(CF(\mathcal{E}))$. This yields the claim.

b) Using lemma 7.9 we see that the canonical linear map $U_x : CF(HM(F))_x = HM(F)/(HM(F)I_x) \to \mathcal{H}_x$ given by $U_x(\xi) = \xi(x)$ is an isometry for every $x \in X$. From condition b) in definition 7.5 it follows that U_x is surjective. Hence the fibre of $CF(HM(F))$ over x is unitarily isomorphic to $HM(F)/(HM(F)I_x)$.

Using these isomorphisms we may view both F and $CF(HM(F))$ as linear subspaces of $\prod_{x\in X}\mathcal{H}_x$. Let $\xi \in F$ and $x \in X$. Then for every $\epsilon > 0$ there exists a neighborhood U of x and $\eta \in HM(F)$ such that $||\xi(y)-\eta(y)|| < \epsilon$ for all $y \in U$. By the definition of $CF(HM(F))$ it follows that $\xi \in CF(HM(F))$. Conversely, let $\xi \in CF(HM(F))$ and $x \in X$. Then we find for every $\epsilon > 0$ a neighborhood U of x and an element $\eta \in HM(F)$ such that $||\xi(y)-\eta(y)|| < \epsilon$ for all $y \in U$. Since F is a continuous field and $HM(F) \subset F$ we conclude $\xi \in F$. This proves part *b*).

The remaining assertion concerning full Hilbert $C_0(X)$-modules is obvious. □

Appendix B: Braided tensor product and the Drinfeld double

Let H be a finite dimensional Hopf algebra and consider

$$\hat{w} = \sum_{j=1}^n S^{-1}(e^j)\otimes e_j \in (H^*)^{\mathsf{cop}} \otimes H$$

where $e_1,\dots,e_n$ is a basis of H with dual basis $e^1,\dots,e^n$ of $(H^*)^{\mathsf{cop}} = H^*$. We write Δ for the comultiplication in H and $\hat{\Delta}$ for the comultiplication in $(H^*)^{\mathsf{cop}}$.

Definition 7.11. *Let H be a finite dimensional Hopf algebra. The Drinfeld double of H is*

$$\mathsf{D}_H = H \otimes (H^*)^{\mathsf{cop}}$$

with the tensor product algebra structure, the counit $\epsilon(f\otimes x) = \epsilon(f)\epsilon(x)$ and the comultiplication

$$\Delta_{\mathsf{D}}(f\otimes x) = f_{(1)} \otimes \hat{w}^{-1}(x_{(1)}\otimes f_{(2)})\hat{w}\otimes x_{(2)}$$

where $\hat{w} \in (H^*)^{\mathsf{cop}}\otimes H$ as above.

We note that the element $\hat{w}$ is a bicharacter, or skew-copairing, of $(H^*)^{\mathsf{cop}}$ and H in the following sense, compare page 358 in [9].

Definition 7.12. *A bicharacter between bialgebras K and H is an element $w \in K\otimes H$ such that*

$$(\epsilon_K\otimes \mathrm{id})(w) = 1, \qquad (\mathrm{id}\otimes\epsilon_H)(w) = 1$$

and

$$(\Delta_K\otimes\mathrm{id})(w) = w_{13}w_{23}, \qquad (\mathrm{id}\otimes\Delta_H)(w) = w_{13}w_{12}$$

Lemma 7.13. *The element $\hat{w}$ is a bicharacter between $(H^*)^{\mathsf{cop}}$ and H. Explicitly, $\hat{w}$ is invertible and the formulas*

$$(\epsilon\otimes\mathrm{id})(\hat{w}) = 1, \qquad (\mathrm{id}\otimes\epsilon)(\hat{w}) = 1$$

and

$$(\hat{\Delta}\otimes\mathrm{id})(\hat{w}) = \hat{w}_{13}\hat{w}_{23}, \qquad (\mathrm{id}\otimes\Delta)(\hat{w}) = \hat{w}_{13}\hat{w}_{12}$$

hold.

Proof. We claim that

$$\hat{w}^{-1} = \sum_{j=1}^{n} e^j \otimes e_j$$

is inverse to $\hat{w}$. To check this let us write

$$\langle x, f\rangle = x(f) = \langle f, x\rangle$$

for the canonical evaluation of $x \in (H^*)^{\mathsf{cop}} = H^*$ and $f \in H$. Note that S^{-1} is the antipode of $(H^*)^{\mathsf{cop}}$ in the above formula for $\hat{w}$, and that $\langle S^{-1}(x), f\rangle = \langle x, S(f)\rangle$ for all $x \in (H^*)^{\mathsf{cop}}$ and $f \in H$. We compute

$$\begin{aligned}
\langle \hat{w}\hat{w}^{-1}, f \otimes x\rangle &= \sum_{j,k=1}^{n} \langle S^{-1}(e^j)e^k \otimes e_j e_k, f \otimes x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j)e^k, f\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j) \otimes e^k, \Delta(f)\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j), f_{(1)}\rangle\langle e^k, f_{(2)}\rangle\langle e_j e_k, x\rangle \\
&= \langle S(f_{(1)})f_{(2)}, x\rangle = \epsilon(f)\epsilon(x)
\end{aligned}$$

