Introduction to cyclic homology

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CHAPTER 1

Preliminaries

Throughout the text we will work over the field $\mathbb{C}$ of complex numbers. In particular, vector spaces, linear maps and algebras will always be defined over the complex numbers. This is convenient for the purposes of noncommutative geometry, however, there are interesting applications of Hochschild and cyclic homology in the setting of more general commutative ground rings. Actually, most of the material we discuss in chapter 3 may be developped in the same way over arbitrary commutative rings.
We point out that in our terminology an algebra will not be required to possess a unit. Again, this is convenient for noncommutative geometry, but it is important to note that this terminology is not commonly accepted.

1. Algebras and modules

The basic object of study in cyclic homology are algebras. We shall thus begin with the definition of an algebra.

DEFINITION 1.1. *An algebra is a vector space A together with a bilinear map $\mu : A \times A \to A$ written as $\mu(a, b) = ab$ and called multiplication such that*

$$(ab)c = a(bc)$$

for all $a, b, c \in A$. A unital algebra is an algebra with an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.
An algebra homomorphism $f : A \to B$ between algebras is a linear map such that $f(ab) = f(a)f(b)$ for all $a, b \in A$. A unital homomorphism $f : A \to B$ between unital algebras is a homomorphism such that $f(1) = 1$.

The easiest example of an algebra is the zero vector space $A = 0$. More generally, one may equip any vector space with the zero multiplication to obtain an algebra. We will discuss more interesting examples of algebras below.
There are a few standard construction with algebras. Let us have a look at two of them. Firstly, given two algebras A and B their direct sum $A \oplus B$ is the algebra defined by the multiplication $(a_1, b_1)(a_2, b_2) = (a_1a_2, b_1b_2)$.
Secondly, there is an easy way to adjoin a unit element to an algebra A. One defines $A^+ = A \oplus \mathbb{C}$ as a vector space but with the multiplication

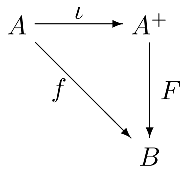
$$(a, \alpha)(b, \beta) = (ab + \alpha b + \beta a, \alpha\beta).$$

It is easy to check that A^+ becomes a unital algebra with unit element $(0, 1)$ in this way. The algebra A^+ is called the unitarization of A. We have a natural algebra homomorphism $\iota : A \to A^+$. Remark that the unit element of A^+ is different from the unit of A if the algebra A itself is unital.
If A happens to be unital, the algebra A^+ can be described as follows.

EXCERCISE 1.2. *Let A be a unital algebra with unit element 1_A. Then the map $\phi : A^+ \to A \oplus \mathbb{C}$ given by $\phi(a, \alpha) = (a + \alpha \cdot 1_A, \alpha)$ is an isomorphism of unital algebras.*

The unitarization of an algebra is characterized by the following property.

Excercise 1.3. *Let A be an algebra and let B be a unital algebra. For every algebra homomorphism $f : A \to B$ there exists a unique unital algebra homomorphism $F : A^+ \to B$ such that the diagram*



is commutative.

Let us discuss some examples of algebras.

1.1. Matrix algebras. We denote by $M_n(\mathbb{C})$ the vector space of $n \times n$-matrices with entries in $\mathbb{C}$ with the usual addition and multiplication. It is easy to check that $M_n(\mathbb{C})$ is a unital algebra. More generally, if A is an arbitrary algebra we obtain the algebra $M_n(A) =$ of $n \times n$-matrices with entries in A. This algebra is unital iff A is unital.

1.2. Smooth functions on manifolds. Let M be a smooth manifold and let $C^\infty(M)$ be the linear space of complex-valued smooth functions on M. Then $C^\infty(M)$ becomes a unital algebra with pointwise multiplication of functions. One may also consider the algebra $C^\infty_c(M)$ of smooth functions with compact support. Clearly, $C^\infty_c(M)$ is unital iff M is compact.

1.3. Group rings. Let Γ be a discrete group and let $\mathbb{C}\Gamma$ be the vector space with basis Γ. Elements in $\mathbb{C}\Gamma$ can be written as finite sums

$$\sum_{j=1}^{n} \alpha_j t_j$$

with $\alpha_j \in \mathbb{C}$ and $t_j \in \Gamma$. One defines a multiplication on $\mathbb{C}\Gamma$ by extending the group multiplication $\Gamma \times \Gamma \to \Gamma$ to a bilinear map $\mathbb{C}\Gamma \times \mathbb{C}\Gamma \to \mathbb{C}\Gamma$. It is easy to check that $\mathbb{C}\Gamma$ becomes a unital algebra in this way. Associativity of the multiplication follows from associativity of the group law and the unit element is given by $e = 1e \in \mathbb{C}\Gamma$ where $e \in \Gamma$ is the unit element.

Returning to the general theory, we come now to the definition of modules over an algebra.

Definition 1.4. *Let A be an algebra. A (left) module over A is a vector space M together with a bilinear map $A \times M \to M$ such that*

$$(ab)m = a(bm)$$

for all $a, b \in A$ and $m \in M$. A unitary (left) module over a unital algebra A is an A-module M such that $1m = m$ for every $m \in M$. An A-module homomorphism $f : M \to N$ between (unitary) A-modules is a linear map which satisfies $f(am) = af(m)$ for all $a \in A$ and $m \in M$.
In a similar way one defines (unitary) right A-modules and their homomorphisms.

If M and N are left A-modules we write $\mathrm{Hom}_A(M, N)$ for the vector space of A-module homomorphisms between M and N. We will frequently write ${}_AM$ of M_A to indicate that M is a left or right A-module, respectively. Every algebra A can be viewed as a left or right module over itself using the multiplication map.
Let $\mathrm{End}(M)$ denote the unital algebra of linear endomorphisms of the vector space M. An A-module structure on M may be described as a homomorphism $\phi : A \to \mathrm{End}(M)$ such that $\phi(a)(m) = am$. Having this in mind, the following statement is a consequence of excercise 1.3.

EXCERCISE 1.5. *Let $_AM$ be a module over A. Then M becomes a unitary module over A^+ by declaring $1m = m$ for all $m \in M$. Conversely, every unitary A^+-module can be viewed as an A-module by restricting the action to A.*

Let us discuss another operation with algebras. The opposite algebra A^{op} of an algebra A has the same underlying vector space as A and is equipped with the opposite multiplication

$$a \bullet b = b \cdot a$$

where $b \cdot a$ denotes the multiplication in A.
An algebra A is called *commutative* if $ab = ba$ for all $a, b \in A$. In this case the opposite algebra A^{op} is equal to A.
Next we shall see that it is in principle sufficient to consider only left modules.

EXCERCISE 1.6. *Let M be a left module over A. Then M is a right A^{op}-module by setting*

$$ma = am$$

for all $m \in M$ and $a \in A^{op}$.

However, usually modules over an algebra A appear naturally as left or right modules and it is convenient not to work with the algebra A^{op}.
We conclude this section with the definition of a bimodule.

DEFINITION 1.7. *Let A and B be algebras. An A-B-bimodule is a vector space M which is both a left A-module and a right B-module such that*

$$(am)b = a(mb)$$

for all $a \in A, m \in M$ and $b \in B$. If A and B are unital, a unitary A-B-bimodule is an A-B-bimodule M such that $1m = m = m1$ for every $m \in M$.
A bimodule homomorphism $f : M \to N$ between (unitary) A-B-bimodules is a linear map which is both a (unitary) A-module homomorphism and a (unitary) B-module homomorphism.

A basic example of an A-A-bimodule is the algebra A itself with the left and right action by multiplication.
A submodule N of an A-module M is a linear subspace $N \subset M$ such that $an \in N$ for all $n \in N$, that is, if it is an A-module with the restricted action. The quotient M/N of an A-module by a submodule is the ordinary quotient space with the A-module structure induced by M. Similar definitions are made for bimodules.

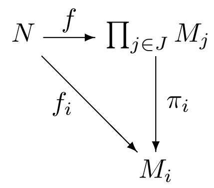
2. Projective and inductive limits

In this section we discuss projective and inductive limits of modules over an algebra.
We begin with direct products. Let A be an algebra and let $(M_j)_{j\in J}$ be a family of A-modules. The direct product of this family is the vector space

$$\prod_{j\in J} M_j$$

with componentwise action of A. For every $i \in J$ the canonical projection $\pi_i : \prod_{j\in J} M_j \to M_i$ is an A-module map. The direct product is a unitary A-module iff all modules M_j are unitary. Moreover we have the following universal property.

EXCERCISE 1.8. *Let $(M_j)_{j\in J}$ be a family of A-modules. For every A-module N and every family $(f_j)_{j\in J}$ of A-module homomorphisms $f_j : N \to M_j$ there exists a* *unique A-module homomorphism $f : N \to \prod_{j\in J} M_j$ such that the diagrams*



are commutative for all $i \in J$.

As a generalization of direct products one defines projective limits. First recall the definition of a partially ordered set.

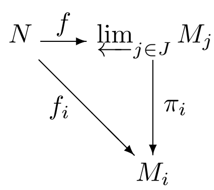
Definition 1.9. *A set J is partially ordered if there is a relation $\leq$ defined on J such that*
a) $j \leq j$ for all $j \in J$ (reflexivity).
b) $j \leq i$ and $i \leq j$ implies $i = j$ (symmetry).
c) $j \leq i$ and $k \leq j$ implies $k \leq i$ (transitivity).
A partially ordered set is called directed if for all $i, j \in J$ there exists $k \in J$ such that $i \leq k$ and $j \leq k$.

Every set is partially ordered using the trivial relation stipulating only $j \leq j$ for all $j \in J$. Note that this partial ordering is directed only if J consists of a single element. An easy example of a directed set is the set $\mathbb{N}$ of natural numbers with its natural ordering. Actually, for our purposes this will be the most important example of a directed set.
An inverse system of A-modules over a partially ordered set J is a family $(M_j)_{j\in J}$ of A-modules together with A-module maps $\pi_{ji} : M_i \to M_j$ for all $j \leq i$ such that $\pi_{ii} = \mathrm{id}$ for all i and $\pi_{kj}\pi_{ji} = \pi_{ki}$ whenever $k \leq j \leq i$. The projective limit of an inverse system is the A-submodule

$$\varprojlim_{j\in J} M_j \subset \prod_{j\in J} M_j$$

consisting of all families $(m_j)_{j\in J}$ such that $m_j = \pi_{ji}(m_i)$ whenever $j \leq i$. Again, for every $i \in J$ the canonical projection $\pi_i : \varprojlim_{j\in J} M_j \to M_i$ is an A-module map. The inverse limit is a unitary A-module if all modules M_j are unitary and we have the following universal property.

Excercise 1.10. *Let $(M_j)_{j\in J}$ be an inverse system of A-modules over the directed set J. For every A-module N and every family $(f_j)_{j\in J}$ of A-module homomorphisms $f_j : N \to M_j$ satisfying $f_j = \pi_{ji} f_i$ for all $j \leq i$ there exists a unique A-module homomorphism $f : N \to \varprojlim_{j\in J} M_j$ such that the diagrams*



are commutative for all $i \in J$.

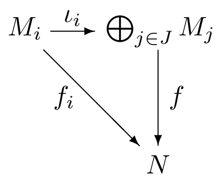
In the special case where J is partially ordered with the trivial partial order relation discussed above we reobtain the definition and characterization of direct products.
Dual to the notion of a direct product one defines direct sums. Let again $(M_j)_{j\in J}$

be a family of A-modules over an algebra A. The direct sum of this family is the vector space

$$\bigoplus_{j\in J} M_j = \{(x_j)_{j\in J} \in \prod_{j\in J} M_j | \, x_j = 0 \text{ for all but finitely many } j \in J\}$$

with addition and module action inherited from $\prod_{j\in J} M_j$. For every $i \in J$ there is a canonical A-module map $\iota_i : M_i \to \bigoplus_{j\in J} M_j$.

EXCERCISE 1.11. *Let $(M_j)_{j\in J}$ be a family of A-modules. For every A-module N and every family $(f_j)_{j\in J}$ of A-module homomorphisms $f_j : M_j \to N$ there exists a unique A-module homomorphism $f : \bigoplus_{j\in J} M_j \to N$ such that the diagrams*



are commutative for all $i \in J$.

An important special case arises if all modules M_j are equal to A^+.

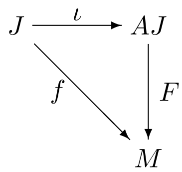
DEFINITION 1.12. *Let J be a set and A be an algebra. The free A-module over J is the direct sum*

$$AJ = \bigoplus_{j\in J} A^+$$

of copies of A^+.

The next excercise describes the universal property of free modules.

EXCERCISE 1.13. *Let AJ be the free A-module over the set J and let M be any A-module. For every map $f : J \to M$ there exists a unique A-module map $F : AJ \to M$ such that the diagram*



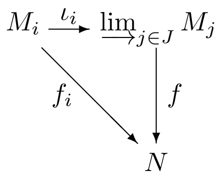
is commutative.

As a generalization of direct sums one defines inductive limits. Essentially this consists of reversing the order of arrows in all statements in the definition of projective limits. Let J be a partially ordered set. An inductive system of A-modules is a family $(M_j)_{j\in J}$ of A-modules together with A-module maps $\pi_{ji} : M_i \to M_j$ for all $i \leq j$ such that $\pi_{kj}\pi_{ji} = \pi_{ki}$ whenever $i \leq j \leq k$. The inductive limit of an inductive system is the quotient A-submodule

$$\bigoplus_{j\in J} M_j \to \varinjlim_{j\in J} M_j$$

obtained by dividing out the subspace generated by all elements of the form $m_j - \pi_{ji}(m_i)$ whenever $i \leq j$. For every $i \in J$ the canonical map $\iota_i : M_i \to \varinjlim_{j\in J} M_j$ is an A-module map. The inductive limit is a unitary A-module if all modules M_j are unitary and we have the following universal property.

Excercise 1.14. *Let $(M_j)_{j\in J}$ be an inductive system of A-modules over the directed set J. For every A-module N and every family $(f_j)_{j\in J}$ of A-module homomorphisms $f_j : M_j \to N$ satisfying $f_j\pi_{ji} = f_i$ for all $i \le j$ there exists a unique A-module homomorphism $f : \varinjlim_{j\in J} M_j \to N$ such that the diagrams*



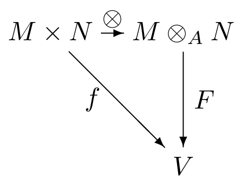
are commutative for all $i \in J$.

As above, in the special case where J is partially ordered with the trivial partial order relation we reobtain the definition and characterization of direct sums.
We point out that in the context of projective and inductive limits the terminology is not unique in the literature. Sometimes projective limits are called inverse limits and inductive limits are called direct limits. An inductive system of modules is also called a directed system.
Finally we remark that in the special case $A = 0$ we (re-)obtain the definitions of direct products, sums as well as projective and inductive limits of vector spaces.

3. Tensor products

In this section we define and study tensor products of modules over algebras. We begin with the tensor product of modules. Let M_A and ${}_AN$ be modules over an algebra A and let V be a vector space. A bilinear map $f : M \times N \to V$ is called A-bilinear if $f(ma, n) = f(m, an)$ for all $m \in M, n \in N, a \in A$.

Definition 1.15. *Let M_A and ${}_AN$ be A-modules. A vector space $M \otimes_A N$ together with an A-bilinear map $\otimes : M \times N \ni (m, n) \mapsto n \otimes n \in M \otimes_A N$ is called tensor product of M and N over A if for every vector space V and every A-bilinear map $f : M \times N \to V$ there exists a unique linear map $F : M \otimes_A N \to V$ such that the diagram*



is commutative.

Lemma 1.16. *The tensor product $M \otimes_A N$ is uniquely determined up to isomorphism by M_A and ${}_AN$.*

Proof. Let $M \otimes_A N$ and $M \boxtimes_A N$ be tensor products of M and N and let $\otimes : M \times N \to M \otimes_A N$ and $\boxtimes : M \times N \to M \boxtimes_A N$ be the corresponding bilinear maps. By the universal property, there exist linear maps $h : M \otimes_A N \to M \boxtimes_A N$ and $k : M \boxtimes_A N \to M \otimes_A N$ such that $\boxtimes = h\otimes$ and $\otimes = k\boxtimes$. Hence $\boxtimes = hk\boxtimes$ and $\otimes = kh\otimes$. By the uniqueness assertion we deduce $hk = \mathrm{id}$ and $kh = \mathrm{id}$. Hence $M \otimes_A N$ and $M \boxtimes_A N$ are isomorphic. □
Even without knowing existence of the tensor product one may prove the following properties directly from the definition.

Excercise 1.17. *Let $M \otimes_A N$ be a tensor product. Then*

a) *$M \otimes_A N$ is generated as a vector space by elementary tensors $m \otimes n$ with $m \in M$, $n \in N$.*

b) $(w + x) \otimes y = w \otimes y + x \otimes y$ *for* $w, x \in M$ *and* $y \in N$.
c) $x \otimes (y + z) = x \otimes y + x \otimes z$ *for* $x \in M$ *and* $y, z \in N$.
d) $xa \otimes y = x \otimes ay$ *for* $x \in M$, $y \in N$ *and* $a \in A$.

EXCERCISE 1.18. *Let* M_1 *and* M_2 *be right* A*-modules and let* N_1 *and* N_2 *be left* A*-modules. If* $f_1 : M_1 \to M_2$ *and* $f_2 : N_1 \to N_2$ *are* A*-module maps there exists a unique linear map* $f_1 \otimes f_2 : M_1 \otimes_A N_1 \to M_2 \otimes N_2$ *such that* $(f_1 \otimes f_2)(m \otimes n) = f_1(m) \otimes f_2(n)$.

We shall now show that tensor products always exist.

PROPOSITION 1.19. *Let* M_A *and* ${}_AN$ *be modules. Then there exists a tensor product* $M \otimes_A N$.

PROOF. We let $M \otimes_A N$ be the quotient of the vector space P with basis $M \times N$ by the relations

$$(w + x, z) = (w, z) + (x, z), \quad (x, y + z) = (x, y) + (x, z), \quad (xa, y) = (x, ay)$$

for all $w, x \in M$, $y, z \in N$ and $a \in A^+$. The map $\otimes : M \times N \to M \otimes_A N$ is induced by the canonical map $\iota : M \times N \to P$. Now let $f : M \times N \to V$ be an A-bilinear map. Then there exists a unique linear map $F : P \to V$ such that $F\iota = f$. Since f is assumed to be A-bilinear we see that F induces a linear map $F : M \otimes_A N \to V$ which satisfies $F\otimes = f$. Now assume that $G : M \otimes_A N \to V$ is another linear map such that $G\otimes = f$. It follows that the resulting map $G\pi : P \to V$ is equal to $F\pi : P \to V$ where $\pi : P \to M \otimes_A N$ is the canonical projection. Since π is surjective this implies $F = G$. □

We may view modules If M_A and ${}_AN$ as unitary modules M_{A^+} and ${}_{A^+}N$ in a natural way. It is straightforward to check that the natural map $M \otimes_A N \to M \otimes_{A^+} N$ is an isomorphism. Hence we do not have to care wether we consider modules over A or unitary modules over A^+.
Next we show that in some cases the tensor product of A-modules can be described in a more concrete way.

PROPOSITION 1.20. *Let* M_A *be an arbitrary* A*-module and let* $N = AJ$ *be the free left* A*-module over* J*. Then*

$$M \otimes_A N \cong \bigoplus_{j \in J} M.$$

PROOF. An A-bilinear map $f : M \times AJ \to V$ is uniquely determined by the linear maps $f_j : M \to V$ given by $f_j(m) = f(m, e_j)$ where e_j denotes the element of $\bigoplus_{j \in J} M$ determined by the unit element of A^+ in the jth position. It follows that $\bigoplus_{j \in J} M$ satisfies the universal property of a tensor product. Hence the assertion is a consequence of lemma 1.16. □

A similar assertion holds if M_A is a free right A-module and N is arbitrary. As a consequence we obtain the following result.

COROLLARY 1.21. *Let* $M_A = IA$ *and* ${}_AN = AJ$ *be free modules. Then*

$$M \otimes_A N \cong \bigoplus_{(i \times j) \in I \times J} A^+.$$

PROOF. This follows from proposition 1.20 and the natural isomorphism

$$\bigoplus_{j \in J} \Big(\bigoplus_{i \in I} A^+ \Big) = \bigoplus_{(i \times j) \in I \times J} A^+$$

which in turn is a consequence of the universal property of direct sums. □

In the special case $A = 0$ we simply write $M \otimes N$ for the tensor product over the zero algebra. Recall that a module over the zero algebra is simply a vector space.

COROLLARY 1.22. *Let V and W be vector spaces with bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$, respectively. Then $M \otimes N$ is a vector space with basis $(u_i \otimes v_j)_{(i,j)\in I\times J}$.*

Let us come back to the general situation. If M and N happen to be bimodules then the same holds true for their tensor product. More precicsely, let ${}_AM_B$ and ${}_BN_C$ be bimodules. We want to define an A-C-bimodule structure on $M \otimes_B N$ using the formulas

$$a(m \otimes n) = am \otimes n, \qquad (m \otimes n)c = m \otimes nc.$$

However, to be precise we define for each $a \in A$ a map $l_a : M \times N \to M \otimes_B N$ by

$$l_a(m, n) = am \otimes n$$

and verify that l_a is B-bilinear. Let $L_a : M \otimes_B N \to M \otimes_B N$ be the corresponding linear map. Then we define the left module structure $A \times (M \otimes_B N) \to M \otimes_B N$ by $(a, x) \mapsto L_a(x)$. Similarly one has to proceed for the right action of C.

EXCERCISE 1.23. *Verify that ${}_AM \otimes_B N_C$ becomes an A-C-bimodule in this way.*

PROPOSITION 1.24. *Let $M_A, {}_AN_B$ and ${}_BP$ be modules. Then there exists a natural isomorphism*

$$(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P).$$

PROOF. Both spaces are universal for trilinear maps $f : M \times N \times P \to V$ which satisfy $f(ma, n, p) = f(m, an, p)$ and $f(m, nb, p) = f(m, n, bp)$ for all elements in M, N, P and A, B, respectively. The assertion follows easily from this. □

EXCERCISE 1.25. *If ${}_AM_B$ and ${}_AN_C$ are bimodules the vector space $\mathrm{Hom}_A(M, N)$ of A-module homomorphisms between M and N becomes an B-C-bimodule using the formula $(bfc)(m) = f(mb)c$.*

We shall now formulate an important property of tensor products.

PROPOSITION 1.26. *Let ${}_AM_B, {}_BN_C$ and ${}_AP_D$ be bimodules. Then there exists a natural isomorphism*

$${}_C\mathrm{Hom}_A(M \otimes_B N, P)_D = {}_C\mathrm{Hom}_B(N, \mathrm{Hom}_A(M, P))_D$$

of C-D-bimodules.

PROOF. One defines a map $\phi : \mathrm{Hom}_A(M \otimes_B N, P) \to \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, P))$ by $\phi(f)(m)(n) = f(m \otimes n)$. Conversely, one defines $\psi : \mathrm{Hom}_B(N, \mathrm{Hom}_A(M, P)) \to \mathrm{Hom}_A(M \otimes_B N, P)$ by $\psi(f)(m \otimes n) = f(n)(m)$. We leave it as an excercise to check that these maps are well-defined inverse isomorphisms. □

We conclude this section with a discussion of the tensor product of algebras. Let A and B be algebras and consider the tensor product $A \otimes B$.

EXCERCISE 1.27. *There is a multiplication on $A \otimes B$ given by $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2$ which turns $A \otimes B$ into an algebra.*

Using excercise 1.5 and excercise 1.6 we see that an A-B-bimodule is the same thing as a unitary left $A^+ \otimes (B^{op})^+$-module. One should however be careful with the unitarizations at this point. The following example shows that not every $A \otimes B^{op}$-module is the restriction of an A-B-bimodule.
Consider the commutative algebra

$$A = B = \{f \in C^\infty[0, 1] | f(0) = 0\}$$

with pointwise multiplication and let M be the linear space of all functions $f \in C^\infty[0, 1]$ such that $f'(0) = 0$. We claim that M is an $A \otimes B$-module using the action

$$(f \otimes g)h = fgh$$

given by pointwise multiplication of functions. The essential point here is to check that this action is well-defined. However, using the Leibniz rule we get

$$\frac{\partial}{\partial x}(fgh)(0) = \frac{\partial}{\partial x}(f)(0)g(0)h(0) + f(0)\frac{\partial}{\partial x}(g)(0)h(0) + f(0)g(0)\frac{\partial}{\partial x}(h)(0) = 0$$

provided $f \in A, g \in B$ and $h \in M$. This shows that $(f \otimes g)h$ is an element of M as required. Assume that M is the restriction of an A-B-bimodule and consider the element

$$m = f \cdot \chi \cdot 1$$

in M where $\chi \in M$ is the constant function with value 1 and 1 denotes the unit of B^+. According to the definitions, we have

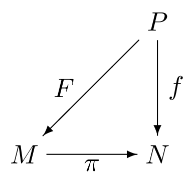
$$(mg)(x) = f(x)g(x)$$

for all $g \in B$ and $x \in [0,1]$. Choosing functions $g_\epsilon \in B$ such that $g_\epsilon(x) = 1$ for $x > \epsilon$ for all $\epsilon > 0$ we deduce $m(x) = f(x)$ for all $x > 0$. Since m is continuous this implies $m = f \in M$ which is a contradiction since the derivative of f at zero does not necessarily vanish.

4. Projective modules

Projective modules play an important role in homological algebra. In this section we discuss their basic properties.

Definition 1.28. *Let A be an algebra. A module ${}_AP$ is called projective if for every epimorphism $\pi : M \to N$ of A-modules and every A-module map $f : P \to N$ there exists an A-module map $F : P \to M$ such that the diagram*



is commutative.

Excercise 1.29. *For every algebra A the A-module A^+ is projective. Moreover, for every set J the free module AJ is projective. More generally, a direct sum of projective modules is projective.*

An A-submodule M of an A-module P is called a *direct summand* if there exists an A-submodule N in P such that the natural map $M \oplus N \to P$ is an isomorphism. Equivalently, there exists an A-module map $\pi : P \to M$ such that $\pi\iota = \mathrm{id}$ where $\iota : M \to P$ is the natural inclusion.

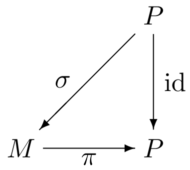
Excercise 1.30. *If M is isomorphic to a direct summand in a projective module, then M is itself projective.*

An epimorphism $\pi : M \to N$ of A-modules is called *split* if there exists an A-module homomorphism $\sigma : N \to M$ such that $\pi\sigma = \mathrm{id}$.

Proposition 1.31. *Let ${}_AP$ be a module. The following are equivalent:*

a) P is projective.
b) Every epimorphism $\pi : M \to P$ splits.
c) P is isomorphic to a direct summand in a free module.

PROOF. *a)* ⇒ *b)* Let $\pi : M \to P$ be an epimorphism. According to the projectivity of P there exists an A-module map $\sigma : P \to M$ such that the diagram



is commutative. This shows that π splits
b) ⇒ *c)* Consider the free A-module over the set P. There exists a canonical A-module map $\pi : \bigoplus_{p\in P} A^+ \to P$ characterized by $\pi\iota_p(1) = p$ where 1 denotes the unit in A^+. Since π is clearly surjective, there exists a splitting σ for π. This shows that P is a direct summand in the free module $\bigoplus_{p\in P} A^+$.
c ⇒ *a)* This implication is contained in excercise 4.14 and excercise 4.15. □
If A is unital we see using excercise 1.2 that there is a natural A-module isomomorphism $A^+ \cong A \oplus \mathbb{C}_\tau$ where $\mathbb{C}_\tau$ is the zero module. It follows that A is a projective A-module in this case.

PROPOSITION 1.32. *Let A be a unital algebra. Then a unitary A-module P is projective iff it is a direct summand in $\bigoplus_{j\in J} A$ for some index set J.*

PROOF. According to the previous considerations and excercise 4.15, direct summands in the A-module $\bigoplus_{j\in J} A$ are projective for every index set J. Note that such direct summands are automatically unitary. Conversely, assume that P is projective and unitary. Then the natural A-module homomorphism

$$f : \bigoplus_{p\in P} A \to P$$

determined by $f\iota_p(a) = ap$ is surjective. Using that P is projective there exists a splitting σ for π and it follows that P is a direct summand in $\bigoplus_{p\in P} A$. □
We next prove the dual basis lemma. For this we need some more terminology. A (unitary) module M over a (unital) algebra A is called finitely generated if there exist elements $m_1, \dots, m_n \in M$ for some $n \in \mathbb{N}$ such that the smallest A-submodule containing $m_1, \dots, m_n$ is equal to M.
For every module M_A denote by M^* the right A-module $\mathrm{Hom}_A(M, A)$. Then there is a natural map

$$db : M \otimes_A M^* \to \mathrm{End}_A(M)$$

defined by $db(m\otimes f)(x) = mf(x)$. This map is called the dual-basis homomorphism.

