Equivariant cyclic homology

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Introduction

In the general framework of noncommutative geometry cyclic homology plays the role of de Rham cohomology [**26**]. It was introduced by Connes [**25**] as the target of the noncommutative Chern character. Besides cyclic cohomology itself Connes also defined periodic cyclic cohomology. The latter is particularly important because it is the periodic theory that gives de Rham cohomology in the commutative case.

The original definition of cyclic cohomology given by Connes is very explicit and convenient for applications. However, there is no geometric picture like in classical de Rham theory and it is difficult to establish general homological properties of cyclic homology starting from this definition. In a series of papers Cuntz and Quillen developed a different approach to cyclic homology theories based on the X-complex [**27**], [**28**], [**29**], [**30**]. They were able to prove excision in bivariant periodic cyclic homology in this framework. Moreover the Cuntz-Quillen formalism provides a more conceptual and geometric definition of the theory and is the basis for analytic versions of cyclic homology [**53**], [**55**].

In this thesis we develop a general framework in which cyclic homology can be extended to the equivariant context. Special cases of our theory have been defined and studied by various authors [**14**], [**17**], [**18**], [**19**], [**20**], [**47**], [**48**]. All these approaches are limited to actions of compact Lie groups or even finite groups. Hence a substantial open problem was how to treat non-compact groups. Even for compact Lie groups an important open question was how to give a correct definition of equivariant cyclic cohomology (in contrast to homology) apart from the case of finite groups.

We will define and study bivariant equivariant periodic cyclic homology $HP^G_*(A, B)$. In order to explain the main features in a clear way we restrict ourselves to the case that G is a discrete group. However, we remark that a large part of the general theory can be developed as well for totally disconnected groups or Lie groups, for instance. As a technical ingredient we have chosen to work in the setting of bornological vector spaces. In this way we obtain the purely algebraic approach as well as a topological version of the theory in a unified fashion.

Our account follows the Cuntz-Quillen approach to cyclic homology. In fact a certain part of the Cuntz-Quillen machinery can be carried over to the equivariant situation without change. However, a completely new feature in the equivariant theory is that the basic objects are not complexes in the sense of homological algebra. More precisely, we introduce an equivariant version X_G of the X-complex but the differential ∂ in X_G does not satisfy $\partial^2 = 0$ in general. To describe this behaviour we say that X_G is a paracomplex. It turns out that in order to obtain ordinary complexes it is crucial to work in the bivariant setting from the very beginning. Although many tools from homological algebra are not available anymore the resulting theory is computable to some extent. We point out that the occurence of paracomplexes is also the reason why we only define and study the periodic theory HP^G_*. It seems to be unclear how ordinary equivariant cyclic homology HC^G_* can be defined correctly in general.

An important ingredient in the definition of HP^G_* is the algebra $\mathcal{K}_G$ of finite rank operators on $\mathbb{C}G$. The elements of $\mathcal{K}_G$ are finite matrices indexed by G. In particular the ordinary Hochschild homology and cyclic homology of this algebra are rather trivial. However, in the equivariant setting $\mathcal{K}_G$ carries homological information of the group G if it is viewed as a G-algebra equipped with the action induced from the regular representation. This should be compared with the properties of the total space EG of the universal principal bundle over the classifying space BG. As a topological space EG is contractible but its equivariant cohomology is the group cohomology of G. Moreover, in the classical theory an arbitrary action of G on a space X can be turned into a free action by replacing X with the G-space $EG \times X$. In our theory tensoring with the algebra $\mathcal{K}_G$ is used to associate to an arbitrary G-algebra another G-algebra which is free as a G-module. Roughly speaking, for a discrete group G the algebra $\mathcal{K}_G$ can be viewed as a noncommutative substitute for the space EG used in topology.

Let us now explain how the text is organized. In the first chapter we present some background material that allows to put our approach into a general perspective. We begin with a brief account to classical equivariant cohomology which is usually also referred to as equivariant Borel cohomology. After this we describe the fundamental work of Cartan which provides an alternative approach to equivariant cohomology in the case of smooth actions of compact Lie groups on manifolds. This is important for a conceptual understanding of our constructions since equivariant cyclic homology may be viewed as a noncommutative (and delocalized) version of the Cartan model. Moreover we give a basic introduction to cyclic homology. We review those aspects of the theory which have been extended to the equivariant context before and which have influenced our approach in an essential way. Finally we describe briefly various constructions of equivariant cohomology theories and equivariant Chern characters in the literature and explain how our constructions fit in there. We remark that all the results in this chapter are stated without proof and are not used later on.

The second chapter contains basic definitions and results which are needed in the sequel. First we give an introduction to the theory of bornological vector spaces. A bornology on a vector space V is a collection of subsets of V satisfying some conditions. The guiding example is given by the collection of bounded subsets of a locally convex vector space. For our purposes it is convenient to work with bornological vector spaces right from the beginning. In particular we describe the natural concept of a group action in this context. After this we introduce the category of covariant modules and explain in detail how covariant modules are related to equivariant sheaves. Moreover we study the structure of morphisms between covariant modules. Next we review some general facts about pro-categories. Since the work of Cuntz and Quillen [**30**] it is known that periodic cyclic homology is most naturally defined for pro-algebras. The same holds true in the equivariant situation where one has to consider pro-G-algebras. We introduce the pro-categories needed in our framework and fix some notation. Finally we define paracomplexes and paramixed complexes. As explained above, paracomplexes play an important role in our theory.

The third chapter is the central part of this thesis. It contains the definition of equivariant periodic cyclic homology and results about the general homological properties of this theory. First we define and study quasifree pro-G-algebras. This discussion extends in a straightforward way the theory of quasifree algebras introduced by Cuntz and Quillen. After this we define equivariant differential forms for pro-G-algebras and show that one naturally obtains paramixed complexes in this way. Equivariant differential forms are used to construct the equivariant X-complex $X_G(A)$ for a pro-G-algebra A. As we have mentioned before this leads to a paracomplex. We show that the paracomplexes obtained from the equivariant X-complex and from the Hodge tower associated to equivariant differential forms are homotopy equivalent. In this way we generalize one of the main results of Cuntz and Quillen to the equivariant setting. After these preparations we define bivariant equivariant periodic cyclic homology $HP^G_*(A, B)$ for pro-G-algebras A and B. We show that HP^G_* is homotopy invariant with respect to smooth equivariant homotopies and stable in a natural sense in both variables. Moreover we prove that HP^G_* satisfies excision in both variables. This shows on a formal level that HP^G_* shares important properties with equivariant KK-theory [**46**].

In the fourth chapter we continue our study of HP^G_*. First we discuss the special case of finite groups. As a result we see that our theory generalizes the constructions known before in this case. Moreover we prove a universal coefficient theorem which clarifies the structure of HP^G_* for finite groups and provides a tool for attacking computations using suitable SBI-sequences. In the second part of the chapter we compute HP^G_* in two special cases. More precisely, we prove homological versions of the Green-Julg theorem $HP^G_*(\mathbb{C}, A) \cong HP_*(A \rtimes G)$ for finite groups and its dual $HP^G_*(A, \mathbb{C}) \cong HP^*(A \rtimes G)$ for arbitrary discrete groups. This shows that HP^G_* behaves as expected from equivariant KK-theory.

In the final chapter we present a more concrete computation of HP^G_* by looking at group actions on simplicial complexes. First we have to discuss carefully the appropriate notion of smooth functions on a simplicial complex X. If the group G acts simplicially on X the corresponding algebra of smooth functions with compact support is a G-algebra in a natural way. Roughly speaking, it turns out that the bivariant cyclic theory HP^G_* for the resulting class of G-algebras is closely related to the bivariant equivariant cohomology theory introduced by Baum and Schneider [**7**]. Together with the results of Baum and Schneider this shows that our theory gives a completely new description of various constructions which existed in the literature. It also shows that HP^G_* behaves as expected in connection with the Baum-Connes conjecture.

At this point it is natural to ask if there exists a bivariant Chern character from equivariant KK-theory to (an appropriate version of) bivariant equivariant cyclic homology. However, this question will not be addressed here. We point out that, once the equivariant cyclic theory is modified appropriately in order to give reasonable results also for G-C^*-algebras, the existence of such a character should follow essentially from the universal property of equivariant KK-theory [**60**]. In the non-equivariant case the construction of a bivariant Chern character has been achieved by Puschnigg using local cyclic homology [**56**]. We also remark that for compact Lie groups and finite groups partial Chern characters have been defined before [**14**], [**48**].

Contents

CHAPTER 1

Background material

In this chapter we describe some background material related to the homology theory developed in this thesis. We begin with a brief discussion of equivariant cohomology in topology and review the fundamental results of Cartan on equivariant de Rham cohomology. In particular the Cartan model is important for our purposes since equivariant cyclic homology can be viewed as a noncommutative (and delocalized) version of the Cartan model. Apart from these classical topics we have included a brief introduction to cyclic homology. We review the work of Block and Getzler on equivariant differential forms and cyclic homology which has influenced our approach in many respects. In the last section we briefly discuss various constructions of equivariant Chern characters relating equivariant K-theory and K-homology to appropriate equivariant homology and cohomology theories. Although we do not touch upon the relations between equivariant cyclic homology and K-theory in this work it will become clear that our theory has strong connections to equivariant K-theory and KK-theory.

1. Borel cohomology

In this section we want to review the definition and some basic properties of equivariant Borel cohomology. Throughout we will be working with complex coefficients.
Let G be a topological group. The equivariant cohomology functor H^*_G assigns to every space X equipped with a continuous G-action a complex vector space $H^*_G(X)$. Before giving the definition of this functor let us discuss the motivating idea behind it.
In general the structure of the quotient X/G of a space by a group action is very poor. The quotient topology can be completely pathological as already simple examples show. From a modern point of view quotient spaces by group actions are in fact basic examples of noncommutative spaces [**26**].
For the definition of equivariant cohomology one first restricts attention to actions where the quotient is a nice space. More precisely, one considers the situation where the action of G on X is principal in the sense that the natural projection $X \to X/G$ defines a G-principal bundle. Roughly speaking, the quotient space X/G will then have similar topological properties as the space X itself. For instance, if a Lie group G acts smoothly on a smooth manifold X then X/G will again be a smooth manifold. The basic requirement on H^*_G is that for such actions the equivariant cohomology is given by

$$H^*_G(X) = H^*(X/G),$$

that is, the ordinary singular cohomology of the quotient space X/G.
In order to treat arbitrary actions the idea is as follows. One should no longer consider $H^*(X/G)$ since the structure of the quotient might be poor. Instead on first has to replace the G-space X in a natural way by a space of the same homotopy type equipped with a

principal action. This is motivated by the fact that cohomology is a homotopy invariant functor. The solution is to look at

$$EG \times X$$

where EG is the total space of the universal principal bundle over the classifying space BG of G. Since EG is contractible the spaces $EG \times X$ and X are homotopy equivalent. However, the cohomology of the corresponding quotients is in general quite different.

DEFINITION 1.1. *Let G be a topological group and let X be a G-space. The equivariant (Borel) cohomology of X with respect to G is defined by*

$$H^*_G(X) = H^*(EG \times_G X)$$

where EG denotes the universal principal bundle over the classifying space BG of G and H^* is singular cohomology.

Remark that one could use in principle any cohomology theory in definition 1.1 to obtain an associated equivariant (Borel) theory.
It is clear from the definition that H^*_G is in fact a functor on the category of G-spaces and equivariant maps. The equivariant cohomology of a point is equal to the cohomology $H^*(BG)$ of the classifying space of G and already quite interesting. Observe that there is a unique G-map from X to the one point space for every G-space X. Using the cup product in cohomology it is clear that this map induces a ring homomorphism $H^*(BG) \to H^*_G(M)$. This implies that the equivariant cohomology groups are modules over $H^*(BG)$. The importance of this module structure is due to the fact that $H^*(BG)$ is usually large and therefore its module category is interesting.
The functor H^*_G inherits many properties from singular cohomology. In particular it is easy to see that H^*_G is homotopy invariant with respect to equivariant homotopies. If $A \subset X$ is a G-subspace we define the relative cohomology group $H^*_G(X, A) = H^*(X \times_G EG, A \times_G EG)$. Using this definition we obtain a long exact sequence

$$\cdots \longrightarrow H^n_G(X, A) \longrightarrow H^n_G(X) \longrightarrow H^n_G(A) \longrightarrow H^{n+1}_G(X, A) \longrightarrow \cdots$$

in equivariant cohomology.
We do not discuss the general features of H^*_G further since our main interest is more special. In fact we want to focus on smooth actions of compact Lie groups on manifolds and look at the de Rham type description of equivariant cohomology in this context.

2. Equivariant de Rham cohomology

We describe the fundamental results of Cartan [**22**], [**23**] on equivariant de Rham cohomology. For a detailed modern exposition of the theory we refer to [**40**].
Let G be a compact Lie group acting smoothly on a smooth manifold M. In this situation it is natural to ask if there exists a de Rham model for the equivariant cohomology $H^*_G(M)$. Of course such a de Rham model should generalize the description of the cohomology of manifolds using differential forms.
Looking at the definition of H^*_G the main problem is how to define differential forms on $EG \times M$. Remark that in general the space EG cannot be a finite dimensional manifold. Hence this problem occurs already in the case that M is a point.
The solution is to give an abstract algebraic characterisation of the expected properties

of EG at the level of differential forms. This is similar to the fact that EG itself is determined by a universal property. First of all the desired substitute $\mathcal{A}(EG)$ for the algebra of differential forms on EG should be a commutative DG algebra. This reflects the basic structure of the algebra $\mathcal{A}(M)$ of (complex-valued) differential forms on a smooth manifold M. The fact that EG is contractible corresponds to the requirement that the cohomology of $\mathcal{A}(EG)$ is given by

$$H^j(\mathcal{A}(EG)) = \begin{cases} \mathbb{C} & j = 0 \\ 0 & j \geq 1. \end{cases}$$

Moreover one has to encode the action of G in an appropriate way. Recall that a smooth action of G on a manifold M gives rise to interior products $\iota(X) : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$ and Lie derivatives $\mathcal{L}(X) : \mathcal{A}^k(M) \to \mathcal{A}^k(M)$ for all elements X in the Lie algebra $\mathfrak{g}$ of G. Hence it is natural to require that $\mathcal{A}(EG)$ should be equipped with an action of G and such operators in a compatible way. Using the interior products $\iota(X)$ one can formulate a substitute for the condition that the action of G on EG is free. Namely, for each $X \in \mathfrak{g}$ there should exist an element $\omega \in \mathcal{A}^1(EG)$ such that $\iota(X)\omega$ is not zero.

It can be shown that for every compact Lie group G such a DG algebra $\mathcal{A}(EG)$ exists. Taking this for granted we shall describe how equivariant de Rham cohomology is defined. If M is a smooth G-manifold one uses $\mathcal{A}(EG) \otimes \mathcal{A}(M)$ as a substitute for differential forms on the product $EG \times M$ in the Borel construction. Since the goal is to describe the cohomology of the quotient $EG \times_G M$ one should consider only basic forms in $\mathcal{A}(EG) \otimes \mathcal{A}(M)$. As usual a form ω is called basic if it is G-invariant and satisfies $\iota(X)\omega = 0$ for all $X \in \mathfrak{g}$. We recall that if $p : P \to B$ is a smooth G-principal bundle this condition is satisfied precisely by those forms in $\mathcal{A}(P)$ which are pull-backs of forms on the base space B. Hence a natural candidate for the equivariant de Rham cohomology of M is the cohomology $H^*(\mathrm{Basic}(\mathcal{A}(EG) \otimes \mathcal{A}(M)))$ of the complex of basic forms in $\mathcal{A}(EG) \otimes \mathcal{A}(M)$. It can be checked that this cohomology is independent of the particular choice of $\mathcal{A}(EG)$. This is analogous to the fact that H^*_G does not depend on the specific model for EG used in its definition. Most importantly one can prove that there exists a natural isomorphism

$$H^*_G(M) \cong H^*(\mathrm{Basic}(\mathcal{A}(EG) \otimes \mathcal{A}(M)))$$

which can be referred to as equivariant de Rham theorem.