so that $\hat{w}\hat{w}^{-1} = 1$. Similarly one obtains $\hat{w}^{-1}\hat{w} = 1$.
The equations

$$(\epsilon \otimes \mathrm{id})(\hat{w}) = \sum_{j=1}^{n} \epsilon(S^{-1}(e^j))e_j = \epsilon(e^j)e_j = \epsilon = 1$$

and $(\mathrm{id} \otimes \epsilon)(\hat{w})$ are easy. We compute

$$\begin{aligned}
\langle(\hat{\Delta} \otimes \mathrm{id})(\hat{w}), f \otimes g \otimes x\rangle &= \sum_{j=1}^{n}\langle\hat{\Delta}(S^{-1}(e^j)) \otimes e_j, f \otimes g \otimes x\rangle \\
&= \sum_{j=1}^{n}\langle S^{-1}(e^j), gf\rangle\langle e_j, x\rangle \\
&= x(S(gf)) \\
&= \sum_{j,k=1}^{n} \langle e^j, S(f)\rangle\langle e^k, S(g)\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j), f\rangle\langle S^{-1}(e^k), g\rangle\langle e_j e_k, x\rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j) \otimes S^{-1}(e^k) \otimes e_j e_k, f \otimes g \otimes x\rangle \\
&= \langle \hat{w}_{13}\hat{w}_{23}, f \otimes g \otimes x\rangle
\end{aligned}$$

for all $f, g \in H$ and $x \in H^*$. Similarly we have

$$\begin{aligned}
\langle (\mathrm{id} \otimes \hat{\Delta})(\hat{w}), f \otimes x \otimes y \rangle &= \sum_{j=1}^{n} \langle S^{-1}(e^j) \otimes \Delta(e_j), f \otimes x \otimes y \rangle \\
&= \sum_{j=1}^{n} \langle S^{-1}(e^j), f \rangle \langle (e_j)_{(1)}, x \rangle \langle (e_j)_{(2)}, y \rangle \\
&= \langle S(f)_{(1)}, x \rangle \langle S^{-1}(f)_{(2)}, y \rangle \\
&= \langle S(f), xy \rangle \\
&= \sum_{j,k=1}^{n} \langle e^k e^j, S(f) \rangle \langle e_k, x \rangle \langle e_j, y \rangle \\
&= \sum_{j,k=1}^{n} \langle S^{-1}(e^j) S^{-1}(e^k) \otimes e_k \otimes e_j, f \otimes x \otimes y \rangle \\
&= \langle \hat{w}_{13} \hat{w}_{12}, f \otimes x \otimes y \rangle
\end{aligned}$$

as required. □

The assertion that $\hat{w}$ is invertible can be viewed in a more conceptual way as follows. Consider the canonical map $c : H \otimes (H^*)^{\mathsf{cop}} \to \mathrm{End}(H)$ given by

$$c(f \otimes x)(h) = |f\rangle\langle x|(h) = f\langle x, h\rangle = f x(h).$$

We view $\mathrm{End}(H)$ as an algebra with the convolution multiplication

$$(T_1 \cdot T_2)(h) = \mu(T_1 \otimes T_2)\Delta(h).$$

Then we compute

$$\begin{aligned}
(c(f \otimes x) \cdot c(g \otimes y))(h) &= \mu(|f\rangle\langle x| \otimes |g\rangle\langle y|)\Delta(h) \\
&= fg\langle x, h_{(1)}\rangle\langle y, h_{(2)}\rangle \\
&= fg\langle x \otimes y, \Delta(h)\rangle \\
&= fg\langle xy, h\rangle = c(fg \otimes xy)(h)
\end{aligned}$$

and conclude that c is an algebra homomorphism. The element $\hat{w} \in (H^*)^{\mathsf{cop}} \otimes H \cong H \otimes (H^*)^{\mathsf{cop}}$ considered above corresponds to $S \in \mathrm{End}(H)$, and the basic Hopf algebra axioms say that S is invertible with respect to convolution. Similarly, the relation $(\mathrm{id} \otimes \Delta)(\hat{w}) = \hat{w}_{13}\hat{w}_{12}$ can be reduced to the condition $\Delta S = (S \otimes S)\Delta^{\mathsf{cop}}$ for the antipode, for instance.

We shall now discuss comodules for D_H.

Definition 7.14. *Let H be a Hopf algebra. An H-Yetter-Drinfeld module M is a vector space which is both a left H-module via $H \otimes M \to M, f \otimes m \mapsto f \cdot m$ and a left H-comodule via $M \to H \otimes M, m \mapsto m_{(-1)} \otimes m_{(0)}$ such that*

$$f_{(1)} m_{(-1)} S(f_{(3)}) \otimes f_{(2)} \cdot m_{(0)} = (f \cdot m)_{(-1)} \otimes (f \cdot m)_{(0)}$$

for all $f \in H$ and $m \in M$.