PROPOSITION 1.33 (Dual basis lemma). *Let P_A be a unitary module over the unital algebra A. The following are equivalent:*
a) P is finitely generated and projective.
b) There exist $f_1, \dots, f_n \in \mathrm{Hom}_A(P, A)$ and $p_1, \dots, p_n \in P$ such that

$$p = \sum_{j=1}^{n} p_j f_j(p).$$

c) The dual basis homomorphism $db : P \otimes_A P^* \to \mathrm{End}_A(P, P)$ is an isomorphism.

PROOF. *a)* ⇒ *b)* According to proposition 1.32 there are A-module homomorphisms $\pi : A^n = \bigoplus_{j=1}^n A \to P$ and $\sigma : P \to A^n$ such that $\pi\sigma = \mathrm{id}$. Define $f_j = \pi_j\sigma$ where $\pi_j : A^n \to A$ is the projection onto the j-th component and $p_j = \pi(\iota_j(1))$. These elements satisfy the desired relation.
b) ⇒ *a)* Since $p = \sum p_j f_j(p)$ for all P the module P is finitely generated. Moreover the homomorphism $\pi : \bigoplus_{j=1}^n A \to P$ determined by $p_i = \pi\iota_i(1)$ is surjective. Let

$\sigma : P \to \bigoplus_{j=1}^n A$ be defined by $\sigma(p) = \sum p_j f_j(p)$. Then $\pi\sigma(p) = p$ for all $p \in P$. Hence P is a direct summand in $\sum_{j=1}^n A$ and thus projective according to proposition 1.32.
b) $\Rightarrow$ *c)* Consider the map $\phi : \mathrm{End}_A(P) \to P \otimes_A P^*$ given by $\phi(f) = \sum f(p_j) \otimes f_j$. We have

$$db\phi(f)(x) = \sum f(p_j) f_j(x) = f(x).$$

Hence $db\phi = \mathrm{id}$. Conversely,

$$\phi db(p \otimes f) = \sum p f(p_j) \otimes f_j = \sum p \otimes f(p_j) f_j = p \otimes f.$$

This shows $\phi db = \mathrm{id}$. Hence db is an isomorphism.
c) $\Rightarrow$ *b)* If db is an isomorphism choose elements p_j and f_j such that $db(\sum p_j \otimes f_j) = \mathrm{id}$. Then p_j and f_j satisfy the required relation. □
Let us have a look at some examples.

PROPOSITION 1.34. *Let $\mathbb{C}_\tau$ be the vector space $\mathbb{C}$ with the zero multiplication. Then $\mathbb{C}_\tau$ is not a projective $\mathbb{C}_\tau$-module.*

PROOF. Assume that $\mathbb{C}_\tau$ is projective. Then there exists a $\mathbb{C}_\tau$-linear splitting σ for the multiplication map $\mathbb{C}_\tau^+ \otimes \mathbb{C}_\tau \to \mathbb{C}_\tau$. Consider

$$\sigma(1) = \sum_{j=1}^n (a_j, \alpha_j) \otimes \beta_j$$

in $\mathbb{C}_\tau^+ \otimes \mathbb{C}_\tau$. Then $\sum_{j=1}^n \alpha_j \beta_j = 1$ since $\pi\sigma(1) = 1$. By $\mathbb{C}_\tau$-linearity of σ we have

$$\sum_{j=1}^n \gamma(a_j, \alpha_j) \otimes \beta_j = \sum_{j=1}^n (\gamma\alpha_j, 0) \otimes \beta_j = 0$$

for all $\gamma \in \mathbb{C}_\tau$. This is a contradiction. □
The following unitary example is more interesting.

PROPOSITION 1.35. *Consider the algebra $A = C^\infty[0,1]$ of smooth functions on the interval $[0,1]$ and let $\mathbb{C}$ be the unitary A-module defined by $f \cdot \alpha = f(0)\alpha$. Then $\mathbb{C}$ is not a projective A-module.*

PROOF. Assume that the A-module $\mathbb{C}$ is projective. Then there exists a section $\sigma : \mathbb{C} \to A$ for the natural A-linear projection $\pi : A \to \mathbb{C}$ given by $\pi(f) = f(0)$. Consider the function $\sigma(1) \in A$. Since $f(x)\sigma(1)(x) = f(0)\sigma(1)(x)$ for all $f \in A$ by A-linearity we see that $\sigma(1)(x) = 0$ for all $x > 0$. Since $\sigma(1)$ is continuous this implies $\sigma(1) = 0$ which is a contradiction to $\pi\sigma = \mathrm{id}$. □
Finally, let us have a look at an example of a projective module which is not free. Consider a discrete group Γ and the unital homomorphism $\epsilon : \mathbb{C}\Gamma \to \mathbb{C}$ defined by $\epsilon(t) = 1$ for all $t \in \Gamma$. The map ϵ is called the augmentation homomorphism of $\mathbb{C}\Gamma$.

PROPOSITION 1.36. *Let Γ be a finite group different from the trivial group. Then the unitary $\mathbb{C}\Gamma$-module $\mathbb{C}$ defined by the augmentation homomorphism is projective but not free.*

PROOF. It is clear that $\mathbb{C}$ is not free by dimension reasons. There exists a $\mathbb{C}\Gamma$-linear splitting σ for the surjection $\epsilon : \mathbb{C}\Gamma \to \mathbb{C}$ given by

$$\sigma(1) = \sum_{t \in \Gamma} \frac{1}{n} t$$

where n is the number of elements in Γ. Hence $\mathbb{C}$ is projective according to proposition 1.32. □
In fact, the $\mathbb{C}\Gamma$-module $\mathbb{C}$ defined by the augmentation homomorphism is projective iff the group Γ is finite.

5. Morita theory

In this section we describe the notion of Morita equivalence of unital algebras. Morita equivalence is an important concept for noncommutative geometry since many natural examples of algebras modelling noncommutative spaces are only determined up to Morita equivalence.

Definition 1.37. *Let A and B be unital algebras. A Morita context for A and B consists of two unitary bimodules ${}_AP_B$ and ${}_BQ_A$ and bimodule maps*

$$\langle -,-\rangle_A : P\otimes_B Q\to A, \qquad \langle -,-\rangle_B : Q\otimes_A P\to B$$

such that

$$\langle p_1,q\rangle_A\, p_2 = p_1\langle q,p_2\rangle_B$$
$$\langle q_1,p\rangle_B\, q_2 = q_1\langle p,q_2\rangle_A$$

for all $p_i,p\in P$ and $q_i,q\in Q$. A Morita context is called strict if the maps $\langle -,-\rangle_A : P\otimes_B Q\to A, \langle -,-\rangle_B : Q\otimes_A P\to B$ are surjective.

Definition 1.38. *Two unital algebras A and B are called Morita equivalent if there exists a strict Morita context for A and B.*

Clearly every unital algebra A is Morita equivalent to itself and the relation of Morita equivalence is symmetric. The next excercise shows that this relation is transitive.

Excercise 1.39. *Let A, B and C be unital algebras. If A is Morita equivalent to B and B is Morita equivalent to C, then A is Morita equivalent to C.*

Hence Morita equivalence satisfies the axioms of an equivalence relation.
The modules P and Q in a strict Morita context are often called equivalence bimodules.
We will now prove a basic result on Morita equivalent algebras.

Theorem 1.40. *Let A and B be Morita equivalent unital algebras and let P and Q be equivalence bimodules. Then*

a) The maps $\langle -,-\rangle_A$ and $\langle -,-\rangle_B$ are isomorphisms.
b) P is finitely generated projective as left A-module and as right B-module. Q is finitely generated projective as left B-module and as right A-module.
c) There are isomorphisms

$$A\cong \mathrm{End}_B(Q,Q)\cong \mathrm{End}_B(P,P) \qquad B\cong \mathrm{End}_A(P,P)\cong \mathrm{End}_A(Q,Q)$$

as algebras.

Proof. $a)$ Since $\langle -,-\rangle_A$ is surjective there exists $\sum x_i\otimes y_i\in P\otimes_B Q$ such that $\sum\langle x_i,y_i\rangle_A = 1$. Now assume that $\sum\langle v_j,w_j\rangle_A = 0$. Then we have

$$\sum v_j\otimes w_j = \sum v_j\otimes w_j\langle x_i,y_i\rangle_A = \sum v_j\langle w_j,x_i\rangle_B\, y_i = \sum\langle v_j,w_j\rangle_A\, x_i\otimes y_i = 0$$

and hence $\langle -,-\rangle_A$ is injective. By symmetry, the same holds true for $\langle -,-\rangle_B$.
$b)$ The isomorphism in proposition 1.26 shows that $\langle -,-\rangle_A\in \mathrm{Hom}_{A-A}(P\otimes_B Q,A)$ corresponds to a map $\phi : Q\to \mathrm{Hom}_A(P,A) = P^*$. Since $\langle -,-\rangle_B$ is surjective there exist elements $q_i\in Q$ and $p_i\in P$ such that $\sum\langle q_j,p_j\rangle_B = 1$. Then

$$p = p\sum\langle q_j,p_j\rangle_B = \sum\langle p,q_j\rangle_A p_j = \sum\phi(q_i)(p)p_j$$

for all $p\in P$. According to the dual basis lemma 1.33 (or rather its version for left modules) this shows that ${}_AP$ is finitely generated and projective. The assertions for $P_B, {}_BQ$ and Q_A are proved in a similar way.
$c)$ The right A-module structure of Q induces a unital algebra homomorphism

$\phi : A \to \mathrm{End}_B(Q)$ given by $\phi(a)(q) = qa$. Let us show that f is an isomorphism. If $\phi(a)(q) = qa = 0$ for all $q \in Q$ we have

$$a = \sum \langle x_i, y_i \rangle_A a = \sum \langle x_i, y_i a \rangle_A = 0.$$

Hence f is injective. For $f \in \mathrm{End}_B(Q)$ we have

$$\begin{aligned} f(q) = f(q1) = \sum f(q\langle x_i, y_i \rangle_A) &= \sum f(\langle q, x_i \rangle_B y_i) \\ &= \sum \langle q, x_i \rangle_B f(y_i) = \sum q \langle x_i, f(y_i) \rangle_A \end{aligned}$$

and hence $f = \phi(\sum \langle x_i, f(y_i) \rangle_A)$ is in the image of ϕ. Again, the remaining assertions follow by symmetry. □

The most important example of a Morita equivalence is given as follows. Consider a unital algebra A and the algebra $M_n(A)$ of matrices over A. Moreover let ${}_A P_{M_n(A)}$ be the space A^n of all row vectors and ${}_{M_n(A)} Q_A$ be the space A^n of all column vectors of length n with entries in A. The module actions are given by matrix multiplication.

Excercise 1.41. *Show that P and Q define equivalence bimodules for A and $M_n(A)$.*

Assume that A is a unital algebra such that every finitely generated projective unitary A-module is isomorphic to a finite direct sum of copies of A. Then it follows from theorem 1.40 that every Morita equivalence between A and another algebra is of the form described before. This applies in particular to the algebra $\mathbb{C}$ of complex numbers.

CHAPTER 2

Homological algebra

Homological algebra is a set of tools to study the homology of chain complexes. We will need some of these tools for our study of cyclic homology. There are several good textbooks on homological algebra, we follow closely the treatment in the book of Weibel [**14**].

1. Chain complexes

Definition 2.1. *Let A be an algebra. A chain complex of A-modules is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of A-modules C_n together with module homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A chain map $f : C \to D$ between chain complexes is a family $f_n : C_n \to D_n$ of A-module homomorphisms such that the diagrams*

$$\begin{array}{ccc} C_n & \xrightarrow{d} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

are commutative for all $n \in \mathbb{N}$.

A chain complex of A-modules for $A = 0$ is simply called a chain complex. Also in the general case we will occasionally omit the algebra A in our terminology.
A chain complex is called bounded below if there exists $N \in \mathbb{Z}$ such that $C_n = 0$ for all $n < N$. Similarly, it is called bounded above if there exists $N \in \mathbb{Z}$ such that $C_n = 0$ for all $n > N$. A chain complex is called bounded if it is bounded below and above. In the sequel we will meet mainly bounded below chain complexes.
It is common to write $d : C_n \to C_{n-1}$ instead of d_n. We will also do this, in this way the important algebraic property of the differential is $d^2 = dd = 0$. The elements $x \in C_n$ of a chain complex C are called n-chains or simply chains. Elements of the form $d(x)$ for some $x \in C_{n+1}$ are called n-boundaries. The space of all n-boundaries is denoted by $B_n(C)$. Similarly, elements $x \in C_n$ satisfying $d(x) = 0$ are called n-cycles. The space of all cycles is denoted by $Z_n(C)$. The relation $d^2 = 0$ implies $B_n \subset Z_n$ for all n.

Definition 2.2. *The n-th homology group of a chain complex C is the space $H_n(C) = Z_n/B_n$.*

Note that the homology $H_n(C)$ of a chain complex (of A-modules) is in fact a vector space (even an A-module). It is easy to check that a chain map $f : C \to D$ induces a map $H_n(f) : H_n(C) \to H_n(D)$ on homology for all n. An important situation is when these induced maps are isomorphisms.

Definition 2.3. *A chain map $f : C \to D$ is called a quasiisomorphism if the induced maps $H_n(f) : H_n(C) \to H_n(D)$ are isomorphisms for all n.*

A chain complex is called *acyclic* if $H_n(C) = 0$ for all n. Clearly, a chain complex C is acyclic iff the trivial chain map $0 \to C$ is a quasiisomorphism. If $f : C \to D$ is a chain map, then $\ker(f)$ and $\mathrm{im}(f)$ are again chain complexes.

Moreover $C/\ker(f)$ and $D/\operatorname{im}(f)$ become chain complexes in a natural way. We will see below that the induced map $H_n(f) : H_n(C) \to H_n(D)$ of an injective (surjective) chain map is not injective (surjective) in general.

DEFINITION 2.4. *Two chain maps $f, g : C \to D$ are called homotopic if there exists a map $h : C \to D$ of degree 1, that is, a family of homomorphisms $h_n : C_n \to D_{n+1}$ such that $dh_n + h_{n-1}d = f_n - g_n$ for all n, or simply*

$$dh + hd = f - g.$$

A chain map $f : C \to D$ is called a homotopy equivalence if there exists a chain map $g : D \to C$ such that fg is homotopic to the identity map on D and gf is homotopic to the identity map on D. Two chain complexes C and D are called homotopy equivalent if there exists a homotopy equivalence between C and D.
A chain complex C is called contractible if it is chain homotopy equivalent to the trivial chain complex 0.

We write $f \sim g$ if the chain maps f and g are homotopic. If $f : C \to D$ is a homotopy equivalence then a map $g : D \to C$ satisfying $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$ is called a homotopy inverse of f. Note that a homotopy inverse is in general not uniquely determined.
The following excercise shows that homotopy is an equivalence relation.

EXCERCISE 2.5. *Let $f, g, h : C \to D$ be chain maps. We have the following implications.*

a) $f \sim f$.
a) $f \sim g$ *implies* $g \sim f$.
b) $f \sim g$ *and* $g \sim h$ *implies* $f \sim h$.

LEMMA 2.6. *Let $f, g : C \to D$ be homotopic. Then we have $H_n(f) = H_n(g)$ for all n. In particular, every homotopy equivalence is a quasiisomorphism.*

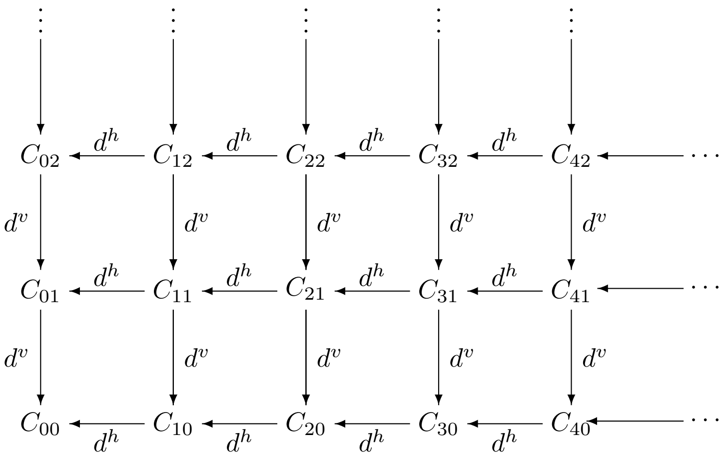
PROOF. Let $x \in C_n$ be a cycle. Then $f(x) - g(x) = dh(x) + hd(x) = dh(x) = 0 \in H_n(D)$. Hence $H_n(f) = H_n(g)$. If $f : C \to D$ is a homotopy equivalence with homotopy inverse $g : D \to C$ the previous assertion implies that $H_n(f)$ is an isomorphism with inverse $H_n(g)$ for all n. □
Apart from ordinary complexes we will also need bicomplexes for the definition of cyclic homology.

DEFINITION 2.7. *A bicomplex is a family $C = (C_{mn})_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ of modules C_{mn} together with horizontal differentials $d^h : C_{mn} \to C_{m-1n}$ and vertical differentials $d^v : C_{mn} \to C_{mn-1}$ such that*

$$d^h d^h = 0, \qquad d^v d^v = 0, \qquad d^h d^v + d^v d^h = 0.$$

A chain map $f : C \to D$ between bicomplexes is a family $f_{mn} : C_{mn} \to D_{mn}$ of maps which commute with both differentials d^h and d^v.

It is convenient to visualize bicomplexes in the plane. To do this one inserts the module C_{mn} in the point $(m, n) \in \mathbb{R}^2$ and connects adjacent points by arrows representing the differentials. Motivated by such a picture, one says that C is a first quadrant bicomplex if $C_{mn} = 0$ if $m < 0$ or $n < 0$. Hence a first quadrant double complex looks like this:



Note that the squares occuring here are not commutative. In fact, they are anti-commutative in the sense that we have $d^h d^v = -d^v d^h$ according to the definition of a bicomplex.
Apart from considering particular rows and columns in a bicomplex C there are essentially two canonical ways to associate an ordinary complex to C. The *direct product total complex* of C is defined by

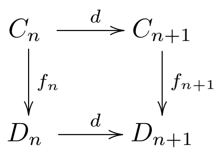
$$\mathrm{Tot}(C)_n = \prod_{p+q=n} C_{pq}$$

with differential $d = d^h + d^v$. The *direct sum total complex* of C is defined by

$$\mathrm{tot}(C)_n = \bigoplus_{p+q=n} C_{pq}$$

and equipped with the same differential. Clearly there is a natural chain map $\mathrm{tot}(C) \to \mathrm{Tot}(C)$. This map is an isomorphism, for instance, if C is a first quadrant bicomplex. In general however, it is not even a quasiisomorphism. The homologies of $\mathrm{Tot}(C)$ and $\mathrm{tot}(C)$ may differ drastically.
In connection with Hochschild cohomology and cyclic cohomology we will also use the concept of a cochain complex.

Definition 2.8. *A cochain complex is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of modules C_n together with homomorphisms $d_n : C_n \to C_{n+1}$ such that $d_{n+1}d_n = 0$ for all $n \in \mathbb{Z}$. A chain map $f : C \to D$ between cochain complexes is a family $f_n : C_n \to D_n$ such that the diagrams*



are commutative for all $n \in \mathbb{N}$.

A cochain complex is called bounded below if there exists $N \in \mathbb{Z}$ such that $C_n = 0$ for all $n < N$. It is called bounded above if there exists $N \in \mathbb{Z}$ such that $C_n = 0$ for all $n > N$.
The elements $x \in C^n$ of a cochain complex C are called n-cochains or simply cochains. Elements of the form $d(x)$ for some $x \in C_{n-1}$ are called n-coboundaries. The space of all n-coboundaries is denoted by $B^n(C)$. Similarly, elements $x \in C^n$

satisfying $d(x) = 0$ are called n-cocycles. The space of all cocycles is denoted by $Z^n(C)$. The relation $d^2 = 0$ implies $B^n \subset Z^n$ for all n.

DEFINITION 2.9. *The n-th cohomology group of a cochain complex C is the space $H^n(C) = Z^n/B^n$.*

Similarly, one may define bicomplexes in the cohomological framework. We shall not write down explicitly the corresponding definitions.
Every cochain complex C can be transformed into a chain complex and vice versa by setting $C_n = C_{-n}$ with the corresponding differential. Hence the concepts of a chain complex and a cochain complexes are essentially equivalent. However, most of the time certain constructions are most naturally viewed as chain complexes or cochain complexes. We leave it to the reader to adapt the notions and results on chain complexes presented in this chapter to the case of cochain complexes.

2. Exact sequences

In this section we discuss the notion of an exact sequence and some fundamental results of homological algebra.
Let A be an algebra. A sequence

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d} C_n \xrightarrow{d} C_{n-1} \longrightarrow \cdots$$

of A-modules and homomorphisms is called exact if $\mathrm{im}(d) = \ker(d) \subset C_n$ for all $n \in \mathbb{Z}$. One also speaks about a *long exact sequence* in this case. Note that a long exact sequence may be viewed as an acyclic complex.
If $C_n = 0$ exept for three consecutive numbers, such a sequence is called a *short exact sequence* and written as

$$K \rightarrowtail^{i} E \xrightarrow{p} Q.$$

Explicitly, a short exact sequence consists of A-modules K, E and Q and A-module homomorphisms i, p such that i is a monomorphism, p is an epimorphism and $\mathrm{im}(i) = \ker(p)$.
A chain map $f : C \to D$ is called a monomorphism (epimorphism) if all maps $f_n : C_n \to D_n$ are monomorphisms (epimorphisms). A short exact sequence of complexes is a diagram

$$K \rightarrowtail^{i} E \xrightarrow{p} Q$$

of chain complexes and chain maps such that i is a monomorphism, p is an epimorphism and $\mathrm{im}(i) = \ker(p)$. Equivalently, in each degree the associated short exact sequence of modules is exact.
The following result is of fundamental importance in homological algebra.

PROPOSITION 2.10. *Let $K \rightarrowtail^{i} E \xrightarrow{p} Q$ be a short exact sequence of chain complexes. Then there exists natural connecting homomorphisms $\partial : H_n(Q) \to H_{n-1}(K)$ for all n such that the sequence*

$$\cdots \longrightarrow H_n(K) \xrightarrow{H_n(i)} H_n(E) \xrightarrow{H_n(p)} H_n(Q) \xrightarrow{\partial} H_{n-1}(K) \longrightarrow \cdots$$

is exact.

The connecting homomorphism $\partial : H_n(Q) \to H_{n-1}(K)$ is constructed as follows. If $x \in Q_n$ is a cycle with homology class $[x]$ we lift x to a chain $y \in E_n$ and apply d. The resulting element $z = dy \in E_{n-1}$ satisfies $pdy = 0$ and hence lies in fact in K_{n-1}. Moreover we clearly have $dz = 0$ and thus z defines a homology class $[z] \in H_{n-1}(K)$.

EXCERCISE 2.11. *The homology class* $[z] \in H_{n-1}(K)$ *depends only on the homology class of* x. *That is, it is independent of the representative for* $[x]$ *and of the choice of* y.

Hence we may define $\partial : H_n(Q) \to H_{n-1}(K)$ by $\partial([x]) = [z]$.
For the proof of proposition 2.10 we shall use the snake lemma.

LEMMA 2.12 (Snake lemma). *Let* A *be an algebra and consider a diagram of* A*-modules*

$$\begin{array}{ccccccc} & & K_1 & \longrightarrow & E_1 & \longrightarrow & Q_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & K_2 & \longrightarrow & E_2 & \longrightarrow & Q_2 & & \end{array}$$

with exact rows. Then there is an exact sequence

$$\ker(f) \longrightarrow \ker(g) \longrightarrow \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \longrightarrow \operatorname{coker}(g) \longrightarrow \operatorname{coker}(h).$$

Here the connecting map $\partial : \ker(h) \to \operatorname{coker}(f)$ is defined in the same way as above. The proof of the snake lemma is done by checking case by case and left to the reader.
Let us now show that the sequence in proposition 2.10 is exact. Consider the diagram

$$\begin{array}{ccccccccc} & & K_n/dK_{n+1} & \longrightarrow & E_n/dE_{n+1} & \longrightarrow & Q_n/dQ_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\ 0 & \longrightarrow & Z_{n-1}(K) & \longrightarrow & Z_{n-1}(E) & \longrightarrow & Z_{n-1}(Q) & & \end{array}$$

where $Z_{n-1}(K)$ denotes the space of $(n-1)$-cycles in the complex K, and similarly for E and Q. It is easy to check directly that the rows in this diagram are exact. Applying the snake lemma 2.12 yields an exact sequence

$$H_n(K) \longrightarrow H_n(E) \longrightarrow H_n(Q) \xrightarrow{\partial} H_{n-1}(K) \longrightarrow H_{n-1}(E) \longrightarrow H_{n-1}(Q)$$

where the maps are given as described in proposition 2.10. Pasting together the exact sequences thus obtained yields the assertion.
Another result which will be used in many situations is the five lemma.

LEMMA 2.13 (Five lemma). *Consider a diagram of* A*-modules of the form*

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

and assume that both rows are exact. Then f *is an isomorphism.*

PROOF. This is a diagram chase best done visually. Let us only show that f is injective. Assume that $f(x) = 0$. Then the image of x in M_4 is zero. Hence there exists x_2 in M_2 which maps to x. The image of x_2 in N_2 comes from an element y_1 of N_1. Hence there exists $x_1 \in M_1$ such that the image of x_1 in M_2 is equal to x_2. It follows that the image of x_2 in M_3 is zero. Hence x is zero. $\square$

3. Projective resolutions and derived functors

In this section study projective resolutions of modules and the derived functor of the tensor product functor.

Definition 2.14. *Let A be an algebra and let M be an A-module. A projective resolution P of M consists of a long exact sequence*

$$M \xleftarrow{\epsilon} P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3 \longleftarrow \cdots$$

of A-modules such that all P_j are projective.

Our first aim is to show that every module has a projective resolution.

Lemma 2.15. *Let M be an A-module. Then there exists a projective resolution for M.*

Proof. Let $P_0 = AM$ be the free module over the set M and let $\epsilon : P_0 \to M$ be the natural A-module map. By construction, ϵ is surjective and P_0 is free, hence projective. Now let $P_1 = A\ker(\epsilon)$ be the free module over the set $\ker(\epsilon)$ and let $d_1 : P_1 \to P_0$ be the natural map. Then $\mathrm{im}(d_1) = \ker(\epsilon)$ and P_1 is again projective. We may next consider $\ker(d_1)$ and continue in this way to obtain a projective resolution of M by free A-modules. □

In many cases one may construct smaller projective resolutions of a module. If P is already a projective module then the most evident projective resolution of P is given by $P_0 = 0$, $P_j = 0$ for $j > 0$ and $\epsilon = \mathrm{id}$.

For the general theory it is important that projective resolutions may be compared.

Proposition 2.16. *Let M and N be A-modules and let P and Q be projective resolutions of M and N, respectively. If $f : M \to N$ is an A-module homomorphism there exist A-module homomorphisms $f_j : P_j \to Q_j$ for all j such that the diagram*

$$\begin{array}{ccccccccccc}
M & \xleftarrow{\epsilon} & P_0 & \longleftarrow & P_1 & \longleftarrow & P_2 & \longleftarrow & P_3 & \longleftarrow & \cdots \\
\downarrow f & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\
N & \xleftarrow{\epsilon} & Q_0 & \longleftarrow & Q_1 & \longleftarrow & Q_2 & \longleftarrow & Q_3 & \longleftarrow & \cdots
\end{array}$$

is commutative. Moreover, if $(g_j)_{j\geq 0}$ is another family of such homomorphisms, then the chain maps f and g thus defined are homotopic.

Proof. The maps f_j are constructed inductively. Since $\epsilon : Q_0 \to N$ is surjective and P_0 is projective, there exists an A-module map $f_0 : P_0 \to Q_0$ such that $f\epsilon = \epsilon f_0$. Now assume that f_j has been constructed. Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & \longleftarrow & P_{j-1} & \xleftarrow{d_j} & P_j & \xleftarrow{d_{j+1}} & P_{j+1} & \longleftarrow & \cdots \\
 & & \downarrow f_{j-1} & & \downarrow f_j & & & & \\
\cdots & \longleftarrow & Q_{j-1} & \xleftarrow{d_j} & Q_j & \xleftarrow{d_{j+1}} & Q_{j+1} & \longleftarrow & \cdots
\end{array}$$

The image of the map $f_j d_{j+1}$ is contained in $\ker(d_j) = \mathrm{im}(d_{j+1})$. Again, since $d_{j+1} : Q_{j+1} \to \mathrm{im}(d_{j+1})$ is surjective and P_{j+1} is projective, there exists f_{j+1} such that $d_{j+1} f_{j+1} = f_j d_{j+1}$.