The advantage of this description clearly depends on whether there exists a convenient model for $\mathcal{A}(EG)$. We shall describe now the Weil complex which provides such a model. In order to do this recall that a connection on a smooth G-principal bundle $p : P \to B$ is given by a $\mathfrak{g}$-valued 1-form ω on P such that $\omega(X) = X$ for each fundamental vector field and $R_s^*(\omega) = (\mathrm{Ad}s^{-1})\omega$ for all $s \in G$ where R_s denotes right translation by s. We have the fundamental equations

$$d\omega + \frac{1}{2}[\omega, \omega] = \Omega, \qquad d\Omega = [\Omega, \omega]$$

where Ω is the curvature of the connection. The connection form $\omega \in \mathcal{A}^1(P, \mathfrak{g})$ induces a map $\mathfrak{g}^* \to \mathcal{A}^1(P)$ which will be denoted by $k(\omega)$. It can be continued to a map $k(\omega) : \Lambda(\mathfrak{g}^*) \to \mathcal{A}(P, \mathbb{R})$ of graded algebras where $\Lambda(\mathfrak{g}^*)$ is the exterior algebra over the real vector space $\mathfrak{g}^*$ and $\mathcal{A}(P, \mathbb{R})$ is the algebra of real-valued differential forms on P. Similarly, the curvature form $\Omega \in \mathcal{A}^2(P, \mathfrak{g})$ induces a map $\mathfrak{g}^* \to \mathcal{A}^2(P, \mathbb{R})$ which will be denoted by $k(\Omega)$.

This map can be extended to a map $k(\Omega) : S(\mathfrak{g}^*) \to \mathcal{A}(P)$ of graded algebras where $S(\mathfrak{g}^*)$ denotes the symmetric algebra over $\mathfrak{g}^*$. The Weil algebra $W(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is defined by

$$W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes_{\mathbb{R}} \Lambda(\mathfrak{g}^*).$$

There is a grading on $W(\mathfrak{g})$ given by $\deg(\xi \otimes \eta) = 2\deg(\xi) + \deg(\eta)$ for homogenous elements ξ and η. Given a connection on P with connection form ω the tensor product of the maps $k(\omega)$ and $k(\Omega)$ described above induces a linear map

$$\chi(\omega) : W(\mathfrak{g}) \to \mathcal{A}(P, \mathbb{R})$$

which preserves grading and product. One can define interior products $\iota(X)$ and Lie derivatives $\mathcal{L}(X)$ on $W(\mathfrak{g})$ for all $X \in \mathfrak{g}$, moreover an action of G and exterior differentiation in such a way that the map $\chi(\omega)$ preserves these structures.

Proposition 1.2. *There is a bijective correspondence between connections on P and DG algebra maps $W(\mathfrak{g}) \to \mathcal{A}(P, \mathbb{R})$ compatible with the Lie derivatives and the interior products.*

Consider the complexified Weil algebra $W(\mathfrak{g})_{\mathbb{C}} = W(\mathfrak{g}) \otimes_{\mathbb{R}} \mathbb{C}$. It can be shown that $W(\mathfrak{g})_{\mathbb{C}}$ is a model for $\mathcal{A}(EG)$. As a consequence of our discussion above we obtain the following theorem.

Theorem 1.3. *Let G be a compact Lie group. There exists a natural isomorphism*

$$H_G^*(M) \cong H^*(\mathrm{Basic}(W(\mathfrak{g})_{\mathbb{C}} \otimes \mathcal{A}(M)))$$

for all G-manifolds M.

The description of equivariant cohomology using the complex $\mathrm{Basic}(W(\mathfrak{g})_{\mathbb{C}} \otimes \mathcal{A}(M))$ is also referred to as *Weil model.* However, the computation of $\mathrm{Basic}(W(\mathfrak{g})_{\mathbb{C}} \otimes \mathcal{A}(M))$ is still difficult. Hence it is important that there exists an automorphism of $W(\mathfrak{g})_{\mathbb{C}} \otimes \mathcal{A}(M)$ which simplifies this computation. This leads to the *Cartan model* which we want to describe now. We identify $S(\mathfrak{g}^*) \otimes_{\mathbb{R}} \mathbb{C}$ with the algebra $\mathbb{C}[\mathfrak{g}]$ of polynomial functions on $\mathfrak{g}$. For a G-manifold M consider the algebra $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ of complex polynomial functions on $\mathfrak{g}$ with values in $\mathcal{A}(M)$. There is a G-action on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ obtained from the transpose of the adjoint action on $\mathbb{C}[\mathfrak{g}]$ and the natural action on $\mathcal{A}(M)$. Explicitly we have

$$(s \cdot p)(X) = s \cdot p(\mathrm{Ad}_{s^{-1}}(X))$$

for any polynomial p. The space

$$\mathcal{A}_G(M) = (\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M))^G$$

of invariants with respect to this G-action is called the space of equivariant differential forms on M. There exists a differential d_G on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$ defined by

$$(d_G p)(X) = d(p(X)) - \iota(X)p(X).$$

The Cartan relation $\mathcal{L}(X) = d\iota(X) + \iota(X)d$ yields $(d_G^2 p)(X) = -\mathcal{L}(X)p(X)$ on $\mathbb{C}[\mathfrak{g}] \otimes \mathcal{A}(M)$. This implies $d_G^2 = 0$ on $\mathcal{A}_G(M)$.

THEOREM 1.4. *Let G be a compact Lie group. There exists a natural isomorphism (called Mathai-Quillen isomorphism)*

$$H^*(\mathrm{Basic}(W(\mathfrak{g})_{\mathbb{C}} \otimes \mathcal{A}(M))) \cong H^*(\mathcal{A}_G(M), d_G)$$

for all G-manifolds M.

In combination with theorem 1.3 one obtains

THEOREM 1.5. *Let G be a compact Lie group. There exists a natural isomorphism*

$$H^*_G(M) \cong H^*(\mathcal{A}_G(M), d_G)$$

for all smooth G-manifolds M.

The Cartan model provides a convenient description of $H^*_G(M)$. For instance it is easy to compute the equivariant cohomology of a point in this framework. In fact sometimes the equivariant cohomology of manifolds is defined using the Cartan model. The geometry of equivariant differential forms is important for applications [**8**].

3. Cyclic homology

In this section we give a brief introduction to cyclic homology. Moreover we review some results of Brylinski on equivariant cyclic homology and related work of Block and Getzler.

Let A be an algebra over the complex numbers, possibly without unit. The noncommutative differential n-forms over A are defined by $\Omega^0(A) = A$ and

$$\Omega^n(A) = A^+ \otimes A^{\otimes n}$$

for $n > 0$ where A^+ denotes the unitarization of A. Typical elements in this tensor product are denoted by $x_0dx_1 \cdots dx_n$ or $dx_1 \cdots dx_n$ with $x_j \in A$. Apart from the obvious left A-module structure on $\Omega^n(A)$ there is a right A-module structure obtained from the Leibniz rule $d(xy) = dxy + xdy$. In this way $\Omega^n(A)$ becomes an A-bimodule.

There are some natural operators on noncommutative differential forms. First of all we have the analogue of the exterior differential d which is given by

$$d(x_0dx_1 \cdots dx_n) = dx_0dx_1 \cdots dx_n, \qquad d(dx_1 \cdots dx_n) = 0.$$

It is clear from the definition that $d^2 = 0$. Another important operator is the Hochschild boundary b. Using the A-bimodule structure of $\Omega^n(A)$ this operator is defined by

$$b(\omega dx) = (-1)^{n-1}(\omega x - x\omega), \qquad b(x) = b(dx) = 0$$

for $\omega \in \Omega^{n-1}(A)$ and $x \in A$. Explicitly we have

$$b(x_0dx_1 \cdots dx_n) = x_0x_1dx_2 \cdots dx_n + \sum_{j=1}^{n-1}(-1)^j x_0dx_1 \cdots d(x_jx_{j+1}) \cdots dx_n + (-1)^n x_nx_0dx_1 \cdots dx_{n-1}$$

for $x_0dx_1 \cdots dx_n \in \Omega^n(A)$. Combining d and b one obtains in addition two operators κ and B. The Karoubi operator κ is the map of degree zero defined by

$$\kappa = \mathrm{id} - (db + bd).$$

Explicitly we have

$$\kappa(\omega dx) = (-1)^{n-1} dx\omega$$

for $\omega \in \Omega^{n-1}(A)$ and $x \in A$. Finally, the operator B is given on $\Omega^n(A)$ by

$$B = \sum_{j=0}^{n} \kappa^j d.$$

The explicit formula for B is

$$B(x_0 dx_1 \cdots dx_n) = \sum_{j=0}^{n} (-1)^{nj} dx_{n+1-j} \cdots dx_n dx_0 \cdots dx_{n-j}.$$

Elementary computations yield the relations $b^2 = 0$, $B^2 = 0$ and $Bb + bB = 0$. In other words, noncommutative differential forms together with the operators b and B constitute a mixed complex [**28**], [**49**]. We obtain a bicomplex

$$\begin{tikzcd}
\arrow[d, "b"'] & \arrow[d, "b"'] & \arrow[d, "b"'] & \arrow[d, "b"'] \\
\Omega^3(A) \arrow[d, "b"'] & \Omega^2(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^1(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] \\
\Omega^2(A) \arrow[d, "b"'] & \Omega^1(A) \arrow[l, "B"'] \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] & \\
\Omega^1(A) \arrow[d, "b"'] & \Omega^0(A) \arrow[l, "B"'] & & \\
\Omega^0(A) & & &
\end{tikzcd}$$

which is usually called the (B, b)-bicomplex. The Hochschild homology $HH_*(A)$ of A is by definition the homology of the first column of this bicomplex. The cyclic homology $HC_*(A)$ of the algebra A is the homology of the total complex of the (B, b)-bicomplex. From the periodicity of the (B, b)-bicomplex it follows that Hochschild homology and cyclic homology are related by a long exact sequence

$$\cdots \longrightarrow HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} HC_{n-1}(A) \longrightarrow \cdots$$

which is referred to as the SBI-sequence. This relationship between $HH_*(A)$ and $HC_*(A)$ is very important for computations. Using periodicity the (B, b)-bicomplex can be continued to the left. The homology of the resulting total complex obtained by taking direct products over all terms of a fixed total degree is by definition the periodic cyclic homology $HP_*(A)$ of A. The periodic theory is $\mathbb{Z}_2$-graded and there exists a short exact sequence

$$0 \longrightarrow \varprojlim\nolimits^1_S HC_{2n+*+1}(A) \longrightarrow HP_*(A) \longrightarrow \varprojlim\nolimits_S HC_{2n+*}(A) \longrightarrow 0$$

relating $HP_*(A)$ to ordinary cyclic homology $HC_*(A)$.

We remark that all definitions can be adapted to the topological setting. If A is a locally convex algebra with jointly continuous multiplication one simply replaces algebraic tensor products by completed projective tensor products. With this in mind we can formulate the following fundamental result due to Connes [**25**].

THEOREM 1.6. *Let $C^\infty(M)$ be the Fréchet algebra of smooth functions on a smooth compact manifold M. Then there exists a natural isomorphism*

$$HP_*(C^\infty(M)) = \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_{dR}(M).$$

We point out that it is actually the periodic theory that gives de Rham cohomology. This is the reason why periodic cyclic homology usually is the theory of main interest.
The main ingredient in the proof of theorem 1.6 is an adaption of the classical Hochschild-Kostant-Rosenberg theorem [**41**] to the setting of smooth manifolds. More precisely, it can be shown that there is a natural isomorphism

$$HH_*(C^\infty(M)) \cong \mathcal{A}^*(M)$$

between the Hochschild homology of $C^\infty(M)$ and differential forms on M.
In the remaining part of this section we review some results which are important for the theory developed in this thesis. First we discuss equivariant cyclic homology essentially in the way it has been introduced in the work of Brylinski [**17**], [**18**]. Let A be a complete locally convex algebra with jointly continuous multiplication. Assume that A is equipped with a smooth action of a compact Lie group G in the sense of [**12**]. Actually we are mainly interested in the example of a smooth action of G on a compact manifold M which leads to a smooth action of G on the Fréchet algebra $C^\infty(M)$. Consider the space $\Omega^n_G(A) = C^\infty(G)\hat{\otimes}_\pi \Omega^n(A)$ where $\hat{\otimes}_\pi$ denotes the completed projective tensor product. The group G acts on $\Omega^n_G(A)$ using the diagonal action

$$t\cdot(f(s)\otimes x_0dx_1\cdots dx_n) = f(t^{-1}st)\otimes(t\cdot x_0)d(t\cdot x_1)\cdots d(t\cdot x_n).$$

The equivariant analogues of the operators d and b are given by

$$d(f(s)\otimes\omega) = f(s)\otimes d\omega$$

and

$$b_G(f(s)\otimes\omega dx) = (-1)^{n-1}(f(s)\otimes(\omega x - (s^{-1}\cdot x)\omega))$$

for $\omega\in\Omega^{n-1}(A)$ and $x\in A$. Using the same formulas as in the non-equivariant case one obtains the equivariant Karoubi operator $\kappa_G = \mathrm{id} - (b_Gd + db_G)$ and the equivariant analogue B_G of the operator B given by

$$B_G = \sum_{j=0}^n \kappa_G^j d$$

on $\Omega^n_G(A)$. It can be checked that on the *invariant subspace* $\Omega_G(A)^G\subset\Omega_G(A)$ the relations $b_G^2 = 0$, $B_G^2 = 0$ and $B_Gb_G + b_GB_G$ hold. Hence $\Omega_G(A)^G$ is a mixed complex complex and one defines equivariant Hochschild homology $HH^G_*(A)$, equivariant cyclic homology $HC^G_*(A)$ and equivariant periodic cyclic homology $HP^G_*(A)$ in the same way as in the non-equivariant situation.
It turns out that equivariant periodic cyclic homology is closely related to equivariant

K-theory. The following theorem was obtained by Brylinski [**17**] and independently by Block [**13**].

Theorem 1.7. *Let G be a compact Lie group acting smoothly on a smooth compact manifold M. There exists an equivariant Chern character*

$$ch_G : K^*_G(M) \to HP^G_*(C^\infty(M))$$

which induces an isomorphism

$$HP^G_*(C^\infty(M)) \cong \mathcal{R}(G) \otimes_{R(G)} K^*_G(M)$$

where $R(G)$ is the representation ring of G and $\mathcal{R}(G) = C^\infty(G)^G$ is the algebra of smooth conjugation invariant functions on G.

In order to clarify the notation we remark that the character map induces a natural ring homomorphism $R(G) \to \mathcal{R}(G)$.
Apart from this Block and Getzler have obtained a description of $HP^G_*(C^\infty(M))$ in terms of equivariant differential forms [**14**]. More precisely, there exists a G-equivariant sheaf $\Omega(M,G)$ over the group G itself viewed as a G-space with the adjoint action. The stalk $\Omega(M,G)_s$ at a group element $s \in G$ is given by germs of G_s-equivariant smooth maps from $\mathfrak{g}^s$ to $\mathcal{A}(M^s)$. Here $M^s = \{x \in M \,|\, s \cdot x = x\}$ is the fixed point set of s, G^s is the centralizer of s in G and $\mathfrak{g}^s$ is the Lie algebra of G_s. In particular the stalk $\Omega(M,G)_e$ at the identity element e is given by

$$\Omega(M,G)_e = C^\infty_0(\mathfrak{g}, \mathcal{A}(M))^G$$

where C^∞_0 is the notation for smooth germs at 0. Hence $\Omega(M,G)_e$ can be viewed as a certain completion of the classical Cartan model $\mathcal{A}_G(M)$. The global sections $\Gamma(G, \Omega(M,G))$ of the sheaf $\Omega(M,G)$ are called global equivariant differential forms and will be denoted by $\mathcal{A}(M,G)$. There exists a natural differential on $\mathcal{A}(M,G)$ extending the Cartan differential. Block and Getzler establish an equivariant version of the Hochschild-Kostant-Rosenberg theorem and deduce the following result.

Theorem 1.8. *Let G be a compact Lie group acting smoothly on a smooth compact manifold M. Then there is a natural isomorphism*

$$HP^G_*(C^\infty(M)) \cong H^*(\mathcal{A}(M,G)).$$

This theorem shows that equivariant cyclic homology can be viewed as a "delocalized" noncommutative version of the Cartan model. We refer to the next section for more information on delocalized cohomology theories. Theorem 1.8 also shows that the language of equivariant sheaves over the group G is necessary to describe equivariant cyclic homology correctly. Combining theorem 1.7 with theorem 1.8 yields the following result.

Theorem 1.9. *Let G be a compact Lie group acting smoothly on a smooth compact manifold M. Then there exists a natural isomorphism*

$$\mathcal{R}(G) \otimes_{R(G)} K^*_G(M) \cong H^*(\mathcal{A}(M,G)).$$

Hence, up to an "extension of scalars", the equivariant K-theory of manifolds can be described using global equivariant differential forms.