We need the following lemma.

Lemma 7.15. *Let $h \otimes y \in H \otimes (H^*)^{\mathsf{cop}} = \mathsf{D}_H$. Then*

$$\begin{aligned}
(\epsilon \otimes f \otimes \mathrm{id} \otimes \epsilon)(\Delta_{\mathsf{D}}(h \otimes y)) &= \langle f_{(2)}, y\rangle f_{(1)} h S(f_{(3)}) \\
&= f_{(1)}(\mathrm{id} \otimes \epsilon \otimes \epsilon \otimes f_{(2)})(\Delta_{\mathsf{D}}(h \otimes y)) S(f_{(3)})
\end{aligned}$$

Proof. We compute

$$
\begin{aligned}
(\epsilon \otimes f \otimes \mathrm{id} \otimes \epsilon)(\Delta_{\mathsf{D}}(h \otimes y)) &= (\epsilon \otimes f \otimes \mathrm{id} \otimes \epsilon)(h_{(1)} \otimes \hat{w}^{-1}(y_{(1)} \otimes h_{(2)} \hat{w} \otimes y_{(2)}) \\
&= \sum_{j,k=1}^{n} (\epsilon \otimes f \otimes \mathrm{id} \otimes \epsilon)(h_{(1)} \otimes e^j y_{(1)} S^{-1}(e^k) \otimes e_j h_{(2)} e_k \otimes y_{(2)}) \\
&= \sum_{j,k=1}^{n} (f \otimes \mathrm{id})(e^j y S^{-1}(e^k) \otimes e_j h e_k) \\
&= \langle f_{(2)}, y \rangle f_{(1)} h S(f_{(3)}) \\
&= f_{(1)} (\mathrm{id} \otimes \epsilon \otimes \epsilon \otimes f_{(2)})(\Delta_{\mathsf{D}}(h \otimes y)) S(f_{(3)})
\end{aligned}
$$

as claimed. □

The following basic result explains the relation between Yetter-Drinfeld modules and comodules for D_H.

Proposition 7.16. *Let H be a finite dimensional Hopf algebra. Then there is a bijective correspondence between H-Yetter-Drinfeld modules and left D_H-comodules.*

Proof. Let M be a left D_H-comodule with coaction $\lambda : M \to \mathsf{D}_H \otimes M$. We obtain a left H-comodule structure $\mu : M \to H \otimes M$ by restriction, that is $\mu = (\pi \otimes \mathrm{id})\lambda$ where $\pi : \mathsf{D}_H \to H, \pi(f \otimes x) = f\epsilon(x)$. We write $\mu(m) = m_{(-1)} \otimes m_{(0)}$. In a similar way we obtain a comodule structure $\hat{\mu} : M \to (H^*)^{\mathsf{cop}} \otimes M$ given by $\hat{\mu} = (\hat{\pi} \otimes \mathrm{id})\lambda$ where $\hat{\pi} : \mathsf{D}_H \to (H^*)^{\mathsf{cop}}, \hat{\pi}(f \otimes x) = \epsilon(f)x$. We may view M as a left H-module by setting

$$f \cdot m = (f \otimes \mathrm{id})\hat{\mu}(m)$$

for $f \in H$.

Consider the equation

$$(\mathrm{id} \otimes \lambda)\lambda(m) = (\Delta_{\mathsf{D}} \otimes \mathrm{id})\lambda(m)$$

in $H \otimes (H^*)^{\mathsf{cop}} \otimes H \otimes (H^*)^{\mathsf{cop}} \otimes M$. If we apply $(\epsilon \otimes f \otimes \mathrm{id} \otimes \epsilon \otimes \mathrm{id})$ on the left hand side we obtain $(f \cdot m)_{(-1)} \otimes (f \cdot m)_{(0)}$. According to lemma 7.15 this is the same as $f_{(1)} m_{(-1)} S(f_{(3)}) \otimes f_{(2)} \cdot m_{(0)}$. Hence we obtain a YD-module in this way.

Conversely, assume that we start from a Yetter-Drinfeld module with coaction $\gamma : M \to H \otimes M$ and left action $\mu : H \otimes M \to M$. We dualize μ to a right coaction $\hat{\mu} : M \to M \otimes H^*$ by $\hat{\mu}(m) = \mu(e_j \otimes m) \otimes e^j$ and view this as a left coaction of $(H^*)^{\mathsf{cop}}$. Now define a linear map $\lambda : M \to \mathsf{D}_H \otimes M$ by

$$\lambda = (\mathrm{id} \otimes \hat{\mu})\gamma.$$

Writing $\mu(f \otimes m) = f \cdot m$ and $\gamma(m) = m_{(-1)} \otimes m_{(0)}$ we compute

$$
\begin{aligned}
(\mathrm{id} \otimes \lambda)\lambda(m) &= (\mathrm{id} \otimes \mathrm{id} \otimes \lambda)(m_{(-1)} \otimes e^j \otimes e_j \cdot m_{(0)}) \\
&= m_{(-1)} \otimes e^j \otimes (e_j \cdot m_{(0)})_{(-1)} \otimes e^k \otimes e_k \cdot (e_j \cdot m_{(0)})_{(0)} \\
&= m_{(-2)} \otimes e^j \otimes (e_j)_{(1)} m_{(-1)} S((e_j)_{(3)}) \otimes e^k \otimes e_k \cdot (e_j)_{(2)} \cdot m_{(0)}
\end{aligned}
$$