To prove the second assertion, it suffices to consider the case $f = 0$. We have to show that any family of maps f_j as above is homotopic to zero. Again, the contracting homotopy h will be constructed inductively. We define $h_{-1} = 0 : M \to Q_0$. Since f is a chain map, the image of f_0 is contained in $\ker(\epsilon) = \mathrm{im}(d_1)$. By projectivity of P_0, there exists $h_0 : P_0 \to Q_1$ such that $d_1 h_0 = f_0$. Now assume that $h_j : P_j \to Q_{j+1}$ has been constructed such that $d_{j+1}h_j + h_{j-1}d_j = f_j$. Then $f_{j+1} - h_j d_{j+1}$ maps into $\ker(d_{j+1}) = \operatorname{im}(d_{j+2})$ since

$$d_{j+1}f_{j+1} - d_{j+1}h_j d_{j+1} = d_{j+1}f_{j+1} + h_{j-1}d_j d_{j+1} - f_j d_{j+1} = 0.$$

By projectivity of P_{j+1}, there exists $h_{j+1} : P_{j+1} \to Q_{j+2}$ such that $d_{j+2}h_{j+1} = f_{j+1} - h_j d_{j+1}$. This yields the claim. □

As a consequence, one has the following result.

Excercise 2.17. *Two projective resolutions of a module M are homotopy equivalent.*

We will now define the derived functor of the tensor product.

Definition 2.18. *Let M_A and ${}_AN$ be modules over an algebra A and choose a projective resolution P of ${}_AN$. Then*

$$\operatorname{Tor}_n^A(M, N) = H_n(M \otimes_A P).$$

Using excercise 4.26 we see that, up to natural isomorphism, the definition of $\operatorname{Tor}(M, N)$ is independent of the resolution P.

Excercise 2.19. *Let ${}_AN$ be a module and let*

$$0 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of left A-modules. Then the induced sequence of vector spaces

$$0 \longrightarrow \operatorname{Hom}_A(N, K) \longrightarrow \operatorname{Hom}_A(N, E) \longrightarrow \operatorname{Hom}_A(N, Q)$$

is exact.
If N_A is projective then in addition the map $\operatorname{Hom}_A(N, E) \to \operatorname{Hom}_A(N, Q)$ is surjective.

Proposition 2.20. *Let ${}_AN$ be a module and let*

$$0 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 0$$

be a short exact sequence of right A-modules. Then the induced sequence

$$K \otimes_A N \longrightarrow E \otimes_A N \longrightarrow Q \otimes_A N \longrightarrow 0$$

is exact.

Proof. According to proposition 1.26 we have

$$\operatorname{Hom}(M \otimes_A N, V) \cong \operatorname{Hom}_A(N, \operatorname{Hom}(M, V))$$

for every module M_A and every vector space V. According to excercise 2.19 this implies that the sequence

$$0 \longrightarrow \operatorname{Hom}(Q \otimes_A N, V) \longrightarrow \operatorname{Hom}(E \otimes_A N, V) \longrightarrow \operatorname{Hom}(K \otimes_A N, V)$$

is exact for every vector space V. Since we are now only dealing with vector spaces it follows that already the sequence

$$K \otimes_A N \longrightarrow E \otimes_A N \longrightarrow Q \otimes_A N \longrightarrow 0$$

is exact. This yields the claim. □

As a consequence of proposition 2.20 one obtains in particular the following statement.

Excercise 2.21. *Let M_A and ${}_AN$ be modules over an algebra A. Then*

$$\operatorname{Tor}_0^A(M, N) \cong M \otimes_A N$$

All constructions in this section can as well be carried out for unitary modules over unital algebras. This is in fact the standard way to define derived functors of modules. We could thus give the following definition. Let M_A and ${}_AN$ be unitary modules over a unital algebra A and choose a projective resolution P of ${}_AN$ of unitary A-modules. Then the unitary derived functor of the tensor product is

$$\mathrm{tor}_n^A(M,N) = H_n(M \otimes_A P).$$

In the same way as above one may prove that this does not depend on the choice of P up to isomorphism. However, using proposition 1.32 one obtains even the following statement.

Excercise 2.22. *Let A be a unital algebra and let M and N be unitary A-modules. Then there is a natural isomorphism*

$$\mathrm{Tor}_n^A(M,N) \cong \mathrm{tor}_n^A(M,N)$$

for all n.

Hence it makes essentially no difference if we work with unital algebras and unitary modules or with arbitrary algebras and arbitrary modules.

4. Inductive and projective limits of chain complexes

In this section we study the homology of inductive and projective limits of chain complexes.
Let J be a partially ordered set and let $(C^j)_{j\in J}$ be an inductive system of chain complexes. That is, we are given chain complexes C^j and a compatible family of chain maps $f_{ji}: C^i \to C^j$ for all $i \leq j$. Then we may form the inductive limit

$$C = \varinjlim_{j\in J} C^j$$

by letting $C_n = \varinjlim_{j\in J} C_n^j$ be the inductive limits in each degree. It is straightfoward to check that the inductive limit C is again a chain complex. As a special case one may consider direct sums.

Excercise 2.23. *Let $(C^j)_{j\in J}$ be a family of chain complexes. Then the natural map*

$$\bigoplus_{j\in J} H^*(C^j) \to H_*\Big(\bigoplus_{j\in J} C^j\Big)$$

is an isomorphism.

Recall that a partially ordered set J is directed if for every $i, j \in J$ there exists $k \in J$ such that $i \leq k$ and $j \leq k$.

Lemma 2.24. *Let $(C^j)_{j\in J}$ be an inductive system of chain complexes over a directed set J. Then the natural map*

$$\varinjlim_{j\in J} H_*(C^j) \to H_*\Big(\varinjlim_{j\in J} C^j\Big)$$

is an isomorphism.

Proof. Let C be the inductive limit of the complexes C_j. There is a compatible family of chain maps $\iota_j : C^j \to C$ and hence an induced map $\iota : \varinjlim H_*(C^j) \to H_*(C)$. Let us show that this map is injective and surjective. If $c \in C$ is a cycle there exists $j \in J$ such that $c = \iota_j(c_j)$ for $c_j \in C^j$. Moreover $dc = 0$ implies $\iota_{kj}(dc_j) = 0 \in C^k$ for some $k \geq j$. Hence $\iota_{kj}(c_j) \in C^k$ is a cycle and $\iota([\iota_{kj}(c_j)]) = [c]$. This show that ι is surjective. If $\iota([c_j]) = 0$ in $H_*(C)$ there exists $b \in C^k$ and $l \in J$ such that $\iota_{lj}(c_j) = \iota_{lk}(db) = d\iota_{lk}b$ in C^l. Here we use the fact that J is directed. It follows that $\iota_{lj}([c_j]) = 0$ in $H_*(C^l)$ and hence the class $[c_j]$ in $\varinjlim H(C^j)$ is zero. This shows that ι is injective. $\square$

For projective limits one start with the dual definitions. Let J be a partially ordered set and let $(C^j)_{j\in J}$ be a projective system of chain complexes. That is, we are given chain complexes C^j and a compatible family of chain maps $f_{ji} : C^i \to C^j$ for all $j \le i$. We form the projective limit

$$C = \varprojlim_{j\in J} C^j$$

componentwise and obtain a chain complex C. Let us consider the special case of direct products.

EXCERCISE 2.25. *Let $(C^j)_{j\in J}$ be a family of chain complexes. Then the natural map*

$$\prod_{j\in J} H^*(C^j) \to H_*\Big(\prod_{j\in J} C^j\Big)$$

is an isomorphism.

The case of general projective limits is more complicated. We consider only the special case where the index set J is the set of natural numbers with the canonical ordering.

First we have to explain what $\varprojlim^1_{j\in\mathbb{N}} M^j$ for a projective system $(M^j)_{j\in\mathbb{N}}$ of modules or chain complexes over $\mathbb{N}$ is. Consider the map $\sigma : \prod_{j\in\mathbb{N}} M^j \to \prod_{j\in\mathbb{N}} M^j$ given by

$$\sigma((x_j)_{j\in\mathbb{N}}) = (\pi_{j,j+1}(x_{j+1}))_{j\in\mathbb{N}}.$$

The kernel of $\mathrm{id} - \sigma$ can be identified with $\varprojlim_{j\in\mathbb{N}} M^j$. By definition $\varprojlim^1_{j\in\mathbb{N}} M^j$ is the cokernel of $\mathrm{id} - \sigma$. Hence we have a short exact sequence

$$\varprojlim_{j\in\mathbb{N}} M^j \rightarrowtail \prod_{j\in\mathbb{N}} M^j \xrightarrow{\mathrm{id}-\sigma} \prod_{j\in\mathbb{N}} M^j \twoheadrightarrow \varprojlim^1_{j\in\mathbb{N}} M^j.$$

In favorable circumstances, the term $\varprojlim^1_{j\in\mathbb{N}} M^j$ vanishes.

LEMMA 2.26. *Let $(M^j)_{j\in\mathbb{N}}$ be an inverse system with surjective structure maps. Then $\varprojlim^1_{j\in\mathbb{N}} M^j = 0$.*

PROOF. If all structure maps are surjective, the map $\mathrm{id} - \sigma$ is surjective as well. Hence $\varprojlim^1_{j\in\mathbb{N}} M^j = 0$ by the definition of $\varprojlim^1_{j\in\mathbb{N}}$. $\square$

In the hypothesis of the previous lemma the condition of having surjective structure maps can be relaxed. One says that the projective system $(M_j)_{j\in\mathbb{N}}$ satisfies the *Mittag-Leffler condition* if for all $j \in \mathbb{N}$ there exists k such that the images of the maps $\pi_{jl} : M^l \to M^j$ are equal for all $l \ge k$. We remark that the assertion of lemma 2.26 remains true for projective systems satisfying the Mittag-Leffler condition.

PROPOSITION 2.27. *Let $(C^n)_{n\in\mathbb{N}}$ be a projective system of chain complexes with surjective structure maps. If we denote by C the projective limit of the system $(C^n)_{n\in\mathbb{N}}$, there is a short exact sequence*

$$\varprojlim^1_{j\in\mathbb{N}} H_{n+1}(C_j) \rightarrowtail H_n(C) \twoheadrightarrow \varprojlim_{j\in\mathbb{N}} H_n(C_j)$$

for each n.

PROOF. We have an exact sequence

$$\varprojlim_{j\in\mathbb{N}} C^j \rightarrowtail \prod_{j\in\mathbb{N}} C^j \xrightarrow{\mathrm{id}-\sigma} \prod_{j\in\mathbb{N}} C^j \twoheadrightarrow \varprojlim^1_{j\in\mathbb{N}} C^j$$

of chain complexes. According to the assumption, the term $\varprojlim^1_{j\in\mathbb{N}} C^j$ is zero and we get a short exact sequence of chain complexes

$$\varprojlim_{j\in\mathbb{N}} C^j \rightarrowtail \prod_{j\in\mathbb{N}} C^j \xrightarrow{\mathrm{id}-\sigma} \prod_{j\in\mathbb{N}} C^j$$

which induces a long exact sequence

$$\cdots \longrightarrow H_n(C) \longrightarrow \prod_{j\in\mathbb{N}} H_n(C^j) \xrightarrow{\mathrm{id}-\sigma} \prod_{j\in\mathbb{N}} H_n(C^j) \longrightarrow \cdots$$

in homology. Here we use excercise 2.25 to describe the homology of the direct product complexes. By definition, $\varprojlim^1_{j\in\mathbb{N}} H_{n+1}(C^j)$ is the cokernel of the map $\mathrm{id}-\sigma$ at the $(n+1)$th stage in this sequence. It maps injectively into $H_n(C)$ by the boundary map. Moreover, the kernel of $\mathrm{id}-\sigma$ at the nth stage is equal to $\varprojlim_{j\in\mathbb{N}} H_n(C^j)$. By exactness, this yields the assertion. □

5. Presimplicial modules

For the description of Hochschild homology it is convenient to use the following concept. Historically, it originates from the study of simplicial homology and singular homology.

Definition 2.28. *A presimplicial module C is a sequence of vector spaces C_n for $n \geq 0$ together with maps*

$$d_j : C_n \to C_{n-1} \qquad \textit{for} \quad j = 0, \dots, n$$

called face maps such that

$$d_i d_j = d_{j-1} d_i \qquad \textit{for} \quad 0 \leq i < j \leq n$$

and all n. A presimplicial map $f : C \to D$ between presimplicial modules is a family of linear maps $C_n \to D_n$ such that $d_i f = f d_i$ for all face maps d_i.

Of course, one might as well consider A-modules over an algebra together with module homomorphisms satisfying the above relations. We will not need this, though.

Historically, the following observation was one of the starting points of homological algebra.

Excercise 2.29. *Let C be a presimplicial module. Then C becomes a complex with boundary operators $d : C_n \to C_{n-1}$ given by*

$$d = \sum_{j=0}^{n} (-1)^j d_j.$$

Observe that every map $f : C \to D$ of presimplicial modules induces a chain map between the associated complexes.

Definition 2.30. *Let $f, g : C \to D$ be presimplicial maps. A presimplicial homotopy between f and g is a family of linear maps $h_j : C_n \to D_{n+1}$ for $j = 0, \dots, n$ such that $d_0 h_0 = f$ and $d_{n+1} h_n = g$ while*

$$d_i h_j = \begin{cases} h_{j-1} d_i, & 0 \leq i < j \leq n \\ d_i h_{i-1}, & 0 < i = j \leq n \\ h_j d_{i-1}, & 1 \leq j+1 < i \leq n+1 \end{cases}$$

for all n.

We write $f \sim g$ if two maps f and g are connected by a presimplicial homotopy. The following excercise shows in particular that presimplicial homotopy is an equivalence relation.

EXCERCISE 2.31. *Consider presimplicial maps from C to D. Then*
a) $f \sim f$.
b) $f \sim g$ *implies* $g \sim f$ *and* $-f \sim -g$.
c) $f \sim g$ *and* $g \sim h$ *implies* $f \sim h$.
c) $f_1 \sim g_1$ *and* $f_2 \sim g_2$ *implies* $(f_1 + f_2) \sim (g_1 + g_2)$.

EXCERCISE 2.32. *Let $f, g : C \to D$ be maps of presimplicial modules which are connected by a presimplicial homotopy. Then the associated chain maps are homotopic.*

Let us remark that there exists also the notion of a *simplicial module*. It is obtained from the definition of a presimplicial module by requiring in addition the existence of certain *degeneracy maps* satisfying some conditions. For the applications we have in mind, such degeneracy maps exist only if we work with unital algebras. Since we do not want to restrict attention to unital algebras it is more natural to consider presimplicial modules.

CHAPTER 3

Hochschild homology and cyclic homology

In this chapter we define Hochschild homology and cyclic homology and study their basic properties. There are different approaches to these theories, and each of these approaches has its particular virtues. We will begin with the definition using the mixed complex of noncommutative differential forms and then deduce the description of Hochschild homology for unital algebras as a derived functor. We also define cyclic homology based on mixed complexes. After having treated the SBI-sequence we will introduce Hochschild cohomology and cyclic cohomology. We discuss periodic cyclic homology and cohomology and the relation of these theories to ordinary cyclic homology and cohomology.

1. Noncommutative differential forms

In this section we define and study a noncommutative replacement of the algebra of differential forms $\mathcal{A}(M)$ on a smooth manifold M.

DEFINITION 3.1. *Let A be an algebra. For $n > 0$ we let $\Omega^n(A) = A^+ \otimes A^{\otimes n}$ be the space of noncommutative n-forms over A. In addition we set $\Omega^0(A) = A$.*

Here we have used the notation

$$A^{\otimes n} = A \otimes A \otimes \cdots \otimes A$$

to denote the tensor product of n copies of A. Elements in $\Omega^n(A)$ are written in the suggestive form $a^0 da^1 \cdots da^n$ for $a^0 \in A^+$ and $a^1, \dots, a^n$ in A. We also write $da^1 \cdots da^n$ if $a^0 = 1 \in A^+$.

Let us first consider the case $n = 1$. We define a left A-module structure on $\Omega^1(A)$ by setting

$$a(a^0 da^1) = aa^0 da^1.$$

A right A-module structure on $\Omega^1(A)$ is defined according to the Leibniz rule $d(ab) = dab + adb$ by

$$(a^0 da^1)a = a^0 d(a^1 a) - a^0 a^1 da.$$

EXCERCISE 3.2. *Verify that $\Omega^1(A)$ becomes an A-A-bimodule in this way.*

The next statement shows that higher differential forms may be constructed out of the bimodule $\Omega^1(A)$. We will use the notation

$$\Omega^1(A)^{\otimes_A n} = \Omega^1(A) \otimes_A \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A)$$

for the tensor product of n copies of $\Omega^1(A)$ over A.

EXCERCISE 3.3. *There is a natural isomorphism*

$$\Omega^n(A) \cong \Omega^1(A) \otimes_A \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A) = \Omega^1(A)^{\otimes_A n}$$

for every $n \geq 1$.

As a consequence, the spaces $\Omega^n(A)$ are equipped with an A-A-bimodule structure in a natural way. Explicitly, the left A-module structure on $\Omega^n(A)$ is given by

$$a(a^0 da^1 \cdots da^n) = aa^0 da^1 \cdots da^n$$

and the right A-module structure may be written as

$$(a^0da^1\cdots da^n)a = a_0da_1\cdots da_{n-1}d(a_na)$$
$$+\sum_{j=1}^{n-1}(-1)^{n-j}a^0da^1\cdots d(a^ja^{j+1})\cdots da^nda + (-1)^n a^0a^1da^2\cdots da^nda.$$

Moreover we view $\Omega^0(A) = A$ as an A-bimodule in the obvious way using the multiplication in A.
According to excercise 3.3 one may define a map $\Omega^n(A)\otimes\Omega^m(A)\to\Omega^{n+m}(A)$ by considering the natural projection

$$\Omega^1(A)^{\otimes_A n}\otimes\Omega^1(A)^{\otimes_A m}\to\Omega^1(A)^{\otimes_A n}\otimes_A\Omega^1(A)^{\otimes_A m} = \Omega^1(A)^{\otimes_A(m+n)}.$$

Let us denote by $\Omega(A)$ the direct sum of the spaces $\Omega^n(A)$ for $n\geq 0$. Then the maps $\Omega^n(A)\otimes\Omega^m(A)\to\Omega^{n+m}(A)$ assemble to a map $\Omega(A)\otimes\Omega(A)\to\Omega(A)$.

Excercise 3.4. *In this way the space $\Omega(A)$ becomes an algebra. Actually, $\Omega(A)$ is a graded algebra if one considers the natural grading given by the degree of a differential form.*

Let us now define a linear operator $d:\Omega^n(A)\to\Omega^{n+1}(A)$ by

$$d(a^0da^1\cdots da^n) = da^0\cdots da^n,\qquad d(da^1\cdots da^n) = 0$$

for $a^0,\dots,a^n\in A$. It follows immediately from the definition that $d^2 = 0$.
A differential form in $\Omega(A)$ is called homogenous of degree n if it is contained in the subspace $\Omega^n(A)$.

Excercise 3.5. *The graded Leibniz rule*

$$d(\omega\eta) = d\omega\eta + (-1)^{|\omega|}\omega d\eta$$

holds on $\Omega(A)$ for homogenous forms ω and η.

Hence the operator d has similar properties like the exterior differential on ordinary differential forms. Actually one might think of this operator as an analogue of the exterior differential.
At this point it would be tempting to define the de Rham homology of an algebra A to be the homology of $\Omega(A)$ with respect to the differential d. However, it is easy to check that $\Omega(A)$ is contractible with respect to this boundary operator. A contracting homotopy $h:\Omega^n(A)\to\Omega^{n-1}(A)$ is given by

$$h(a^0da^1\cdots da^n) = 0,\qquad h(da^1\cdots da^n) = a^1da^2\cdots da^n$$

for $a^0,a^1\dots,a^n\in A$. Hence we do not obtain any interesting information in this way. Instead we have to consider more interesting boundary operators.
We define a linear operator $b:\Omega^n(A)\to\Omega^{n-1}(A)$ by

$$b(a^0da^1\cdots da^n) = (-1)^{n-1}(a^0da^1\cdots da^{n-1}a^n - a^na^0da^1\cdots da^{n-1})$$
$$= (-1)^{n-1}[a^0da^1\cdots da^{n-1},a^n]$$

for $a^0\in A^+$ and $a^1\cdots a^n\in A$. Here $[x,y] = xy-yx$ denotes the ordinary commutator for elements in the algebra $\Omega(A)$. The operator b is called the *Hochschild operator*. Using the explicit formula for the right A-module structure of $\Omega(A)$ we obtain

$$b(a^0da^1\cdots da^n) = a^0a^1da^2\cdots da^n$$
$$+\sum_{j=1}^{n-1}(-1)^j a^0da^1\cdots d(a^ja^{j+1})\cdots da^n + (-1)^n a^na^0da^1\cdots da^{n-1}.$$

which is closer to the traditional form of the definition of the Hochschild boundary map.

LEMMA 3.6. *The Hochschild operator b satisfies $b^2 = 0$.*

PROOF. We compute for $\omega = a^0 da^1 \cdots da^n$ and $x, y \in A$

$$\begin{aligned} b^2(\omega dx dy) &= b((-1)^{n+1}(\omega dxy - y\omega dx)) = b((-1)^{n+1}(\omega d(xy) - \omega x dy - y\omega dx)) \\ &= (-1)^n(-1)^{n+1}(\omega xy - xy\omega - (\omega xy - y\omega x) - (y\omega x - xy\omega)) = 0 \end{aligned}$$

which yields the claim. □

We proceed to construct more operators as follows. The *Karoubi operator* $\kappa : \Omega^n(A) \to \Omega^n(A)$ is given by

$$\kappa = \mathrm{id} - (bd + db)$$

and the *Connes operator* $B : \Omega^n(A) \to \Omega^{n+1}(A)$ is defined by

$$B = \sum_{j=0}^{n} \kappa^j d.$$

Using $d^2 = 0$ we obtain $\kappa d = d\kappa$. Moreover this implies immediately $B^2 = 0$. Let us record explicit formulas for the operators κ and B on $\Omega^n(A)$. Clearly one has $\kappa(a) = a$ for $a \in \Omega^0(A) = A$.

EXCERCISE 3.7. *For all $n > 0$ one has*

$$\kappa(a^0 da^1 \cdots da^n) = (-1)^{n-1} da^n a^0 da^1 \cdots da^{n-1}$$

on $\Omega^n(A)$.

For the Connes operator we compute

$$B(a^0 da^1 \cdots da^n) = \sum_{i=0}^{n} (-1)^{ni} da^{n+1-i} \cdots da^n da^0 \cdots da^{n-i}$$

using excercise 3.7. We need the following lemma concerning relations between the operators constructed above.

LEMMA 3.8. *On $\Omega^n(A)$ the following relations hold:*

a) $\kappa^{n+1} d = d$
b) $\kappa^n = \mathrm{id} + b\kappa^n d$
c) $b\kappa^n = b$
d) $\kappa^{n+1} = \mathrm{id} - db$
e) $(\kappa^{n+1} - \mathrm{id})(\kappa^n - \mathrm{id}) = 0$
f) $Bb + bB = 0$

PROOF. a) follows directly from the explicit formula for κ obtained in excercise 3.7. b) Using again the formula for κ we compute

$$\begin{aligned} \kappa^n(a^0 da^1 \cdots da^n) &= da^1 \cdots da^n a^0 \\ &= a^0 da^1 \cdots da^n + (-1)^n b(da^1 \cdots da^n da^0) \\ &= a^0 da^1 \cdots da^n + b\kappa^n d(a^0 da^1 \cdots da^n). \end{aligned}$$

c) follows by applying the Hochschild boundary b to both sides of b). d) Apply κ to b) and use a). e) is a consequence of b) and d). f) We compute

$$\begin{aligned} Bb + bB &= \sum_{j=0}^{n-1} \kappa^j db + \sum_{j=0}^{n} b\kappa^j d = \sum_{j=0}^{n-1} \kappa^j (db + bd) + \kappa^n bd \\ &= \mathrm{id} - \kappa^n + \kappa^n bd = \mathrm{id} - \kappa^n(\mathrm{id} - bd) = \mathrm{id} - \kappa^n(\kappa + db) = 0 \end{aligned}$$

where we use d) and b). □

We can rephrase parts of this discussion using the following definition.

Definition 3.9. *A mixed complex M is a sequence of vector spaces M_n together with differentials b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and*

$$[b, B] = bB + Bb = 0.$$

on M_n for all n.

Proposition 3.10. *Let A be an algebra. The space $\Omega(A)$ of noncommutative differential forms together with the operators b and B is a mixed complex.*

2. Hochschild homology

In this section we define and study the Hochschild homology of an algebra.

Definition 3.11. *Let A be an algebra. The Hochschild homology of A is the homology of $\Omega(A)$ with respect to the Hochschild boundary b. We denote by $HH_n(A)$ the n-th Hochschild homology group of A.*

Let us identify the homology group $HH_0(A)$.

Lemma 3.12. *Let A be an algebra. Then $HH_0(A)$ is the quotient $A/[A, A]$ of A by the linear span of all commutators.*

Proof. The image of the Hochschild boundary $b : \Omega^1(A) \to A$ is equal to $[A, A]$ since $b(a_0 da_1) = a_0 a_1 - a_1 a_0$. □

As a consequence we obtain immediately

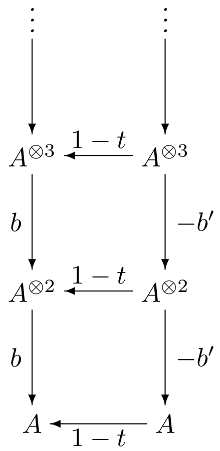
Corollary 3.13. *Let A be a commutative algebra. Then $HH_0(A) = A$.*

Let A be an arbitrary algebra and consider the direct sum decomposition

$$\Omega^n(A) = A^+ \otimes A^{\otimes n} = (A \oplus \mathbb{C}) \otimes A^{\otimes n} = A^{\otimes n+1} \oplus A^{\otimes n}.$$

for $n > 0$.

Excercise 3.14. *Using this decomposition the Hochschild complex $\Omega(A)$ of A can be identified with the total complex of the bicomplex*



where the vertical operators are defined by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

and

$$b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n.$$

The horizontal operator is constructed using the map t given by

$$t(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

We denote by $C(A)$ the complex given by the first column of this bicomplex. That is, $C_n(A) = A^{\otimes n+1}$ with boundary operator b. This complex will be called the *unital Hochschild complex* of A. Remark that we use the letter b both for the boundary operator in $\Omega(A)$ and the boundary operator in $C(A)$. However, this should not lead to confusion - the operator b on $C(A)$ is just the restriction of the operator b on $\Omega(A)$ to the first column.
The second column of the above bicomplex is denoted by $\mathrm{Bar}(A)$ and called the *Bar-complex* of A. We have $\mathrm{Bar}_n(A) = A^{\otimes n}$ with boundary operator $-b'$. By construction, we have a short exact sequence

$$C(A) \rightarrowtail \Omega(A) \twoheadrightarrow \mathrm{Bar}(A)$$

of complexes. Using this exact sequence we shall now obtain a different description of Hochschild homology in the case that the algebra A is unital.

Excercise 3.15. *If A is a unital algebra the Bar-complex $Bar(A)$ is contractible using the contracting homotopy $s : Bar_n(A) \to Bar_{n+1}(A)$ given by*

$$s(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}.$$

Proposition 3.16. *Let A be a unital algebra. Then the inclusion of the unital Hochschild complex $C(A)$ into $\Omega(A)$ is a homotopy equivalence.*

Proof. Define a map $\rho : \Omega(A) \to C(A)$ by

$$\rho(a_0 da_1 \cdots da_n) = a_0 \otimes a_1 \otimes \cdots \otimes a_n, \quad \rho(da_1 \cdots da_n) = -(1-t)s(a_1 \otimes \cdots \otimes a_n)$$

in degree n. Then one has

$$b\rho(a_0 da_1 \cdots da_n) = \rho b(a_0 da_1 \cdots da_n)$$

for $a_0, a_1, \ldots, a_n \in A$ and and

$$\begin{aligned} b\rho(da_1 \cdots da_n) &= -b(1-t)s(a_1 \otimes \cdots \otimes a_n) = (1-t)b's(a_1 \otimes \cdots \otimes a_n) \\ &= (1-t)(1-sb')(a_1 \otimes \cdots \otimes a_n) = \rho b(da_1 \cdots da_n) \end{aligned}$$

which shows that ρ is a chain map. If $\iota : C(A) \to \Omega(A)$ denotes the canonical inclusion then $\rho\iota = \mathrm{id}$. Moreover $\iota\rho$ is homotopic to the identity using the homotopy given by s on $\mathrm{Bar}(A)$ and by 0 on $C(A)$. □
Hence we obtain

Proposition 3.17. *Let A be a unital algebra. Then the Hochschild homology of A is naturally isomorphic to the homology of the unital Hochschild complex $C(A)$.*

Recall that the opposite algebra A^{op} of A has the same underlying vector space and the opposite multiplication $a \bullet b = ba$. Let us form the tensor product algebra $A^e = A \otimes A^{op}$. If A is unital then A^e is again unital and called the *extended algebra* of A. Note that there is a bijective correspondence between unitary (left) A^e-modules and unitary A-A-bimodules.
Let us define an A-bimodule structure on $\mathrm{Bar}_n(A)$ by the formula

$$a(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1})b = aa_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}b.$$

If A is unital the complex $\mathrm{Bar}(A)$ consists of projective unitary A^e-modules and due to excercise 3.15 we obtain the following statement.