4. Equivariant Chern characters

In the previous section we have seen that equivariant periodic cyclic homology for compact Lie groups is connected to equivariant K-theory by a Chern character. In this section we want to describe briefly different approaches to the construction of equivariant cohomology theories and equivariant Chern characters which exist in the literature.
The classical result in this direction is the completion theorem of Atiyah and Segal [**2**] which describes the relation between equivariant K-theory and Borel cohomology. Consider the representation ring $R(G)$ of a compact Lie group G. Associating to a virtual representation its dimension yields a natural ring homomorphism $R(G) \to \mathbb{Z}$. The kernel of this homomorphism is referred to as the augmentation ideal. Remark that using the character map we may view $R(G)$ as a ring of complex-valued functions on G. In this picture the augmentation ideal is the kernel of the homomorphism which is given by evaluation at the identity. Now let M be a compact manifold equipped with a smooth action of G. The Atiyah-Segal completion theorem states (in particular) that the Borel cohomology of M is isomorphic to the localisation and completion of the complexified equivariant K-theory of M with respect to the augmentation ideal.
The geometric picture behind this phenomenon is that Borel cohomology is localized in the identity element of the group G. To obtain a theory closer to equivariant K-theory one should take into account contributions from all elements of G.
We have already seen that equivariant periodic cyclic homology can be viewed as such a delocalized equivariant cohomology theory. Preceeding the work of Block and Getzler, Baum, Brylinski and MacPherson defined a delocalized cohomology theory $H^*(G, M)$ for actions of abelian Lie groups [**3**] on manifolds. This theory is, roughly speaking, a combination of de Rham theory and the representation theory of closed subgroups of G. An important property of this theory is that there exists a Chern character $K^*_G(M) \to H^*(G, M)$ which becomes an isomorphism after tensoring with $\mathbb{C}$.
More closely related to cyclic homology is the theory $\mathcal{K}^*_G(M)$ developed by Duflo and Vergne [**31**]. In fact there exists a natural map $HP^G_*(C^\infty(M)) \to \mathcal{K}^*_G(M)$ provided G is a compact Lie group and M a smooth compact G-manifold. Duflo and Vergne conjecture that this natural map is an isomorphism.
In the context of discrete groups Lück and Oliver define Chern characters for equivariant K-theory with values in certain equivariant Bredon cohomology groups [**51**]. In [**50**] Lück has given a general construction of equivariant Chern characters for proper equivariant homology theories. These methods can be applied to obtain computations of the rationalized sources in the Farrel-Jones conjectures and the Baum-Connes conjecture.
An equivariant Chern character for discrete groups was defined earlier by Baum and Connes [**5**] using quite different methods.
Baum and Schneider introduced a delocalized bivariant equivariant cohomology theory $H^*_G(X, Y)$ for totally disconnected groups [**7**] which generalizes the construction of Baum and Connes. Under certain assumptions Baum and Schneider construct a bivariant equivariant Chern character

$$ch_G : KK^G_*(C_0(X), C_0(Y)) \to \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X, Y)$$

for profinite groups G which becomes an isomorphism after tensoring $KK^G_*(C_0(X), C_0(Y))$ with $\mathbb{C}$. Here KK^G_* denotes Kasparov's equivariant KK-theory [**46**].
It turns out that the theory of Baum and Schneider is closely related to bivariant equivariant cyclic homology HP^G_*. We will see that there exists a natural isomorphism

$$\bigoplus_{j\in\mathbb{Z}} H_G^{*+2j}(X,Y) \cong HP^G_*(C^\infty_c(X), C^\infty_c(Y))$$

provided X and Y are simplicial complexes and the action of G on X is proper. For the precise formulation of this statement and much more details we refer to chapter 5. Here we only remark that this result shows that HP^G_* fits nicely into existing constructions. Moreover we see that the theory is closely related to equivariant KK-theory.

CHAPTER 2

Basic definitions

As indicated in the title this chapter contains basic notions and results. We have ommitted many proofs and refer to the literature. However, important aspects are described in detail.
First we recall some material on bornological vector spaces and algebras. The framework of bornological vector spaces provides the foundation for all constructions in the sequel. In the second section we define complete bornological G-modules and G-algebras for a discrete group G. We study crossed products and look at some examples. This includes a discussion of the algebra $\mathcal{K}_G$ of finite rank operators on the regular representation of G. In the third section we introduce covariant modules. These objects play an important role in equivariant cyclic homology and are closely related to certain equivariant sheaves. In the fourth section we discuss some standard material on projective systems and pro-categories. The general constructions are applied to the category of G-modules and the category of covariant modules. Finally we introduce the notion of a paracomplex. This concept is needed to describe correctly the structure of equivariant cyclic homology.

1. Bornological vector spaces

In this section we recall basic definitions and results of the theory of bornological vector spaces and bornological algebras. We follow closely the treatment of [**53**] and refer also to [**42**], [**43**] for more information.
We begin with the definition of a convex bornological vector space. A subset S of a complex vector space V is called a disk if it is circled and convex. The disked hull $S^\diamond$ is the circled convex hull of S. To a disk $S \subset V$ we associate the semi-normed space V_S which is defined as the linear span of S endowed with the semi-norm $\|\cdot\|_S$ given by the Minkowski functional. The disk S is called norming if V_S is a normed space and completant if V_S is a Banach space.

Definition 2.1. *A collection $\mathfrak{S}$ of subsets of the vector space V is called a (convex) bornology on V if the following conditions are satisfied:*
a) $\{v\} \in \mathfrak{S}$ for all $v \in V$,
b) if $S \in \mathfrak{S}$ and $T \subset S$ then $T \in \mathfrak{S}$,
c) if $S_1, S_2 \in \mathfrak{S}$ then $S_1 + S_2 \in \mathfrak{S}$,
d) if $S \in \mathfrak{S}$ then $S^\diamond \in \mathfrak{S}$.
A vector space V together with a bornology $\mathfrak{S}$ is called a bornological vector space.

If $\mathfrak{S}$ is a bornology we call the elements $S \in \mathfrak{S}$ small sets. The bornological vector space V is called separated if all disks $S \in \mathfrak{S}$ are norming. It is called complete if each $S \in \mathfrak{S}$ is contained in a completant small disk $T \in \mathfrak{S}$. A complete bornological vector space is always separated. A linear map $f : V \to W$ between bornological vector spaces is called bounded if it maps small sets to small sets. The space of bounded linear maps from V to W is denoted by $\mathrm{Hom}(V, W)$.

We will usually only work with complete bornological vector spaces. To any bornological vector space V one can associate a complete bornological vector space V^c and a bounded linear map $\natural : V \to V^c$ such that composition with $\natural$ induces a bijective correspondence between bounded linear maps $V^c \to W$ with complete target W and bounded linear maps $V \to W$. In contrast to the completion in the category of locally convex vector spaces the natural map $\natural$ from a bornological vector space V into the completion V^c need not be injective.

Given an arbitrary collection X of subsets of a vector space V the bornology $\mathfrak{S}(X)$ generated by X is defined as the smallest collection of subsets of V containing X which satisfies the axioms for a bornology on V. The resulting bornological vector space may not be separated or complete. In order to obtain a complete bornological vector space one can complete V with respect to $\mathfrak{S}(X)$ in a second step.

In the category of complete bornological vector spaces direct sums, direct products, projective limits and inductive limits exist. In all these cases one has characterizations by universal properties.

Next we want to study subspaces and quotients of bornological vector spaces. In order to describe the properties of these constructions we need some more definitions. A sequence $(v_n)_{n\in\mathbb{N}}$ in a bornological vector space V is bornologically convergent towards $v \in V$ iff there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of scalars that converges to zero in the usual sense and a small subset $S \subset \mathfrak{S}(V)$ such that $v_n - v \in \epsilon_n S$ for all $n \in \mathbb{N}$. A subset of a bornological vector space is bornologically closed iff it is sequentially closed for bornologically convergent sequences.

If $V \subset W$ is a linear subspace of a complete bornological vector space W it inherits a natural bornology from W. A subset $S \subset V$ is declared to be bounded in the subspace bornology iff $S \in \mathfrak{S}(W)$. Equipped with the subspace bornology V is complete iff it is bornologically closed in W. The quotient space W/V also carries a natural bornology. A subset $S \subset W/V$ is small in the quotient bornology iff $S = T \mod W$ for some $T \in \mathfrak{S}(W)$. Equipped with the quotient bornology W/V is complete iff V is bornologically closed in W. Occasionally we will have to consider non-separated quotients. This will always be indicated carefully.

Now let us describe the tensor product of complete bornological vectors spaces. A bilinear map $l : V_1 \times V_2 \to W$ between bornological vector spaces is called bounded if $l(S_1 \times S_2) \in \mathfrak{S}(W)$ for all $S_j \in \mathfrak{S}(V_j)$. The completed bornological tensor product $V_1 \hat{\otimes} V_2$ is determined up to isomorphism by the universal property that every bounded bilinear map $V_1 \times V_2 \to W$ with complete target W corresponds uniquely to a bounded linear map $V_1 \hat{\otimes} V_2 \to W$ and the requirement that $V_1 \hat{\otimes} V_2$ is complete. It can be shown that the tensor product $\hat{\otimes}$ is associative in the sense that there is a natural isomorphism $(U \hat{\otimes} V) \hat{\otimes} W \cong U \hat{\otimes} (V \hat{\otimes} W)$ for all U, V, W. Moreover the tensor product is compatible with arbitrary inductive limits. The complete bornological tensor product $V_1 \hat{\otimes} V_2$ is constructed by completing the algebraic tensor product $V_1 \otimes V_2$ with respect to the bornology generated by the sets $S_1 \otimes S_2 = \{v_1 \otimes v_2 | v_1 \in S_1, v_2 \in S_2\}$ for small sets $S_1 \in \mathfrak{S}(V_1)$ and $S_2 \in \mathfrak{S}(V_2)$.

We will always use the following convention for the definition of bounded linear maps on tensor products. If for instance a bounded linear map $f : V_1 \hat{\otimes} V_2 \to W$ shall be defined we will simply write down a formula for f on the algebraic tensor product $V_1 \otimes V_2$. It is understood that this formula gives a bounded linear map and extends to the desired map f after completing the left hand side.

Definition 2.2. *A complete bornological algebra is a complete bornological vector space A with an associative multiplication given as a bounded linear map $m : A \hat{\otimes} A \to A$. A homomorphism between complete bornological algebras is a bounded linear map $f : A \to B$ which is compatible with multiplication.*

Remark that complete bornological algebras are not assumed to have a unit. Even if A and B have units a homomorphisms $f : A \to B$ need not preserve the unit of A. A homomorphism $f : A \to B$ between unital bornological algebras satisfying $f(1) = 1$ will be called a unital homomorphism.
We denote the unitarization of a complete bornological algebra A by A^+. It is the complete bornological algebra with underlying vector space $A \oplus \mathbb{C}$ and multiplication defined by $(a, \alpha) \cdot (b, \beta) = (ab + \alpha b + \beta a, \alpha\beta)$. If $f : A \to B$ is a homomorphism between complete bornological algebras there exists a unique extension to a unital homomorphism $f^+ : A^+ \to B^+$.
Let us discuss briefly the definition of modules over a complete bornological algebra A. A left A-module is a complete bornological vector space M together with a bounded linear map $\lambda : A \hat{\otimes} M \to M$ satisfying the axiom $\lambda(\mathrm{id} \hat{\otimes} \lambda) = \lambda(m \hat{\otimes} \mathrm{id})$ for an action. A homomorphisms $f : M \to N$ of A-modules is a bounded linear map commuting with the action of A. We denote by $\mathrm{Hom}_A(M, N)$ the space of all A-module homomorphisms. Let V be any complete bornological vector space. An A-module of the form $M = A^+ \hat{\otimes} V$ with action given by left multiplication is called the free A-module over V. If an A-module P is a direct summand in a free A-module it is called projective. Projective modules are characterized by the following property. If P is projective and $f : M \to N$ a surjective A-module homomorphism with a bounded linear splitting $s : N \to M$ then any A-module homomorphism $g : P \to N$ can be lifted to an A-module homomorphism $h : P \to M$ such that $fh = g$.
In a similar way one can define and study right A-modules and A-bimodules. We can also work in the unital category starting with a unital complete bornological algebra A. A unitary module M over a unital complete bornological algebra A is an A-module such that $\lambda(1 \otimes m) = m$ for all $m \in M$. In the category of unitary modules the modules of the form $A \hat{\otimes} V$ where V is a complete bornological vector space are free. Projective modules are again direct summands of free modules and can be characterized by a lifting property as before.
After this brief introduction to the general theory of bornological vector spaces and algebras we shall have a look at examples.

1.1. Fine spaces. Let V be an arbitrary complex vector space. The fine bornology $\mathfrak{Fine}(V)$ is the smallest possible bornology on V. This means that $S \subset V$ is contained in $\mathfrak{Fine}(V)$ iff there are finitely many vectors $v_1, \dots, v_n \in V$ such that S is contained in the disked hull of $\{v_1, \dots, v_n\}$. Equipped with the fine bornology V becomes a complete bornological vector space.
It follows immediately from the definitions that all linear maps $f : V \to W$ from a fine

space V into any bornological space W are bounded. In particular we obtain a fully faithful functor $\mathfrak{Fine}$ from the category of complex vector spaces into the category of complete bornological vector spaces. This embedding is compatible with tensor products. If V_1 and V_2 are fine spaces the completed bornological tensor product $V_1 \hat{\otimes} V_2$ is the algebraic tensor product $V_1 \otimes V_2$ equipped with the fine bornology. In follows in particular that every algebra A over the complex numbers can be viewed as a complete bornological algebra with the fine bornology.
Since the completed bornological tensor product is compatible with direct sums we see that $V_1 \hat{\otimes} V_2$ is as a vector space simply the algebraic tensor product $V_1 \otimes V_2$ provided V_1 or V_2 is a fine space. However, the bornology on the tensor product is in general not the fine bornology.

1.2. Locally convex spaces. The most important examples of bornological vector spaces are obtained from locally convex vector spaces. If V is any locally convex vector space one can associate two natural bornologies $\mathfrak{Bound}(V)$ and $\mathfrak{Comp}(V)$ to V which are called the bounded bornology and the precompact bornology, respectively.
The elements in $\mathfrak{Bound}(V)$ are by definition the bounded subsets of V. Equipped with the bornology $\mathfrak{Bound}(V)$ the space V is separated if its topology is Hausdorff and complete if the topology of V is sequentially complete.
The bornology $\mathfrak{Comp}(V)$ consists of all precompact subsets of V. This means that $S \in \mathfrak{Comp}(V)$ iff for all neighborhoods U of the origin there is a finite subset $F \subset V$ such that $S \subset F + U$. If V is complete then $S \subset V$ is precompact iff its closure is compact. One checks that the precompact subsets indeed form a bornology. Equipped with the bornology $\mathfrak{Comp}(V)$ the space V is separated if the topology of V is Hausdorff and complete if V is a complete topological vector space. We mention that the precompact bornology is particularly important for local cyclic homology.

1.3. Fréchet spaces. In the case of Fréchet spaces the properties of the bounded bornology and the precompact bornology can be described more in detail. Let V and W be Fréchet spaces endowed both with the bounded or the precompact bornology. A linear map $f : V \to W$ is bounded if and only if it is continuous. This is due to the fact that a linear map between metrizable topological spaces is continuous iff it is sequentially continuous. Hence the functors $\mathfrak{Bound}$ and $\mathfrak{Comp}$ from the category of Fréchet spaces into the category of complete bornological vector spaces are fully faithful.
The following theorem describes the completed bornological tensor product of Fréchet spaces and is proved in [**53**].

THEOREM 2.3. *Let V and W be Fréchet spaces and let $V \hat{\otimes}_\pi W$ be their completed projective tensor product. Then there are natural isomorphisms*

$$(V, \mathfrak{Comp}) \hat{\otimes} (W, \mathfrak{Comp}) \cong (V \hat{\otimes}_\pi W, \mathfrak{Comp})$$
$$(V, \mathfrak{Bound}) \hat{\otimes} (W, \mathfrak{Bound}) \cong (V \hat{\otimes}_\pi W, \mathfrak{S})$$

of complete bornological vector spaces where the bornology $\mathfrak{S}$ can be described more explicitly in certain cases. In particular $\mathfrak{S}$ is equal to the bounded bornology $\mathfrak{Bound}(V \hat{\otimes}_\pi W)$ if V or W is nuclear.