Evaluating with $f \in H$ on the second tensor factor and $g \in H$ on the fourth factor gives

$$m_{(-2)} \otimes f_{(1)} m_{(-1)} S(f_{(3)}) \otimes g \cdot (f_{(2)} \cdot m_{(0)})$$

Conversely, we have

$$
\begin{aligned}
(\Delta_{\mathsf{D}} \otimes \mathrm{id})\lambda(m) &= (\Delta_{\mathsf{D}} \otimes \mathrm{id})(m_{(-1)} \otimes e^j \otimes e_j \cdot m_{(0)}) \\
&= m_{(-2)} \otimes \hat{w}^{-1}((e^j)_{(1)} \otimes m_{(-1)})\hat{w} \otimes (e^j)_{(2)} \otimes e_j \cdot m_{(0)} \\
&= m_{(-2)} \otimes e^k (e^j)_{(1)} S^{-1}(e^l) \otimes e_k m_{(-1)} e_l \otimes (e^j)_{(2)} \otimes e_j \cdot m_{(0)}
\end{aligned}
$$

Evaluating with $f \in H$ on the second tensor factor and $g \in H$ on the fourth factor gives

$$\begin{aligned}&m_{(-2)} \otimes \langle f_{(2)}, (e^j)_{(1)}\rangle f_{(1)} m_{(-1)} S(f_{(3)}) \langle g, (e^j)_{(2)}\rangle \otimes e_j \cdot m_{(0)})\\&= m_{(-2)} \otimes f_{(1)} m_{(-1)} S(f_{(3)}) \otimes (g f_{(2)}) \cdot m_{(0)})\end{aligned}$$

This shows that λ is coassociative, and it is clearly counital. $\square$

Definition 7.17. *A bialgebra H is called coquasitriangular if there exists a linear form $r : H \otimes H \to \mathbb{C}$ such that r is invertible with respect to the convolution product and*

a) $m^{\mathsf{opp}} = r * m * r^{-1}$
a) $r(m \otimes \mathrm{id}) = r_{13} r_{23}$ *and* $r(\mathrm{id} \otimes m) = r_{13} r_{12}$

Proposition 7.18. *The Drinfeld codouble D_H is coquasitriangular with universal r-form $r : \mathsf{D}_H \otimes \mathsf{D}_H \to \mathbb{C}$ given by*

$$r(f \otimes x \otimes g \otimes y) = \epsilon(f) g(x) \epsilon(y)$$

Proof. We claim that

$$\overline{r}(f \otimes x \otimes g \otimes y) = \epsilon(f) g(S(x)) \epsilon(y)$$

is the inverse of r. We compute

$$\begin{aligned}(r * \overline{r})(f^1 \otimes x^1 \otimes f^2 \otimes x^2) &= (r_{13} \overline{r}_{24})(\Delta_{\mathsf{D}}(f^1 \otimes x^1) \otimes \Delta_{\mathsf{D}}(f^2 \otimes x^2))\\&= (r_{13} \overline{r}_{24})(f^1_{(1)} \otimes \hat{w}^{-1}(x^1_{(1)} \otimes f^1_{(2)}) \hat{w} \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes \hat{w}^{-1}(x^2_{(1)} \otimes f^2_{(2)}) \hat{w} \otimes x^2_{(2)})\\&= \epsilon(f^1) \langle x^1_{(1)}, f^2_{(1)}\rangle \langle S(x^1_{(2)}), f^2_{(2)}\rangle \epsilon(x^2)\\&= (\epsilon_{\mathsf{D}} \otimes \epsilon_{\mathsf{D}})(f^1 \otimes x^1 \otimes f^2 \otimes x^2)\end{aligned}$$

and similarly

$$\begin{aligned}(\overline{r} * r)(f^1 \otimes x^1 \otimes f^2 \otimes x^2) &= (\overline{r}_{13} r_{24})(\Delta_{\mathsf{D}}(f^1 \otimes x^1) \otimes \Delta_{\mathsf{D}}(f^2 \otimes x^2))\\&= (\overline{r}_{13} r_{24})(f^1_{(1)} \otimes \hat{w}^{-1}(x^1_{(1)} \otimes f^1_{(2)}) \hat{w} \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes \hat{w}^{-1}(x^2_{(1)} \otimes f^2_{(2)}) \hat{w} \otimes x^2_{(2)})\\&= \epsilon(f^1) \langle S(x^1_{(1)}), f^2_{(1)}\rangle \langle x^1_{(2)}, f^2_{(2)}\rangle \epsilon(x^2)\\&= (\epsilon_{\mathsf{D}} \otimes \epsilon_{\mathsf{D}})(f^1 \otimes x^1 \otimes f^2 \otimes x^2)\end{aligned}$$