Lemma 3.18. *Let A be a unital algebra. Then $Bar(A)$ is a projective resolution of the A-bimodule A.*

EXCERCISE 3.19. *Let A be unital. The map $\phi : C(A) \to A \otimes_{A^e} Bar(A)$ given by*

$$\phi(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$$

is an isomorphism of chain complexes.

According to the definition of Tor in section 3 we now deduce the following result.

PROPOSITION 3.20. *Let A be a unital algebra. Then there is a natural isomorphism*

$$HH_n(A) \cong \mathrm{Tor}_n^{A^e}(A, A)$$

for all n.

This result is important since it allows to compute the Hochschild homology of a unital algebra using arbitrary projective resolutions of the A-bimodule A. Often the computation of Hochschild homology groups relies on finding particularly nice such resolutions. As a very simple example we will illustrate this in calculating the Hochschild homology of the complex numbers.

LEMMA 3.21. *The Hochschild homology of $\mathbb{C}$ is given by*

$$HH_0(\mathbb{C}) = \mathbb{C}$$

and $HH_n(\mathbb{C}) = 0$ for $n > 0$.

PROOF. Observe that we have an algebra isomorphism $\mathbb{C}^e \cong \mathbb{C}$. Hence the $\mathbb{C}^e$-module $\mathbb{C}$ is projective. It follows that there exists a projective resolution of length 0 for $\mathbb{C}$. As a consequence $HH_n(\mathbb{C}) = 0$ for $n > 0$. We have $HH_0(\mathbb{C}) = \mathbb{C}$ since $\mathbb{C}$ is commutative. □

We will see below a more interesting example of a computation based on a specific projective resolution. Before we proceed we remark that the definition in the non-unital case is made in such a way that the following result holds.

LEMMA 3.22. *Let A be an algebra. Then there is a split short exact sequence*

$$HH_n(A) \rightarrowtail HH_n(A^+) \twoheadrightarrow HH_n(\mathbb{C})$$

for all n.

PROOF. Consider the normalized bar complex of A^+ defined by

$$\mathrm{bar}_n(A) = A^+ \otimes A^{\otimes n} \otimes A^+$$

for $n \geq 0$. Then, as above, $\mathrm{bar}(A)$ is a projective resolution of A^+ by unitary A^+-bimodules. Since A^+ is unital, the Hochschild homology $HH(A^+)$ may be computed by $A^+ \otimes_{(A^+)^e} \mathrm{bar}(A)$. A straightforward calculation shows $A^+ \otimes_{(A^+)^e} \mathrm{bar}_0(A) = A^+$ and

$$A^+ \otimes_{(A^+)^e} \mathrm{bar}_n(A) \cong \Omega(A)$$

for $n > 0$. Moreover the differential is precisely the Hochschild boundary of $\Omega(A)$ under this identification. It follows that the natural projection $A^+ \to \mathbb{C}$ induces isomorphisms

$$HH_n(A) \cong HH_n(A^+)$$

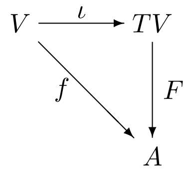
for $n > 0$ and $HH_0(A^+) = HH_0(A) \oplus \mathbb{C} = HH_0(A) \oplus HH_0(\mathbb{C})$. □

In the remaining part of this section we shall consider tensor algebras. Let V be a vector space. The *tensor algebra* TV over V is defined by

$$TV = \bigoplus_{j=1}^{\infty} V^{\otimes j}$$

with multiplication given by concatenation of tensors. Here one uses the canonical isomorphisms $V^{\otimes n} \otimes V^{\otimes m} \cong V^{\otimes n+m}$ for all $n, m \in \mathbb{N}$. There is an obvious map $\iota : V \to TV$ given by inclusion of tensors of length one. Remark that, according to our definition, the tensor algebra TV does not possess a unit element. The tensor algebra satisfies the following universal property.

EXCERCISE 3.23. *Let TV be the tensor algebra over a vector space V. For every algebra A and any linear map $f : V \to A$ there exists a unique homomorphism $F : TV \to A$ such that the diagram*



is commutative.

We shall now calculate the Hochschild homology of the unitarized tensor algebra $(TV)^+$. Define $P_0 = (TV)^+ \otimes (TV)^+$ and $P_1 = (TV)^+ \otimes V \otimes (TV)^+$ and consider the complex

$$(TV)^+ \xleftarrow{\ \mu\ } P_0 \xleftarrow{\ d\ } P_1 \longleftarrow 0$$

where μ denotes the multiplication map and d is defined by

$$d(x \otimes v \otimes y) = (x \otimes v) \otimes y - x \otimes (v \otimes y).$$

In addition we set $P{-1} = (TV)^+$. Evidently, the maps μ and d are $(TV)^+$-$(TV)^+$-bimodule homomorphisms. Let us show that this complex is exact. We define a map $s_{-1} : (TV)^+ \to P_0$ by $s_{-1}(x) = x \otimes 1$. Moreover we define $s_0 : P_0 \to P_1$ by $s_0(x \otimes 1) = 0$ and

$$\begin{aligned} s_0(x \otimes v_1 \otimes \cdots \otimes v_n) &= -(x \otimes v_1 \otimes \cdots \otimes v_{n-1}) \otimes v_n \otimes 1 \\ &- \sum_{j=2}^{n-1} (x \otimes v_1 \otimes \cdots \otimes v_{j-1}) \otimes v_j \otimes (v_{j+1} \cdots \otimes v_n) - x \otimes v_1 \otimes (v_2 \otimes \cdots \otimes v_n) \end{aligned}$$

for $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n} \subset TV$ for $n > 0$. Clearly one has $\mu s_{-1} = \mathrm{id}$.

EXCERCISE 3.24. *Verify the relations $s_{-1}\mu + ds_0 = \mathrm{id}$ and $s_0 d = \mathrm{id}$.*

It follows that P defines a projective resolution of length 1 of the bimodule $(TV)^+$. Algebras A allowing for resolutions of length ≤ 1 of A^+ by projective unitary A^+-A^+-bimodules are called *quasifree*. Hence TV is a quasifree algebra. According to the general theory, the complexes $\mathrm{Bar}((TV)^+)$ and P are homotopy equivalent. Let us explicitly write down a homotopy equivalence $f : \mathrm{Bar}((TV)^+) \to P$. We let $f_0 : \mathrm{Bar}_0((TV)^+) = (TV)^+ \otimes (TV)^+ \to P_0$ be the identity map. In degree one we define

$$f_1(x \otimes (v_1 \otimes \cdots \otimes v_n) \otimes y) = \sum_{j=1}^{n} (x \otimes v_1 \otimes \cdots \otimes v_{j-1}) \otimes v_j \otimes (v_{j+1} \otimes \cdots \otimes y).$$

Let us also define $g : P \to \mathrm{Bar}((TV)^+)$ by $g_0 = \mathrm{id}$ and $g_1 : (TV)^+ \otimes V \otimes (TV)^+ \to (TV)^+ \otimes (TV)^+ \otimes (TV)^+$ by

$$g_1(x \otimes v \otimes y) = x \otimes v \otimes y.$$

Clearly one has $fg = \mathrm{id}$.

EXCERCISE 3.25. *The maps f and g are chain maps.*

Since these maps cover the identity in degree -1 it follows already by the general theory that f and g are inverse homotopy equivalences.

Excercise 3.26. *Construct explicitly a homotopy between gf and the identity map on $Bar((TV)^+)$.*

According to proposition 3.20 we may compute the Hochschild homology of $(TV)^+$ using the resolution P. Tensoring this resolution over the extended algebra $((TV)^+)^e$ with $(TV)^+$ it follows that the Hochschild homology of $(TV)^+$ is the homology of the complex

$$0 \longleftarrow (TV)^+ \longleftarrow (TV)^+ \otimes V \longleftarrow 0$$

where the boundary maps an element $x \otimes v$ to $x \otimes v - v \otimes x$.
Let us denote by $\tau : TV \to TV$ the linear map given by $\tau(v_1 \otimes \cdots \otimes v_n) = v_n \otimes v_1 \otimes \cdots \otimes v_{n-1}$. We denote by $(TV)^\tau$ the space of elements fixed by τ and let $(TV)_\tau$ be the quotient of TV by all elements $x - \tau(x)$ with $x \in TV$. With these definitions we obtain immediately the following result.

Proposition 3.27. *The Hochschild homology of $(TV)^+$ is given by*

$$\begin{aligned} HH_0((TV)^+) &= \mathbb{C} \oplus (TV)_\tau \\ HH_1((TV)^+) &= (TV)^\tau \\ HH_n((TV)^+) &= 0 \qquad \textit{for} \quad n > 1. \end{aligned}$$

Under this identification the copy of $\mathbb{C}$ in degree zero corresponds to multiples of the unit element of $(TV)^+$. Using lemma 3.22 we obtain the following result.

Proposition 3.28. *The Hochschild homology of the tensor algebra TV is given by*

$$\begin{aligned} HH_0(TV) &= (TV)_\tau \\ HH_1(TV) &= (TV)^\tau \\ HH_n(TV) &= 0 \qquad \textit{for} \quad n > 1. \end{aligned}$$

3. Cyclic homology

In this section we define cyclic homology and study some of its basic properties. Let A be an algebra. According to proposition 3.10 we can form the bicomplex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^3(A) & \xleftarrow{B} & \Omega^2(A) & \xleftarrow{B} & \Omega^1(A) & \xleftarrow{B} & \Omega^0(A) \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
\Omega^2(A) & \xleftarrow{B} & \Omega^1(A) & \xleftarrow{B} & \Omega^0(A) & & \\
\downarrow b & & \downarrow b & & & & \\
\Omega^1(A) & \xleftarrow{B} & \Omega^0(A) & & & & \\
\downarrow b & & & & & & \\
\Omega^0(A) & & & & & &
\end{array}$$

which is by definition the (B,b)-bicomplex of A.

DEFINITION 3.29. *Let A be an algebra. The cyclic homology of A is the homology of the total complex of the (B,b)-bicomplex of A. We denote by $HC_n(A)$ the n-th cyclic homology group of A.*

Remark that we do not have to specify wether we use direct products or direct sums to define the total complex since the (B,b)-bicomplex is located in the first quadrant.
It is easy to describe the cyclic homology group $HC_0(A)$.

LEMMA 3.30. *Let A be an algebra. Then $HC_0(A) = HH_0(A)$ is equal to $A/[A,A]$.*

PROOF. This follows immediately by an inspection of the (B,b)-bicomplex. $\square$
Observe that the first column of the (B,b)-bicomplex is precisely the Hochschild complex of A. Moreover, the quotient of the (B,b)-bicomplex by the first column is naturally isomorphic to another copy of the (B,b)-bicomplex. Taking into account the corresponding degree shifts on the total complexes, proposition 2.10 immediately implies the following result.

PROPOSITION 3.31. *For every algebra A there is a natural long exact sequence*

$$\cdots \longrightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(Q) \xrightarrow{B} HH_{n-1}(A) \longrightarrow \cdots$$

This long exact sequence is called the SBI-sequence. The SBI-sequence is an important tool to compute cyclic homology groups.
It is often useful to have some information on the boundary map in the SBI-sequence. Actually, the boundary map $B : HC_n(A) \to HH_{n+1}(A)$ is closely related to the operator B on differential forms.

LEMMA 3.32. *Let A be an algebra. The map $B : HC_n(A) \to HH_{n+1}(A)$ is induced by the map $B : \Omega^n(A) \to \Omega^{n+1}(A)$.*

PROOF. Consider a cycle $z = (z_{n-2j})_{j\geq 0}$ of degree n in the total complex of the cyclic bicomplex where $z_k \in \Omega^k(A)$. We may lift z to a cycle of dimension $n+2$ by adding 0 in $\Omega^{n+2}(A)$. Then, by definition of the boundary map B we obtain the cycle $B(z_n) \in \Omega_{n+1}(A)$ representing a Hochschild homology class of degree $n+1$. This proves the claim. $\square$
It is clear that the definition of Hochschild homology and cyclic homology for algebras can be extended to arbitrary mixed complexes. We write $HH_n(M)$ and $HC_n(M)$ for the Hochschild and cyclic homology of a mixed complex M. As above, these theories are related by an SBI-sequence. One says that a map of mixed complexes $f : M \to N$ induces an isomorphism in Hochschild homology if $HH_n(f) : HH_n(M) \to HH_n(N)$ is an isomorphism for all n. The same terminology is used for cyclic homology.

LEMMA 3.33. *Let $f : M \to N$ be a map of mixed complexes. Then f induces an isomorphism in Hochschild homology iff it induces an isomorphism in cyclic homology.*

PROOF. This is a consequence of the five lemma 2.13. Remark that in the SBI-sequence there are much more entries containing cyclic homology groups then entries with Hochschild homology. However, consider the last part of the SBI-sequence

$$\begin{array}{ccccccccccc}
\longrightarrow HH_1(M) & \longrightarrow & HC_1(M) & \longrightarrow & 0 & \longrightarrow & HH_0(M) & \longrightarrow & HC_0(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\longrightarrow HH_1(N) & \longrightarrow & HC_1(N) & \longrightarrow & 0 & \longrightarrow & HH_0(N) & \longrightarrow & HC_0(N) & \longrightarrow & 0
\end{array}$$

for M and N. If f induces an isomorphism in Hochschild homology it follows that f also induces an isomorphism on HC_0 and HC_1. Inductively, we see that f induces an isomorphism on HC_n for all n. □
In particular we obtain the following result.

COROLLARY 3.34. *Let $f : A \to B$ be an algebra homomorphism. Then f induces an isomorphism in Hochschild homology iff it induces an isomorphism in cyclic homology.*

Let us now apply the SBI-sequence to determine the cyclic homology of the complex numbers.

PROPOSITION 3.35. *The cyclic homology of the complex numbers is given by*

$$HC_{2n}(\mathbb{C}) = \mathbb{C}, \qquad HC_{2n+1}(\mathbb{C}) = 0$$

for all n.

PROOF. Clearly $HC_0(\mathbb{C}) = HH_0(\mathbb{C}) = \mathbb{C}$ and exactness of the SBI-sequence shows $HC_1(\mathbb{C}) = 0$. Moreover $HH_n(\mathbb{C}) = 0$ for $n > 0$ implies that $S : HC_{n+2}(\mathbb{C}) \to HC_n(\mathbb{C})$ is an isomorphism for all n. This proves the claim. □
Note that the isomorphism $\mathbb{C} \cong HC_{2n+2}(\mathbb{C}) \to HC_{2n}(\mathbb{C}) \cong \mathbb{C}$ implemented by S is the identity map under the above identifications.
Let us also consider tensor algebras.

PROPOSITION 3.36. *Let V be a vector space. The cyclic homology of the tensor algebra TV is given by*

$$HC_0(TV) = (TV)_\tau, \qquad HC_n(TV) = 0$$

for all $n > 0$.

PROOF. Clearly $HC_0(TV) = HH_0(TV) = (TV)_\tau$ according to proposition 3.28. Let us show that the boundary map $B : HC_0(TV) \to HH_1(TV)$ is an isomorphism. Due to lemma 3.32 this map is induced by $d : TV \to \Omega^1(TV)$. Using the map ρ in proposition 3.16 the element $d(v_1 \otimes \cdots \otimes v_n)$ is mapped to $v_1 \otimes \cdots \otimes v_n \otimes 1 - 1 \otimes v_1 \otimes \cdots \otimes v_n$ in $C_1((TV)^+)$. This, in turn, is mapped to

$$-\sum_{j=1}^{n} (v_{j+1} \otimes \cdots \otimes v_n \otimes v_1 \otimes \cdots \otimes v_{j-1}) \otimes v_j$$

in $(TV)^+ \otimes V$ under chain map $C_1((TV)^+) \to (TV)^+ \otimes V$ induced by the map f definded in the previous section. Hence, on homology the map $B : (TV)_\tau \to (TV)^\tau$ is given by

$$B(v_1 \otimes \cdots \otimes v_n) = -\sum_{j=1}^{n} v_{j+1} \otimes \cdots \otimes v_n \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes v_j.$$

It is easy to see that this map is surjective. To check injectivity observe that

$$B(x) + nx = (\tau - \mathrm{id}) \sum_{j=0}^{n-1} (j+1)\tau^j(x)$$

for $x \in TV$ homogenous of degree n.
Now exactness of the SBI-sequence shows $HC_1(TV) = 0$ and $HC_2(TV) = 0$. Since $HH_n(TV) = 0$ for $n > 1$ it follows that $S : HC_{n+2}(TV) \to HC_n(TV)$ is an isomorphism for all $n > 0$. This proves the claim. □
We record the following statement concerning the behaviour of cyclic homology with respect to unitarizations.

Lemma 3.37. *Let A be an algebra. Then there is a natural split short exact sequence*

$$HC_n(A) \rightarrowtail HC_n(A^+) \twoheadrightarrow HC_n(\mathbb{C})$$

for all n. That is, there are natural isomorphisms $HC_n(A^+) \cong HC_n(A)$ if n is odd and $HC_n(A^+) \cong HC_n(A) \oplus \mathbb{C}$ if n is even.

Proof. According to lemma 3.22 the natural homomorphisms $A \to A^+$ and $\mathbb{C} \to A^+$ determine a map of mixed complexes $\Omega(A) \oplus \Omega(\mathbb{C}) \to \Omega(A^+)$ which induces an isomorphism in Hochschild homology. Hence this map of mixed complexes determines an isomorphism in cyclic homology as well according to lemma 3.33. $\square$
In particular we obtain according to proposition 3.38 the following description of the cyclic homology for unitarized tensor algebras.

Proposition 3.38. *Let V be a vector space. The cyclic homology of the unitarized tensor algebra $(TV)^+$ is given by*

$$HC_0((TV)^+) = \mathbb{C} \oplus (TV)_\tau, \qquad HC_{2n}((TV)^+) = \mathbb{C}, \qquad HC_{2n-1}((TV)^+) = 0$$

for all $n > 0$.

It is frequently useful to describe cyclic homology by other complexes. We shall be interested in particular in the cyclic bicomplex. The cyclic bicomplex $CC(A)$ of an algebra A is given by

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \longleftarrow & \cdots \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \longleftarrow & \cdots \\
b\downarrow & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\
A & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & A & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & A & \longleftarrow & \cdots
\end{array}$$

Here the operators b, b' and t already have been defined in section 2. The operator $N : A^{\otimes n+1} \to A^{\otimes n+1}$ is given by

$$N(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} a_{n+1-j} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{n-j}.$$

It is easy to check that $N(\mathrm{id} - t) = 0$ and $(\mathrm{id} - t)N = 0$.

Excercise 3.39. *For all n the relation $Nb = b'N$ holds on $A^{\otimes n+1}$.*

Excercise 3.39 shows together with excercise 3.14 that the cyclic bicomplex is indeed a first quadrant bicomplex. Moreover, we have already seen in excercise 3.14 that the total complex of the first two columns of this bicomplex is naturally isomorphic to the Hochschild complex $\Omega(A)$ of A. We may use this observation to identify the total complexes of the (B, b)-bicomplex and the cyclic bicomplex of A.

Excercise 3.40. *Under this identification of the total complex of $CC(A)$ with the total complex of the (B, b)-bicomplex of A, the operator N is corresponds to the boundary operator B.*

Hence we obtain the following statement.

PROPOSITION 3.41. *For every algebra A the cyclic homology $HC_*(A)$ is equal to the homology of the total complex associated to $CC(A)$.*

Finally remark that the homology of the rows in the cyclic bicomplex vanishes except in degree zero. More precisely, we have $\operatorname{im}(\operatorname{id}-t) = \ker(N)$ and $\ker(\operatorname{id}-t) = \operatorname{im}(N)$. This is seen using the maps $h_0 : A^{\otimes n+1} \to A^{\otimes n+1}$ and $h_1 : A^{\otimes n+1} \to A^{\otimes n+1}$ given by

$$h_0(x) = \frac{1}{n+1}\,x, \qquad h_1(x) = -\frac{1}{n+1}\sum_{j=0}^{n}(j+1)t^j(x)$$

for $n \geq 0$. We have $Nh_0(x) = x$ for $x \in \ker(\operatorname{id}-t)$ and $(\operatorname{id}-t)h_1(x) = x$ for $x \in \ker(N)$. In a slightly different form the latter relation already appeared in the proof of proposition 3.38.

4. Hochschild cohomology and cyclic cohomology

In algebraic topology one considers the (singular) homology of a topological space as well as its cohomology. Singular cohomology is obtained by dualizing the chain complex defining singular homology. In a similar way there are dual theories to Hochschild homology and cyclic homology. These theories will be discussed in this section.
If V is a vector space we denote by $V' = \operatorname{Hom}(V, \mathbb{C})$ its dual space. If $f : V \to W$ is a linear map then it induces a linear map $W' \to V'$ which will be denoted by f'. Applying the dual space functor to the Hochschild complex $\Omega(A)$ of an algebra A we obtain by definition the Hochschild cochain complex $\Omega(A)'$.

DEFINITION 3.42. *Let A be an algebra. The Hochschild cohomology of A is the cohomology of the Hochschild cochain complex $\Omega(A)'$. We denote by $HH^n(A)$ the n-th Hochschild cohomology group of A.*

It is easy to identify the cohomology group $HH^0(A)$.

LEMMA 3.43. *Let A be an algebra. Then $HH^0(A)$ is the linear space of traces on A.*

PROOF. The kernel of the Hochschild coboundary $b : A' \to \Omega^1(A)'$ is the space of all linear maps $\tau : A \to \mathbb{C}$ such that $b\tau(a_0da_1) = \tau(a_0a_1) - \tau(a_1a_0) = 0$. This means precisely that τ is a trace. □
In particular, if A is a commutative algebra we have $HH^0(A) = A'$.
Since $\mathbb{C}$ is a field the dual space functor is *exact*, that is, if

$$K \rightarrowtail^{i} E \twoheadrightarrow^{p} Q$$

is an exact sequence of vector spaces then the induced sequence

$$Q' \rightarrowtail^{p'} E' \twoheadrightarrow^{i'} K'$$

is again exact. This implies the following result.

PROPOSITION 3.44. *Let A be an algebra. The Hochschild cohomology group $HH^n(A)$ is canonically isomorphic to $HH_n(A)'$.*

PROOF. The assertion follows from the observation that the exact sequence

$$\operatorname{im}(b) \rightarrowtail \ker(b) \twoheadrightarrow HH_n(A)$$

induces an exact sequence

$$HH_n(A)' \rightarrowtail \ker(b)' \twoheadrightarrow \operatorname{im}(b)'$$

and the fact that $\ker(b) \cong \Omega^n(A)'/\operatorname{im}(b')$ and $\operatorname{im}(b)' = \Omega^n(A)'/\ker(b')$ where now $b' : \Omega(A)' \to \Omega(A)'$ denotes the transposed of the Hochschild boundary. It is easy to check that the isomorphism $\phi : HH^n(A) \to HH_n(A)'$ arising in this way is given explicitly by by $\phi(f)(z) = f(z)$. □

Hence one may easily obtain a description of the Hochschild cohomology of an algebra as soon as its Hochschild homology is known.

Let us now come to cyclic cohomology. In the same way as above we construct the dual of the (B, b)-bicomplex of an algebra A. Explicitly, we have

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Omega^3(A)' & \xrightarrow{B} & \Omega^2(A)' & \xrightarrow{B} & \Omega^1(A)' & \xrightarrow{B} & \Omega^0(A)' \\
b\uparrow & & b\uparrow & & b\uparrow & & \\
\Omega^2(A)' & \xrightarrow{B} & \Omega^1(A)' & \xrightarrow{B} & \Omega^0(A)' & & \\
b\uparrow & & b\uparrow & & & & \\
\Omega^1(A)' & \xrightarrow{B} & \Omega^0(A)' & & & & \\
b\uparrow & & & & & & \\
\Omega^0(A)' & & & & & &
\end{array}$$

and this is again a bicomplex.

Definition 3.45. *Let A be an algebra. The cyclic cohomology of A is the cohomology of the total complex of the dual (B, b)-bicomplex of A. We denote by $HC^n(A)$ the n-th cyclic cohomology group of A.*

As for homology we have the following result for the group $HC^0(A)$.

Lemma 3.46. *Let A be an algebra. Then $HC^0(A) = HH^0(A)$ is the space of traces on A.*

The SBI-sequence relates Hochschild cohomology and cyclic cohomology.

Proposition 3.47. *There is a long exact sequence*

$$\cdots \longleftarrow HH^n(A) \xleftarrow{I} HC^n(A) \xleftarrow{S} HC^{n-2}(Q) \xleftarrow{B} HH^{n-1}(A) \longleftarrow \cdots$$

for every algebra A.

Corollary 3.48. *Let $f : A \to B$ be an algebra homomorphism. Then f induces an isomorphism in Hochschild cohomology iff it induces an isomorphism in cyclic cohomology.*

We may also consider the dual of the cyclic bicomplex $CC(A)$ which looks as follows.

$$\begin{array}{ccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
(A^{\otimes 3})' & \xrightarrow{1-t} & (A^{\otimes 3})' & \xrightarrow{N} & (A^{\otimes 3})' & \xrightarrow{1-t} & (A^{\otimes 3})' & \xrightarrow{N} & (A^{\otimes 3})' & \longrightarrow & \cdots \\
b\uparrow & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \uparrow b & & \\
(A^{\otimes 2})' & \xrightarrow{1-t} & (A^{\otimes 2})' & \xrightarrow{N} & (A^{\otimes 2})' & \xrightarrow{1-t} & (A^{\otimes 2})' & \xrightarrow{N} & (A^{\otimes 2})' & \longrightarrow & \cdots \\
b\uparrow & & \uparrow -b' & & \uparrow b & & \uparrow -b' & & \uparrow b & & \\
A' & \xrightarrow[1-t]{} & A' & \xrightarrow[N]{} & A' & \xrightarrow[1-t]{} & A' & \xrightarrow[N]{} & A' & \longrightarrow & \cdots
\end{array}$$

This is a first quadrant bicomplex. Again, the cohomology of the rows of this bicomplex is located in degre zero.

5. Periodic cyclic homology and cohomology

Since the cyclic bicomplex is periodic it may be continued to the left using this periodicity. More precisely, we consider the infinite product total complex of this periodic bicomplex and obtain by definition the periodic cyclic complex $PC(A)$ given by

$$\prod_{j\in\mathbb{Z}} \Omega^{2j}(A) \underset{B+b}{\overset{B+b}{\rightleftarrows}} \prod_{j\in\mathbb{Z}} \Omega^{2j+1}(A)$$

which is a $\mathbb{Z}_2$-graded complex. Equivalently, we may also view $PC(A)$ as a periodic complex indexed by the integers.

Definition 3.49. *Let A be an algebra. The periodic cyclic homology of A is the homology of the periodic cyclic complex $PC(A)$ of A.*

It follows from the definitions that there are natural surjective chain maps $\pi_{2n} : PC(A) \to CC(A)$ which are compatible with the S-operator in the sense that $S\pi_{2n+2} = \pi_{2n}$ for all n. Actually one has

$$PC(A) \cong \varprojlim_{j\in\mathbb{N}} CC[2j](A)$$

where $CC[2j](A)$ is the bicomplex $CC(A)$ shifted by degree $2j$, that is

$$CC[2j](A)_{pq} = CC(A)_{(p+2j)q}$$

and the limit is taken using the S-operator. The projective system $(CC[2j](A))_{j\in\mathbb{N}}$ clearly has surjective structure maps. The following statement then follows from proposition 2.27.

Proposition 3.50. *Let A be an algebra. Then there is a short exact sequence*

$$\varprojlim\nolimits^1_{j\in\mathbb{Z}} HC_{*+2j+1}(A) \rightarrowtail HP_*(A) \twoheadrightarrow \varprojlim\nolimits_{j\in\mathbb{Z}} HC_{*+2j}(A)$$

Here projective limits over $\mathbb{Z}$ are taken in order to describe the (derived) projective limit of the homologies $H_n(CC[2j](A))$ in terms of cyclic homology.
As a consequence we may determine the periodic cyclic homology of the complex numbers.