1.4. LF-spaces. More generally we can consider LF-spaces. A locally convex vector space V is an LF-space if there exists an increasing sequence of subspaces $V_n \subset V$ with union equal to V such that each V_n is a Fréchet space in the subspace topology and V carries the corresponding inductive limit topology. A linear map $V \to W$ from the LF-space V into an arbitrary locally convex space W is continuous iff its restriction to the subspaces V_n is continuous for all n. From the definition of the inductive limit topology it follows that a bounded subset of an LF-space V is contained in a Fréchet subspace V_n. If V_1 and V_2 are LF-spaces endowed with the bounded or the precompact bornology a bilinear map $b : V_1 \times V_2 \to W$ is bounded iff it is separately continuous. This implies that an LF-space equipped with a separately continuous multiplication becomes a complete bornological algebra with respect to the bounded or the precompact bornology.
From theorem 2.3 one obtains the following description of tensor products of LF-spaces [**53**].

Theorem 2.4. *Let V and W be nuclear LF-spaces endowed with the bounded bornology. Then $V \hat{\otimes} W$ is isomorphic to the inductive tensor product $V \hat{\otimes}_\iota W$ endowed with the bounded bornology.*

As an example consider the nuclear LF-space $C^\infty_c(M)$ of smooth functions with compact support on a smooth manifold M. We endow $C^\infty_c(M)$ with the bounded bornology which is equal to the precompact bornology in this case. Theorem 2.4 yields a natural bornological isomorphism

$$C^\infty_c(M) \hat{\otimes} C^\infty_c(N) \cong C^\infty_c(M \times N)$$

for all smooth manifolds M and N. In chapter 5 we will have to work with similar examples of LF-spaces obtained by considering smooth functions on simplicial complexes.

2. Actions and crossed products

We begin with the definition of the category of G-modules for the discrete group G.

Definition 2.5. *A G-module is a complete bornological vector space V with a given (left) action of the group G by bounded linear automorphisms. A bounded linear map $f : V \to W$ between two G-modules is called equivariant if $f(s \cdot v) = s \cdot f(v)$ for all $v \in V$ and $s \in G$.*

We denote by G-Mod the category of G-modules and equivariant linear maps. If we view the group algebra $\mathbb{C}G$ of G as a complete bornological algebra with the fine bornology it is easy to check that the category G-Mod is equivalent to the category of unitary modules over $\mathbb{C}G$.
It is clear that the direct sum of a family of G-modules is again a G-module. The tensor product $V \hat{\otimes} W$ of two G-modules becomes a G-module using the diagonal action $s \cdot (v \otimes w) = s \cdot v \otimes s \cdot w$ for $v \in V$ and $w \in W$. For every group the trivial one-dimensional G-module $\mathbb{C}$ behaves like a unit with respect to the tensor product. In this way G-Mod becomes an additive monoidal category.
Next we want to specify the class of G-algebras that we are going to work with. Expressed in the language of category theory our definition amounts to saying that a G-algebra is an algebra in the monoidal category G-Mod. Let us formulate this more explicitly in the following definition.

DEFINITION 2.6. *Let G be a discrete group. A G-algebra is a complete bornological algebra A which is at the same time a G-module such that the multiplication satisfies*

$$s \cdot (xy) = (s \cdot x)(s \cdot y)$$

for all $x, y \in A$ and $s \in G$. An equivariant homomorphism $f : A \to B$ between G-algebras is an algebra homomorphism which is equivariant.

If A is unital we say that A is a unital G-algebra if $s \cdot 1 = 1$ for all $s \in G$. The unitarisation A^+ of a G-algebra A is a unital G-algebra in a natural way. We will occasionally also speak of an action of G on A to express that A is a G-algebra.
There is a natural way to enlarge any G-algebra to a G-algebra where all group elements act by inner automorphisms. This is the crossed product construction which we study next.

DEFINITION 2.7. *Let G be a discrete group and let A be a G-algebra. The crossed product $A \rtimes G$ of A by G is $A \hat{\otimes} \mathbb{C}G = C_c(G, A)$ with multiplication given by*

$$(f * g)(t) = \sum_{s \in G} f(s) s \cdot g(s^{-1}t)$$

for $f, g \in C_c(G, A)$. Here $\mathbb{C}G$ is equipped with the fine bornology.

It is easy to check that $A \rtimes G$ is a complete bornological algebra. The crossed product $A \rtimes G$ has a unit iff the algebra A is unital.
Let us have a look at some basic examples of G-algebras and the associated crossed products. In particular the algebra $\mathcal{K}_G$ introduced below will play an important role in our theory.

2.1. Trivial actions. The simplest example of a G-algebra is the algebra of complex numbers with the trivial G-action. More generally one can equip any complete bornological algebra A with the trivial action to obtain a G-algebra. The corresponding crossed product algebra $A \rtimes G$ is simply a tensor product,

$$A \rtimes G \cong A \hat{\otimes} \mathbb{C}G.$$

This explains why one may view crossed products in general as twisted tensor products.

2.2. Commutative algebras. Let M be a smooth manifold on which the group G acts by diffeomorphisms and let $C^\infty_c(M)$ be the LF-algebra of compactly supported smooth functions on M. Then we get an action of G on $A = C^\infty_c(M)$ by defining

$$(s \cdot f)(x) = f(s^{-1} \cdot x)$$

for all $s \in G$ and $f \in A$. Of course this algebra is unital iff M is compact. The associated crossed product $A \rtimes G$ may be described as the smooth convolution algebra of the translation groupoid $M \rtimes G$ associated to the action of G on M.

2.3. Algebras associated to representations of G. Let $\mathcal{H}$ be a G-pre-Hilbert space, a unitary representation of the group G on a not necessarily complete inner product space. Such a representation induces an action of G on the algebra $l(\mathcal{H})$ of finite rank operators on $\mathcal{H}$ by the formula

$$(s \cdot T)(\xi) = s \cdot T(s^{-1} \cdot \xi)$$

for $s \in G$ and $\xi \in \mathcal{H}$. Of course $l(\mathcal{H})$ is spanned linearly by the rank one operators $|\chi\rangle\langle\eta|$ defined by $|\chi\rangle\langle\eta|(\xi) = \chi\langle\eta,\xi\rangle$ for $\chi,\eta \in \mathcal{H}$. We equip $l(\mathcal{H})$ with the fine bornology and remark that this algebra is unital iff $\mathcal{H}$ is finite dimensional.
In particular we may consider the case of the regular representation of G on $\mathbb{C}G \subset l^2(G)$. We shall look at this example more closely in the next subsection.

2.4. The algebra $\mathcal{K}_G$. We view $\mathbb{C}G$ as subspace of the Hilbert space $l^2(G)$ of square-summable functions on G and consider the left regular representation λ_s given by

$$\lambda_s(f)(t) = f(s^{-1}t)$$

for $s \in G$ and $f \in \mathbb{C}G$. The corresponding algebra of finite rank operators with the G-action described in the previous paragraph will be denoted by $\mathcal{K}_G$. The algebra $\mathcal{K}_G$ is spanned linearly by the rank one operators $|r\rangle\langle s|$ with $r, s \in G$ defined by

$$|r\rangle\langle s|(t) = \delta_{st} r$$

for $t \in G \subset \mathbb{C}G$ where δ_{st} is the Kronecker delta. The multiplication rule in $\mathcal{K}_G$ for such operators is

$$|r\rangle\langle s| \cdot |p\rangle\langle q| = \delta_{sp}\, |r\rangle\langle q|$$

and the G-action is described by

$$t \cdot |r\rangle\langle s| = |tr\rangle\langle ts|.$$

The following easy fact will be important.

Lemma 2.8. *Let A be any G-algebra. Then the n-fold tensor power $(A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes} n}$ is a free G-module for all $n \geq 1$.*

Proof. Consider first the case $n = 1$. We view $V = \mathbb{C}G\hat{\otimes}A$ as trivial G-module and construct an equivariant isomorphism $A\hat{\otimes}\mathcal{K}_G \cong \mathbb{C}G\hat{\otimes}V$ as follows: Define $\alpha : A\hat{\otimes}\mathcal{K}_G \to \mathbb{C}G\hat{\otimes}V, \alpha(a \otimes |r\rangle\langle s|) = r \otimes r^{-1}s \otimes r^{-1} \cdot a$. One checks that α is a G-module map with inverse $\beta : \mathbb{C}G\hat{\otimes}V \to A\hat{\otimes}\mathcal{K}_G, \beta(r \otimes s \otimes a) = r \cdot a \otimes |r\rangle\langle rs|$. To deal with higher tensor powers observe that we have an isomorphism

$$(A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes} n} \cong ((A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes} n-1}\hat{\otimes}A)\hat{\otimes}\mathcal{K}_G$$

of G-algebras. The claim follows by applying the previous argument to the G-algebra $B = (A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes} n-1}\hat{\otimes}A$. $\square$

3. Covariant modules and equivariant sheaves

In this section we introduce covariant modules. Since covariant modules are closely related to equivariant sheaves we first recall the general definition of the latter [**9**].
Let G be a topological group and let X be a G-space. Define the maps $d_j : G \times X \to X$ for $j = 0, 1$ by

$$d_0(s, x) = s^{-1} \cdot x, \qquad d_1(s, x) = x.$$

Moreover let $s_0 : X \to G \times X$ be given by

$$s_0(x) = (e, x)$$

where $e \in G$ is the identity element. In a similar way define the maps $d_j : G \times G \times X \to G \times X$ for $j = 0, 1, 2$ by

$$d_0(s, t, x) = (t, s^{-1} \cdot x), \qquad d_1(s, t, x) = (st, x), \qquad d_2(s, t, x) = (s, x).$$

If $f : X \to Y$ is a continuous map and $\mathcal{F}$ is a sheaf on Y we denote its inverse image sheaf on X by $f^*\mathcal{F}$.

Definition 2.9. *Let G be a topological group and let X be a G-space. A G-equivariant sheaf on X is a sheaf $\mathcal{F}$ on X together with an isomorphism $\theta : d_1^*\mathcal{F} \to d_0^*\mathcal{F}$ satisfying the conditions*

$$d_0^*\theta \, d_2^*\theta = d_1^*\theta, \qquad s_0^*\theta = \mathrm{id}\,.$$

A morphism of equivariant sheaves is a sheaf homomorphism $\phi : \mathcal{F} \to \mathcal{G}$ that commutes with θ.

We will only consider sheaves of complex vector spaces. The category of equivariant sheaves on X is denoted by $\mathrm{Sh}_G(X)$.
The action of G on an equivariant sheaf $\mathcal{F}$ is encoded in a rather indirect way in the morphism θ. If the group G is discrete a G-equivariant sheaf $\mathcal{F}$ can be described more concretely by specifying sheaf homomorphisms $\alpha_s : \mathcal{F} \to s^*\mathcal{F}$ for all $s \in G$ which satisfy the natural axioms for an action.
If in addition the space X is discrete there is a purely algebraic way to define equivariant sheaves. We need some general terminology. If A is a not necessarily unital algebra equipped with the fine bornology a (left) module M over A is called nondegenerate if AM, the linear span of all elements $a \cdot m$ with $a \in A$ and $m \in M$, is equal to M. Let us assume that A is a G-algebra. A covariant system for A is a nondegenerate A-module M which is at the same time a G-module such that

$$s \cdot (f \cdot m) = (s \cdot f) \cdot (s \cdot m)$$

for all $s \in G, f \in A$ and $m \in M$. A morphism of covariant systems is a map $\phi : M \to N$ which is both A-linear and equivariant. The category of covariant systems for A is denoted by $\mathsf{Mod}(A, G)$.
Now let X be a discrete G-space and denote by $C_c(X)$ the algebra of functions on X with finite support equipped with the fine bornology. There is a natural G-action on this algebra coming from the action on X. We denote by $\mathsf{Mod}(C_c(X), G, \mathfrak{Fine})$ the full subcategory of $\mathsf{Mod}(C_c(X), G)$ consisting of those covariant systems M for $C_c(X)$ where M is a fine space.

PROPOSITION 2.10. *Let X be a discrete G-space for the discrete group G. Taking global sections with compact support induces an equivalence between the categories* $\mathrm{Sh}_G(X)$ *and* $\mathsf{Mod}(C_c(X), G, \mathfrak{Fine})$.

This assertion remains true in the setting of totally disconnected groups and spaces. For a proof in the non-equivariant case see [**10**].
In our context we are interested in the discrete group G itself viewed as a G-space using the adjoint action. Denote by $\langle G\rangle$ the set of conjugacy classes in G. For an element $s \in G$ let $\langle s\rangle$ be the conjugacy class of s. Of course the orbits of the adjoint action are just the conjugacy classes.
We equip the algebra $C_c(G)$ of functions with finite support on G with the fine bornology and the adjoint action. The resulting G-algebra will be denoted by $\mathcal{O}_G$. Explicitly we have $t \cdot f(s) = f(t^{-1}st)$ for $f \in \mathcal{O}_G$ and $s \in G$. Remark that $\mathcal{O}_G$ is unital iff the group G is finite.
If G is a finite group the invariant part of $\mathcal{O}_G$ will be denoted by $\mathcal{R}(G)$. The elements of $\mathcal{R}(G)$ are precisely the class functions, that is the functions f satisfying $f(t^{-1}st) = f(s)$ for all $s, t \in G$. Since the group G is finite the algebra $\mathcal{R}(G)$ is just the complexification of the representation ring $R(G)$ of G.
Let us now introduce covariant modules. Using the terminology from above a covariant module is simply a covariant system for the G-algebra $\mathcal{O}_G$. We give the following explicit definition.

DEFINITION 2.11. *Let G be a discrete group. A G-covariant module is a complete bornological vector space M which is both a nondegenerate $\mathcal{O}_G$-module and a G-module such that*
$$s \cdot (f \cdot m) = (s \cdot f) \cdot (s \cdot m)$$
for all $s \in G, f \in \mathcal{O}_G$ and $m \in M$. A bounded linear map $\phi : M \to N$ between covariant modules is called covariant if it is $\mathcal{O}_G$-linear and equivariant.

From the previous discussion it is clear that covariant modules are essentially equivariant sheaves except that we include bornologies as extra information. Usually we will not mention the group explicitly in our terminology and simply speak of covariant modules and covariant maps. We remark that there is a bijective correspondence between covariant modules and nondegenerate modules for the crossed product $\mathcal{O}_G \rtimes G$.
Covariant modules and covariant maps constitute a category G-$\mathfrak{Mod}$. The space of covariant maps between two covariant modules M and N will be denoted by $\mathfrak{Hom}_G(M, N)$. We let $\mathfrak{Hom}(M, N)$ be the collection of maps that are only $\mathcal{O}_G$-linear and $\mathrm{Hom}_G(M, N)$ will be the set consisting of equivariant maps. When dealing with covariant modules we will mainly be interested in covariant maps.
The basic example of a covariant module is the algebra $\mathcal{O}_G$ itself. More generally, let V be a G-module. We obtain an associated covariant module by considering $\mathcal{O}_G \hat{\otimes} V$ with the diagonal G-action and the obvious $\mathcal{O}_G$-module structure given by multiplication.
Now we want to discuss in detail the structure of covariant maps between covariant modules. We begin with an arbitrary covariant module M. Given a conjugacy class $\langle s\rangle \in \langle G\rangle$ we associate to M the localized module at $\langle s\rangle$ defined by $M_s = M/\mathfrak{p}_{\langle s\rangle} M$ where $\mathfrak{p}_{\langle s\rangle} \subset \mathcal{O}_G$ denotes the G-invariant ideal of all functions vanishing on $\langle s\rangle$. The space $M_{\langle s\rangle}$ is still a G-module and a module over $\mathcal{O}_G$. In particular it is a complete bornological vector space.

Following the terminology introduced in [**21**] we call the localisations $M_{\langle s\rangle}$ at conjugacy classes of elements of finite order the elliptic components and the localisations at conjugacy classes of elements of infinite order the hyperbolic components of M.
We may also localize M at a point $s \in G$. This is defined in a similar way, but now $\mathfrak{p}_{\langle s\rangle}$ is replaced by the maximal ideal $\mathfrak{m}_s$ of all functions vanishing in s. The ideal $\mathfrak{m}_s$ is in general not G-invariant. On the corresponding localization M_s we get consequently only a G_s-module structure where G_s denotes the centralizer of the element s in G. Consider for every conjugacy class $\langle s\rangle$ the characteristic function $\chi_{\langle s\rangle}$ on $\langle s\rangle$. These functions are multipliers of $\mathcal{O}_G$. Since every covariant module M is a nondegenerate $\mathcal{O}_G$-module these multiplier act also on M. With this in mind it is easy to see that there is a natural isomorphism $M_{\langle s\rangle} \cong \chi_{\langle s\rangle} M$. Similarly we get $M_s \cong \chi_s M$ where χ_s is the characteristic function on $s \in G$.
In the following proposition localized modules are used to describe covariant maps.