We compute

$$
\begin{aligned}
(m^{\mathsf{opp}} * r)(f^1 \otimes x^1 \otimes f^2 \otimes x^2) &= (m_{13}^{\mathsf{opp}} \otimes r_{24})(\Delta_{\mathsf{D}}(f^1 \otimes x^1) \otimes \Delta_{\mathsf{D}}(f^2 \otimes x^2)) \\
&= (m_{13}^{\mathsf{opp}} \otimes r_{24})(f^1_{(1)} \otimes \hat{w}^{-1}(x^1_{(1)} \otimes f^1_{(2)})\hat{w} \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes \hat{w}^{-1}(x^2_{(1)} \otimes f^2_{(2)})\hat{w} \otimes x^2_{(2)}) \\
&= \sum_{j,k} (m_{13}^{\mathsf{opp}} \otimes r_{24})(f^1_{(1)} \otimes x^1_{(1)} \otimes f^1_{(2)} \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes e^j x^2_{(1)} S^{-1}(e^k) \otimes e_j f^2_{(2)} e_k \otimes x^2_{(2)}) \\
&= \sum_{j,k} f^2_{(1)} f^1 \otimes e^j x^2 S^{-1}(e^k) x^1_{(1)} \langle x^1_{(2)}, e_j f^2_{(2)} e_k \rangle \\
&= \sum_{j,k} f^2_{(1)} f^1 \otimes e^j x^2 S^{-1}(e^k) x^1_{(1)} \langle x^1_{(4)}, e_j \rangle \langle x^1_{(3)}, f^2_{(2)} \rangle \langle x^1_{(2)}, e_k \rangle \\
&= f^2_{(1)} f^1 \otimes x^1_{(4)} x^2 S^{-1}(x^1_{(2)}) x^1_{(1)} \langle x^1_{(3)}, f^2_{(2)} \rangle \\
&= f^2_{(1)} f^1 \otimes x^1_{(2)} x^2 \langle x^1_{(1)}, f^2_{(2)} \rangle \\
&= f^2_{(1)} f^1 S(f^2_{(3)}) f^2_{(4)} \otimes x^1_{(2)} x^2 \langle x^1_{(1)}, f^2_{(2)} \rangle \\
&= \sum_{j,k} \langle e^j, f^2_{(1)} \rangle \langle x^1_{(1)}, f^2_{(2)} \rangle \langle S^{-1}(e^k), f^2_{(3)} \rangle e_j f^1 e_k f^2_{(4)} \otimes x^1_{(2)} x^2 \\
&= \sum_{j,k} \langle e^j x^1_{(1)} S^{-1}(e^k), f^2_{(1)} \rangle e_j f^1 e_k f^2_{(2)} \otimes x^1_{(2)} x^2 \\
&= \sum_{j,k} (r_{13} \otimes m_{24})(f^1_{(1)} \otimes e^j x^1_{(1)} S^{-1}(e^k) \otimes e_j f^1_{(2)} e_k \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes x^2_{(1)} \otimes f^2_{(2)} \otimes x^2_{(2)}) \\
&= (r_{13} \otimes m_{24})(f^1_{(1)} \otimes \hat{w}^{-1}(x^1_{(1)} \otimes f^1_{(2)})\hat{w} \otimes x^1_{(2)} \otimes f^2_{(1)} \otimes \hat{w}^{-1}(x^2_{(1)} \otimes f^2_{(2)})\hat{w} \otimes x^2_{(2)}) \\
&= (r_{13} \otimes m_{24})(\Delta_{\mathsf{D}}(f^1 \otimes x^1) \otimes \Delta_{\mathsf{D}}(f^2 \otimes x^2)) \\
&= (r * m)(f^1 \otimes x^1 \otimes f^2 \otimes x^2)
\end{aligned}
$$

Moreover we have

$$
\begin{aligned}
r(m \otimes \mathrm{id})(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3) &= r(f^1 f^2 \otimes x^1 x^2 \otimes f^3 \otimes x^3) \\
&= \epsilon(f^1)\epsilon(f^2) f^3(x^1 x^2)\epsilon(x^3) \\
&= \epsilon(f^1)\langle x^1, f^3_{(1)} \rangle \epsilon(f^2) \langle x^2, f^3_{(2)} \rangle \epsilon(x^3) \\
&= \epsilon(f^1)\langle x^1, f^3_{(1)} \rangle \epsilon(x^3_{(1)}) r(f^2 \otimes x^2 \otimes f^3_{(2)} \otimes x^3_{(2)}) \\
&= r_{13} r_{24}(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes f^3_{(1)} \otimes \hat{w}^{-1}(x^3_{(1)} \otimes f^3_{(2)})\hat{w} \otimes x^3_{(2)})) \\
&= r_{13} r_{24}(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes \Delta_{\mathsf{D}}(f^3 \otimes x^3)) \\
&= r_{13} r_{23}(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3)
\end{aligned}
$$