LEMMA 3.51. *The periodic cyclic homology of the complex numbers is given by*

$$HP_0(\mathbb{C}) = \mathbb{C}, \qquad HP_1(\mathbb{C}) = 0.$$

PROOF. Due to proposition 3.35 we have $HC_{2n}(\mathbb{C}) = \mathbb{C}$ and $HC_{2n+1}(\mathbb{C}) = 0$ for all n. Moreover, under this identification the operator $S : HC_{2n+2}(\mathbb{C}) \to HC_{2n}(\mathbb{C})$ is the identity map. It follows that the structure maps in the inverse system $(HC_{*+2j}(\mathbb{C}))_{j\in\mathbb{Z}}$ are all surjective and hence

$$HP_*(\mathbb{C}) = \varprojlim_{j\in\mathbb{N}} HC_{*+2j}(\mathbb{C}).$$

This proves the claim. □

PROPOSITION 3.52. *Let V be a vector space. The periodic cyclic homology of the tensor algebra TV is given by*

$$HP_0(TV) = 0, \qquad HP_1(TV) = 0.$$

PROOF. According to proposition 3.38 the S-operator is zero on $HC_*(TV)$. This implies $\varprojlim HC_*(TV) = 0$ and $\varprojlim^1 HC_*(TV) = 0$ as well. Now the claim follows from proposition 3.50. □

Actually, the projective system given by the cyclic homology of a tensor algebra is an easy example of a projective system satisfying the Mittag-Leffler condition.

LEMMA 3.53. *Let A be an algebra. Then there are natural split short exact sequences*

$$HP_j(A) \rightarrowtail HP_j(A^+) \twoheadrightarrow HP_j(\mathbb{C})$$

for $j = 0$ and $j = 1$.

PROOF. This follows from proposition 3.50 and lemma 3.37. □

As a consequence, proposition 3.54 implies the following statement.

PROPOSITION 3.54. *Let V be a vector space. The periodic cyclic homology of the unitarized tensor algebra $(TV)^+$ is given by*

$$HP_0((TV)^+) = \mathbb{C}, \qquad HP_1((TV)^+) = 0.$$

As for Hochschild homology and cyclic homology, periodic cyclic homology may be defined for arbitrary mixed complexes. Moreover, the analogue of proposition 3.50 holds also in this more general situation. According to the five lemma and lemma 3.33, proposition 3.50 implies the following result.

PROPOSITION 3.55. *Let $f : M \to N$ be a map of mixed complexes which induces an isomorphism in Hochschild homology. Then the induced map $HP_*(M) \to HP_*(N)$ is an isomorphism as well.*

We remark that the converse of proposition 3.55 is not true. A map of mixed complexes which induces an isomorphism in periodic cyclic homology is not necessarily a quasiisomorphism on the level of Hochschild homology or cyclic homology. According to proposition 3.54 an easy example is the map $\Omega(0) \to \Omega(TV)$ induced by the homomorphism $0 \to TV$ for some nonzero vector space V.

For cohomology we have to take the dual $PC(A)'$ of the periodic cyclic complex $PC(A)$ using direct sums. More precisely, $PC(A)'$ is defined by

$$\bigoplus_{j\in\mathbb{Z}} \Omega^{2j}(A)' \underset{B+b}{\overset{B+b}{\rightleftarrows}} \bigoplus_{j\in\mathbb{Z}} \Omega^{2j+1}(A)'$$

which is again a $\mathbb{Z}_2$-graded complex.

DEFINITION 3.56. *Let A be an algebra. The periodic cyclic cohomology of A is the homology of $PC(A)'$.*

LEMMA 3.57. *Let A be an algebra. Then there is a natural isomorphism*

$$\varinjlim_{j\in\mathbb{N}} HC^{*+2j}(A) \cong HP^*(A).$$

We conclude this section by defining the pairing between periodic cyclic cohomology and periodic cyclic homology. Let A be an algebra and consider the direct product complex $PC(A)' \times PC(A)$. It follows immediately from the definition of the differential in $PC(A)'$ that the obvious map

$$\langle -,- \rangle : PC(A)' \times PC(A) \to \mathbb{C}[0], \qquad \langle \phi, c\rangle = \phi(c)$$

is a chain map. Here $\mathbb{C}[0]$ denotes the trivial supercomplex with $\mathbb{C}$ in degree zero and 0 and degree one. Hence we obtain an induced map

$$HP^*(A) \times HP_*(A) \to \mathbb{C}[0]$$

in homology. This is the pairing between periodic cyclic homology and cohomology.

6. Morita invariance

In this section we shall show that Morita equivalent algebras have isomorphic Hochschild homology and cyclic homology.

PROPOSITION 3.58. *Let A and B be Morita equivalent unital algebras. Then there is a natural isomorphism $HH_*(A) \cong HH_*(B)$.*

PROOF. Let ${}_AP_B$ and ${}_BQ_A$ be equivalence bimodules. Since $\langle -,-\rangle_A : P \otimes_B Q \to A$ is an isomorphism there exist elements $p_i \in P$ and $q_i \in Q$ for $i = 1, \dots, m$ and some $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{m} \langle p_i, q_i\rangle_A = 1$$

We construct a chain map $\phi : C(A) \to C(B)$ as follows. On chains of degree k we define

$$\phi(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{i_0,\dots,i_k=1}^{m} \langle q_{i_0}, a_0 p_{i_1}\rangle_B \otimes \langle q_{i_1}, a_1 p_{i_2}\rangle_B \otimes \cdots \otimes \langle q_{i_k}, a_k p_{i_0}\rangle_B$$

and using the equation

$$\langle q_i, a_i p_j\rangle_B \langle q_j, a_j p_k\rangle_B = \langle q_i, a_i p_j \langle q_j, a_j p_k\rangle_B\rangle_B = \langle q_i, a_i \langle p_j, q_j\rangle_A a_j p_k\rangle_B$$

it is easy to check that ϕ is a chain map. In the same way we obtain elements $x_j \in Q$ and y_j in P for $j = 1, \dots, n$ such that

$$\sum_{i=1}^{m} \langle x_i, y_i\rangle_B = 1$$

and a chain map $\psi : C(B) \to C(A)$ by

$$\psi(b_0 \otimes b_1 \otimes \cdots \otimes b_n) = \sum_{j_0,\dots,j_k=1}^{n} \langle x_{j_0}, b_0 y_{j_1}\rangle_A \otimes \langle x_{j_1}, b_1 x_{j_2}\rangle_A \otimes \cdots \otimes \langle x_{j_k}, b_k y_{j_0}\rangle_A.$$

The composition $\psi\phi : C(A) \to C(A)$ is given by

$$\psi\phi(a_0 \otimes a_1 \otimes \cdots \otimes a_k) =$$

$$\sum_{j_0,\dots,j_k=1}^{n} \sum_{i_0,\dots,i_k=1}^{m} \langle x_{j_0}, \langle q_{i_0}, a_0 p_{i_1}\rangle_B y_{j_1}\rangle_A \otimes \langle x_{j_1}, \langle q_{i_1}, a_1 p_{i_2}\rangle_B y_{j_2}\rangle_A \otimes \cdots$$

$$\cdots \otimes \langle x_{j_k}, \langle q_{i_k}, a_k p_{i_0}\rangle_B y_{j_0}\rangle_A.$$

We shall construct a presimplicial homotopy h on $C(A)$ between id and $\psi\phi$ as follows. In degree k we define

$$h_r(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{j_0,\dots,j_r=1}^{n} \sum_{i_0,\dots,i_r=1}^{m} a_0\langle p_{i_0}, x_{j_0}\rangle_A \otimes \langle y_{j_0}, q_{i_0}\rangle_A a_1 \langle p_{i_1}, x_{j_1}\rangle_A \otimes \cdots$$
$$\cdots \otimes \langle y_{j_{r-1}}, q_{i_{r-1}}\rangle_A a_r \langle p_r, x_r\rangle_A \otimes \langle y_{j_r}, q_{i_r}\rangle_A \otimes a_{r+1} \otimes \cdots \otimes a_k$$

for $r = 0, \dots, k$. Let us check that this is indeed a presimplicial homotopy in degree $k = 1$. We have to prove

$$d_0h_1 = h_0d_0, \quad d_1h_1 = d_1h_0, \quad d_0h_0 = \mathrm{id}, \qquad d_2h_1 = \psi\phi.$$

For the first equation we calculate

$$d_0h_1(a_0 \otimes a_1) = \sum_{j_0,j_1=1}^{n} \sum_{i_0,i_1=1}^{m} a_0\langle p_{i_0}, x_{j_0}\rangle_A \langle y_{j_0}, q_{i_0}\rangle_A a_1 \langle p_{i_1}, x_{j_1}\rangle_A \otimes \langle y_{j_1}, q_{i_1}\rangle_A$$
$$= \sum_{j_1=1}^{n} \sum_{i_1=1}^{m} a_0 a_1 \langle p_{i_1}, x_{j_1}\rangle_A \otimes \langle y_{j_1}, q_{i_1}\rangle_A = h_0d_0(a_0 \otimes a_1)$$

The second equation follows in the same way. For the third equation we have

$$d_0h_0(a_0 \otimes a_1) = \sum_{j_0=1}^{n} \sum_{i_0=1}^{m} a_0\langle p_{i_0}, x_{j_0}\rangle_A \langle y_{j_0}, q_{i_0}\rangle_A \otimes a_1 = a_0 \otimes a_1$$

and the last equation is verified by calculating

$$d_2h_1(a_0 \otimes a_1) = \sum_{j_0,j_1=1}^{n} \sum_{i_0,i_1=1}^{m} \langle y_{j_1}, q_{i_1}\rangle_A a_0 \langle p_{i_0}, x_{j_0}\rangle_A \otimes \langle y_{j_0}, q_{i_0}\rangle_A a_1 \langle p_{i_1}, x_{j_1}\rangle_A$$
$$= \psi\phi(a_0 \otimes a_1).$$

The fact that h satisfies the relations for a presimplicial homotopy in other degrees is proved in a similar way. We leave the verification to the reader.
As a consequence we deduce that the complexes $C(A)$ and $C(B)$ are homotopy equivalent. This proves the claim. □

COROLLARY 3.59. *Let A and be B be Morita equivalent unital algebras. Then there are natural isomorphisms $HC_*(A) \cong HC_*(B)$ and $HP_*(A) \cong HP_*(B)$.*

PROOF. It is evident that the chain map $\phi : C(A) \to C(B)$ constructed in proposition 3.58 is a map of cyclic modules. Hence it induces a map $HC_*(A) \to HC_*(B)$ which an isomorphism according to lemma 3.33. The assertion for the periodic theory follows from proposition 3.55. □

It is instructive to consider explicitly the case of matrix algebras. Let A be a unital algebra and let $\iota : A \to M_n(A)$ be the algebra homomorphism given by $\iota(a) = ae_{11}$. Here ae_{ij} for $1 \leq i, j \leq n$ denotes the matrix with the only nonzero entry a in degree (i, j). Remark that the homomorphism ι does not preserve the units. Still ι induces a chain map $C(A) \to C(M_n(A))$ which will again be denoted by ι.
Conversely, define the trace map $\tau : C(M_n(A)) \to C(A)$ by

$$\tau(A^0 \otimes A^1 \otimes \cdots \otimes A^k) = \sum_{i_0,\dots,i_k=1}^{n} A^0_{i_0i_1} \otimes A^1_{i_1i_2} \otimes \cdots \otimes A^k_{i_ni_0}$$

in degree k where A_{ij} denotes the (i, j)-th entry of the matrix A. It can be easily checked that τ is a chain map. This also follows from the following excercise.

EXCERCISE 3.60. *Let $M_n(A)$ be the algebra of $n \times n$-matrices over a unital algebra A and let $P = A^n$ and $Q = A^n$ be the natural Morita context relating A and $M_n(A)$. Then the maps ϕ and ψ constructed above are equal to ι and τ.*

7. The Chern character in K-theory

In this section we define the K-group K_0 of an algebra A and a natural homomorphism $\mathrm{ch}_0 : K_0(A) \to HP_0(A)$.
Let A be a unital algebra. In order to be definінite, we shall work with unitary left modules in the sequel. However, we will see that one could equally well work with right modules. Recall that a unitary A-module P is finitely generated and projective iff it is a direct summand in A^n for some n. In particular, if P and Q are unitary finitely generated projective modules then the direct sum $M \oplus N$ is again finitely generated and projective. It follows that isomorphism classes of unitary finitely generated projective modules form an abelian semigroup $P(A)$ with direct sum as addition and neutral element the zero module. Remark that, in contrast to isomorphism classes of arbitrary modules, isomorphism classes of finitely generated projective unitary modules form a set since every finitely generated projective unitary module is isomorphic to a submodule of A^n for some n.

DEFINITION 3.61. *Let H be an abelian semigroup with neutral element. An abelian group $G(H)$ together with an semigroup homomorphism $\iota : H \to G(H)$ is called a Grothendieck group of H if for every abelian group M and every semigroup homomorphism $f : H \to M$ there is a unique group homomorphism $F : G(H) \to M$ such that $F\iota = f$.*

As usual, it is easy to see that a Grothendieck group $G(H)$ of H is uniquely determined up to isomorphism.

LEMMA 3.62. *For every abelian semigroup with neutral element there exists a Grothendieck group.*

PROOF. Consider the free abelian group $F(H)$ generated by H and let $\iota : H \to F(H)$ be the natural map. We let $G(H)$ be the quotient of $F(H)$ by the relations $\iota(x + y) = \iota(x) + \iota(y)$ for all $x, y \in H$. Then the induced map $\iota : H \to G(H)$ is a semigroup homomorphism and it is easy to verify the universal property.
However, it is often important to work with a more concrete realization of the Grothendieck group. More precisely, an alternative definition is $G(H) = H \times H/\sim$ where $(a_1, b_1) \sim (a_2, b_2)$ iff there exists $c \in H$ such that $a_1 + b_2 + c = b_1 + a_2 + c$. It is straightforward to check that componentwise addition defines turns $G(H)$ into an abelian group. The neutral element is $(0, 0)$ and the inverse of an element (a, b) is given by (b, a). The natural map $\iota : H \to G(H)$ is defined by $\iota(a) = (a, 0)$. We leave it as an excercise to verify the universal property. One should think of elements (a, b) as formal differences $a - b$. □
As an example consider the semigroup $\mathbb{N}_0$ of nonnegative integers with addition.

EXCERCISE 3.63. *The Grothendieck group $G(\mathbb{N}_0)$ is isomorphic to $\mathbb{Z}$ and $\iota : \mathbb{N}_0 \to \mathbb{Z}$ is the obvious inclusion in this case.*

We now define the K-group of a unital algebra.

DEFINITION 3.64. *Let A be a unital algebra. The K-group $K_0(A)$ of A is the Grothendieck group of the semigroup $P(A)$.*

First we shall discuss the functoriality of this construction.

EXCERCISE 3.65. *Let A and B be unital algebras and let ${}_AM$ be an A-module. If M is finitely generated, then $B \otimes_A M$ is finitely generated as well. If M is projective, then $B \otimes_A M$ is projective as well.*

Every unital algebra homomorphism $f : A \to B$ induces a semigroup homomorphism $P(A) \to P(B)$ by sending a finitely generated projective (left) A-module P to the B-module $B \otimes_A P$. By the universal property of the Grothendieck group, this induces a group homomorphism $K_0(f) : K_0(A) \to K_0(B)$.

EXCERCISE 3.66. *Let A and B be unital algebras. Then the natural projections $\pi_A : A \oplus B \to A$ and $\pi_B : A \oplus B \to B$ induce an isomorphism $K_0(A \oplus B) \to K_0(A) \oplus K_0(B)$.*

In order to extend the definition of K_0 to arbitrary algebras we have to proceed as follows.

LEMMA 3.67. *Let A be a unital algebra. Then $K_0(A)$ is naturally isomorphic to the kernel of the augmentation homomorphism $K_0(A^+) \to K_0(\mathbb{C})$.*

PROOF. Since A is unital we have an isomorphism $A^+ \cong A \oplus \mathbb{C}$ of unital algebras. Now the assertion follows easily from excercise 3.66. □

DEFINITION 3.68. *Let A be an algebra. The group $K_0(A)$ is the kernel of the natural map $K_0(A^+) \to K_0(\mathbb{C})$ induced by the augmentation homomorphism.*

According to lemma 3.67 this definition is compatible with the previous one for unital algebras.
Consider for instance the case $A = \mathbb{C}$. Since every finitely generated projective module over $\mathbb{C}$ is isomorphic to $\mathbb{C}^n$ for some n we obtain $P(\mathbb{C}) = \mathbb{N}_0$. Hence we get $K_0(\mathbb{C}) = \mathbb{Z}$.
We will now define an additive map $\mathrm{ch}_0 : K_0(A) \to HP_0(A)$ for an augmented algebra A. For an idempotent $e \in M_n(A)$ set

$$\mathrm{ch}_0(e) = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{k!} \operatorname{tr}\left(\left(e - \frac{1}{2}\right)(dede)^k\right)$$

viewed as an element in the periodic cyclic complex $PC(A)$ where $\operatorname{tr} : PC(M_n(A)) \to PC(A)$ is the trace map defined by

$$\operatorname{tr}(M^0 dM^1 \cdots dM^k) = \sum_{1 \le i_0, \ldots, i_k \le n} M^0_{i_0 i_1} dM^1_{i_1 i_2} \cdots dM^k_{i_k i_0}$$
$$\operatorname{tr}(dM^1 \cdots dM^k) = \sum_{1 \le i_1, \ldots, i_k \le n} dM^1_{i_1 i_2} \cdots dM^k_{i_k i_1}$$

for differential forms of degree k. Here M_{ij} denotes the (i,j)th entry of a matrix $M \in M_n(A)$. Note that tr is actually just the trace map occuring in the proof of Morita invariance for matrix algebras over unital algebras. Remark also that the map $\operatorname{tr} : PC(M_n(A)) \to PC(A)$ is a chain map for arbitrary algebras A.

LEMMA 3.69. *The element $\mathrm{ch}_0(e)$ is a cycle and defines a class in $HP_0(A)$.*

PROOF. We compute

$$B\left((-1)^k \frac{(2k)!}{k!}\left(e - \frac{1}{2}\right)(dede)^k\right) = (-1)^k (2k+1) \frac{(2k)!}{k!} de(dede)^k$$

and

$$b\left((-1)^{k+1}\frac{(2k+2)!}{(k+1)!}\left(e-\frac{1}{2}\right)(dede)^k\right) = (-1)^{k+1}\frac{(2k+2)!}{(k+1)!}\left(ede(dede)^k - \right.$$
$$\left. -\frac{1}{2}ede(dede)^k + \frac{1}{2}de(dede)^k - \frac{1}{2}ede(dede)^k\right)$$
$$= (-1)^{k+1}\frac{(2k+1)!2(k+1)}{(k+1)!}\,\frac{1}{2}de(dede)^k$$
$$= (-1)^{k+1}\frac{(2k+1)!}{k!}\,de(dede)^k.$$

Hence these terms cancel and since tr is a chain map we deduce $(B+b)\,\mathrm{ch}_0(e) = 0$. It follows that $\mathrm{ch}_0(e)$ is a cycle and hence defines an element in $HP_0(A)$. □

Let $e \in M_n(A)$ and $f \in M_m(A)$ be idempotents and consider their direct sum $e \oplus f \in M_{n+m}(A)$. By the definition of the trace map we see

$$\mathrm{ch}_0(e \oplus f) = \mathrm{ch}(e) + \mathrm{ch}(f).$$

It follows that ch_0 defines an additive map from $P(A)$ to $HP_0(A)$. By the universal property of the Grothendieck group we therefore obtain the following result.

PROPOSITION 3.70. *Let A be a unital algebra. The Chern character* ch_0 *defines a natural transformation* $K_0(A) \to HP_0(A)$.

It remains to extend the Chern character to arbitrary algebras. This is done using the commutative diagram

$$\begin{array}{ccccc} K_0(A) & \rightarrowtail & K_0(A^+) & \twoheadrightarrow & K_0(\mathbb{C}) \\ \downarrow & & \downarrow \mathrm{ch}_0 & & \downarrow \mathrm{ch}_0 \\ HP_0(A) & \rightarrowtail & HP_0(A^+) & \twoheadrightarrow & HP_0(\mathbb{C}) \end{array}$$

which is obtained using lemma 3.53.

CHAPTER 4

The Hochschild-Kostant-Rosenberg theorem

In this chapter we calculate the Hochschild and cyclic homology of the Fréchet algebra $C^\infty(M)$ of smooth functions on a smooth manifold M. First we discuss some background material from functional analysis in section 1. More precisely, we explain the concept of a locally convex vector space and the projective tensor product. In section 2 we discuss how to adapt the tools of homological algebra to locally convex spaces. Section 3 contains a review of standard constructions with differential form on smooth manifolds including the exterior derivative, Lie derivatives and interior products. In section 4 we formulate the Hochschild-Kostant-Rosenberg theorem which computes the Hochschild homology of the Fréchet algebra $C^\infty(M)$. We prove this theorem first in the special case of an open convex neighborhood of zero in $\mathbb{R}^n$. The proof for arbitrary manifolds is carried out in section 5 using an appropriate localization procedure. Section 6 contains the computation of cyclic and periodic cyclic homology for $C^\infty(M)$. Finally, in section 7 we recall the classical Chern-Weil construction of characteristic classes using connections on vector bundles. If M is compact, the Chern character from K-theory to periodic cyclic homology for the algebra $C^\infty(M)$ is identified with the classical Chern character with values in de Rham cohomology.

1. Locally convex vector spaces and tensor products

Let M be a smooth manifold. For the purposes of cyclic homology it is not appropriate to consider $C^\infty(M)$ as a complex algebra without further structure. Actually, the purely algebraic Hochschild and cyclic homology groups of $C^\infty(M)$ as defined in chapter 3 are not known in general. The main problem is that, apart from trivial cases, the algebraic tensor product $C^\infty(M)\otimes C^\infty(N)$ is not isomorphic to $C^\infty(M\times N)$ for smooth manifolds M,N.
It is more natural to consider $C^\infty(M)$ as a locally convex algebra. Accordingly, the algebraic tensor product is replaced by the completed projective tensor product. The completed projective tensor product has the property that $C^\infty(M)\hat{\otimes}_\pi C^\infty(N)$ is naturally isomorphic to $C^\infty(M\times N)$.
In this section we explain some of the concepts and results from functional analysis involved here. For more details we refer to [**10**], [**15**], [**7**].
The complex numbers are always equipped with the natural topology coming from the metric $d(\lambda,\mu)=|\lambda-\mu|$.

DEFINITION 4.1. *A topological vector space is a vector space V which is equipped with a Hausdorff topology such that the addition $V\times V\to V,(v,w)\mapsto v+w$ and the scalar multiplication $\mathbb{C}\times V\to V(\alpha,v)\mapsto\alpha v$ are continuous.*

In a topological vector space the translation maps $T_v: V\to V$ given by $T_v(w)=v+w$ are homeomorphisms for every $v\in V$. As a consequence, to describe the topology of a topological vector space it suffices to specify a basis of neighborhoods of the origin.
Let V be a vector space. A seminorm on V is a map $p: V\to\mathbb{R}^+$ such that

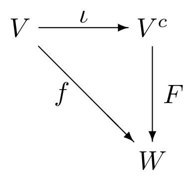
$$p(\lambda v)=|\lambda|p(v),\qquad p(v+w)\le p(v)+p(w)$$

for all $v, w \in V$ and $\lambda \in \mathbb{C}$. Note that $p(0) = 0$ for every seminorm. A seminorm is a norm if $p(v) = 0$ implies $v = 0$.
Assume that p is a seminorm on V. By definition, the (open) ball $B_p(v; r)$ with radius r around $v \in V$ consists of all vectors $w \in V$ such that $p(w - v) < r$. If the seminorm is clear from the context we simply write $B(v, r)$.
We are interested in the following class of topological vector spaces.

Definition 4.2. *A locally convex vector space is a topological vector space V with topology given by a family $(p_i)_{i \in I}$ of seminorms $p_i : V \to \mathbb{C}$. That is, a basis of neighborhoods around the origin is given by the balls $B_{p_i}(0, r)$ with $r > 0$ and $i \in I$.*

Accordingly, the basis of neighborhoods around an arbitrary point v in a locally convex vector space V is given by the balls $B_{p_i}(v, r)$ with $r > 0$ and $i \in I$. Since V is assumed to be Haussdorff there exists for every nonzero vector $v \in V$ a seminorm p_i such that $p_i(v) > 0$.
A subset K of a vector space V is called convex if $\lambda v + (1-\lambda)w \in K$ for all $v, w \in K$ and $0 < \lambda < 1$. We remark that locally convex vector spaces can be characterized as those topological vector spaces V in which every point $v \in V$ has a neighborhood base of convex sets.
Examples of locally convex vector spaces are normed spaces or Banach spaces. These spaces are special in the sense that the topology is determined by a single norm.
As for normed spaces, the concept of completeness plays an important role for locally convex spaces. A net $(v_j)_{j \in J}$ in a locally convex space V is called a Cauchy net if for every defining seminorm p_i and every $\epsilon > 0$ there exists $k \in J$ such that $p(v_m - v_n) \leq \epsilon$ for all $m, n \geq k$. A net $(v_j)_{j \in J}$ in V is convergent to $v \in V$ iff for every $\epsilon > 0$ and every defining seminorm p there exists $k \in J$ such that $p(v - v_n) < \epsilon$ for all $n \geq k$. Note that the limit v is uniquely determined since V is Hausdorff. Clearly every convergent net is a Cauchy net. A locally convex vector space V is called complete if every Cauchy net in V is convergent.

Definition 4.3. *Let V be a locally convex vector space. A completion of V is a complete locally convex vector space V^c together with a continuous linear map $\iota : V \to V^c$ such that for every complete locally convex vector space W and every continuous linear map $f : V \to W$ there exists a unique continuous linear map $F : V^c \to W$ such that the diagram*



is commutative.

Being defined by a universal property, the completion is uniquely determined up to isomorphism.

Theorem 4.4. *For every locally convex vector space there exists a completion.*

An important class of locally convex vector spaces is the class of Fréchet spaces.

Definition 4.5. *A Fréchet space is a complete locally convex vector space V such that the topology can be defined by a countable family of seminorms.*

We remark that a locally convex vector space V is metrizable iff its topology is defined by a countable family of seminorms.

Many general constructions with vector spaces extend easily to the setting of (complete) locally convex spaces. For instance, direct products, direct sums, projective and inductive limits are defined by the analogous universal properties.
A more subtle point is the notion of a tensor product. Similarly to the algebraic setting, tensor products of locally convex vector spaces are determined by considering bilinear maps with certain continuity properties. The most evident continuity property for a bilinear map $b : V \times W \to X$ is to require b to be continuous for the product topology on $V \times W$. We say that b is jointly continuous in this case.

Definition 4.6. *Let p be a seminorm on V and let q be a seminorm on W. The tensor product $p \otimes q : V \otimes W \to \mathbb{R}^+$ is defined by*

$$(p \otimes q)(z) = \inf\Big(\sum_{j=1}^{n} p(v_j)q(w_j) \mid z = \sum_{j=1}^{n} v_j \otimes w_j\Big).$$

The following result summarizes basic properties of the tensor product of two seminorms.

Proposition 4.7. *Let p and q be seminorms on V and W, repectively. Then $p \otimes q$ is a seminorm on $V \otimes W$. Moreover*

$$(p \otimes q)(v \otimes w) = p(v)q(w)$$

for all simple tensors $v \otimes w \in V \otimes W$.

Proof. It is easy to check that $p \otimes q$ is a seminorm. From the definition of $p \otimes q$ it is immediate that

$$(p \otimes q)(v \otimes w) \leq p(v)q(w)$$

for $v \in V$ and $w \in W$. For the other inequality choose $v' \in V'$ such that $v'(v) = p(v)$ and $|v'(x)| \leq p(x)$ for all $x \in V$. Here V' denotes the space of linear maps from V to $\mathbb{C}$ which are bounded for the seminorm p. The existence of v' follows from the classical Hahn-Banach theorem for the normed space $V'/\ker(p)$. In the same way one obtains $w' \in W'$ such that $w'(w) = q(w)$ and $|w'(y)| \leq q(y)$ for all $y \in W$. Consider the linear form $v' \otimes w'$ on $V \otimes W$ and let $v \otimes w$ be represented as a linear combination $\sum x_i \otimes y_i$ with $x_i \in V$ and $y_i \in W$. Then we have

$$|v' \otimes w'(v \otimes w)| \leq \sum |v'(x_i)w'(y_i)| \leq \sum p(x_i)q(y_i)$$

for every such representation. By the definition of $p \otimes q$ we thus obtain

$$p(v)q(w) = v'(v)w'(w) = |v' \otimes w'(v \otimes w)| \leq (p \otimes q)(v \otimes w)$$

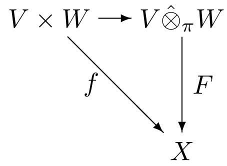
which yields the claim. □
If V and W are locally convex vector spaces with defining seminorms $(p_i)_{i \in I}$ and $(q_j)_{j \in J}$ the projective topology on $V \otimes W$ is the locally convex topology defined by the seminorms $p_i \otimes q_j$ for all $i \in I$ and $j \in J$. We write $V \otimes_\pi W$ for the algebraic tensor product equipped with the projective topology. It can be shown that the projective topology is again Hausdorff. Moreover it follows from proposition 4.7 that the canonical bilinear map $V \times W \to V \otimes_\pi W$ is jointly continuous.