PROPOSITION 2.12. *Let M and N be covariant modules. Choose a representative s for every conjugacy class $\langle s\rangle$ of elements in G. Then there are natural isomorphisms*

$$\mathfrak{Hom}_G(M,N) \cong \prod_{\langle s\rangle\in\langle G\rangle} \mathfrak{Hom}_G(M_{\langle s\rangle}, N_{\langle s\rangle}) \cong \prod_{\langle s\rangle\in\langle G\rangle} \mathrm{Hom}_{G_s}(M_s, N_s).$$

PROOF. Obviously there are natural maps

$$\mathfrak{Hom}_G(M,N) \longrightarrow \prod\nolimits_{\langle s\rangle\in\langle G\rangle} \mathfrak{Hom}_G(M_{\langle s\rangle}, N_{\langle s\rangle}) \longrightarrow \prod\nolimits_{\langle s\rangle\in\langle G\rangle} \mathrm{Hom}_{G_s}(M_s, N_s).$$

It is easy to check that the first map is injective. To see that it is also surjective let $(\phi_{\langle s\rangle}) \in \prod_{\langle s\rangle\in\langle G\rangle} \mathfrak{Hom}_G(M_{\langle s\rangle}, N_{\langle s\rangle})$ be given. With the notation as above we define a covariant map ϕ by

$$\phi(m) = \sum_{\langle s\rangle\in\langle G\rangle} \chi_{\langle s\rangle}\phi_{\langle s\rangle}(\chi_{\langle s\rangle} m).$$

This is well-defined because covariant modules are nondegenerate $\mathcal{O}_G$-modules. It is clear that ϕ maps to the family $\phi_{\langle s\rangle}$ under the natural map. To see that the second map is an isomorphism it suffices to consider a fixed conjugacy class $\langle s\rangle$. For every $t \in \langle s\rangle$ choose $r(t)$ such that $\chi_t = r(t)\cdot\chi_s$. Again injectivity is easy and it remains to show surjectivity. Given $\phi_s \in \mathrm{Hom}_{G_s}(M_s, N_s)$ we define $\phi \in \mathfrak{Hom}_G(M_{\langle s\rangle}, N_{\langle s\rangle})$ by

$$\phi(m) = \sum_{t\in\langle s\rangle} r(t)\cdot\phi_s(r(t)^{-1}\cdot(\chi_t m))$$

This is independent of the choice of $r(t)$ since ϕ_s is supposed to be G_s-linear. It is easy to check that ϕ maps to ϕ_s under the natural map. $\square$

COROLLARY 2.13. *Let M be a covariant module and let V be a G-module. Choose a representative s for every conjugacy class $\langle s\rangle$. Then there is a natural isomorphism*

$$\mathfrak{Hom}_G(\mathcal{O}_G\hat{\otimes}V, M) \cong \prod_{\langle s\rangle\in\langle G\rangle} \mathrm{Hom}_{G_s}(V, M_s).$$

PROOF. This follows from proposition 2.12 and the fact that $(\mathcal{O}_G\hat{\otimes}V)_s$ is isomorphic to V for all $s \in G$. $\square$
A covariant module is called projective if it satisfies the lifting property for surjective

covariant maps $M \to N$ with bounded linear splitting. It is not hard to check that one can construct out of any bounded linear splitting for the surjection $M \to N$ a bounded linear splitting which is in addition $\mathcal{O}_G$-linear. Hence it is equivalent to require the lifting property for surjections $M \to N$ of covariant modules with bounded $\mathcal{O}_G$-linear splitting to define the class of projective covariant modules.

COROLLARY 2.14. *Let V be a free G-module. Then $\mathcal{O}_G \hat{\otimes} V$ is a projective covariant module.*

PROOF. Let $\phi : M \to N$ be a surjection of covariant modules and let $\sigma : N \to M$ be a bounded $\mathcal{O}_G$-linear splitting for ϕ. Then by proposition 2.12 ϕ corresponds to a family of maps $\phi_s \in \mathrm{Hom}_{G_s}(M_s, N_s)$. The map σ gives splittings $\sigma_s \in \mathrm{Hom}(N_s, M_s)$ for the maps ϕ_s. Now $(\mathcal{O}_G \hat{\otimes} V)_s \cong V$ is free as a G_s-module since V is a free G-module. Hence any map $\psi : \mathcal{O}_G \hat{\otimes} V \to N$ corresponding to a family of maps $\psi_s \in \mathrm{Hom}_{G_s}((\mathcal{O}_G \hat{\otimes} V)_s, N_s)$ can be lifted to a family $\eta_s \in \mathrm{Hom}_{G_s}((\mathcal{O}_G \hat{\otimes} V)_s, M_s)$ such that $\phi_s \eta_s = \psi_s$. This means $\phi\eta = \psi$ for the covariant map η corresponding to the family η_s. □

4. Projective systems

Since the work of Cuntz and Quillen [**30**] it is known that periodic cyclic homology is most naturally defined on the category of pro-algebras. Similarly, the correct way to define equivariant periodic cyclic homology is to work in the category of pro-G-algebras. This means that we have to consider projective systems of G-modules and covariant modules. In this section we shall explain these notions and fix our notation.
First we review the general construction of pro-categories. To an additive category $\mathcal{C}$ one associates the pro-category $\mathrm{pro}(\mathcal{C})$ of projective systems over $\mathcal{C}$ as follows.
A projective system over $\mathcal{C}$ consists of a directed index set I, objects V_i for all $i \in I$ and morphisms $p_{ij} : V_j \to V_i$ for all $j \geq i$. The morphisms are assumed to satisfy $p_{ij} p_{jk} = p_{ik}$ if $k \geq j \geq i$. These conditions are equivalent to saying that we have a contravariant functor from the small category I to $\mathcal{C}$. The class of objects of $\mathrm{pro}(\mathcal{C})$ consists by definition of all projective systems over $\mathcal{C}$.
The space of morphisms between projective systems $(V_i)_{i \in I}$ and $(W_j)_{j \in J}$ is defined by

$$\mathrm{Mor}((V_i), (W_j)) = \varprojlim_j \varinjlim_i \mathrm{Mor}_{\mathcal{C}}(V_i, W_j)$$

where the limits are taken in the category of abelian groups. The morphisms between projective systems will be called pro-morphisms. Of course one has to check that the composition of pro-morphisms can be defined in a consistent way. We refer to [**1**] for this and further details.
It is useful to study pro-objects by comparing them to constant pro-objects. A constant pro-object is by definition a pro-object where the index set consists only of one element. If $V = (V_i)_{i \in I}$ is any pro-object a morphism $V \to C$ with constant range C is given by a morphism $V_i \to C$ for some i. This follows directly from the definition of the morphisms in $\mathrm{pro}(\mathcal{C})$.
In the category $\mathrm{pro}(\mathcal{C})$ projective limits always exist. This is due to the fact that a projective system of pro-objects $(V_j)_{j \in J}$ can be identified naturally with a pro-object in the following way. Let $V_j = (V_{ji})_{i \in I_j}$ be a projective system of objects in $\mathcal{C}$. Consider the

pro-object $(V_{ji})_{j\in J,i\in I_j}$ with structure maps induced by the structure maps in V_j for $j \in J$ and the transition maps coming from the projective system $(V_j)_{j\in J}$. It can be checked that $(V_{ji})_{j\in J,i\in I_j}$ is the projective limit of the projective system $(V_j)_{j\in J}$ in $\mathrm{pro}(\mathcal{C})$.
Since there are finite direct sums in $\mathcal{C}$ we also have finite direct sums in $\mathrm{pro}(\mathcal{C})$. Explicitly, the direct sum of $V = (V_i)_{i\in I}$ and $W = (W_j)_{j\in J}$ is given by

$$(V_i)_{i\in I} \oplus (W_j)_{j\in J} = (V_i \oplus W_j)_{(i,j)\in I\times J}$$

where the index set $I \times J$ is ordered using the product ordering. The structure maps of this projective system are obtained by taking direct sums of the structure maps of $(V_i)_{i\in I}$ and $(W_j)_{j\in J}$. With this notion of direct sums the category $\mathrm{pro}(\mathcal{C})$ becomes an additive category.
If we apply these general constructions to the category of G-modules we obtain the category of pro-G-modules. A morphism in $\mathrm{pro}(G\text{-}\mathsf{Mod})$ will be called an equivariant pro-linear map. Similarly we have the category of covariant pro-modules as the pro-category of G-$\mathfrak{Mod}$. Morphisms in $\mathrm{pro}(G\text{-}\mathfrak{Mod})$ will simply be called covariant maps again.
Let us come back to the general situation. Assume in addition that $\mathcal{C}$ is monoidal such that the tensor product functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is bilinear. In this case we define the tensor product $V \otimes W$ for pro-objects $V = (V_i)_{i\in I}$ and $W = (W_j)_{j\in J}$ by

$$(V_i)_{i\in I} \otimes (W_j)_{j\in J} = (V_i \otimes W_j)_{(i,j)\in I\times J}$$

where again $I \times J$ is ordered using the product ordering. The structure maps are obtained by tensoring the structure maps of $(V_i)_{i\in I}$ and $(W_j)_{j\in J}$. Observe that any morphism $f : V \otimes W \to C$ with constant range C factors through $V_i \otimes W_j$ for some $i \in I, j \in J$. This means that we can write f in the form $f = g(f_V \otimes f_W)$ where $f_V : V \to C_V$ and $f_W : W \to C_W$ are morphisms with constant range and $g : C_V \otimes C_W \to W$ is a morphism of constant pro-objects.
Equipped with this tensor product the category $\mathrm{pro}(\mathcal{C})$ is additive monoidal and we obtain a faithful additive monoidal functor $\mathcal{C} \to \mathrm{pro}(\mathcal{C})$ in a natural way.
The existence of a tensor product in $\mathrm{pro}(\mathcal{C})$ yields a natural notion of algebras and algebra homomorphisms in this category. These algebras will be called pro-algebras and their homomorphism will be called pro-algebra homomorphisms. Moreover we can consider pro-modules for pro-algebras and their homomorphisms. The definitions are straightforward.
The category G-Mod is monoidal in the sense explained above. To indicate that we use completed bornological tensor products in G-Mod we will denote the tensor product of two pro-G-modules V and W by $V \hat{\otimes} W$.
In order to fix terminology we give the following definition.

Definition 2.15. *A pro-G-algebra A is an algebra in the category* $\mathrm{pro}(G\text{-}\mathsf{Mod})$. *An algebra homomorphism $f : A \to B$ in* $\mathrm{pro}(G\text{-}\mathsf{Mod})$ *is called an equivariant homomorphism of pro-G-algebras.*

Occasionally we will consider unital pro-G-algebras. The unitarisation A^+ of a pro-G-algebra A is defined in the same way as for G-algebras.
We also include a short discussion of extensions. This will be important in particular in connection with excision in equivariant periodic cylic homology. Let again $\mathcal{C}$ be any additive category and let K, E and Q be objects in $\mathrm{pro}(\mathcal{C})$. An admissible extension is a

diagram of the form

$$K \underset{\iota}{\overset{\rho}{\rightleftarrows}} E \underset{\pi}{\overset{\sigma}{\rightleftarrows}} Q$$

in $\mathrm{pro}(\mathcal{C})$ such that $\rho\iota = \mathrm{id}$, $\pi\sigma = \mathrm{id}$ and $\iota\rho + \sigma\pi = \mathrm{id}$. In other words we require that E decomposes into a direct sum of K and Q. Note that the morphisms ρ and σ determine each other uniquely.
Let us also give the following definition in the concrete situation $\mathcal{C} = \mathrm{pro}(G\text{-}\mathsf{Mod})$.

Definition 2.16. *Let K, E and Q be pro-G-algebras. An admissible extension of pro-G-algebras is an admissible extension*

$$K \underset{\iota}{\overset{\rho}{\rightleftarrows}} E \underset{\pi}{\overset{\sigma}{\rightleftarrows}} Q$$

in $\mathrm{pro}(G\text{-}\mathsf{Mod})$ *where ι and π are equivariant algebra homomorphisms.*

Usually we will simply write $(\iota, \pi) : 0 \to K \to E \to Q \to 0$ or simply $0 \to K \to E \to Q \to 0$ for an admissible extension of pro-G-algebras and assume tacitly that a splitting for the quotient map is given.
Working with pro-G-modules or covariant pro-modules may seem somewhat difficult because there are no longer concrete elements to manipulate with. Nevertheless we will write down explicit formulas involving "elements" in subsequent sections. This can be justified by noticing that these formulas are concrete expressions for identities between abstractly defined morphisms.

5. Paracomplexes

In this section we introduce the concept of a paramixed complex which will play a central role in our theory. Our terminology is motivated from [**34**] but it is slightly different. The related notion of a paracyclic module is well-known in the study of the cyclic homology of crossed products and smooth groupoids [**32**], [**34**], [**54**].
Whereas cyclic modules and mixed complexes constitute the fundamental framework for cyclic homology theory, paracyclic modules are mainly regarded as a tool in computations. However, in the equivariant situation the point of view is changed completely! Here the fundamental objects are paramixed complexes. Conversely, mixed complexes show up only in concrete calculations. We will see this in particular in chapters 4 and 5 below.
Let us give the following general definition.

Definition 2.17. *Let $\mathcal{C}$ be an additive category. A paracomplex P in $\mathcal{C}$ is a sequence of objects P_n and morphisms $d : P_n \to P_{n-1}$ in $\mathcal{C}$ (not necessarily satisfying $d^2 = 0$). A chain map $f : P \to Q$ between two paracomplexes is a sequence of morphisms $f_n : P_n \to Q_n$ commuting with the differentials.*

We did not specify the grading in this definition, we may consider $\mathbb{Z}$-graded paracomplexes or $\mathbb{N}$-graded paracomplexes or parasupercomplexes. The morphisms d in a paracomplex will be called differentials although this contradicts the usual definition of a differential.
In general it does not make sense to speak about the homology of a paracomplex. Given a paracomplex P in an abelian category with differentials d one could force it to become a complex by dividing out the subspace $d^2(P)$ and then take homology. However, it turns

out that this procedure is not appropriate in the situations we will consider.
Although there is no reasonable definition of homology we can nevertheless give a meaning to the statement that two paracomplexes are homotopy equivalent: Let $f, g : P \to Q$ be two chain maps between paracomplexes. A chain homotopy connecting f and g is a map $s : P \to Q$ of degree 1 satisfying the usual relation $ds + sd = f - g$. It should be noted that the map $ds + sd$ for a general morphism $s : P \to Q$ of degree 1 is a chain map iff d^2 commutes with s. In particular it follows from the homotopy relation $ds + sd = f - g$ that a chain homotopy s commutes with d^2. Of course two paracomplexes P and Q are called homotopy equivalent if there exist chain maps $f : P \to Q$ and $g : Q \to P$ which are inverse to each other up to chain homotopy.
Nevertheless, at this level of generality paracomplexes are not very interesting. The paracomplexes we have in mind are rather special. They arise from paramixed complexes that we are going to define now.

DEFINITION 2.18. *Let $\mathcal{C}$ be an additive category. A paramixed complex M in $\mathcal{C}$ is a bounded below sequence of objects M_n together with differentials b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and*

$$[b, B] = bB + Bb = \mathrm{id} - T$$

with an invertible morphism T.

For $T = \mathrm{id}$ we reobtain the notion of a mixed complex. Since the Hochschild operator b in a paramixed complex satisfies $b^2 = 0$ one can define and study Hochschild homology in the usual way. On the other hand we do not intend to define the cyclic homology of a paramixed complex. We will see in chapter 3 how bivariant periodic cyclic homology of certain paramixed complexes can be defined in a natural way. Observe that in a paramixed complex we have control about the failure of the usual differential $\partial = B + b$ to satisfy $\partial^2 = 0$. For the moment we only remark that this is the reason why it is possible to work with these objects.

CHAPTER 3

Equivariant periodic cyclic homology

In this chapter we define bivariant equivariant periodic cyclic homology HP^G_* and study its general homological properties.
Our treatment follows closely the axiomatic machinery for cylic homology theories developed by Meyer in [**53**]. In fact a large part of the necessary adaptions is straightforward and carried out here in detail only to keep our exposition self-contained.
In the first section we study quasifree pro-G-algebras and universal locally nilpotent extensions. This can be done as in the non-equivariant case. In the second section we construct noncommutative equivariant differential forms. As indicated in the introduction the definitions are related to the Cartan model in the classical setting. We show that noncommutative equivariant differential forms satisfy the axioms of a paramixed complex. In the third section we study the equivariant X-complex which can be thought of as the noncommutative equivariant de Rham complex. With these preparations we can define bivariant equivariant periodic cyclic homology $HP^G_*(A, B)$. The rest of the chapter is devoted to proving general properties of these bivariant homology groups. We show that HP^G_* is homotopy invariant, stable and satisfies excision in both variables. The proofs from the non-equivariant context can be adapted with some modifications.