and

$$
\begin{aligned}
r(\mathrm{id} \otimes m)(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3) &= r(f^1 \otimes x^1 \otimes f^2 f^3 \otimes x^2 x^3) \\
&= \epsilon(f^1)(f^2 f^3)(x^1)\epsilon(x^2 x^3) \\
&= \epsilon(f^1)\langle x^1_{(1)}, f^3 \rangle \langle x^1_{(2)}, f^2 \rangle \epsilon(x^2)\epsilon(x^3) \\
&= \epsilon(f^1_{(1)})\langle x^1_{(1)}, f^3 \rangle \epsilon(x^3) \epsilon(f^1_{(2)}) \langle x^1_{(2)}, f^2 \rangle \epsilon(x^2) \\
&= r(f^1_{(1)} \otimes x^1_{(1)} \otimes f^3 \otimes x^3) \epsilon(f^1_{(2)}) \langle x^1_{(2)}, f^2 \rangle \epsilon(x^2) \\
&= r_{14} r_{23}(f^1_{(1)} \otimes \hat{w}^{-1}(x^1_{(1)} \otimes f^1_{(2)})\hat{w} \otimes x^1_{(2)} \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3) \\
&= r_{14} r_{23}(\Delta_{\mathsf{D}}(f^1 \otimes x^1) \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3) \\
&= r_{13} r_{12}(f^1 \otimes x^1 \otimes f^2 \otimes x^2 \otimes f^3 \otimes x^3)
\end{aligned}
$$

where we note that $x_{(1)} \otimes x_{(2)}$ is Sweedler notation for the comultiplication in $(H^*)^{\mathsf{cop}}$. ☐

Note that if $\gamma_{MN} : M \otimes N \to N \otimes M$ is the braiding in a braided monoidal category $\mathcal{C}$, then also $\phi_{MN} = \gamma_{NM}^{-1} : M \otimes N \to N \otimes M$ is a braiding for the monoidal category $\mathcal{C}$. This follows from the symmetry in the hexagon diagrams, see [13], page 430. The braiding ϕ is indeed different from γ in general, since in particular the maps ϕ_{MM} and γ_{MM} for $M \in \mathcal{C}$ differ unless $\mathcal{C}$ is symmetric.
If X, Y are left D_H-comodules we obtain a braiding $\gamma_{XY} : X \otimes Y \to Y \otimes X$ by

$$\gamma_{XY}(x \otimes y) = r(y_{(-1)} \otimes x_{(-1)}) y_{(0)} \otimes x_{(0)}.$$

Indeed, we compute

$$\begin{aligned}
\lambda_{Y \otimes X} \gamma_{XY}(x \otimes y) &= r(y_{(-2)} \otimes x_{(-2)}) y_{(-1)} x_{(-1)} \otimes y_{(0)} \otimes x_{(0)} \\
&= r(y_{(-2)} \otimes x_{(-2)}) m(y_{(-1)} \otimes x_{(-1)}) \otimes y_{(0)} \otimes x_{(0)} \\
&= m^{\mathsf{opp}}(y_{(-2)} \otimes x_{(-2)}) r(y_{(-1)} \otimes x_{(-1)}) \otimes y_{(0)} \otimes x_{(0)} \\
&= x_{(-2)} y_{(-2)} r(y_{(-1)} \otimes x_{(-1)}) \otimes y_{(0)} \otimes x_{(0)} \\
&= (\mathrm{id} \otimes \gamma_{XY}) \lambda_{X \otimes Y}(x \otimes y).
\end{aligned}$$

Moreover we obtain

$$\begin{aligned}
(\mathrm{id} \otimes \gamma_{XZ})(\gamma_{XY} \otimes \mathrm{id})(x \otimes y \otimes z) &= r(y_{(-1)} \otimes x_{(-1)})(\mathrm{id} \otimes \gamma_{XZ})(y_{(0)} \otimes x_{(0)} \otimes z) \\
&= r(y_{(-1)} \otimes x_{(-2)}) r(z_{(-1)} \otimes x_{(-1)}) y_{(0)} \otimes z_{(0)} \otimes x_{(0)} \\
&= r_{13} r_{23}(y_{(-1)} \otimes z_{(-1)} \otimes x_{(-1)}) y_{(0)} \otimes z_{(0)} \otimes x_{(0)} \\
&= r(m \otimes \mathrm{id})(y_{(-1)} \otimes z_{(-1)} \otimes x_{(-1)}) y_{(0)} \otimes z_{(0)} \otimes x_{(0)} \\
&= r(y_{(-1)} z_{(-1)} \otimes x_{(-1)}) y_{(0)} \otimes z_{(0)} \otimes x_{(0)} \\
&= \gamma_{X, Y \otimes Z}(x \otimes y \otimes z)
\end{aligned}$$