Definition 4.8. *Let V and W be locally convex vector spaces. The completed projective tensor product $V \hat{\otimes}_\pi W$ is the completion of $V \otimes_\pi W$.*

The completed projective tensor product is determined by the following universal property.

Proposition 4.9. *Let V and W be locally convex vector spaces. For every complete locally convex vector space X and every jointly continuous bilinear map*

$f : V \times W \to X$ there exists a unique continuous linear map $F : V\hat{\otimes}_\pi W \to X$ such that the diagram



is commutative.

Proof. Let $f : V \times W \to X$ be a continuous bilinear map and let $F : V \otimes W \to X$ be the associated linear map. For every defining seminorm q on X there exist defining seminorms p_V and p_W on V and W, respectively, such that $q(f(v, w)) \leq p_V(v)p_W(w)$ for all $v \in V$ and $w \in W$. Consequently, for $x = \sum v_i \otimes w_i \in V \otimes W$ we have

$$q(F(x)) = q\Big(\sum F(v_i \otimes w_i)\Big) \leq \sum p_V(v_i)p_W(w_i)$$

and hence $q(F(x)) \leq (p_V \otimes p_W)(x)$. This shows that F is continuous. By the universal property of the algebraic tensor product the map F is uniquely determined by f. The proof is finished using the universal property of the completion. □

In the sequel we write $V\hat{\otimes}W$ instead of $V\hat{\otimes}_\pi W$ for the completed projective tensor product of two locally convex vector spaces. Moreover, we will assume for simplicity that all locally convex spaces are complete.

Let us carry over some definitions of chapter 1 to the setting of locally convex vector spaces.

Definition 4.10. *A locally convex algebra is a locally convex vector space A together with a continuous bilinear map $\mu : A \times A \to A$ such that*

$$a(bc) = (ab)c$$

for all $a, b, c \in A$. A unital locally convex algebra is a locally convex algebra with an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

An algebra homomorphism $f : A \to B$ between locally convex algebras is a continuous linear map such that $f(ab) = f(a)f(b)$ for all $a, b \in A$. A unital homomorphism $f : A \to B$ between unital locally convex algebras is a homomorphism such that $f(1) = 1$.

Note that the multiplication in a (complete) locally convex algebra A can equivalently be described by a continuous linear map $\mu : A\hat{\otimes}A \to A$.

It is clear that every locally convex algebra is in particular an algebra in the sense of chapter 1. The standard constructions with algebras described in chapter 1 extend easily to locally convex algebras.

Definition 4.11. *Let A be a locally convex algebra. A locally convex (left) module over A is a locally convex vector space M together with a continuous bilinear map $A \times M \to M$ such that*

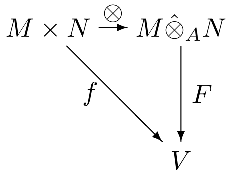
$$(ab)m = a(bm)$$

for all $a, b \in A$ and $m \in M$. A unitary locally convex (left) module over a unital locally convex algebra A is an A-module M such that $1m = m$ for every $m \in M$. An A-module homomorphism $f : M \to N$ between locally convex (unitary) A-modules is a continuous linear map which satisfies $f(am) = af(m)$ for all $a \in A$ and $m \in M$. In a similar way one defines locally convex (unitary) right modules, (unitary) bimodules and their homomorphisms.

We will frequently speak of modules instead of locally convex modules for simplicity.

Let us discuss the projective tensor product of locally convex modules. Assume M_A and ${}_AN$ are locally convex modules over the locally convex algebra A and let V be a locally convex vector space. A jointly continuous bilinear map $f : M \times N \to V$ is called A-bilinear if $f(ma, n) = f(m, an)$ for all $m \in M, n \in N, a \in A$.

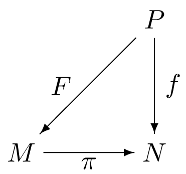
Definition 4.12. *Let M_A and ${}_AN$ be locally convex A-modules. A complete locally vector space $M\hat{\otimes}_A N$ together with a jointly continuous A-bilinear map $\otimes$: $M \times N \ni (m, n) \mapsto n \otimes n \in M\hat{\otimes}_A N$ is called tensor product of M and N over A if for every complete locally convex vector space V and every jointly continuous A-bilinear map $f : M \times N \to V$ there exists a unique continuous linear map $F : M\hat{\otimes}_A N \to V$ such that the diagram*



is commutative.

As in the algebraic case, the tensor product $M\hat{\otimes}_A N$ is uniquely determined up to isomorphism by M_A and ${}_AN$. It is constructed as the quotient of the projective tensor product $M\hat{\otimes}N$ by the closed linear subspace generated by all tensors of the form $ma \otimes n - m \otimes an$.
Let A be a locally convex algebra. A surjective continuous A-module homomorphism $\pi : M \to N$ is called strict if there exists a continuous linear map $\sigma : N \to M$ such that $\pi\sigma = \mathrm{id}$.

Definition 4.13. *Let A be a locally convex algebra. A locally convex module ${}_AP$ is called projective if for every strict epimorphism $\pi : M \to N$ of A-modules and every A-module homomorphism $f : P \to N$ there exists an A-module homomorphism $F : P \to M$ such that the diagram*



is commutative.

As in the algebraic case one has the following result.

Excercise 4.14. *For every locally convex algebra A the A-module A^+ is projective. Direct sums of projective modules are projective.*

An A-submodule M of an A-module P is called a *direct summand* if there exists an A-submodule N in P such that the natural map $M \oplus N \to P$ is an isomorphism. Equivalently, there exists an A-module homomorphism $\pi : P \to M$ such that $\pi\iota = \mathrm{id}$ where $\iota : M \to P$ is the natural inclusion.

Excercise 4.15. *If M is isomorphic to a direct summand in a projective module, then M is itself projective.*

We need some more terminology. An epimorphism $\pi : M \to N$ of A-modules is called *split* if there exists an A-module homomorphism $\sigma : N \to M$ such that $\pi\sigma = \mathrm{id}$. If V is any locally convex vector space, then $A^+\hat{\otimes}V$ with the obvious left A-module structure is called the free A-module over V. In general, a locally convex A-module of this form for some locally convex space V is called free.

Proposition 4.16. *Let $_AP$ be a locally convex module. The following are equivalent:*
a) P is projective.
b) Every strict epimorphism $\pi : M \to P$ splits.
c) P is isomorphic to a direct summand in a free module.

Let us now have a look at the locally convex vector spaces we are interested in, namely spaces of smooth functions on manifolds. First let $K \subset \mathbb{R}^n$ be a compact subset. We define seminorms on $C^\infty(K)$ by

$$||f||_\alpha^K = \sup_{x \in K} |D^\alpha f(x)|$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex and D^α denotes the derivative

$$D^\alpha(f) = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

In other words, a sequence $(f_j)_{j\in\mathbb{N}}$ of smooth functions on K converges to f with respect to $||-||_\alpha^K$ iff the functions $D^\alpha f_j$ converge to $D^\alpha f$ uniformly on the compact set K.
Now let M be a (second countable) smooth manifold. Choose a sequence $(K_j)_{j\in\mathbb{N}}$ of compact subsets K_j of M contained in chart domains U_j such that $\bigcup_{j\in\mathbb{N}} I_j = M$ where I_j is the interior of K_j. The locally convex topology on $C^\infty(M)$ is given by all seminorms $||f||_\alpha^i = ||f_{|K_i}||_\alpha^{K_i}$ where $f_{|K_i}$ denotes the restriction of f to K_i and the seminorms $||f_{|K_i}||_\alpha^{K_i}$ are those for $C^\infty(K_i)$ where K_i is viewed as a compact subset of $\mathbb{R}^n$. In this way $C^\infty(M)$ becomes a locally convex topological vector space. It is not difficult to show that the topology does not depend on the choice of the compact subsets K_j.

Excercise 4.17. *Let M be a smooth manifold. Then $C^\infty(M)$ is a Fréchet space. A sequence $(f_n)_{n\in\mathbb{N}}$ in $C^\infty(M)$ converges to $f \in C^\infty(M)$ iff $D(f_n)$ converges uniformly on compact subsets to $D(f)$ for all differential operators D on M. Moreover $C^\infty(M)$ is a locally convex algebra with pointwise multiplication of functions.*

One may generalize the construction of the locally convex topology on $C^\infty(M)$ to vector bundles as follows. Let M be a smooth manifold and let E be a smooth complex vector bundle over M. Then the space $C^\infty(M, E)$ of smooth sections of E is a unitary locally convex module over $C^\infty(M)$. The topology is defined by requiring uniform convergence of all derivatives on compact subsets. For this one uses the fact that locally $E_{|U} = U \times \mathbb{C}^k$ for some $k \in \mathbb{N}$.
Let E be a locally convex vector space and let $U \subset \mathbb{R}^n$ be open. A function $f : U \to E$ is called differentiable at $x^0 \in U$ if there are vectors $e_1, \ldots, e_n \in E$ such that

$$\frac{f(x) - f(x^0) - \sum_{j=1}^n (x_j - x_j^0) e_j}{|x - x^0|}$$

converges to 0 in E as $|x - x^0|$ converges to zero. The vectors e_j are then called the first partial derivatives of f at x^0 and one writes

$$e_j = \frac{\partial f}{\partial x_j}(x^0)$$

for $j = 1, \ldots, n$. As usual, a function is called differentiable on U if it is differentiable at every point in U. Note that a differentiable function is continuous. A function $f : U \to E$ is called smooth if all iterated partial derivatives of f exist.
More generally, let M be a smooth manifold. A function $f : M \to E$ is called smooth if for every coordinate domain $U \subset M$ the induced mapping $f_{|U} : U \to E$

is smooth in the sense explained before. We denote by $C^\infty(M, E)$ the linear space of all smooth maps from M to E. The space $C^\infty(M, E)$ becomes a locally convex vector space with the topology of uniform convergence on compact subsets of the iterated partial derivatives.

PROPOSITION 4.18. *Let M and N be smooth manifolds. Then there are natural topological isomorphisms*

$$C^\infty(M \times N) \cong C^\infty(M, C^\infty(N)) \cong C^\infty(N, C^\infty(M)).$$

PROOF. It clearly suffices to prove $C^\infty(M \times N) \cong C^\infty(M, C^\infty(N))$. Define a linear map $\phi : C^\infty(M \times N) \to C^\infty(M, C^\infty(N))$ by $\phi(f)(x)(y) = f(x, y)$. To see that this map is well-defined observe first that $\phi(f)(x) \in C^\infty(N)$ for all $x \in M$ since f is smooth. Moreover it follows easily from the definitions that $\phi(f)$ is a smooth map from M to $C^\infty(N)$. Conversely, define $\psi : C^\infty(M, C^\infty(N)) \to C^\infty(M \times N)$ by $\psi(f)(x, y) = f(x)(y)$. Again, it is straightforward to check that $\psi(f)$ is indeed a smooth function. The maps ϕ and ψ are obviously inverse to each other. Moreover one checks that both maps are continuous for the natural topologies. □

THEOREM 4.19. *Let M be a smooth manifold and V be a complete locally convex vector space. Then there is a natural topological isomorphism*

$$C^\infty(M)\hat{\otimes}V \cong C^\infty(M, V).$$

We will not discuss the proof of theorem 4.19. Let us only note that a combination of this theorem with proposition 4.18 yields the following result.

THEOREM 4.20. *Let M and N be smooth manifolds. Then there is a natural topological isomorphism*

$$C^\infty(M)\hat{\otimes}C^\infty(N) \cong C^\infty(M \times N).$$

As a consequence, the abstractly defined completed tensor product has a very concrete realization for spaces of smooth functions on manifolds.

2. Homological algebra with locally convex spaces

In this section we explain how the homological algebra developped in chapter 2 may be adapted to the framework of locally convex spaces. Again, we shall assume for simplicitly that all locally convex spaces are complete.

DEFINITION 4.21. *Let A be a locally convex algebra. A chain complex of A-modules is a sequence $C = (C_n)_{n\in\mathbb{Z}}$ of locally convex A-modules C_n together with A-module homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. A chain map $f : C \to D$ between chain complexes is a family $f_n : C_n \to D_n$ of A-module homomorphisms such that the diagrams*

$$\begin{array}{ccc} C_n & \xrightarrow{d} & C_{n-1} \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} \\ D_n & \xrightarrow{d} & D_{n-1} \end{array}$$

are commutative for all $n \in \mathbb{N}$.

There are two possibilities to define the homology of a chain complex of locally convex vector spaces. Namely, one may divide the space of cycles Z_n by the space of boundaries B_n as in the algebraic case or by the *closure* of B_n. Observe that the space of cycles, being the kernel of a continuous linear map, is always closed.

DEFINITION 4.22. *The n-th homology group of a chain complex C is the space $H_n(C) = Z_n/B_n$.*

If V is a topological vector space and W is a linear subspace one may equip V/W with the quotient topology. It is not hard to show that the quotient topology on V/W is Hausdorff iff the subspace V is closed. Hence, according to definition 4.22, the homology groups of a complex of locally convex vector spaces may fail to be separated for the quotient topology.
Although this does not happen in the case we study below, we shall view the homology of a complex of locally convex vector spaces always as an abstract vector space without topology. Since we forget the topology of a complex when considering homology, the machinery of exact sequences can be applied without change.
Homotopies and homotopy equivalences for complexes of locally convex modules are defined in the obvious way by requiring continuity of the involved maps. The same applies to bicomplexes and their associated total complexes.
Let us now discuss the appropriate notion of a projective resolution.

Definition 4.23. *Let A be a locally convex algebra and let M be a locally convex A-module. A projective resolution P of M consists of a long exact sequence*

$$M \xleftarrow{\epsilon} P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow P_3 \longleftarrow \cdots$$

of locally convex A-modules which is split exact as a sequence of locally convex vector spaces such that all P_j are projective.

Recall that projective locally convex modules were introduced in the previous section. As in the algebraic setting one proves that every locally convex module has a projective resolution.

Lemma 4.24. *Let M be a locally convex A-module. Then there exists a projective resolution for M.*

Similarly, the comparison result for projective resolutions holds.

Proposition 4.25. *Let M and N be locally convex A-modules and let P and Q be projective resolutions of M and N, respectively. If $f : M \to N$ is an A-module homomorphism there exist A-module homomorphisms $f_j : P_j \to Q_j$ for all j such that the diagram*

$$\begin{array}{ccccccccccc} M & \xleftarrow{\epsilon} & P_0 & \longleftarrow & P_1 & \longleftarrow & P_2 & \longleftarrow & P_3 & \longleftarrow & \cdots \\ \downarrow f & & \downarrow f_0 & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ N & \xleftarrow{\epsilon} & Q_0 & \longleftarrow & Q_1 & \longleftarrow & Q_2 & \longleftarrow & Q_3 & \longleftarrow & \cdots \end{array}$$

is commutative. Moreover, if $(g_j)_{j\geq 0}$ is another family of such homomorphisms, then the chain maps f and g thus defined are continuously homotopic.

We leave the proof of this assertion to the reader. As a consequence, one has the following result.

Excercise 4.26. *Two projective resolutions of a locally convex module M are continuously homotopy equivalent.*

Let us proceed to and define the derived functor of the tensor product.

Definition 4.27. *Let M_A and ${}_AN$ be complete locally convex modules over a locally convex algebra A and choose a projective resolution P of ${}_AN$. Then*

$$\mathrm{Tor}_n^A(M, N) = H_n(M \hat{\otimes}_A P).$$

Using excercise 4.26 we see that, up to natural isomorphism, the definition of $\mathrm{Tor}(M, N)$ is independent of the resolution P.
Finally let us discuss how Hochschild and cyclic homology are defined for locally

convex algebras. For a locally convex algebra it is natural to consider the space of *completed* noncommutative differential forms.

DEFINITION 4.28. *Let A be a complete locally convex algebra. For $n > 0$ we let $\Omega^n(A)^c = A^+ \hat{\otimes} A^{\hat{\otimes} n}$ be the space of completed noncommutative n-forms over A. In addition we set $\Omega^0(A)^c = A$.*

All operators on noncommutative differential forms defined in chapter 3 and their algebraic relations carry over to the locally convex setting. In particular we obtain the following statement.

PROPOSITION 4.29. *Let A be a complete locally convex algebra. The space $\Omega(A)^c$ of completed noncommutative differential forms together with the operators b and B is a mixed complex.*

As a consequence, the definition of Hochschild and cyclic homology is straightforward.

DEFINITION 4.30. *Let A be a complete locally convex algebra. The continuous Hochschild (cyclic, periodic cyclic) homology of A is the Hochschild (cyclic, periodic cyclic) homology of the mixed complex $\Omega(A)^c$ of completed noncommutative differential forms over A.*

We denote by $HH_n(A)$ the n-th continuous Hochschild homology group of A and accordingly for the other theories. Of course this notation is imprecise since one should distinguish the continuous homology groups from the purely algebraic ones defined in chapter 3. For simplicity we shall not do this since we are only interested in the continuous homology groups in this chapter. Similarly, we will also write $\Omega(A)$ instead of $\Omega(A)^c$ for the space of completed differential forms in the sequel.

3. Differential forms and de Rham cohomology

In this section we review some constructions and results related to differential forms on a manifold.
Let M be a smooth manifold. We denote by $\mathcal{A}^k(M)$ the space of smooth complex-valued k-forms on M and write $\mathcal{A}(M)$ for the direct sum of the spaces $\mathcal{A}^k(M)$. Since $\mathcal{A}^k(M) = 0$ for $k > n = \dim(M)$ this is a finite direct sum. An element of $\mathcal{A}(M)$ is a section of the complexified exterior algebra bundle of the cotangent bundle of M. In particular, there is a natural $C^\infty(M)$-module structure on the spaces $\mathcal{A}^k(M)$. A smooth map $\phi : M \to N$ induces a linear map $\phi^* : \mathcal{A}(N) \to \mathcal{A}(M)$. Explicitly, one has

$$\phi^*(\omega)(X_1, \ldots, X_k) = \omega(T(\phi)X_1, \ldots, T(\phi)X_k)$$

if $T(\phi) : T(M) \to T(N)$ denotes the corresponding map of the tangent bundles.
Locally in a coordinate domain U, every differential form $\omega \in \mathcal{A}^k(M)$ can be written as

$$\omega = f(x)dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

for some smooth function $f \in C^\infty(U)$. If $\eta = g dx_{j_1} \wedge \cdots \wedge dx_{j_l}$ is another differential form expressed locally in this form, the exterior product $\omega \wedge \eta \in \mathcal{A}^{k+l}(M)$ is given by

$$\omega \wedge \eta = fg dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$$

on U. The exterior product is graded commutative, that is,

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

for $\omega \in \mathcal{A}^k(M), \eta \in \mathcal{A}^l(M)$. One has the explicit formula

$$\begin{aligned}(\omega\wedge\eta)&(X_1,\dots,X_{k+l})\\ &= \sum_{\sigma\in S_{k+l}} (-1)^{\mathrm{sign}(\sigma)}\frac{1}{k!\,l!}\,\omega(X_{\sigma(1)},\dots,X_{\sigma(k)})\,\eta(X_{\sigma(k+1)},\dots,X_{\sigma(k+l)})\end{aligned}$$

for the exterior product.
The exterior differential $d : \mathcal{A}^0(M) \to \mathcal{A}^1(M)$ is the linear map defined by

$$d(f)(X) = X(f)$$

for all vector fields X on M. The map d is extended to a linear map $d : \mathcal{A}(M) \to \mathcal{A}(M)$ of degree 1 in a unique way such that $d^2 = 0$ and such that the Leibniz rule

$$d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^{|\omega|}\omega\wedge d\eta$$

holds for homogenous forms ω and η. If $\omega \in \mathcal{A}^k(M)$ and $X_0,\dots,X_k$ are vector fields on M one has the explicit formula

$$\begin{aligned}(d\omega)(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i X_i(\omega(X_0,\dots,X_{i-1},X_{i+1},\dots,X_k))\\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j],X_0,\dots,X_{i-1},X_{i+1},\dots,X_{j-1},X_{j+1},\dots,X_k)\end{aligned}$$

and in local coordinates the exterior differential is given by

$$d(f(x)\,dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = \sum_{j=1}^{n}\frac{\partial f}{\partial x_j}\,dx_j\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

According to the relation $d^2 = 0$ one obtains a cochain complex

$$\mathcal{A}^0(M) \xrightarrow{d} \mathcal{A}^1(M) \xrightarrow{d} \mathcal{A}^2(M) \xrightarrow{d} \mathcal{A}^3(M) \xrightarrow{d} \cdots$$

which is called the de Rham complex of M.

Definition 4.31. *Let M be a smooth manifold. The de Rham cohomology of M is the cohomology of the de Rham complex $\mathcal{A}(M)$ and denoted by $H^*_{dR}(M)$.*

A smooth map $f : M \to N$ induces an algebra homomorphism $f^* : \mathcal{A}(N) \to \mathcal{A}(M)$ which commutes with the exterior differential. Hence one also obtains induced maps $H_{dR}(f) : H^*_{dR}(N) \to H^*_{dR}(M)$.
For later reference we note the homotopy invariance of de Rham cohomology. Two smooth maps $f_0, f_1 : M \to N$ between manifolds are smoothly homotopic if there exists a smooth map $f : M \times [0,1] \to N$ restricting to f_0 and f_1 at 0 and 1, respectively.

Proposition 4.32. *Let $f_0, f_1 : M \to N$ be smoothly homotopic smooth maps between manifolds M and N. Then the induced maps $f_0^*, f_1^* : H^*_{dR}(N) \to H^*_{dR}(M)$ are equal.*

Let X be a vector field on M. There is a unique operator $\iota_X : \mathcal{A}(M) \to \mathcal{A}(M)$ of degree -1 such that

$$\iota_X(\omega) = \omega(X)$$

for all $\omega \in \mathcal{A}^1(M)$ and

$$\iota_X(\omega\wedge\eta) = (\iota_X\omega)\wedge\eta + (-1)^{|\omega|}\omega\wedge(\iota_X\eta)$$

for all homogenous forms ω and η. The operator ι_X is called contraction with the vector field X and one has the explicit formula

$$\iota_X(\omega)(X_1,\dots,X_k) = \omega(X,X_1,\dots X_k)$$

for $\omega \in \mathcal{A}^{k+1}(M)$. In particular $\iota_X(f) = 0$ and $\iota_X^2 = 0$. In local coordinates the interior product is given by

$$\iota_X(f(x)\,dx_{i_0} \wedge \cdots \wedge dx_{i_k}) = \sum_{j=0}^{k} (-1)^j X(x_{i_j}) f(x)\,dx_{i_0} \wedge \cdots dx_{i_{j-1}} \wedge dx_{i_{j+1}} \cdots \wedge dx_{i_k}.$$

Finally we want to discuss the Lie derivative. If X is a vector field on M then the Lie derivative $\mathcal{L}_X : \mathcal{A}(M) \to \mathcal{A}(M)$ is the linear operator of degree zero defined by

$$\mathcal{L}_X(\omega) = \frac{d}{d\tau} \exp(\tau X)^* \omega|_{\tau=0}$$

where $\exp(\tau X)$ denotes the flow of X. Since pull-back of differential forms commutes with the exterior differential one easily obtains

$$\mathcal{L}_X d = d\mathcal{L}_X.$$

Moreover the Lie derivative is an even derivation on $\mathcal{A}(M)$ in the sense that

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X(\omega) \wedge \eta + \omega \wedge \mathcal{L}_X(\eta)$$

for all $\omega, \eta \in \mathcal{A}(M)$. An important relation between the operators $\mathcal{L}_X$, ι_X and d is the following Cartan homotopy formula.

PROPOSITION 4.33. *Let X be a vector field on M. Then*

$$\mathcal{L}_X = d\iota_X + \iota_X d$$

on $\mathcal{A}(M)$.

PROOF. Since both sides define even derivations on $\mathcal{A}(M)$ it suffices to prove this formula in degree zero and one. In degree zero one has $\mathcal{L}_X = \iota_X d$ and $d\iota_X = 0$. Since d commutes with $\mathcal{L}_X$ we also have $\mathcal{L}_X(df) = (d\iota_X + \iota_X d)(df)$ for all exact one-forms df. Hence the claim in degree one follows from the observation that locally every element in $\mathcal{A}^1(M)$ can be expressed as a sum of one-forms $f_0 df_1$ with $f_0, f_1 \in C^\infty(M)$. □

We now come to an explicit calculation that will be needed in the proof of the Hochschild-Kostant-Rosenberg theorem. Let $U \subset \mathbb{R}^n$ be a convex open neighborhood of zero. The Euler vector field on U is defined by

$$E = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}.$$

The corresponding flow Φ_t^E on U is given by

$$\Phi_t^E(x) = \exp(t)x.$$

For fixed $x \in U$ we consider $\Phi_t^E(x)$ for all t such that $\exp(t)x$ is contained in U. In a similar way we define another flow Φ_t on U by $\Phi_t(x) = tx$. By definition, one has $\Phi_t^E(x) = \Phi_{\exp(t)}(x)$ provided t is small enough.

PROPOSITION 4.34. *Let $U \subset \mathbb{R}^n$ be a convex open neighborhood of zero. Then*

$$\int_0^1 \Phi_t^* \mathcal{L}_E(\omega) \frac{dt}{t} = \omega - i^*\omega$$

for all $\omega \in \mathcal{A}(U)$ where $i : U \to U$ denotes the constant map with value 0.

PROOF. We calculate

$$\int_0^1 \Phi_t^* \mathcal{L}_E(\omega) \frac{dt}{t} = \int_{-\infty}^0 \Phi_{\exp(s)}^* \mathcal{L}_E(\omega) ds = \int_{-\infty}^0 (\Phi_s^E)^* \mathcal{L}_E(\omega) ds$$

using the coordinate change $t = \exp(s)$ and the relation between the flows Φ and Φ^E. By definition of the Lie derivative the last expression is equal to

$$\int_{-\infty}^{0} (\Phi_s^E)^* \frac{d}{d\sigma}(\Phi_\sigma^E)^*(\omega)|_{\sigma=0} ds = \int_{-\infty}^{0} \frac{d}{d\sigma}(\Phi_\sigma^E)^*(\omega)|_{\sigma=s} ds = \omega - i^*\omega$$

which yields the claim. □

Observe that $i^* f = f(0)$ for $f \in \mathcal{A}^0(U)$ and $i^*\omega = 0$ for $\omega \in \mathcal{A}^k(U)$ and $k > 0$.

4. The Hochschild-Kostant-Rosenberg theorem

In this section we formulate the Hochschild-Kostant-Rosenberg theorem describing the Hochschild homology of the algebra of smooth functions on a manifold M. We prove this theorem in the special case where M is a convex open neighborhood of zero in $\mathbb{R}^n$.

Let M be a smooth manifold. We view $\mathcal{A}(M)$ as a mixed complex with b-boundary equal to zero and B-boundary equal to the exterior differential d. The Hochschild-Kostant-Rosenberg map $\alpha : \Omega(C^\infty(M)) \to \mathcal{A}(M)$ is defined by

$$\alpha(a_0 da_1 \cdots da_n) = \frac{1}{n!}\, a_0 da_1 \wedge \cdots \wedge da_n$$

on elementary tensors. It is easy to check that this formula induces a map on the completed tensor products used in the definition of $\Omega(C^\infty(M))$.

Lemma 4.35. *The Hochschild-Kostant-Rosenberg map $\alpha : \Omega(C^\infty(M)) \to \mathcal{A}(M)$ is a map of mixed complexes.*

Proof. We compute

$$\begin{aligned}
\alpha b(a_0 da_1 \cdots da_n) &= \sum_{j=0}^{n-1} (-1)^j \alpha(a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_n) \\
&\quad + (-1)^n \alpha(a_n a_0 da_1 \cdots da_{n-1}) \\
&= \frac{1}{(n-1)!} \Bigg(\sum_{j=0}^{n-1} (-1)^j\, a_0 da_1 \wedge \cdots \wedge d(a_j a_{j+1}) \cdots \wedge da_n \\
&\quad + (-1)^n a_n a_0 da_1 \wedge \cdots \wedge da_{n-1} \Bigg) = 0
\end{aligned}$$

using the Leibniz rule. Moreover we have

$$\begin{aligned}
\alpha B(a_0 da_1 \cdots da_n) &= \sum_{j=0}^{n} (-1)^{nj} \alpha(da_{n-j+1} \cdots da_n da_0 \cdots da_{n-j}) \\
&= \frac{1}{(n+1)!} \sum_{j=0}^{n} (-1)^{nj}\, da_{n-j+1} \wedge \cdots \wedge da_n \wedge da_0 \wedge \cdots \wedge da_{n-j} \\
&= \frac{1}{n!}\, da_0 \wedge \cdots \wedge da_n = d\alpha(a_0 da_1 \cdots da_n)
\end{aligned}$$

which shows that α commutes with the boundary operators as claimed. □

The goal is to show that this natural map induces an isomorphism in Hochschild homology.