1. Quasifree pro-G-algebras

Let G be a discrete group and let A be a pro-G-algebra. The space $\Omega^n(A)$ of noncommutative n-forms over A is defined by $\Omega^n(A) = A^+ \hat{\otimes} A^{\hat{\otimes} n}$ for $n \geq 0$. We recall that A^+ denotes the unitarization of A. From its definition as a tensor product it is clear that $\Omega^n(A)$ becomes a pro-G-module in a natural way. The differential $d : \Omega^n(A) \to \Omega^{n+1}(A)$ and the multiplication of forms $\Omega^n(A) \hat{\otimes} \Omega^m(A) \to \Omega^{n+m}(A)$ are defined as usual [**30**] and it is clear that both are equivariant pro-linear maps. Multiplication of forms yields in particular an A-bimodule structure on $\Omega^n(A)$ for all n. Apart from the ordinary product of differential forms we have the Fedosov product given by

$$\omega \circ \eta = \omega\eta - (-1)^{|\omega|} d\omega d\eta$$

for homogenous forms ω and η. Consider the pro-G-module $\Omega^{\leq n}(A) = A \oplus \Omega^1(A) \oplus \cdots \oplus \Omega^n(A)$ equipped with the Fedosov product where forms above degree n are ignored. It is easy to check that this multiplication is associative and turns $\Omega^{\leq n}(A)$ into a pro-G-algebra. Moreover we have the usual $\mathbb{Z}_2$-grading on $\Omega^{\leq n}(A)$ into even and odd forms. The natural projection $\Omega^{\leq m}(A) \to \Omega^{\leq n}(A)$ for $m \geq n$ is an equivariant homomorphism and compatible with the grading. Hence we get a projective system $(\Omega^{\leq n}(A))_{n \in \mathbb{N}}$ of pro-G-algebras. By definition the periodic differential envelope $\theta\Omega(A)$ of A is the pro-G-algebra obtained as the projective limit of this system. We define the periodic tensor algebra $\mathcal{T}A$ of A to be the even part of $\theta\Omega(A)$. If we set $\mathcal{T}A/(\mathcal{J}A)^n := A \oplus \Omega^2(A) \oplus \cdots \oplus \Omega^{2n-2}(A)$ we can describe $\mathcal{T}A$ as the projective limit of the projective system $(\mathcal{T}A/(\mathcal{J}A)^n)_{n\in\mathbb{N}}$. The natural projection $\theta\Omega(A) \to A$ restricts to an equivariant homomorphism $\tau_A : \mathcal{T}A \to A$. Since the natural inclusions $A \to A \oplus \Omega^2(A) \oplus \cdots \oplus \Omega^{2n-2}(A)$ assemble to give an equivariant pro-linear section σ_A for τ_A we obtain an admissible extension

$$\begin{tikzcd} \mathcal{J}A \arrow[r, shift right, "\iota_A"'] & \mathcal{T}A \arrow[l, shift right, dashed] \arrow[r, shift right, "\tau_A"'] & A \arrow[l, shift right, dashed, "\sigma_A"'] \end{tikzcd}$$

of pro-G-algebras where $\mathcal{J}A$ is by definition the projective limit of the pro-G-algebras $\mathcal{J}A/(\mathcal{J}A)^n := \Omega^2(A) \oplus \cdots \oplus \Omega^{2n-2}(A)$.
A large part of this section is devoted to the study of the pro-G-algebras $\mathcal{T}A$ and $\mathcal{J}A$. We will begin with $\mathcal{J}A$ and need some definitions. Let $m^n : N^{\otimes n} \to N$ be the iterated multiplication in an arbitrary pro-G-algebra N. Then N is called k-nilpotent for $k \in \mathbb{N}$ if the iterated multiplication $m^k : N^{\hat{\otimes} k} \to N$ is zero. It is called nilpotent if N is k-nilpotent for some $k \in \mathbb{N}$. We call N locally nilpotent if for every equivariant pro-linear map $f : N \to C$ with constant range C there exists $n \in \mathbb{N}$ such that $fm^n = 0$. In particular we see that nilpotent pro-G-algebras are locally nilpotent. An admissible extension $0 \to K \to E \to Q \to 0$ of pro-G-algebras is called locally nilpotent (k-nilpotent, nilpotent) if K is locally nilpotent (k-nilpotent, nilpotent).

Lemma 3.1. *The pro-G-algebra $\mathcal{J}A$ is locally nilpotent.*

Proof. Let $l : \mathcal{J}A \to C$ be an equivariant pro-linear map. By the construction of projective limits it follows that there exists $n \in \mathbb{N}$ such that l factors through $\mathcal{J}A/(\mathcal{J}A)^n$. The pro-$G$-algebra $\mathcal{J}A/(\mathcal{J}A)^n$ is n-nilpotent by the definition of the Fedosov product. Hence $lm^n_{\mathcal{J}A} = 0$ as desired. □
Later we will need the following result which shows how local nilpotence is inherited by tensor products.

Lemma 3.2. *Let N be a locally nilpotent pro-G-algebra and let A be any pro-G-algebra. Then the pro-G-algebra $A\hat{\otimes}N$ is locally nilpotent.*

Proof. Let $f : A\hat{\otimes}N \to C$ be an equivariant pro-linear map with constant range. By the construction of tensor products in pro(G-Mod) this map can be written as $g(f_1\hat{\otimes}f_2)$ for equivariant pro-linear maps $f_1 : A \to C_2, f_2 : N \to C_2$ with constant range and an equivariant bounded linear map $g : C_1\hat{\otimes}C_2 \to C$. Since N is locally nilpotent there exists a natural number n such that $f_2m^n_N = 0$. Up to a coordinate flip the n-fold multiplication in $A\hat{\otimes}N$ is given by $m^n_A\hat{\otimes}m^n_N$. This implies $fm^n_{A\hat{\otimes}N} = 0$ for the multiplication $m_{A\hat{\otimes}N}$ in $A\hat{\otimes}N$. Hence $A\hat{\otimes}N$ is locally nilpotent. □
Next we want to have a closer look at the pro-G-algebra $\mathcal{T}A$. Our first goal is to explain the universal property that characterizes $\mathcal{T}A$. In order to formulate this correctly we need another definition. An equivariant pro-linear map $l : A \to B$ between pro-G-algebras is called a lonilcur if its curvature $\omega_l : A\hat{\otimes}A \to B$ defined by $\omega_l(a, b) = l(ab) - l(a)l(b)$ is locally nilpotent, that is, if for every equivariant pro-linear map $f : B \to C$ with constant range C there exists $n \in \mathbb{N}$ such that $fm^n_B\omega_l^{\hat{\otimes} n} = 0$. The term lonilcur is an abbreviation for "equivariant pro-linear map with *lo*cally *nil*potent *cur*vature". It follows immediately from the definitions that every equivariant homomorphism is a lonilcur because the curvature is zero in this case. Using the fact that $\mathcal{J}A$ is locally nilpotent one checks easily that the natural map $\sigma_A : A \to \mathcal{T}A$ is a lonilcur.

PROPOSITION 3.3. *Let A be a pro-G-algebra. The pro-G-algebra $\mathcal{T}A$ and the equivariant pro-linear map $\sigma_A : A \to \mathcal{T}A$ satisfy the following universal property. If $l : A \to B$ is a lonilcur into a pro-G-algebra B there exists a unique equivariant homomorphism $[[l]] : \mathcal{T}A \to B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\sigma_A"] \arrow[dr, "l"'] & \mathcal{T}A \arrow[d, "{[[l]]}"] \\ & B \end{tikzcd}$$

is commutative.

PROOF. Let $\omega_l : A\hat{\otimes}A \to B$ be the curvature of l. For each $k \geq 0$ we define an equivariant pro-linear map $\phi_l^k : \Omega^{2k}(A) \to B$ by

$$\phi_l^k(x_0dx_1 \cdots dx_{2k}) = l(x_0)\omega_l(x_1, x_2) \cdots \omega_l(x_{2k-1}, x_{2k})$$

where l is extended to an equivariant pro-linear map $A^+ \to B^+$ which is again denoted by l and determined by $l(1) = 1$. Now let $f : B \to C$ be an equivariant pro-linear map with constant range and consider the map $h : B^+\hat{\otimes}B \to C$ given by $h(y_0 \otimes y_1) = f(y_0y_1)$. By the definition of tensor products in pro(G-Mod) there exist equivariant pro-linear maps with constant range $f_1 : B^+ \to C_1, f_2 : B \to C_2$ and an equivariant bounded linear map $g : C_1\hat{\otimes}C_2 \to C$ such that $h = g(f_1\hat{\otimes}f_2)$. Since l is a lonilcur there exists a natural number n such that $f_2m_B^n\omega_l^{\hat{\otimes}n} = 0$. Hence we have

$$f\phi_l^k = h(\phi_l^{k-n}\hat{\otimes}m_B^n\omega_l^{\hat{\otimes}n}) = g(f_1\phi_l^{k-n}\hat{\otimes}f_2m_B^n\omega_l^{\hat{\otimes}n}) = 0$$

for all $k \geq n$.
In order to construct $[[l]]$ we write $B = (B_i)_{i\in I}$ as a projective system. For each $i \in I$ we have a natural equivariant pro-linear map $f_i : B \to B_i$ with constant range. By the previous discussion there exists a natural number n_i such that $f_i\phi_l^k = 0$ for all $k \geq n_i$. We define the equivariant pro-linear map $[[l]]_i : \mathcal{T}A/(\mathcal{J}A)^{n_i} \to B_i$ by

$$[[l]]_i = f_i(\phi_l^0 \oplus \cdots \oplus \phi_l^{n_i-1}) : \bigoplus_{j=0}^{n_i-1} \Omega^{2j}(A) \to B_i.$$

Since $f_i\phi_l^k = 0$ for all $k \geq n_i$ the maps $[[l]]_i$ form a morphism of projective systems from $(\mathcal{T}A/(\mathcal{J}A)^n)_{n\in\mathbb{N}}$ to $(B_i)_{i\in I}$. This morphism induces an equivariant pro-linear map $[[l]] : \mathcal{T}A \to B$ of the corresponding projective limits. It is not difficult to check that $[[l]]$ is in fact a homomorphism. Moreover we have $[[l]]\sigma_A = l$ by construction. Using the definition of the Fedosov product we see that $[[l]]$ is the only equivariant homomorphism satisfying this relation. □
Let us now define and study quasifree pro-G-algebras.

DEFINITION 3.4. *A pro-G-algebra R is called quasifree if there exists an equivariant splitting homomorphism $R \to \mathcal{T}R$ for the natural projection τ_R.*

In the following theorem the class of quasifree pro-G-algebras is characterized.

Theorem 3.5. *Let G be a discrete group and let R be a pro-G-algebra. Then the following conditions are equivalent:*

a) R is quasifree.

b) There exists a family of equivariant homomorphisms $v_n : R \to \mathcal{T}R/(\mathcal{J}R)^n$ such that $v_1 = \mathrm{id}$ and v_{n+1} is a lifting of v_n.

c) For every admissible locally nilpotent extension $0 \to K \to E \to Q \to 0$ of pro-G-algebras and every equivariant homomorphism $f : R \to Q$ there exists an equivariant lifting homomorphism $h : R \to E$.

d) For every admissible nilpotent extension $0 \to K \to E \to Q \to 0$ of pro-G-algebras and every equivariant homomorphism $f : R \to Q$ there exists an equivariant lifting homomorphism $h : R \to E$.

e) For every admissible 2-nilpotent extension $0 \to K \to E \to Q \to 0$ of pro-G-algebras and every equivariant homomorphism $f : R \to Q$ there exists an equivariant lifting homomorphism $h : R \to E$.

f) For every admissible 2-nilpotent extension $0 \to K \to E \to R \to 0$ of pro-G-algebras there exists an equivariant splitting homomorphism $R \to E$.

g) There exists an equivariant splitting homomorphism for the natural homomorphism $\mathcal{T}R/(\mathcal{J}R)^2 \to R$.

h) There exists an equivariant pro-linear map $\phi : R \to \Omega^2(R)$ satisfying

$$\phi(xy) = \phi(x)y + x\phi(y) - dxdy$$

for all $x, y \in R$.

i) There exists an equivariant pro-linear map $\nabla : \Omega^1(R) \to \Omega^2(R)$ satisfying

$$\nabla(x\omega) = x\nabla(\omega), \qquad \nabla(\omega x) = \nabla(\omega)x - \omega dx$$

for all $x \in R$ and $\omega \in \Omega^1(R)$.

j) The R-bimodule $\Omega^1(R)$ is projective in $\mathrm{pro}(G\text{-}\mathsf{Mod})$.

k) There exists an admissible projective resolution of the R-bimodule R^+ of length 1 in $\mathrm{pro}(G\text{-}\mathsf{Mod})$.

Proof. $a) \Leftrightarrow b)$ follows directly from the definition of $\mathcal{T}R$ as the projective limit of the inverse system of pro-G-algebras $(\mathcal{T}R/(\mathcal{J}R)^n)_{n\in\mathbb{N}}$.
$a) \Rightarrow c)$ Let $f : R \to Q$ be an equivariant homomorphism and let

$$\begin{tikzcd} K \arrow[r, shift right, "i"'] & E \arrow[l, dashed, shift right] \arrow[r, shift right, "p"'] & Q \arrow[l, dashed, shift right, "s"'] \end{tikzcd}$$

be an admissible locally nilpotent extension as in $c)$. Since f is a homomorphism we compute $\omega_{sf}(x, y) = \omega_s(f(x), f(y))$ for the curvature of sf and it follows that ω_{sf} maps R into K. Since K is locally nilpotent by assumption we see that sf is a lonilcur. By proposition 3.3 there exists an equivariant homomorphism $[[sf]] : \mathcal{T}R \to E$ such that $[[sf]]\sigma_R = sf$. Now let $v : R \to \mathcal{T}R$ be an equivariant splitting homomorphism for the natural projection and put $h = [[sf]]v$. Using the uniqueness assertion in proposition 3.3 we obtain $p[[sf]] = f\tau_R$. This implies $ph = p[[sf]]v = f\tau_R v = f$. Hence h is a lifting homomorphism for f as desired.
$c) \Rightarrow a)$ Apply $c)$ to the admissible locally nilpotent extension $0 \to \mathcal{J}R \to \mathcal{T}R \to R \to 0$ and the identity map $R \to R$.
The implications $c) \Rightarrow d) \Rightarrow e) \Rightarrow f)$ are trivial.