and

$$\begin{aligned}
(\gamma_{XZ} \otimes \mathrm{id})(\mathrm{id} \otimes \gamma_{YZ})(x \otimes y \otimes z) &= r(z_{(-1)} \otimes y_{(-1)})(\gamma_{XZ} \otimes \mathrm{id})(x \otimes z_{(0)} \otimes y_{(0)}) \\
&= r(z_{(-2)} \otimes y_{(-1)}) r(z_{(-1)} \otimes x_{(-1)}) z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \\
&= r_{13} r_{12}(z_{(-1)} \otimes x_{(-1)} \otimes y_{(-1)}) z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \\
&= r(\mathrm{id} \otimes m)(z_{(-1)} \otimes x_{(-1)} \otimes y_{(-1)}) z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \\
&= r(z_{(-1)} \otimes x_{(-1)} y_{(-1)}) z_{(0)} \otimes x_{(0)} \otimes y_{(0)} \\
&= \gamma_{X \otimes Y, Z}(x \otimes y \otimes z)
\end{aligned}$$

which shows that γ satisfies the hexagon relations as on page 430 in [13].
Note that according to our remark above

$$\phi_{YX}(y \otimes x) = \gamma_{XY}^{-1}(y \otimes x) = r^{-1}(y_{(-1)} \otimes x_{(-1)}) x_{(0)} \otimes y_{(0)}$$

defines a braiding as well.
In order to explain the connection with the C^*-algebraic framework we consider the case of finite quantum groups. If G is a finite quantum group then $H = C(G)$ and $(H^*)^{\mathsf{cop}} = C^*(G)$ are finite dimensional Hopf algebras. The GNS-construction is $\mathbb{H}_G = H$ together with the map $\Lambda : H \to \mathbb{H}_G$ given by the identity. The scalar product on $\mathbb{H}_G$ is

$$\langle \Lambda(f), \Lambda(g) \rangle = \phi(f^* g)$$

for $f, g \in H$ where $\phi : H \to \mathbb{C}$ is the Haar state. The fundamental unitary W is given by

$$W^*(\Lambda(f) \otimes \Lambda(g)) = \Lambda(g_{(1)} f) \otimes \Lambda(g_{(2)})$$

which can equivalently be described by

$$W(\Lambda(f) \otimes \Lambda(g)) = \Lambda(S^{-1}(g_{(1)}) f) \otimes \Lambda(g_{(2)})$$

using the inverse of the antipode S of H. The algebra H is contained as subalgebra of $\mathbb{L}(\mathbb{H}_G)$ consisting of all operators $(\mathrm{id}\otimes\omega)W$ for $\omega \in \mathbb{L}(\mathbb{H}_G)^*$. One checks that this are precisely the multiplication operators

$$f \cdot \Lambda(h) = \Lambda(fh)$$

for $f \in H$. The comultiplication of H can be recovered from the formula

$$\begin{aligned}\Delta(h)(\Lambda(f) \otimes \Lambda(g)) &= W^*(1 \otimes h)W(\Lambda(f) \otimes \Lambda(g))\\ &= W^*(1 \otimes h)(\Lambda(S^{-1}(g_{(1)})f) \otimes \Lambda(g_{(2)}))\\ &= W^*(\Lambda(S^{-1}(g_{(1)})f) \otimes \Lambda(hg_{(2)}))\\ &= \Lambda(h_{(1)}f) \otimes \Lambda(h_{(2)}g)\end{aligned}$$

in this representation.

The dual quantum group $C^*(G) = \hat{H} = (H^*)^{\mathsf{cop}}$ consists of all operators of the form $(\omega \otimes \mathrm{id})W$ for $\omega \in \mathbb{L}(\mathbb{H}_G)^*$. It can be identified with the linear dual of H acting on $\mathbb{H}_G$ according to the formula

$$x \cdot \Lambda(h) = x(S^{-1}(h_{(1)}))\Lambda(h_{(2)})$$

for a linear form x on H.

Using Sweedler notation in $(H^*)^{\mathsf{cop}}$, the comultiplication $\hat{\Delta}$ on $\hat{H}$ is given by the formula

$$\begin{aligned}\hat{\Delta}(x)(\Lambda(f) \otimes \Lambda(g)) &= \Sigma W(x \otimes 1)W^*\Sigma(\Lambda(f) \otimes \Lambda(g))\\ &= \Sigma W(x \otimes 1)(\Lambda(f_{(1)}g) \otimes \Lambda(f_{(2)}))\\ &= x(S^{-1}(f_{(1)}g_{(1)}))\Sigma W(\Lambda(f_{(2)}g_{(2)}) \otimes \Lambda(f_{(3)}))\\ &= x_{(1)}(S^{-1}(f_{(1)}))x_{(2)}(S^{-1}(g_{(1)}))\Lambda(f_{(2)}) \otimes \Lambda(g_{(2)})\end{aligned}$$

that is, $\hat{\Delta}$ is precisely the comultiplication in $(H^*)^{\mathsf{cop}}$.