Theorem 4.36 (Hochschild-Kostant-Rosenberg). *For every smooth manifold M the Hochschild-Kostant-Rosenberg map*

$$\alpha : \Omega(C^\infty(M)) \to \mathcal{A}(M)$$

induces an isomorphism in Hochschild homology.

The proof of theorem 4.36 is divided into several steps. We define a continuous map $\beta : C^\infty(M)^{n+1} \to \Omega^n(C^\infty(M))$ by

$$\beta(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sign}(\sigma)} a_0 da_{\sigma(1)} \cdots da_{\sigma(n)}$$

where S_n is the symmetric group on n elements. A straightforward calculation shows that $b\beta(a_0, \ldots, a_n) = 0$ for all $a_0, \ldots, a_n \in C^\infty(M)$. Moreover we have

$$\alpha\beta(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^{\mathrm{sign}(\sigma)} \alpha(a_0 da_{\sigma(1)} \cdots da_{\sigma(n)}) = a_0 da_1 \wedge \cdots \wedge da_n$$

and these relations imply that the map $\alpha : HH_*(C^\infty(M)) \to \mathcal{A}(M)$ is surjective. Consequently, in order to prove theorem 4.36 it suffices to show that α is injective. We will first consider the special case where M is a convex open neighborhood of zero in $\mathbb{R}^n$. The general case will be treated in section 5.

Theorem 4.37. *Let $U \subset \mathbb{R}^n$ be a convex open neighborhood of zero. The Hochschild-Kostant-Rosenberg map*

$$\alpha : \Omega(C^\infty(U)) \to \mathcal{A}(U)$$

induces an isomorphism on the homology with respect to the Hochschild boundary.

Proof. We construct a projective resolution of the $C^\infty(U)$-bimodule $C^\infty(U)$ as follows. Let $\Lambda^k(\mathbb{R}^n)^*$ be the space of complex-valued alternating k-linear maps on $\mathbb{R}^n$. We set

$$P^k = C^\infty(U) \hat{\otimes} \Lambda^k(\mathbb{R}^n)^* \hat{\otimes} C^\infty(U)$$

and equip this space with the obvious $C^\infty(U)$-bimodule structure

$$(f \cdot \omega \cdot g)(x, z) = f(x)\omega(x, z)g(z)$$

using the identification

$$C^\infty(U) \hat{\otimes} \Lambda^k(\mathbb{R}^n)^* \hat{\otimes} C^\infty(U) \cong C^\infty(U \times U, \Lambda^k(\mathbb{R}^n)^*).$$

The differential $\partial : P^{k+1} \to P^k$ is defined by

$$\partial(\omega)(x, z)(y_1, \ldots, y_k) = \omega(x, z)(z - x, y_1, \ldots, y_k)$$

and it is clear that $\partial^2 = 0$. If we let $\mu : P^0 \to C^\infty(U)$ be the multiplication map we obtain a complex

$$C^\infty(U) \xleftarrow{\mu} P_0 \xleftarrow{\partial} P_1 \xleftarrow{\partial} P_2 \xleftarrow{\partial} P_3 \longleftarrow \cdots$$

which we will call the *Koszul complex* for U.

We want to show that the Koszul complex is a projective resolution of $C^\infty(U)$. In order to do this we need a different description of the differential ∂. Consider the isomorphism

$$P^k \cong C^\infty(U \times U, \Lambda^k(\mathbb{R}^n)^*) \cong C^\infty(U, C^\infty(U, \Lambda^k(\mathbb{R}^n)^*) = C^\infty(U, \mathcal{A}^k(U))$$

given by $\gamma(\omega)(x)(z) = \omega(x, z)$. Fix an element $x \in U$ and consider the vector field X_x on U defined by $X_x(z) = z - x$. If $\iota(x) : \mathcal{A}(U) \to \mathcal{A}(U)$ denotes contraction with X_x we have

$$\partial(\omega)(x) = \iota(x)\omega(x)$$

for all $\omega \in P^k$ and $k > 0$. Let us also write $\mathcal{L}(x)$ for the Lie derivative with respect to X_x.

The flow $\Phi_t(x) : \mathbb{R}^n \to \mathbb{R}^n$ given by $\Phi_t(x)(z) = (1-t)x + tz$ preserves U for $t \in [0, 1]$. In particular there are induced maps $\Phi_t(x)^* : \mathcal{A}(U) \to \mathcal{A}(U)$ for all $t \in [0, 1]$. Let us define $h : P^k \to P^{k+1}$ by

$$h(\omega)(x) = \int_0^1 \Phi_t(x)^*(d\omega(x)) \frac{dt}{t}$$

for $\omega \in C^\infty(U, \mathcal{A}^k(U))$ and $k \geq 0$. In addition we define $h : C^\infty(U) \to C^\infty(U \times U)$ by $h(f)(x,z) = f(x)$. Let us check that the map h is well-defined. This is evident in degree -1. Since the differential form $d\omega(x)$ has degree at least one it is easy to check that, when viewed as a function of t, the integrand

$$\frac{1}{t}\ \Phi_t(x)^*(d\omega(x))$$

is bounded on $[0,1]$ for every $\omega \in P^k$. For instance, if $\omega(x)(z) = f(z)$ is a function on U we have

$$\frac{1}{t}\ \Phi_t(x)^*(d\omega(x))(z) = \frac{1}{t}\sum_{j=1}^n \frac{\partial f}{\partial z_j}((1-t)x+tz)d(tz_j) = \sum_{j=1}^n \frac{\partial f}{\partial z_j}((1-t)x+tz)dz_j.$$

Hence the integral defines indeed an element in P^{k+1}. Moreover one checks that h is a continuous map.

Proposition 4.38. *The map h defines a contracting homotopy for the Koszul complex P.*

Proof. Under the coordinate change $\psi_x(z) = z - x$ we have $\Phi_t(x) = \psi_x^{-1}\Phi_t\psi_x$ where $\Phi_t = \Phi_t(0)$ is the flow considered in proposition 4.34 and X_x corresponds to the Euler vector field E. Using proposition 4.33 and proposition 4.34 we thus compute

$$\begin{aligned}(h\partial+\partial h)(\omega)(x) &= \int_0^1 \Phi_t(x)^*(d\iota(x)\omega(x))\frac{dt}{t} + \iota(x)\int_0^1 \Phi_t(x)^*(d\omega(x))\frac{dt}{t}\\ &= \int_0^1 \Phi_t(x)^*(d\iota(x)+\iota(x)d)\omega(x))\frac{dt}{t}\\ &= \int_0^1 \Phi_t(x)^*(\mathcal{L}(x)\omega(x))\frac{dt}{t} = \omega(x) - i_x^*\omega(x) = \omega(x)\end{aligned}$$

on P^k for $k > 0$ where i_x is the constant map with value x. In addition we have

$$(h\mu + \partial h)(f)(x,z) = f(x,x) + f(x,z) - f(x,x) = f(x,z)$$

on P^0 and $(\mu h)(f)(x) = f(x)$ on $C^\infty(U)$ which yields the claim. □

We may thus compute the Hochschild homology of $C^\infty(U)$ using the resolution P. First observe that there are natural isomorphisms

$$\begin{aligned}C^\infty(U)\hat{\otimes}_{C^\infty(U\times U)}P^k &= C^\infty(U)\hat{\otimes}_{C^\infty(U\times U)}C^\infty(U\times U)\hat{\otimes}\Lambda^k(\mathbb{R}^n)^*\\ &= C^\infty(U)\hat{\otimes}\Lambda^k(\mathbb{R}^n)^* \cong \mathcal{A}^k(U).\end{aligned}$$

It remains to determine the boundary operators of this complex. Using the identification

$$C^\infty(U) \otimes_{C^\infty(U\times U)} P^k \cong C^\infty(U)\hat{\otimes}\Lambda^k(\mathbb{R}^n)^*$$

one sees that this map is given by restriction of the boundary operator ∂ to the diagonal Δ in $U \times U$, that is,

$$\partial(\omega)(x)(y_1, \dots, y_k) = \omega(x)(x-x, y_1, \dots, y_k) = 0$$

for all $\omega \in C^\infty(U, \Lambda^{k+1}(\mathbb{R}^n)^*)$. As a consequence we obtain an isomorphism

$$HH_n(C^\infty(U)) \cong \mathcal{A}^n(U)$$

for all n. However, theorem 4.37 claims slightly more, namely, that this isomorphism may be realized using the Hochschild-Kostant-Rosenberg map.

From the general theory we know that there exists a $C^\infty(U\times U)$-linear chain map $f : P \to \mathrm{Bar}(C^\infty(U))$, unique up to homotopy, which induces the above isomorphism after tensoring over $C^\infty(U \times U)$ with $C^\infty(U)$. Let us explicitly write down such a map. In degree zero we have $P_0 = \mathrm{Bar}_0(C^\infty(U)) = C^\infty(U \times U)$ and we let f_0 be the identity map. Define $f_k : P_k \to \mathrm{Bar}_k(C^\infty(U))$ for $k > 0$ by

$$f_k(\omega)(x, y_1, \dots, y_k, z) = \omega(x, z)(X_{y_1}(z), \dots, X_{y_k}(z))$$

for all $\omega \in C^\infty(U \times U, \Lambda^k(\mathbb{R}^n)^*)$. We compute

$$\begin{aligned} b' f(\omega)(x, &y_1, \dots, y_k, z) = \omega(x, z)(X_x(z), X_{y_1}, \dots, X_{y_k}(z)) \\ &+ \sum_{j=1}^{k} (-1)^j \omega(x, z)(X_{y_1}(z), \dots, X_{y_j}(z), X_{y_j}(z), \dots, X_{y_k}(z)) \\ &+ (-1)^{k+1} \omega(x, z)(X_{y_1}(z), \dots, X_{y_k}(z), X_z(z)) \\ = \omega(x, z)&(X_x(z), X_{y_1}(z), \dots, X_{y_k}(z)) = f\partial(\omega)(x, y_1, \dots, y_k, z) \end{aligned}$$

using that $\omega(x, z)$ is alternating and $X_z(z) = 0$. Hence f defines a chain map. We may also view an element $\omega \in C^\infty(U \times U, \Lambda^k(\mathbb{R}^n)^*)$ as a smooth function on U with values in $\mathcal{A}(U)$. Since U is an open subset of $\mathbb{R}^n$ such an element may be written in a unique way as a linear combination of terms of the form

$$\eta(x)(z) = a_0(x, z) dz^{j_1} \wedge \dots \wedge dz^{j_k} = a_0 da_1 \wedge \dots \wedge da_k$$

where z^j denotes the j-th component of z, a_0 is a smooth function on $U \times U$ and $a_i(z) = z^{j_i}$ for $i > 0$. Now observe that for the function a given by $a(z) = z^j - x^j$ for some j we have

$$da(X_y)(x) = da(x)(x - y) = x^j - y^j = -a(y).$$

Moreover $dz^j = d(z^j - x^j)$ if x^j is viewed as a constant function of the variable z. Hence for η in the form above we get

$$\begin{aligned} f(\eta)(x, &y_1, \dots, y_k, x) \\ &= \sum_{\sigma \in S_k} (-1)^{\mathrm{sign}(\sigma)} a_0(x, x) da_1(x)(x - y_{\sigma(1)}) \cdots da_k(x)(x - y_{\sigma(k)}) \\ &= \sum_{\sigma \in S_k} (-1)^k (-1)^{\mathrm{sign}(\sigma)} a_0(x, x) a_1(y_{\sigma(1)}) \cdots a_k(y_{\sigma(k)}). \end{aligned}$$

Consequently, the induced chain map $F : \mathcal{A}(U) \to C(C^\infty(U))$ is given by

$$F(a_0 da_1 \wedge \dots \wedge da_k)(x_0, \dots, x_k) = \sum_{\sigma \in S_k} (-1)^{\mathrm{sign}(\sigma)+k} a_0(x_0) a_{\sigma(1)}(x_1) \cdots a_{\sigma(k)}(x_k)$$

and we obtain

$$\alpha F(a_0 da_1 \wedge \dots \wedge da_k) = (-1)^k a_0 da_1 \wedge \dots \wedge da_k.$$

Since we know that $F : \mathcal{A}(U) \to C(C^\infty(U))$ is a quasiisomorphism it follows that α induces an isomorphism $HH_*(C^\infty(U)) \cong \mathcal{A}(U)$. This finishes the proof of the Hochschild-Kostant-Rosenberg theorem 4.37.

5. The proof in the general case

In section 4 we formulated the Hochschild-Kostant-Rosenberg theorem computing the Hochschild homology of $C^\infty(M)$ and proved it in the special case where M is a convex open neighborhood of zero in $\mathbb{R}^n$. We shall now treat the general case of an arbitrary smooth manifold M. The idea is to reduce the problem to convex open subsets of $\mathbb{R}^n$ by an appropriate localization procedure. We follow the proof of Teleman [**11**], [**12**].

Choose a Riemannian metric on M and let $d : M \times M \to [0, \infty)$ be the associated distance function. For every $k > 0$ we let $\rho : M^{k+1} \to [0, \infty)$ be the smooth map

$$\rho(x_0, \dots, x_k) = d^2(x_0, x_1) + d^2(x_1, x_2) + \dots + d^2(x_{k-1}, x_k) + d^2(x_k, x_0)$$

which measures the square of the distance to the diagonal. Moreover we choose a smooth function $\lambda : [0, \infty) \to [0, 1]$ with support in $[0, 1]$ which takes the value 1 on the intervall $[0, 1/2]$. Let us define

$$\rho_t(x_0, \dots, x_k) = \lambda\left(\frac{\rho(x_0, \dots, x_k)}{t}\right)$$

for every $k > 0$ and $t > 0$. In addition let Δ_t be the set of all points in M^{k+1} with $\rho(x_0, \dots, x_k) \leq t$. We call Δ_t the ($(k+1)$-dimensional) t-neighborhood of the diagonal. By construction, the support of the function ρ_t is contained in the t-neighborhood of the diagonal.

Let us identify Hochschild chains in $C_k(C^\infty(M))$ with smooth functions on M^{k+1}. If a Hochschild chain $f \in C_k(C^\infty(M))$ is zero on the $(k+1)$-dimensional t-neighborhood of the diagonal, then the boundary $b(f) \in C_{k-1}(C^\infty(M))$ vanishes on the k-dimensional t-neighborhood of the diagonal. For $t > 0$ let $C(C^\infty(M))_t$ be the subcomplex of $C(C^\infty(M))$ consisting in every dimension of all chains which are zero on Δ_t. Moreover let $C(C^\infty(M))_0$ be the union of the complexes $C(C^\infty(M))_t$ for all $t > 0$. Then $C(C^\infty(M))_0$ is a subcomplex of $C(C^\infty(M))$ and we obtain a short exact sequence

$$C(C^\infty(M))_0 \rightarrowtail C(C^\infty(M)) \twoheadrightarrow C(C^\infty(M))_\Delta$$

of complexes where $C(C^\infty(M))_\Delta$ denotes the corresponding quotient complex. We will call $C(C^\infty(M))_\Delta$ the complex of germs around the diagonal.

For $t > 0$ we define an operator $E_t : C(C^\infty(M)) \to C(C^\infty(M))$ of degree one by

$$E_t(f)(x_0, \dots x_{k+1}) = \lambda\left(\frac{d^2(x_0, x_1)}{t}\right) f(x_1, \dots, x_{k+1})$$

which has the following property.

Lemma 4.39. *Let $\epsilon > 0$. Then the support of $E_t(f)$ is contained in $\Delta_{3t+3\epsilon}$ provided f is supported in Δ_ϵ. Moreover, the operator E_t maps $C(C^\infty(M))_\epsilon$ into $C(C^\infty(M))_{\frac{\epsilon}{3}}$.*

Proof. Assume $\rho(x_0, \dots, x_{k+1}) > 3t + 3\epsilon$ and $d^2(x_0, x_1) \leq t$. By the triangle inequality we have

$$d^2(x_{k+1}, x_0) \leq d^2(x_{k+1}, x_1) + 2d(x_{k+1}, x_1)d(x_1, x_0) + d^2(x_1, x_0)$$

which implies

$$d^2(x_{k+1}, x_0) \leq 3d^2(x_{k+1}, x_1) + 3d^2(x_1, x_0).$$

Hence we have

$$\begin{aligned}
\rho(x_1, \dots, x_{k+1}) &= d^2(x_1, x_2) + \cdots d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x_1) \\
&\geq \frac{1}{3}(d^2(x_1, x_2) + \cdots d^2(x_k, x_{k+1})) + d^2(x_{k+1}, x_1) \\
&\geq \frac{1}{3}(d^2(x_1, x_2) + \cdots d^2(x_k, x_{k+1})) + d^2(x_{k+1}, x_1) + d^2(x_1, x_0) - t \\
&\geq \frac{1}{3}(d^2(x_1, x_2) + \cdots d^2(x_k, x_{k+1}) + d^2(x_{k+1}, x_0)) - t \\
&= \frac{1}{3}\rho(x_0, \dots, x_{k+1}) - t > t + \epsilon - t = \epsilon
\end{aligned}$$

which yields the first claim. The estimate

$$d(x_{k+1}, x_1)^2 \leq d(x_{k+1}, x_0)^2 + d(x_0, x_1)^2 + 2d(x_{k+1}, x_0)d(x_0, x_1)$$

shows

$$\rho(x_1, \dots, x_{k+1}) \leq \rho(x_0, \dots, x_{k+1}) + 2\rho(x_0, \dots, x_{k+1})$$

which easily implies the second assertion. □

We define an operator $N_t : C(C^\infty(M)) \to C(C^\infty(M))$ of degree zero by

$$N_t(f)(x_0,\dots,x_k) = (-1)^k \lambda\left(\frac{d^2(x_0,x_1)}{t}\right)(f(x_1,\dots,x_k,x_0) - f(x_1,\dots,x_k,x_1))$$

for $t > 0$. We can rewrite this as

$$N_t(f)(x_0,\dots,x_k) = (-1)^k (E_t(f)(x_0,x_1,\dots,x_k,x_0) - E_t(f)(x_0,x_1,\dots,x_k,x_1))$$

using the map E_t.

LEMMA 4.40. *The operator N_t is a chain map and we have*

$$bE_t + E_t b = \mathrm{id} - N_t$$

for every $t > 0$. The support of $N_t(f)$ is contained in $\Delta_{3t+3\epsilon}$ provided f is supported in Δ_ϵ for some $\epsilon > 0$ and N_t maps $C(C^\infty(M))_\epsilon$ into $C(C^\infty(M))_{\frac{\epsilon}{3}}$. Moreover $(N_t)^k = 0$ on $C_k(C^\infty(M))_{(k+k^2)t}$.

PROOF. A straightforward calculation yields the relation $bE_t + E_t b = \mathrm{id} - N_t$. This relation also shows that N_t is a chain map.

Assume that f is supported in Δ_ϵ. Then the first term in the definition of N_t is again supported in Δ_ϵ. For the second term observe that the argument given in lemma 4.39 shows $\rho(x_1,\dots,x_k,x_1) > \epsilon$ provided $\rho(x_0,\dots,x_k) > 3t + 3\epsilon$ and $d^2(x_0,x_1) \leq t$. It follows that the second term is supported in $\Delta_{3t+3\epsilon}$. Hence the support of $N_t(f)$ is contained in $\Delta_{3t+3\epsilon}$ as well. The fact that N_t maps $C(C^\infty(M))_\epsilon$ into $C(C^\infty(M))_{\frac{\epsilon}{3}}$ is proved in the same way as the corresponding assertion for E_t. From the explicit formla for N_t it follows that the operator $(N_t)^k$ is of the form

$$(N_t)^k(f)(x_0,x_1,\dots,x_k) = \prod_{j=0}^{k-1} \lambda\left(\frac{d^2(x_j,x_{j+1})}{t}\right)\mathcal{F}(f)(x_0,x_1,\dots,x_k)$$

where $\mathcal{F}(f)$ is a linear combination of functions constructed out of f by permutation of the arguments and restriction to certain diagonal subsets. For the first factor in this expression to be nonzero at $(x_0,\dots,x_k)$ we necessarily have $d^2(x_j,x_{j+1}) < t$ for $0 \leq j < k$. The triangle inequality implies $d(x_0,x_k) < kt^{\frac{1}{2}}$ in this case. Hence $\rho(x_0,\dots,x_k) < kt + k^2 t$ at such a point. As a consequence we have $(N_t)^k(f) = 0$ for $f \in C(C^\infty(M))_{(k+k^2)t}$. □

Lemma 4.40 implies the following result.

PROPOSITION 4.41. *The natural map $HH_*(C^\infty(M)) \to H_*(C(C^\infty(M))_\Delta)$ is an isomorphism.*

PROOF. It suffices to show that $C(C^\infty(M))_0$ is acyclic. According to lemma 4.39 and lemma 4.40 the operators E_t and N_t define maps from $C(C^\infty(M))_0$ into $C(C^\infty(M))_0$ and we have $(N_t)^k = 0$ on $C_k(C^\infty(M))_{(k+k^2)t}$. Since N_t is a chain map this implies

$$\begin{aligned}\mathrm{id} = \mathrm{id} - (N_t)^k &= \sum_{j=0}^{k-1} (N_t)^j - (N_t)^{j+1} = (\mathrm{id} - N_t)\sum_{j=0}^{k-1}(N_t)^j \\ &= \sum_{j=0}^{k-1} bE_t(N_t)^j + E_t(N_t)^j b = bh_t + h_t b\end{aligned}$$

on $C_k(C^\infty(M))_{(k+k^2)t}$ where

$$h_t = \sum_{j=0}^{k-1} E_t(N_t)^j.$$

Applying this formula to a cycle $f \in C_k(C^\infty(M))_{(k+k^2)t}$ yields $[f]=0$ for the corresponding homology class in $H_k(C(C^\infty(M))_0)$. Since every cycle $f \in C_k(C^\infty(M))_0$ is contained in $C_k(C^\infty(M))_\epsilon$ for some $\epsilon > 0$ this yields the claim. □

The assertion of proposition 4.41 may be rephrased by saying that the homology class of a Hochschild cycle only depends on its germ around the diagonal. We shall refine this statement and show that the homology class actually depends only on the infinite jet at the diagonal.

In order to make this precise we need some preparations. For a smooth function $h \in C^\infty(N)$ on a smooth manifold N let us consider locally the iterated partial differentials

$$\frac{\partial^{|\alpha|} h}{\partial x_\alpha} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_p}\right)^{\alpha_p} (h)$$

in some coordinate system $(x_1, \dots, x_p)$ where $\alpha = (\alpha_1, \dots, \alpha_p)$ is a multiindex. Clearly these partial differentials depend on the choice of the coordinate system, but the statement that all iterated partial differentials are zero on a fixed subset of N is independent of the coordinates.

Let us call two Hochschild chains f and g in $C_k(C^\infty(M)) = C^\infty(M^{k+1})$ equivalent if all iterated partial derivatives of $f-g$ vanish on the diagonal $\Delta \subset M^{k+1}$. By definition, the space $J^\infty C(C^\infty(M))$ of infinite jets at the diagonal is the quotient of $C(C^\infty(M))$ under this equivalence relation. The infinite jet at the diagonal of a Hochschild chain $f \in C^\infty(M^{k+1})$ is the class of f in $J^\infty C(C^\infty(M))$. It is easy to check that the Hochschild boundary induces a map $b : J^\infty C(C^\infty(M)) \to J^\infty C(C^\infty(M))$. Hence $J^\infty C(C^\infty(M))$ is a complex in a natural way and the projection $J^\infty : C(C^\infty(M)) \to J^\infty C(C^\infty(M))$ induces a short exact sequence

$$KC(C^\infty(M)) \rightarrowtail C(C^\infty(M)) \twoheadrightarrow J^\infty C(C^\infty(M))$$

of complexes where $KC(C^\infty(M))$ is the kernel of J^∞. Note that we have canonical chain maps $C(C^\infty(M))_0 \to KC(C^\infty(M))$ and $C(C^\infty(M))_\Delta \to J^\infty C(C^\infty(M))$.

Proposition 4.42. *The map $J^\infty : HH_*(C^\infty(M)) \to H_*(J^\infty C(C^\infty(M)))$ is an isomorphism.*

Proof. It suffices to show that the complex $KC(C^\infty(M))$ is acyclic. Let $[f] \in H_k(KC(C^\infty(M)))$ be a homology class and set $c = 2(k+k^2)$. According to proposition 4.41 we may assume that $[f]$ is represented by a cycle f which is supported in the $c/2$-neighborhood of the diagonal. Consider the chain f_t given by

$$f_t(x_0, \dots, x_k) = \left(\frac{d}{d\tau}\rho_{c\tau}(x_0, \dots, x_k)\right)_{|\tau=t} f(x_0, \dots, x_k)$$

for $t > 0$. It is straightforward to check that f_t is again a cycle. Moreover, the support of f_t is contained in $\Delta_{ct} \setminus \Delta_{ct/2}$. Since $ct/2 = (k+k^2)t$ we may apply the homotopy formula $\mathrm{id} = bh_t + h_t b$ obtained in proposition 4.41 to f_t. If we integrate from ϵ to 1 and use the assumption that f is supported in the $c/2$-neighborhood of the diagonal this yields

$$f - \rho_{c\epsilon} f = b l_\epsilon(f) + l_\epsilon b(f)$$

for every ϵ such that $0 < \epsilon < 1$ where

$$l_\epsilon(f) = \int_\epsilon^1 h_t(f_t)dt.$$

Since f vanishes on the diagonal the limit

$$\lim_{\epsilon \to 0} \rho_{c\epsilon} f$$

exists pointwise and is zero.

Using lemma 4.39, lemma 4.40 and the definition of h_t we see that the support of

$h_t(f_t)$ is contained in $\Delta_{rt} \setminus \Delta_{st}$ for some positive constants $s < r$ independent of t and f. In particular, the function $h_t(f_t)$ vanishes on the diagonal for all $t > 0$. It follows that the limit

$$l(f) = \lim_{\epsilon \to 0} l_\epsilon(f)$$

exists pointwise and defines a function on M^{k+2} which is smooth outside the diagonal Δ.

Let us show that $l(f)$ is in fact a smooth function on M^{k+2}. Fix a point $y \in M$ and choose a compact neighborhood $K \subset M$ of y. We let $K^{k+2} \subset M^{k+2}$ be the corresponding neighborhood of $y_\Delta = (y, \ldots, y)$. Moreover let r be as above and denote by μ the supremum norm of the derivative of λ. According to the chain rule and the definition of h_t we see that there exists a constant $C > 0$ such that

$$|h_t(f_t)(x)| \leq C \frac{\mu}{t^2} \sup_{v \in K^{k+1} \cap \Delta_{ct}} |f(v)|$$

for all $x \in K^{k+2}$. Since the support of $h_t(f_t)$ is contained in Δ_{rt} we may assume $t^2 \geq \rho(x)^2/r^2$ and obtain

$$|h_t(f_t)(x)| \leq C \frac{\mu c^2}{\rho(x)^2} \sup_{v \in K^{k+1} \cap \Delta_{ct}} |f(v)|$$

for all $x \in K^{k+2}$. After possibly shrinking K appropriately, we may apply the Taylor formula to f in a local coordinate system and obtain for every $p > 0$

$$f(v) = \frac{1}{p!} \sum_{|\alpha|=p} (v - v_\Delta)^\alpha \frac{\partial^{|\alpha|} f}{\partial x_\alpha}(v_\Delta + \theta(v - v_\Delta))$$

for $v \in K^{k+1}$ where θ is a real number between zero and one. Here v_Δ denotes the euclidean projection of v onto the diagonal and $w^\alpha = w_1^{\alpha^1} \cdots w_m^{\alpha^m}$ for $w = (w_1, \ldots, w_m)$ and every multiindex α. Note that for the above description of f we use that all partial derivatives of f vanish on the diagonal by assumption.