$f) \Rightarrow g)$ Let $\mathcal{J}R/(\mathcal{J}R)^2$ be the pro-G-module $\Omega^2(R)$ equipped with the trivial multiplication. Consider the admissible 2-nilpotent extension $0 \to \mathcal{J}R/(\mathcal{J}R)^2 \to \mathcal{T}R/(\mathcal{J}R)^2 \to R \to 0$ and apply $f)$.
$g) \Rightarrow e)$ Retaining the notation from above we see as in the proof of the implication $a) \Rightarrow c)$ that there exists an equivariant homomorphism $[[sf]] : \mathcal{T}R \to E$ such that $[[sf]]\sigma_R = sf$. Since K is 2-nilpotent this homomorphism descends to a homomorphism $g : \mathcal{T}R/(\mathcal{J}R)^2 \to E$. Furthermore we have $pg = f\tau$ where $\tau : \mathcal{T}R/(\mathcal{J}R)^2 \to R$ is the natural projection. Let $v : R \to \mathcal{T}R/(\mathcal{J}R)^2$ be a splitting homomorphism for τ. The map $h = gv : R \to E$ is an equivariant homomorphism satisfying $ph = pgv = f\tau v = f$.
$e) \Rightarrow b)$ The homomorphisms v_n are constructed by induction. In the induction step we apply $f)$ to the admissible 2-nilpotent extension $0 \to (\mathcal{J}R)^n/(\mathcal{J}R)^{n+1} \to \mathcal{T}R/(\mathcal{J}R)^{n+1} \to \mathcal{T}R/(\mathcal{J}R)^n \to 0$ to obtain a lifting for $v_n : R \to \mathcal{T}R/(\mathcal{J}R)^n$.
$g) \Leftrightarrow h)$ By definition $\mathcal{T}R/(\mathcal{J}R)^2$ is the pro-G-algebra $R \oplus \Omega^2(R)$ with multiplication given by the Fedosov product. An equivariant pro-linear section v for the natural projection $\mathcal{T}R/(\mathcal{J}R)^2 \to R$ is necessarily of the form $v = \sigma_R + \phi$ for an equivariant pro-linear map $\phi : R \to \Omega^2(R)$. The section v is an algebra homomorphism iff

$$0 = (\sigma_R + \phi)(xy) - (\sigma_R + \phi)(x) \circ (\sigma_R + \phi)(y) = \phi(xy) - x\phi(y) - \phi(x)y + dxdy.$$

This yields the assertion.
$h) \Leftrightarrow i)$ Equivariant pro-linear maps from R to $\Omega^2(R)$ correspond to equivariant pro-linear left R-module homomorphisms $\Omega^1(R) = R^+\hat{\otimes}R \to \Omega^2(R)$. Using this correspondence conditions $h)$ and $i)$ are equivalent.
$h) \Leftrightarrow j)$ Consider the admissible short exact sequence

$$\begin{tikzcd} \Omega^2(R) \arrow[r, shift right, "i"'] & R^+\hat{\otimes}R\hat{\otimes}R^+ \arrow[l, dashed, shift right, "r"'] \arrow[r, shift right, "p"'] & \Omega^1(R) \arrow[l, dashed, shift right, "s"'] \end{tikzcd}$$

of pro-G-modules where i and p are the R-bimodule homomorphisms given by

$$i(x_0dx_1dx_2) = x_0x_1 \otimes x_2 \otimes 1 - x_0 \otimes x_1x_2 \otimes 1 + x_0 \otimes x_1 \otimes x_2$$

and

$$p(x_0 \otimes x_1 \otimes x_2) = x_0dx_1x_2$$

and the equivariant pro-linear maps r and s are defined by $r(x_0 \otimes x_1 \otimes x_2) = x_0dx_1dx_2$ and $s(x_0dx_1) = x_0 \otimes x_1 \otimes 1$. Clearly $\Omega^1(R)$ is a projective R-bimodule iff there exists an R-bimodule homomorphism $\rho : R^+\hat{\otimes}R\hat{\otimes}R^+ \to \Omega^2(R)$ satisfying $\rho i = \mathrm{id}$. Now R-bimodule homomorphisms $R^+\hat{\otimes}R\hat{\otimes}R^+ \to \Omega^2(R)$ correspond to equivariant pro-linear maps $R \to \Omega^2(R)$. Under this correspondence conditions $h)$ and $j)$ are equivalent.
$j) \Rightarrow k)$ If $\Omega^1(R)$ is projective the admissible short exact sequence

$$\begin{tikzcd} \Omega^1(R) \arrow[r, shift right, "i"'] & R^+\hat{\otimes}R^+ \arrow[l, dashed, shift right, "r"'] \arrow[r, shift right, "m"'] & R^+ \arrow[l, dashed, shift right, "s"'] \end{tikzcd}$$

is a projective resolution of the R-bimodule R^+ of length one. Here $m : R^+\hat{\otimes}R^+ \to R^+$ is multiplication, $i : \Omega^1(R) \to R^+\hat{\otimes}R^+$ is given by $i(x_0dx_1) = x_0 \otimes x_1 - x_0x_1 \otimes 1$ and $s : R^+ \to R^+\hat{\otimes}R^+$ is defined by $s(x) = x \otimes 1$.
$k) \Rightarrow j)$ We shall prove the following more general statement. Assume that $(\iota_P, \pi_P) : 0 \to K \to P \to R^+ \to 0$ and $(\iota_Q, \pi_Q) : 0 \to L \to Q \to R^+ \to 0$ are admissible extensions of R-bimodules in pro(G-Mod) with P and Q projective. Consider $M = P \oplus Q$ and the map $\pi = \pi_P \oplus \pi_Q$. Since we have pro-linear isomorphisms $P \cong R^+ \oplus K$ and $Q \cong R^+ \oplus L$

there exist pro-linear isomorphisms $M \cong R^+ \oplus L \oplus P$ and $\ker(\pi) \cong L \oplus P$. We obtain an admissible extension $0 \to L \to \ker(\pi) \to P \to 0$ of R-bimodules. Since P is projective we see that $\ker(\pi) \cong P \oplus L$ as R-bimodules. In the same way we obtain an R-bimodule isomorphism $\ker(\pi) \cong Q \oplus K$. It follows that K is projective iff L is projective. Applying this to the extension $0 \to \Omega^1(R) \to R^+ \hat{\otimes} R^+ \to R^+ \to 0$ and the projective resolution of R^+ of length one given in k) we deduce that $\Omega^1(R)$ is a projective R-bimodule in pro(G-Mod). This yields the assertion. □

A basic example of a quasifree pro-G-algebra is the algebra of complex numbers $\mathbb{C}$ with the trivial G-action. More generally the following easy observation is useful.

LEMMA 3.6. *Let A be a pro-algebra equipped with the trivial G-action. If A is quasifree as a pro-algebra it is quasifree as a pro-G-algebra.*

The following result is important.

PROPOSITION 3.7. *Let A be any pro-G-algebra. The periodic tensor algebra $\mathcal{T}A$ is quasifree.*

PROOF. We have to show that there exists an equivariant splitting homomorphism for the projection $\tau_{\mathcal{T}A} : \mathcal{T}\mathcal{T}A \to \mathcal{T}A$. Let us consider the equivariant pro-linear map $\sigma_A^2 = \sigma_{\mathcal{T}A}\sigma_A : A \to \mathcal{T}\mathcal{T}A$. We want to show that σ_A^2 is a lonilcur. First we compute the curvature $\omega_{\sigma_A^2}$ of σ_A^2 as follows:

$$\begin{aligned}\omega_{\sigma_A^2}(x,y) &= \sigma_A^2(xy) - \sigma_A^2(x) \circ \sigma_A^2(y) \\ &= \sigma_{\mathcal{T}A}(\sigma_A(xy)) - \sigma_{\mathcal{T}A}(\sigma_A(x) \circ \sigma_A(y)) + d\sigma_A^2(x) d\sigma_A^2(y) \\ &= \sigma_{\mathcal{T}A}(\omega_{\sigma_A}(x,y)) + d\sigma_A^2(x) d\sigma_A^2(y).\end{aligned}$$

Next consider the equivariant pro-linear map $\sigma_A = \tau_{\mathcal{T}A}\sigma_A^2$. Since $\tau_{\mathcal{T}A}$ is a homomorphism we obtain $\omega_{\sigma_A} = \tau_{\mathcal{T}A}\omega_{\sigma_A^2}$. Let $l : \mathcal{T}\mathcal{T}A \to C$ be an equivariant pro-linear map with constant range C. Composition with $\sigma_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\mathcal{T}A$ yields a map $k = l\sigma_{\mathcal{T}A} : \mathcal{T}A \to C$ with constant range. Since σ_A is a lonilcur there exists $n \in \mathbb{N}$ such that

$$km^n_{\mathcal{T}A}\omega_{\sigma_A}^{\hat{\otimes} n} = km^n_{\mathcal{T}A}\tau_{\mathcal{T}A}^{\hat{\otimes} n}\omega_{\sigma_A^2}^{\hat{\otimes} n} = k\tau_{\mathcal{T}A}m^n_{\mathcal{T}\mathcal{T}A}\omega_{\sigma_A^2}^{\hat{\otimes} n} = 0.$$

By the construction of $\mathcal{T}\mathcal{T}A$ the map l factors over $\mathcal{T}\mathcal{T}A/(\mathcal{J}(\mathcal{T}A))^m$ for some m. Using the formula for the curvature of σ_A^2 and our previous computation we obtain $lm^{mn}_{\mathcal{T}\mathcal{T}A}\omega_{\sigma_A^2}^{\hat{\otimes} mn} = 0$. Hence σ_A^2 is a lonilcur. By the universal property of $\mathcal{T}A$ there exists a homomorphism $v = [[\sigma_A^2]] : \mathcal{T}A \to \mathcal{T}\mathcal{T}A$ such that $v\sigma_A = \sigma_A^2$. This implies $(\tau_{\mathcal{T}A}v)\sigma_A = \tau_{\mathcal{T}A}\sigma_{\mathcal{T}A}\sigma_A = \sigma_A$. From the uniqueness assertion of proposition 3.3 we deduce $\tau_{\mathcal{T}A}v = \mathrm{id}$. This means that $\mathcal{T}A$ is quasifree. □

In connection with unital algebras the following result is useful.

PROPOSITION 3.8. *Let A be a pro-G-algebra. Then A is quasifree if and only if A^+ is quasifree.*

PROOF. Assume first that A^+ is quasifree and consider the admissible extension

$$\begin{tikzcd} \mathcal{J}A \arrow[r, shift right] & (\mathcal{T}A)^+ \arrow[l, dashed, shift right] \arrow[r, shift right, "\tau^+"'] & A^+ \arrow[l, dashed, shift right] \end{tikzcd}$$

of pro-G-algebras obtained by considering the unitarized version τ^+ of the natural homomorphism $\tau : \mathcal{T}A \to A$. Since $\mathcal{J}A$ is locally nilpotent and A^+ is quasifree there exists an equivariant splitting homomorphism $v : A^+ \to (\mathcal{T}A)^+$ for τ^+. From the relation $\tau^+ v = \mathrm{id}$ it follows that v restricts to a homomorphism $v : A \to \mathcal{T}A$. This implies that A is quasifree. Conversely, assume that A is quasifree. Since $\mathbb{C}$ is quasifree we can lift the canonical equivariant homomorphism $\mathbb{C} \to A^+$ to a homomorphism $e : \mathbb{C} \to \mathcal{T}(A^+)$. Consider the pro-$G$-module $e\mathcal{T}(A^+)e = \mathbb{C}\hat{\otimes}\mathcal{T}(A^+)\hat{\otimes}\mathbb{C}$ with multiplication defined by

$$(\alpha_1 \otimes x_1 \otimes \beta_1) \cdot (\alpha_2 \otimes x_2 \otimes \beta_2) = \alpha_1 \otimes x_1 e(\beta_1) e(\alpha_2) x_2 \otimes \beta_2.$$

It is easy to check that $e\mathcal{T}(A^+)e$ becomes a pro-G-algebra with this multiplication. There is a natural equivariant homomorphism $e\mathcal{T}(A^+)e \to \mathcal{T}(A^+)$ mapping $\alpha \otimes x \otimes \beta$ to $e(\alpha)xe(\beta)$. One should think of $e\mathcal{T}(A^+)e$ as the algebra obtained from $\mathcal{T}(A^+)$ by truncating with the idempotent associated to the homomorphism e. In a similar way we define $e\mathcal{J}(A^+)e$. It is easy to see that we obtain an admissible extension

$$\begin{tikzcd} e\mathcal{J}(A^+)e \arrow[r] & e\mathcal{T}(A^+)e \arrow[l, dashed, shift left] \arrow[r] & A^+ \arrow[l, dashed, shift left] \end{tikzcd}$$

of pro-G-algebras. Moreover one checks that the pro-G-algebra $e\mathcal{J}(A^+)e$ is locally nilpotent. Since A is assumed to be quasifree there exists an equivariant lifting homomorphism $u : A \to e\mathcal{T}(A^+)e$ for the natural homomorphism $A \to A^+$ due to theorem 3.5. We denote by v the composition of u with the canonical homomorphism $e\mathcal{T}(A^+)e \to \mathcal{T}(A^+)$. Now define $w = v \oplus e : A^+ \cong A \oplus \mathbb{C} \to \mathcal{T}(A^+)$. By definition w is an equivariant pro-linear map. Moreover we compute

$$\begin{aligned} w((a \oplus \alpha)(b \oplus \beta)) &= w(ab + \alpha b + \beta a \oplus \alpha\beta) \\ &= v(ab) + e(\alpha)v(b) + e(\beta)v(a) + e(\alpha\beta) = w(a \oplus \alpha)w(b \oplus \beta), \end{aligned}$$

hence w is a homomorphism. Since w lifts the map $\mathcal{T}(A^+) \to A^+$ we see that A^+ is quasifree. ☐

Let us also introduce separable pro-G-algebras. Separable pro-G-algebras constitute a special class of quasifree pro-G-algebras.

Definition 3.9. *A pro-G-algebra S is called separable if S^+ is a projective S-bimodule in* $\mathrm{pro}(G\text{-}\mathsf{Mod})$.

It follows directly from the definition that S is separable iff there exists an equivariant pro-linear S-bimodule splitting for the admissible short exact sequence $0 \to \Omega^1(S) \to S^+\hat{\otimes}S^+ \to S^+ \to 0$. The equivalence of conditions $a)$ and $k)$ in theorem 3.5 shows that separable pro-G-algebras are quasifree.

We will now study universal locally nilpotent extensions of pro-G-algebras. These extensions play an important conceptual role in equivariant periodic cyclic homology.

Definition 3.10. *Let A be a pro-G-algebra. A universal locally nilpotent extension of A is an admissible extension of pro-G-algebras $0 \to N \to R \to A \to 0$ where N is locally nilpotent and R is quasifree.*

We equip the Fréchet algebra $C^\infty[0,1]$ of smooth functions on the interval $[0,1]$ with the bounded bornology which equals the precompact bornology. We view $C^\infty[0,1]$ as a G-algebra with the trivial G-action. An equivariant homotopy is an equivariant homomorphism of pro-G-algebras $h : A \to B\hat{\otimes}C^\infty[0,1]$ where $C^\infty[0,1]$ is viewed as a constant pro-G-algebra. For each $t \in [0,1]$ evalutation at t defines an equivariant homomorphism $h_t : A \to B$. Two equivariant homomorphisms are equivariantly homotopic if they can be connected by an equivariant homotopy. We will also write $B[0,1]$ for the pro-G-algebra $B\hat{\otimes}C^\infty[0,1]$.

PROPOSITION 3.11. *Let $(\iota, \pi) : 0 \to N \to R \to A \to 0$ be a universal locally nilpotent extension of A. If $(i,p) : 0 \to K \to E \to Q \to 0$ is any other locally nilpotent extension and $\phi : A \to Q$ an equivariant homomorphism there exists a commutative diagram of pro-G-algebras*

$$\begin{tikzcd} N \arrow[r, "\iota"] \arrow[d, "\xi"'] & R \arrow[r, "\pi"] \arrow[d, "\psi"] & A \arrow[d, "\phi"] \\ K \arrow[r, "i"'] & E \arrow[r, "p"'] & Q \end{tikzcd}$$

Moreover the equivariant homomorphisms ξ and ψ are unique up to smooth homotopy. More generally let (ξ_t, ψ_t, ϕ_t) for $t = 0, 1$ be equivariant homomorphisms of extensions and let $\Phi : A \to Q[0,1]$ be an equivariant homotopy connecting ϕ_0 and ϕ_1. Then Φ can be lifted to an equivariant homotopy (Ξ, Ψ, Φ) between (ξ_0, ψ_0, ϕ_0) and (ξ_1, ψ_1, ϕ_1).