Let us compare the Yetter-Drinfeld conditions in the algebraic setting and the C^*-algebraic framework. The C^*-algebraic condition amounts to

$$\begin{aligned}&w(a_{(-1)}\otimes(a_{(0)})_{[-1]})w^{-1} \otimes (a_{(0)})_{[0]}\\ &\qquad= \sum_{j,k=1}^{n} e_j a_{(-1)} e_k \otimes e^j (a_{(0)})_{[-1]} S^{-1}(e^k) \otimes (a_{(0)})_{[0]}\\ &\qquad= (a_{[0]})_{(-1)} \otimes a_{[-1]} \otimes (a_{[0]})_{(0)}\end{aligned}$$

where we write

$$\alpha(a) = a_{(-1)} \otimes a_{(0)}, \qquad \lambda(a) = a_{[-1]} \otimes a_{[0]}$$

and use

$$w = \sum_{j=1}^{n} e_j \otimes e^j, \qquad w^{-1} = \sum_{j=1}^{n} e_j \otimes S^{-1}(e^j).$$

Evaluating the above equality on $f \in H$ in the second tensor factor gives

$$\begin{aligned}(f \cdot a)_{(-1)} \otimes (f \cdot a)_{(0)} &= (a_{[0]})_{(-1)}\langle f, a_{[-1]}\rangle (a_{[0]})_{(0)}\\ &= \sum_{j,k=1}^{n} e_j a_{(-1)} e_k \langle e^j (a_{(0)})_{[-1]} S^{-1}(e^k), f\rangle (a_{(0)})_{[0]}\\ &= \sum_{j,k=1}^{n} e_j a_{(-1)} e_k \langle e^j, f_{(1)}\rangle \langle (a_{(0)})_{[-1]}, f_{(2)}\rangle \langle S^{-1}(e^k), f_{(3)}\rangle (a_{(0)})_{[0]}\\ &= f_{(1)} a_{(-1)} S(f_{(3)}) f_{(2)} \cdot a_{(0)}\end{aligned}$$

and thus amounts to the algbraic Yetter-Drinfeld condition.
Now let A be an D_H-algebra and B be an H-algebra. We define the algebraic braided tensor product $A \boxtimes B$ as $A \otimes B$ with the multiplication

$$(a \boxtimes b)(a' \boxtimes b') = a(b_{(-1)} \cdot a') \boxtimes b_{(0)} b'$$

Applying γ_{BA} to $b \otimes a'$ gives

$$r(a'_{[-1]} \otimes b_{[-1]}) a'_{[0]} \otimes b_{[0]} = (b_{(-1)} \cdot a') \otimes b_{(0)},$$

so that the above multiplication is indeed given by

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id} \otimes \gamma_{BA} \otimes \mathrm{id}} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

Let us compare the commutation relations of A and B in the above algebraic setting with the commutation relations in the C^*-algebraic framework used in definition 4.9. We have the correspondence

$$(a \boxtimes 1)(1 \boxtimes b) = a \boxtimes b \sim a_{[-1]} b_{(-1)} \otimes a_{[0]} \otimes b_{(0)} = \lambda(a)_{12} \beta(b)_{13}$$

where we view H and $(H^*)^{\mathsf{cop}}$ as acting on the GNS-space. Moreover

$$\begin{aligned}(1 \boxtimes b)(a \boxtimes 1) &= (b_{(-1)} \cdot a) \boxtimes b_{(0)} \\ &= \langle b_{(-1)}, a_{[-1]} \rangle \, a_{[0]} \boxtimes b_{(0)} \\ &\sim \langle b_{(-2)}, a_{[-2]} \rangle a_{[-1]} b_{(-1)} \otimes a_{[0]} \otimes b_{(0)}\end{aligned}$$

and we should compare this with

$$\beta(b)_{13} \lambda(a)_{12} = b_{(-1)} a_{[-1]} \otimes a_{[0]} \otimes b_{(0)}.$$

The two expressions will be equal provided

$$\langle f_{(1)}, x_{(1)} \rangle x_{(2)} f_{(2)} = fx$$

for all $f \in H$ and $x \in (H^*)^{\mathsf{cop}}$ where we use Sweedler notation in H and $(H^*)^{\mathsf{cop}}$. Indeed,

$$\begin{aligned}f \cdot x \cdot \Lambda(h) &= \langle S^{-1}(h_{(1)}), x \rangle \Lambda(f h_{(2)}) \\ &= \langle S^{-1}(f_{(2)}) f_{(1)}, x_{(1)} \rangle \langle S^{-1}(h_{(1)}), x_{(2)} \rangle \Lambda(f_{(3)} h_{(2)}) \\ &= \langle f_{(1)}, x_{(1)} \rangle \langle S^{-1}(f_{(2)}), x_{(2)} \rangle \langle S^{-1}(h_{(1)}), x_{(3)} \rangle \Lambda(f_{(3)} h_{(2)}) \\ &= \langle f_{(1)}, x_{(1)} \rangle \langle S^{-1}(f_{(2)} h_{(1)}), x_{(2)} \rangle \Lambda(f_{(3)} h_{(2)}) \\ &= \langle f_{(1)}, x_{(1)} \rangle \, x_{(2)} \cdot f_{(2)} \cdot \Lambda(h),\end{aligned}$$

and thus the desired equality holds.

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