Let s be chosen as above. In order to estimate $|h_t(f_t)(x)|$ we may assume in addition $st \leq \rho(x)$ and obtain

$$\sup_{v \in K^{k+1} \cap \Delta_{ct}} |f(v)| \leq \frac{1}{(2p)!} \sum_{|\alpha|=2p} \sup_{v \in K^{k+1} \cap \Delta_{ct}} \left| \frac{\partial^{|\alpha|} f}{\partial x_\alpha}(v) \right| |(v - v_\Delta)^\alpha| \leq c_p\, t^p \leq \frac{c_p}{s^p} \rho(x)^p$$

for every p and some constant c_p. In particular, using this estimate for $p = 3$ and our previous considerations we compute for $x \in K^{k+2}$

$$|l(f)(x)| \leq \lim_{\epsilon \to 0} \int_\epsilon^1 |h_t(f_t)(x)| dt \leq R \int_0^1 \rho(x) dt = R\rho(x)$$

for some constant $R > 0$. This shows that $l(f)$ is continuous in a neighborhood of y_Δ. Similarly, one sees that the partial derivatives of $l(f)$ exist and are continuous functions vanishing on the diagonal. We deduce that $l(f)$ is actually infinitely often differentiable and that all higher partial derivatives of f at the diagonal Δ are zero. Hence $l(f)$ defines an element in $KC_{k+1}(C^\infty(M))$.

The remaining part of the proof is straightforward. The relation $f = (bl + lb)(f)$ implies $[f] = 0$. Hence the complex $KC(C^\infty(M))$ is acyclic. □

We will now finish the proof of theorem 4.36. Let $(U_j)_{j \in J}$ be a locally finite open covering of M. Restriction of functions defines homomorphisms $C^\infty(M) \to C^\infty(U_j)$ and chain maps $p_j : C(C^\infty(M)) \to C(C^\infty(U_j))$ for all $j \in J$. These maps determine a chain map p from $C(C^\infty(M))$ into the direct product of the complexes $C(C^\infty(U_j))$.

PROPOSITION 4.43. *Let $(U_j)_{j\in J}$ be a locally finite open covering of M. Then the natural map*

$$p : HH_*(C^\infty(M)) \to \prod_{j\in J} HH_*(C^\infty(U_j))$$

induced by restriction of functions is injective.

PROOF. Let $[z] \in HH_k(C^\infty(M))$ be a homology class such that $p([z]) = 0$. This means that $p_j([z]) = 0$ in $HH_k(C^\infty(U_j))$ for all $j \in J$. Let $(\chi_j)_{j\in J}$ be a partitition of unity subordinate to the covering $(U_j)_{j\in J}$ of M. That is, $\sum_{j\in J} \chi_j = 1$ and the support K_j of χ_j is contained in U_j for all j. For every j we choose $t_j > 0$ such that $\Delta_{t_j} \cap (K_j \times M^k)$ is contained in U_j^{k+1}. Since $p_j([z]) = 0$ there exists $d_j \in C^\infty(U_j^{k+2})$ such that $b(d_j) = z|_{U_j^{k+1}}$. Define $c_j \in C_c^\infty(U_j^{k+2})$ by

$$c_j(x) = \eta_j(x)h_j(x)d_j(x)$$

where h_j is a function in $C_c^\infty(U_j^{k+2})$ such that $h_j = 1$ on K_j^{k+2} and $\eta_j \in C^\infty(U_j^{k+2})$ is the pull-back of χ_j along the projection onto the first factor. Observe that $\sum_{j\in J} \eta_j z = z$ and that the chains $\eta_j z$ are again cycles. By construction, the functions $b(c_j)$ and $\eta_j z$ coincide on the set K_j^{k+1}. Moreover c_j can be viewed as a chain in $C_{k+1}(C^\infty(M))$ if we extend it by zero outside U_j^{k+2}. We define an element $c \in C_{k+1}(C^\infty(M))$ by

$$c = \sum_{j\in J} c_j.$$

Note that this infinite sum is well-defined since locally only finitely many summands are nonzero. By construction of c the element $d = z - b(c)$ is contained in the kernel of $J^\infty : C_k(C^\infty(M)) \to J^\infty C_k(C^\infty(M))$. It follows that $[J^\infty(d)] = 0$ in $H_k(J^\infty C(C^\infty(M)))$. Since J^∞ is a quasiisomorphism according to proposition 4.42 we deduce $[d] = 0$. Hence $[z] = [d] = 0$ in $HH_k(C^\infty(M))$ which yields the claim. □

Choose a locally finite open covering $(U_j)_{j\in J}$ of M by coordinate domains such that all charts ϕ_i identify U_i with some convex open neighborhood of zero in $\mathbb{R}^n$. Consider the commutative diagram

$$\begin{array}{ccc} HH_*(C^\infty(M)) & \longrightarrow & \prod_{j\in J} HH_*(C^\infty(U_j)) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \prod \alpha} \\ \Omega(M) & \longrightarrow & \prod_{j\in J} \Omega(U_j) \end{array}$$

where the horizontal maps are induced by restriction to the open sets U_j. The upper horizontal arrow is injective by proposition 4.43. The right vertical arrow is an isomorphism according to theorem 4.37. Hence the left vertical arrow α is injective. We have seen in section 4 that the map $\alpha : HH_*(C^\infty(M)) \to \Omega(M)$ is surjective. Hence the Hochschild-Kostant-Rosenberg map for M is an isomorphism. This finishes the proof of the Hochschild-Kostant-Rosenberg theorem 4.36.

6. Cyclic homology and periodic cyclic homology

In this section we calculate the cyclic homology and periodic cyclic homology of $C^\infty(M)$. Using the Hochschild-Kostant-Rosenberg theorem 4.36 this is quite easy. We begin with cyclic homology.

THEOREM 4.44. *Let M be a smooth manifold. Then the cyclic homology of $C^\infty(M)$ is given by*

$$HC_n(C^\infty(M)) \cong \mathcal{A}^n(M)/d\mathcal{A}^{n-1}(M) \oplus \bigoplus_{j>0} H^{n-2j}_{dR}(M).$$

PROOF. According to theorem 4.36 and lemma 3.33 the cyclic homology of $C^\infty(M)$ is isomorphic to the cyclic homology of the mixed complex $\mathcal{A}(M)$ with $b = 0$ and $B = d$. The cyclic homology of this mixed complex is equal to the right hand side of the above formula. □

It is instructive to determine the explicit form of the maps S, B and I relating Hochschild and cyclic homology. For I and S this can immediately be read off from the mixed complex $\mathcal{A}(M)$. The map $I : HH_n(C^\infty(M)) \to HC_n(C^\infty(M))$ is given by the natural projection $\mathcal{A}^n(M) \to \mathcal{A}^n(M)/d\mathcal{A}^{n-1}(M)$. The periodicity operator $S : HC_n(C^\infty(M)) \to HC_{n-2}(C^\infty(M))$ kills the first summand $\mathcal{A}^n(M)/d\mathcal{A}^{n-1}(M)$, is the obvious map $H^{n-2}_{dR}(M) \to \mathcal{A}^{n-2}(M)/d\mathcal{A}^{n-3}(M)$ on the second component and the identity on the remaining summands. Finally, for $B : HC_n(C^\infty(M)) \to HH_{n+1}(C^\infty(M))$ we apply lemma 3.32 and obtain that this homomorphism can be identified with the map $d : \mathcal{A}^n(M)/d\mathcal{A}^{n-1}(M) \to \mathcal{A}^{n+1}(M)$.

Let us now consider periodic cyclic homology.

THEOREM 4.45. *Let M be a smooth manifold. The periodic cyclic homology of $C^\infty(M)$ is given by*

$$HP_*(C^\infty(M)) \cong \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_{dR}(M).$$

PROOF. According to theorem 4.36 and proposition 3.55 the periodic cyclic homology of $C^\infty(M)$ is isomorphic to the periodic cyclic homology of the mixed complex $\mathcal{A}(M)$. The latter is easily seen to be equal to the right hand side of the above formula. □

As a consequence, one may view periodic cyclic homology as a noncommutative analogue of de Rham cohomology. Indeed, in the general framework of noncommutative geometry, cyclic homology plays a role similar to the one of de Rham cohomology in differential geometry.

7. The classical Chern character

In this section we recall the classical Chern-Weil construction of the Chern character and compare it with the noncommutative Chern character introduced in chapter 3. Throughout this section we assume that M is a compact smooth manifold and that all modules over $A = C^\infty(M)$ are unitary. Moreover, we tacitly view A-modules as left, right or bimodules using that A is commutative.

The K-group $K^0(M)$ of the manifold M is equal to $K_0(C^\infty(M))$ provided M is compact. According to the following classical result, the group $K_0(C^\infty(M))$ may be viewed as the group of stable isomorphim classes of smooth complex vector bundles over M.

PROPOSITION 4.46 (Serre-Swan). *Let M be a compact smooth manifold. Then the category of smooth complex vector bundles over M is equivalent to the category of finitely generated projective modules over $C^\infty(M)$.*

PROOF. If V is a smooth vector bundle over M then the space $C^\infty(M, V)$ of smooth sections of V becomes a unitary $C^\infty(M)$-module by pointwise multiplication. Clearly every vector bundle morphism $\phi : V \to W$ induces a module homomorphism $C^\infty(M, \phi) : C^\infty(M, V) \to C^\infty(M, W)$. Since every vector bundle over M is a direct summand in a free bundle $M \times \mathbb{C}^n$ for some n it is easily seen that $C^\infty(M, V)$ is actually a finitely generated projective module.

Conversely, assume that the unitary projective $C^\infty(M)$-module P is represented as $P = C^\infty(M)^n \cdot p \subset C^\infty(M)^n$ for some idempotent matrix $p \in M_n(C^\infty(M))$. Define $V \subset M \times \mathbb{C}^n$ by $V = \bigcup_{m \in M} V_m$ where V_m is the image of the evaluation map $ev_m : C^\infty(M)^n \cdot p \to \mathbb{C}^n$ at m. If the dimension of the vector space V_m is k then there exists a small neighborhood U of m such that $\dim(V_x) \geq k$ for all $x \in U$. Applying the same argument to the projective module corresponding to $1 - p$ we see that the dimension of the fibers is locally constant. Choosing elements $e_1, \dots, e_k \in P$ such that around $e_1(m), \dots, e_k(m)$ form a basis for $V(m)$ yields a trivialization of V in a neighborhood of m. It follows that V defines indeed a smooth vector bundle over M. The module $C^\infty(M, V)$ of sections of this bundle is naturally isomorphic to P. □

Recall that $A = C^\infty(M)$ denotes the algebra of smooth functions on the manifold M. We will write $\mathcal{A}^k(A)$ for the space $\mathcal{A}^k(M)$ of differential k-forms. This notation is motivated by the fact that parts of the discussion in the sequel may be generalized to arbitrary commutative algebras. If B is a commutative algebra, one may actually define a space $\mathcal{A}^k(B)$ of (commutative) differential k-forms over B. In the case $A = C^\infty(M)$ one reobtains the space of differential forms in the usual sense. Although we will not discuss this more general approach here, it is remarkable since it provides a very algebraic description of (ordinary) differential forms.

Let P be an A-module. Then $P \otimes_A \mathcal{A}(A)$ is a graded vector space where the grading is induced by the degree of a differential form. Recall that a linear map $f : V \to W$ of graded vector spaces has degree k if $f(V_n) \subset W_{n+k}$ for all n.

Definition 4.47. *Let P be an A-module. A connection on P is a linear map $\nabla : P \otimes_A \mathcal{A}(A) \to P \otimes_A \mathcal{A}(A)$ of degree 1 which satisfies*

$$\nabla(s\omega) = \nabla(s)\omega + (-1)^n s d\omega$$

for all $s \in P \otimes_A \mathcal{A}^n(A)$ and $\omega \in \mathcal{A}(A)$.

Here $P \otimes_A \mathcal{A}(A)$ is viewed as a right $\mathcal{A}(A)$-module in the obvious way. If $P = C^\infty(M, V)$ for a complex vector bundle V over M we also say that ∇ is a connection on V.

Let us first show that connections exist for all finitely generated projective modules. According to proposition 4.46 this is equivalent to showing that every vector bundle over M admits a connection.

Proposition 4.48. *Let P be a finitely generated projective A-module. Then there exists a connection on P.*

Proof. If $P = A^n$ is a free module of rank n we have $P \otimes_A \mathcal{A}(A) = A^n \otimes_A \mathcal{A}(A) = \mathcal{A}(A)^n$. In this case the map $d^{\oplus n}$ defined by

$$d^{\oplus n}(\omega_1, \dots, \omega_n) = (d\omega_1, \dots, d\omega_n)$$

is a connection where d is the exterior derivative. In general, P is a direct summand of A^n for some n. Hence there exist A-module maps $\iota : P \to A^n$ and $\pi : A^n \to P$ such that $\pi\iota = \mathrm{id}$. We define a map $\nabla : P \otimes_A \mathcal{A}(A) \to P \otimes_A \mathcal{A}(A)$ of degree 1 using the commutative diagram

$$\begin{array}{ccc} A^n \otimes_A \mathcal{A}(A) & \xrightarrow{d^{\oplus n}} & A^n \otimes_A \mathcal{A}(A) \\ \uparrow{\scriptstyle \iota \otimes \mathrm{id}} & & \downarrow{\scriptstyle \pi \otimes \mathrm{id}} \\ P \otimes_A \mathcal{A}(A) & \xrightarrow{\nabla} & P \otimes_A \mathcal{A}(A) \end{array}$$

It is straightforward to check that ∇ is indeed a connection. □

We shall now define the curvature of a connection.

DEFINITION 4.49. *Let $\nabla : P \otimes_A \mathcal{A}(A) \to P \otimes_A \mathcal{A}(A)$ be a connection on an A-module P. The curvature of ∇ is the linear map*

$$\nabla\nabla : P = P \otimes_A \mathcal{A}^0(A) \to P \otimes_A \mathcal{A}^2(A).$$

We will write R or R_∇ for the curvature of a connection ∇.

LEMMA 4.50. *Let ∇ be a connection on the A-module P. Then the map $\nabla\nabla : P \otimes_A \mathcal{A}(A) \to P \otimes_A \mathcal{A}(A)$ is $\mathcal{A}(A)$-linear. In particular, the curvature R of ∇ is an A-module map.*

PROOF. We compute

$$\begin{aligned}\nabla\nabla(s\omega) &= \nabla(\nabla(s)\omega + (-1)^n s d\omega) \\ &= \nabla\nabla(s)\omega + (-1)^{n+1}\nabla(s)d\omega + (-1)^n\nabla(s)d\omega + s \otimes d^2(\omega) = \nabla\nabla(s)\omega\end{aligned}$$

for $s \in P \otimes \mathcal{A}^n(A)$ and $\omega \in \mathcal{A}(A)$. This shows that $\nabla\nabla$ is $\mathcal{A}(A)$-linear. In particular R is A-linear. □

Let P be an A-module. Then there is a natural linear map $\Phi : \mathrm{End}_A(P) \otimes_A \mathcal{A}(A) \to \mathrm{Hom}_A(P, P \otimes_A \mathcal{A}(A))$ defined by

$$\Phi(\phi \otimes \omega)(s) = \phi(s) \otimes \omega.$$

Observe that since A is commutative it does not matter if we view P as a left or right module and wether we use the A-module structure of $\mathrm{End}_A(P) = \mathrm{Hom}_A(P, P)$ coming from the first or second variable.

PROPOSITION 4.51. *Let P be a finitely generated projective A-module. Then the natural map*

$$\Phi : \mathrm{End}_A(P) \otimes_A \mathcal{A}(A) \to \mathrm{Hom}_A(P, P \otimes_A \mathcal{A}(A))$$

is an isomorphism.

PROOF. Let $f_1, \dots f_n \in \mathrm{Hom}_A(P, A)$ and $p_1, \dots, p_n \in P$ be elements satsifying the conditions of the dual basis lemma 1.33. We define a map $\Psi : \mathrm{Hom}_A(P, P \otimes_A \mathcal{A}(A)) \to \mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$ by

$$\Psi(\phi) = \sum_{i,j=1}^{n} db(p_i \otimes f_j) \otimes (f_i \otimes \mathrm{id})\phi(p_j).$$

Then one computes

$$\Phi\Psi(\phi)(s) = \sum p_i f_j(s)(f_i \otimes \mathrm{id})\phi(p_j) = \sum (db(p_i \otimes f_i) \otimes \mathrm{id})\phi(s) = \phi(s)$$

and

$$\Psi\Phi(f \otimes \omega) = \sum db(p_i \otimes f_j) \otimes f_i(f(p_j))\omega = f \otimes \omega$$

using the dual basis lemma. Hence Ψ is inverse to the natural map Φ. □

Assume that ∇ is a connection on the finitely generated projective module P. Using proposition 4.51 we may define a linear map $\mathrm{ad}(\nabla) : \mathrm{End}_A(P) \otimes_A \mathcal{A}(A) \to \mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$ by

$$\mathrm{ad}(\nabla)(\alpha) = \nabla\alpha - (-1)^{|\alpha|}\alpha\nabla$$

for $\alpha \in \mathrm{Hom}_A(P, P \otimes_A \mathcal{A}(A)) \cong \mathrm{Hom}_{\mathcal{A}(A)}(P \otimes_A \mathcal{A}(A), P \otimes_A \mathcal{A}(A))$ equipped with the natural grading. To check that $\mathrm{ad}(\nabla)(\alpha)$ is indeed $\mathcal{A}(A)$-linear we compute

$$\begin{aligned}\mathrm{ad}(\nabla)(\alpha)(s\omega) &= \nabla\alpha(s\omega) - (-1)^{|\alpha|}\alpha\nabla(s\omega) \\ &= \nabla(\alpha(s)\omega) - (-1)^{|\alpha|}\alpha(\nabla(s)\omega) - (-1)^{n+|\alpha|}\alpha(sd\omega) \\ &= \nabla\alpha(s)\omega + (-1)^{|\alpha(s)|}\alpha(s)d\omega - (-1)^{|\alpha|}\alpha\nabla(s)\omega - (-1)^{|\alpha(s)|}\alpha(s)d\omega \\ &= \mathrm{ad}(\nabla)(\alpha)(s)\omega\end{aligned}$$

for a homogenous element $\alpha \in \mathrm{Hom}_{\mathcal{A}(A)}(P \otimes_A \mathcal{A}(A), P \otimes_A \mathcal{A}(A))$ and elements $s \in P \otimes_A \mathcal{A}^n(A), \omega \in \mathcal{A}(A)$.
Moreover we may view the curvature R of ∇ as an element in $\mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$ using lemma 4.50 and proposition 4.51.

Lemma 4.52. *The curvature R of the connection ∇ satisfies* $\mathrm{ad}(\nabla)(R) = 0$ *in* $\mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$.

Proof. Since R has degree 2 we compute

$$\mathrm{ad}(\nabla)(R) = \nabla R - R\nabla = \nabla\nabla^2 - \nabla^2\nabla = \nabla^3 - \nabla^3 = 0$$

in $\mathrm{Hom}_{\mathcal{A}(A)}(P \otimes_A \mathcal{A}(A), P \otimes_A \mathcal{A}(A))$ which proves the claim. □
The dual basis lemma 1.33 yields an isomorphism $\mathrm{End}_A(P) \cong P \otimes P^*$ for every finitely generated projective module P. One may thus define a map $\mathrm{tr} : \mathrm{End}_A(P) \to A$ by $\mathrm{tr}(p \otimes f) = f(p)$. It is easy to check that tr is indeed a trace on the algebra $\mathrm{End}_A(P)$ and that it coincides with the natural trace on $M_n(A)$ if $P = A^n$ is free of finite rank. Moreover $\mathrm{tr} : \mathrm{End}_A(P) \to A$ is A-linear.

Lemma 4.53. *Let P be a finitely generated projective A-module. Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{End}_A(P) \otimes_A \mathcal{A}(A) & \xrightarrow{\mathrm{ad}(\nabla)} & \mathrm{End}_A(P) \otimes_A \mathcal{A}(A) \\ \downarrow{\scriptstyle \mathrm{tr} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{tr} \otimes \mathrm{id}} \\ \mathcal{A}(A) & \xrightarrow{\quad d \quad} & \mathcal{A}(A) \end{array}$$

where d is the exterior differential.

Proof. Observe that the assertion holds for a direct sum $P \oplus Q$ iff it holds for P and Q. Thus it suffices to consider the case of a free module of finite rank which in turn reduces to the case $P = A$. Using that ∇ satisfies the Leibniz rule the calculation

$$\mathrm{ad}(\nabla)(\Omega)(1) = \nabla(\omega) - (-1)^n \Omega\nabla(1) = \nabla(1)\omega + d\omega - (-1)^n \omega\nabla(1) = d\omega$$

yields the claim where $\omega \in \mathcal{A}^n(A)$ is identified with a right $\mathcal{A}(A)$-linear map $\Omega : \mathcal{A}(A) \to \mathcal{A}(A)$ in the obvious way. □
Observe that $\mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$ is an algebra in a natural way. Moreover let ∇ be a connection on P with curvature R. Since $\mathcal{A}^k(A) = 0$ for $k > n = \dim(M)$ and R is homogenous of degree 2 the expression

$$\exp(-R) = \sum_{j=0}^{\infty} \frac{(-1)^j R^j}{j!}$$

reduces to a finite sum and defines an element $\exp(-R) \in \mathrm{End}_A(P) \otimes_A \mathcal{A}(A)$. Using lemma 4.52 and lemma 4.53 one obtains

$$d(\mathrm{tr}(\exp(-R)) = \mathrm{tr}(\mathrm{ad}(\nabla)(\exp(-R))) = 0$$

for this element where we have written tr instead of $\mathrm{tr} \otimes \mathrm{id}$. It follows that

$$\mathrm{ch}(P, \nabla) = \mathrm{tr}(\exp(-R))$$

defines a cohomology class in the even de Rham cohomology $H^{ev}_{dR}(M)$ of M. If $P = C^\infty(M, V)$ for a complex vector bundle V over M we also write $\mathrm{ch}(V, \nabla)$ instead of $\mathrm{ch}(P, \nabla)$.
Let us show that this cohomology class does not dependent on the choice of the connection ∇.

LEMMA 4.54. *Let ∇_0 and ∇_1 be connections on a complex vector bundle V. Then*

$$\mathrm{ch}(V, \nabla_0) = \mathrm{ch}(V, \nabla_1)$$

in $H^*_{dR}(M)$.

PROOF. We denote by $P = C^\infty(M, V)$ the projective module corresponding to V. Consider the compact manifold $M[0,1] = M \times [0,1]$ and let $A[0,1] = C^\infty(M[0,1])$. There is an obvious homomorphism $A \to A[0,1]$ induced by the canonical projection $M[0,1] \to M$. Consider the $A[0,1]$-module $P[0,1] = P \otimes_A A[0,1]$. Geometrically, $P[0,1]$ corresponds to the pull-back bundle of V along the map $M[0,1] \to M$. Let us define a linear map $\nabla : P[0,1] \to P[0,1] \otimes_{A[0,1]} \mathcal{A}(A[0,1]) = P \otimes_A \mathcal{A}(A[0,1])$ by

$$\nabla(s \otimes f) = (1-t)\nabla_0(sf(t)) + t\nabla_1(sf(t)) + s\frac{\partial f}{\partial t}dt$$

for $s \in P$ and $f \in A[0,1]$. Here $\nabla_i(sf(t))$ is viewed as an element of $P \otimes_A \mathcal{A}(A[0,1])$ using the natural map $\mathcal{A}(A) \to \mathcal{A}(A[0,1])$. One has

$$\begin{aligned}\nabla(s \otimes fg)(t) &= (1-t)(\nabla_0(sf(t))g(t) + sf(t)dg(t)) + \\ &\qquad t(\nabla_1(sf(t))g(t) + sf(t)dg(t)) + s\frac{\partial(fg)}{\partial t}(t) \\ &= \nabla(s \otimes f)(t)g(t) + sf(t)(dg)(t)\end{aligned}$$

for all $s \in P[0,1]$ and $f, g \in A[0,1]$. The map ∇ can be extended to a connection $\nabla : P[0,1] \otimes_{A[0,1]} \mathcal{A}(A[0,1]) \to P[0,1] \otimes_{A[0,1]} \mathcal{A}(A[0,1])$ using the Leibniz rule. Now let $\iota_t : M \to M[0,1]$ be the inclusion of M into $M \times [0,1]$ at the point $t \in [0,1]$. The image of $\mathrm{ch}(V[0,1], \nabla)$ under the map $H^*_{dR}(M[0,1]) \to H^*_{dR}(M)$ induced by ι_i is equal to $\mathrm{ch}(V, \nabla_i)$ for $i = 0, 1$. According to proposition 4.32, that is, by homotopy invariance of de Rham cohomology, the maps $H^*_{dR}(M[0,1]) \to H^*_{dR}(M)$ induced by ι_0 and ι_1 are equal. Hence we obtain $\mathrm{ch}(V, \nabla_0) = \mathrm{ch}(V, \nabla_1)$. □

We may now define the classical Chern character.

DEFINITION 4.55. *Let M be a compact manifold and let V be a complex vector bundle over M. The (classical) Chern character of V is the cohomology class*

$$\mathrm{ch}(V) \in H^{ev}_{dR}(M)$$

defined as above using an arbitrary connection on V.

LEMMA 4.56. *Let V be a complex vector bundle over M determined by the idempotent $e \in M_n(A)$ according to $C^\infty(M, V) = eA^n$. Then*

$$\mathrm{ch}(V) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\,\mathrm{tr}(e(dede)^k)$$

in $H^*_{dR}(M)$.

PROOF. The Levi-Civita connection $\nabla : eA^n \to eA^n \otimes_A \mathcal{A}(A)$ is computed by

$$\nabla(a_1, \dots, a_n) = e \cdot (da_1, \dots, da_n) = \left(\sum e_{1j_1}da_{j_1}, \dots, \sum e_{nj_n}da_{j_n}\right)$$

for $a_1, \dots, a_n \in A^n e$ and $e = (e_{ij})$. It follows that the curvature of this connection is given by

$$R(a_1, \dots, a_n) = \left(\sum e_{1i_1}de_{i_1j_1}da_{j_1}, \dots, \sum e_{ni_n}de_{i_nj_n}da_{j_n}\right).$$

We compute for every r

$$\begin{aligned}\sum_{i,j} e_{ri}de_{ij}da_j &= \sum_{i,j,k} e_{ri}de_{ij}de_{jk}a_k + e_{ri}de_{ij}e_{jk}da_k \\ &= \sum_{i,j,k} e_{ri}de_{ij}de_{jk}a_k + e_{ri}de_{ik}da_k - e_{rj}de_{jk}da_k\end{aligned}$$

which implies

$$\sum_{i,j} e_{ri}de_{ij}da_j = \sum_{i,j,k} e_{ri}de_{ij}de_{jk}a_k$$

and thus

$$R(a_1,\dots,a_n) = \Big(\sum e_{1i_1}de_{i_1j_1}de_{j_1k_1}a_{k_1},\dots,\sum e_{ni_n}de_{i_nj_n}de_{j_nk_n}a_{k_n}\Big).$$

If $de \in M_n(A)\otimes_A \mathcal{A}(A)$ is the matrix with entries (de_{ij}) and tr denotes the trace map the relation

$$(edede)^k = e(dede)^k$$

yields the assertion. The latter is easily proved by induction taking into account that e is idempotent. ☐

Proposition 4.57. *The classical Chern character determines an additive map $K^0(M) \to H^{ev}_{dR}(M)$.*

Proof. Using that $\mathrm{tr} : \mathrm{End}_A(P) \to A$ is invariant under conjugation one easily checks that $\mathrm{ch}(P)$ depends only on the isomorphism class of the finitely generated projective module P. The assertion that ch is additive with respect to direct sums follows easily from lemma 4.56 and the additivity of tr. ☐

Proposition 4.58. *Let M be a compact smooth manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K_0(C^\infty(M)) & \xrightarrow{\mathrm{ch}} & H^*_{dR}(M) \\ \big\downarrow{\scriptstyle \mathrm{ch}_0} & & \big\| \\ HP_0(C^\infty(M)) & \xrightarrow{\alpha} & H^*_{dR}(M) \end{array}$$

Hence the Chern character in cyclic homology coincides with the classical Chern character.

Proof. It suffices to compare the images of an idempotent $e \in M_n(A)$ under the maps $\alpha\,\mathrm{ch}_0$ and ch. Composition of the Chern character in cyclic homology with the Hochschild-Kostant-Rosenberg map yields the class

$$\alpha\,\mathrm{ch}_0(e) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(-1)^k\frac{(2k)!}{k!}\,\mathrm{tr}\Big(\Big(e-\frac{1}{2}\Big)(dede)^k\Big) = \sum_{k=0}^{\infty}(-1)^k\frac{1}{k!}\,\mathrm{tr}(e(dede)^k)$$

in $H^*_{dR}(M)$ where we use the fact that the differential form $\mathrm{tr}((dede)^k) \in \mathcal{A}^{2k}(M)$ is closed. According to lemma 4.56 this is precisely the class defining the classical Chern character of e. ☐

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