PROOF. Let $v : R \to \mathcal{T}R$ be a splitting homomorphism for the projection $\tau_R : \mathcal{T}R \to R$ and let $s : Q \to E$ be an equivariant pro-linear section for the projection $p : E \to Q$. Since $p(s\phi\pi) = \phi\pi$ is an equivariant homomorphism the curvature of $s\phi\pi : R \to E$ has values in K. Since by assumption K is locally nilpotent it follows that $s\phi\pi$ is a lonilcur. From the universal property of $\mathcal{T}R$ we obtain an equivariant homomorphism $k = [[s\phi\pi]] : \mathcal{T}R \to E$ such that $k\sigma_R = s\phi\pi$. Define $\psi = kv : R \to E$. We have

$$(pk)\sigma_R = ps\phi\pi = \phi\pi = (\phi\pi\tau_R)\sigma_R$$

and by the uniqueness assertion in proposition 3.3 we get $pk = \phi\pi\tau_R$. Hence $p\psi = pkv = \phi\pi\tau_R v = \phi\pi$ as desired. Moreover ψ maps N into K and restricts consequently to an equivariant homomorphism $\xi : N \to K$ making the diagram commutative.
The assertion that ψ and ξ are uniquely defined up to smooth homotopy follows from the more general statement about the lifting of homotopies. Hence let (ξ_t, ψ_t, ϕ_t) for $t = 0, 1$ and $\Phi : A \to Q[0,1]$ be given as above. Tensoring with $C^\infty[0,1]$ yields an admissible extension $(i[0,1], p[0,1]) : 0 \to K[0,1] \to E[0,1] \to Q[0,1] \to 0$ of pro-G-algebras. An equivariant pro-linear splitting $s[0,1]$ for this extension is obtained by tensoring s with the identity on $C^\infty[0,1]$. Since $\Phi_t\pi = p\psi_t$ for $t = 0, 1$ the equivariant pro-linear map $l : R \to E[0,1]$ defined by

$$l = s[0,1]\Phi\pi + (\psi_0 - s\phi_0\pi) \otimes (1 - t) + (\psi_1 - s\phi_1\pi) \otimes t$$

satisfies $\mathrm{ev}_t\, l = \psi_t$ for $t = 0, 1$ and $p[0,1]l = \Phi\pi$. The map $p[0,1]l = \Phi\pi$ is a homomorphism and hence the curvature of l has values in $K[0,1]$. Due to lemma 3.2 the pro-G-algebra $K[0,1] = K\hat{\otimes}C^\infty[0,1]$ is locally nilpotent. Consequently we get an equivariant homomorphism $[[l]] : \mathcal{T}R \to E[0,1]$ such that $[[l]]\sigma_R = l$. We define $\Psi = [[l]]v$ and in the same way as above we obtain $p[0,1]\Psi = \Phi\pi$. An easy computation shows $\Psi_t = \mathrm{ev}_t\,\Psi = \psi_t$ for $t = 0, 1$. Clearly Ψ restricts to an equivariant homomorphism $\Xi : N \to K[0,1]$ such that (Ξ, Ψ, Φ) becomes an equivariant homomorphism of extensions. $\square$

PROPOSITION 3.12. *Let A be a pro-G-algebra. The extension $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$ is a universal locally nilpotent extension of A. If $0 \to N \to R \to A \to 0$ is any other universal locally nilpotent extension of A it is equivariantly homotopy equivalent over A to $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$. In particular R is equivariantly homotopy equivalent to $\mathcal{T}A$ and N is equivariantly homotopy equivalent to $\mathcal{J}A$.*

PROOF. The pro-G-algebra $\mathcal{J}A$ is locally nilpotent by lemma 3.1. Moreover $\mathcal{T}A$ is quasifree by proposition 3.7. Hence the assertion follows from proposition 3.11. □

2. Equivariant differential forms

In the previous section we have seen that the space of noncommutative n-forms $\Omega^n(A)$ for a pro-G-algebra A is a pro-G-module in a natural way. We begin with the following assertion about the structure of this G-module in the situation where A is an ordinary G-algebra.

LEMMA 3.13. *For every G-algebra A and all n the G-module $\Omega^n(A\hat{\otimes}\mathcal{K}_G)$ is free.*

PROOF. The claim follows from the equivariant isomorphisms

$$\Omega^n(A\hat{\otimes}\mathcal{K}_G) \cong (A\hat{\otimes}\mathcal{K}_G)^+\hat{\otimes}(A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes}n} \cong (A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes}n+1} \oplus (A\hat{\otimes}\mathcal{K}_G)^{\hat{\otimes}n}$$

and lemma 2.8. □

Consider now the covariant module $\Omega^n_G(A) = \mathcal{O}_G\hat{\otimes}\Omega^n(A)$. The G-action on this space is defined by

$$t \cdot f(s) \otimes \omega = f(t^{-1}st) \otimes t \cdot \omega$$

for all $f \in \mathcal{O}_G$ and $\omega \in \Omega^n(A)$ and the $\mathcal{O}_G$-module structure is given by multiplication.

DEFINITION 3.14. *Let A be a pro-G-algebra. The covariant pro-module $\Omega^n_G(A)$ is called the space of equivariant n-forms over A.*

Let us define operators d and b_G on equivariant differential forms by

$$d(f(s) \otimes \omega) = f(s) \otimes d\omega$$

and

$$b_G(f(s) \otimes \omega dx) = (-1)^n(f(s) \otimes (\omega x - (s^{-1} \cdot x)\omega))$$

for $\omega \in \Omega^n(A)$ and $x \in A$. In order to clarify the notation we remark that one may view elements in $\Omega^n_G(A)$ as functions from G to $\Omega^n(A)$. In particular the precise meaning of the last formula is that evaluation of $b_G(f \otimes \omega dx) \in \Omega^n_G(A)$ at the group element $s \in G$ yields $(-1)^n(f(s)(\omega x - (s^{-1} \cdot x)\omega)) \in \Omega^n(A)$.

Having this in mind we want to study the properties of the operators d and b_G. As in the non-equivariant case we have $d^2 = 0$. The operator b_G should be thought of as a twisted version of the ordinary Hochschild boundary. We compute for $\omega \in \Omega^n(A)$ and $x, y \in A$

$$\begin{aligned} b_G^2(f(s) \otimes \omega dxdy) &= b_G((-1)^{n+1}(f(s) \otimes \omega dxy - f(s) \otimes (s^{-1} \cdot y)\omega dx)) \\ &= b_G((-1)^{n+1}(f(s) \otimes \omega d(xy) - f(s) \otimes \omega xdy - f(s) \otimes (s^{-1} \cdot y)\omega dx)) \\ &= (-1)^n(-1)^{n+1}(f(s) \otimes \omega xy - f(s) \otimes s^{-1} \cdot (xy)\omega \\ &\quad - (f(s) \otimes \omega xy - f(s) \otimes (s^{-1} \cdot y)\omega x) \\ &\quad - (f(s) \otimes (s^{-1} \cdot y)\omega x - f(s) \otimes (s^{-1} \cdot x)(s^{-1} \cdot y)\omega)) = 0. \end{aligned}$$

This shows $b_G^2 = 0$ and hence b_G is an ordinary differential.
Similar to the non-equivariant case we construct an equivariant Karoubi operator κ_G and an equivariant Connes operator B_G out of d and b_G. We define

$$\kappa_G = \mathrm{id} - (b_G d + d b_G)$$

and on $\Omega^n_G(A)$ we set

$$B_G = \sum_{j=0}^{n} \kappa_G^j d.$$

Using that κ_G commutes with d and $d^2 = 0$ we obtain $B_G^2 = 0$. Let us record the following explicit formulas on $\Omega^n_G(A)$. For $n \geq 1$ we have

$$\kappa_G(f(s) \otimes \omega dx) = (-1)^{n-1} f(s) \otimes (s^{-1} \cdot dx)\omega$$

and we obtain $\kappa_G(f(s) \otimes x) = f(s) \otimes s^{-1} \cdot x$ for $f(s) \otimes x \in \Omega^0_G(A)$. For the Connes operator we compute

$$B_G(f(s) \otimes x_0 dx_1 \cdots dx_n) = \sum_{i=0}^{n} (-1)^{ni} f(s) \otimes s^{-1} \cdot (dx_{n+1-i} \cdots dx_n) dx_0 \cdots dx_{n-i}.$$

In addition we define the operator T by

$$T(f(s) \otimes \omega) = f(s) \otimes s^{-1} \cdot \omega = s^{-1} \cdot (f(s) \otimes \omega).$$

It is easy to check that all operators constructed so far are covariant.
The following lemma describes an important property of the map T.

Lemma 3.15. *Let A and B be pro-G-algebras and let $\phi : \Omega_G(A)^m \to \Omega_G(B)^n$ be a covariant map for some m, n. Then $\phi T = T \phi$.*

Proof. For an element $s \in G$ we let $\delta_s \in \mathcal{O}_G$ be the function determined by $\delta_s(s) = 1$ and $\delta_s(t) = 0$ for $t \neq s$. We compute

$$(T\phi)(\delta_s \otimes \omega) = s \cdot \phi(\delta_s \otimes \omega) = \phi(s \cdot (\delta_s \otimes \omega)) = (\phi T)(\delta_s \otimes \omega)$$

using the fact that ϕ is $\mathcal{O}_G$-linear and equivariant. Since $\mathcal{O}_G$ is spanned linearly by the functions δ_s with $s \in G$ the claim follows. $\square$
In order to keep the formulas readable we will frequently write b instead of b_G in the sequel and use similar simplifications for the other operators.
We need the following lemma concerning relations between the operators constructed above. See [**28**] for the corresponding assertion in the non-equivariant context.

Lemma 3.16. *On $\Omega^n_G(A)$ the following relations hold:*

a) $\kappa^{n+1} d = T d$
b) $\kappa^n = T + b\kappa^n d$
c) $b\kappa^n = bT$
d) $\kappa^{n+1} = (\mathrm{id} - db)T$
e) $(\kappa^{n+1} - T)(\kappa^n - T) = 0$
f) $Bb + bB = \mathrm{id} - T$

Proof. The proof of a) is easy. b) Using the explicit formula for κ from above we compute

$$\begin{aligned}\kappa^n(f(s)\otimes x_0dx_1\cdots dx_n) &= f(s)\otimes s^{-1}\cdot(dx_1\cdots dx_n)x_0\\ &= f(s)\otimes s^{-1}\cdot(x_0dx_1\cdots dx_n)+(-1)^nb(f(s)\otimes s^{-1}\cdot(dx_1\cdots dx_n)dx_0)\\ &= T(f(s)\otimes x_0dx_1\cdots dx_n)+b\kappa^nd(f(s)\otimes x_0dx_1\cdots dx_n).\end{aligned}$$

c) follows by applying the Hochschild boundary b to both sides of b). d) Apply κ to b) and use a). e) is a consequence of b) and d). f) We compute

$$\begin{aligned}Bb+bB &= \sum_{j=0}^{n-1}\kappa^j db+\sum_{j=0}^{n} b\kappa^j d=\sum_{j=0}^{n-1}\kappa^j(db+bd)+\kappa^n bd\\ &= \mathrm{id}-\kappa^n(\mathrm{id}-bd)=\mathrm{id}-\kappa^n(\kappa+db)=\mathrm{id}-T+dbT-Tdb=\mathrm{id}-T\end{aligned}$$

where we use d) and b) and the fact that T commutes with covariant maps due to lemma 3.15. □

Let us summarize this discussion as follows.

Proposition 3.17. *Let A be a pro-G-algebra. The space $\Omega_G(A)$ of equivariant differential forms is a paramixed complex in the category* $\mathrm{pro}(G\text{-}\mathfrak{Mod})$ *of covariant pro-modules and all the operators constructed above are covariant.*

As for ordinary differential forms we define $\Omega_G^{\leq n}(A)=\Omega_G^0(A)\oplus\Omega_G^1(A)\oplus\cdots\oplus\Omega_G^n(A)$ for all $n\geq 0$. We have the usual $\mathbb{Z}_2$-grading on $\Omega_G^{\leq n}(A)$ into even and odd forms. The natural projection $\Omega_G^{\leq m}(A)\to\Omega_G^{\leq n}(A)$ for $m\geq n$ is a covariant homomorphism and compatible with the grading. Hence we obtain a projective system $(\Omega_G^{\leq n}(A))_{n\in\mathbb{N}}$ and we let $\theta\Omega_G(A)$ be the corresponding projective limit.

We conclude this section with the following fact which is a consequence of corollary 2.14 and lemma 3.13.

Proposition 3.18. *For every G-algebra A and all n the covariant module $\Omega_G^n(A\hat{\otimes}\mathcal{K}_G)$ is projective.*

3. The equivariant X-complex

In this section we define and study the equivariant X-complex. Apart from the periodic tensor algebra introduced in section 3.1 this object is the main ingredient in the definition of equivariant periodic cyclic homology.

Consider the paramixed complex $\Omega_G(A)$ of equivariant differential forms for a pro-G-algebra A which was defined in the previous section. Following Cuntz and Quillen [**28**] we define the n-th level of the Hodge tower associated to $\Omega_G(A)$ by

$$\theta^n\Omega_G(A)=\Omega_G(A)/F^n\Omega_G(A)=:\bigoplus_{j=0}^{n-1}\Omega_G^j(A)\oplus\Omega_G^n(A)/b_G(\Omega_G^{n+1}(A))$$

for $n\geq 0$. It is easy to see that the operators d and b_G descend to $\theta^n\Omega_G(A)$. Consequently the same holds true for κ_G and B_G. Using the natural grading into even and odd forms we see that $\theta^n\Omega_G(A)$ together with the boundary operator B_G+b_G becomes a pro-parasupercomplex.

We emphasize that $\theta^n\Omega_G(A)$ is in general a projective systems of not necessarily separated bornological spaces. However, we will only have to work with these objects either in the case they are in fact projective systems of separated spaces or if the bornology is not taken into account at all.
We define the Hodge filtration on $\theta^n\Omega_G(A)$ by

$$F^k\theta^n\Omega_G(A) = b_G(\Omega_G^{k+1}(A)) \oplus \bigoplus_{j=k+1}^{n-1} \Omega_G^j(A) \oplus \Omega_G^n(A)/b_G(\Omega_G^{n+1}(A)).$$

Clearly $F^k\Omega_G(A)$ is closed under b_G and B_G. The Hodge filtration on $\theta^n\Omega_G(A)$ is a finite decreasing filtration such that $F^{-1}\theta^n\Omega_G(A) = \theta^n\Omega_G(A)$ and $F^n\theta^n\Omega_G(A) = 0$. Remark that these definitions can be extended to arbitrary paramixed complexes in straightforward way. For $m \geq n$ there exists a natural map $\theta^m\Omega_G(A) \to \theta^n\Omega_G(A)$. The Hodge tower of $\Omega_G(A)$ is the projective limit of the projective system $(\theta^n\Omega_G(A))_{n\in\mathbb{N}}$ obtained in this way. Recall from section 3.2 the definition of $\theta\Omega_G(A)$. In the category of projective systems of non-separated covariant modules the Hodge tower of $\Omega_G(A)$ is isomorphic to $\theta\Omega_G(A)$.

Definition 3.19. *Let A be a pro-G-algebra. The equivariant X-complex $X_G(A)$ of A is the pro-parasupercomplex defined by*

$$X_G(A) : \Omega_G^0(A) \underset{b_G}{\overset{\natural d}{\rightleftarrows}} \Omega_G^1(A)/b_G(\Omega_G^2(A))$$

where $\natural : \Omega_G^1(A) \to \Omega_G^1(A)/b_G(\Omega_G^2(A))$ denotes the natural projection.

It follows from the definitions that $X_G(A)$ is the first level $\theta^1\Omega_G(A)$ of the Hodge tower of $\Omega_G(A)$. We will only be interested in the case where A is a quasifree pro-G-algebra. Recall from theorem 3.5 that for a quasifree pro-G-algebra A the A-bimodule $\Omega^1(A)$ is projective in pro(G-Mod). Using this fact it is easy to see that $\Omega_G^1(A)/b_G(\Omega_G^2(A))$ is a projective system of separated spaces. We will view the subspace $b_G(\Omega_G^2(A))$ of $\Omega_G^1(A)$ as the space of "equivariant commutators" and occasionally write $\omega \mod [\ ,\ ]_G$ for elements in $\Omega_G^1(A)/b_G(\Omega_G^2(A))$.
We point out that, despite of our terminology, $X_G(A)$ is usually not a complex. If we denote the differential in $X_G(A)$ by ∂ we obtain $\partial^2 = \mathrm{id} - T$. This shows that in general ∂^2 is not zero.
Let us determine the equivariant X-complex of the complex numbers. This will be useful in computations later on.

Lemma 3.20. *The equivariant X-complex $X_G(\mathbb{C})$ of the complex numbers $\mathbb{C}$ can be identified with the trivial supercomplex $\mathcal{O}_G[0]$.*

Proof. Since the action on $\mathbb{C}$ is trivial we have

$$f(s) \otimes ede = f(s) \otimes ed(e^2) = f(s) \otimes ede + f(s) \otimes edee = 2f(s) \otimes ede$$

and hence $f(s) \otimes ede = 0$ in $X_G^1(\mathbb{C})$. Similarly,

$$f(s) \otimes 1de = f(s) \otimes 1d(e^2) = 2f(s) \otimes ede = 0$$

in $X_G^1(\mathbb{C})$. This implies $X_G^1(\mathbb{C}) = 0$ and the claim follows. □
We are interested in the equivariant X-complex of the periodic tensor algebra $\mathcal{T}A$ of a pro-G-algebra A. The first goal is to relate the covariant pro-module $X_G(\mathcal{T}A)$ to equivariant

differential forms over A. If we denote the even part of $\theta\Omega_G(A)$ by $\theta\Omega_G^{ev}(A)$ we obtain a covariant isomorphism

$$X_G^0(\mathcal{T}A) = \mathcal{O}_G\hat{\otimes}\mathcal{T}A \cong \theta\Omega_G^{ev}(A)$$

according to the definition of $\mathcal{T}A$.
Before we consider $X_G^1(\mathcal{T}A)$ we have to make a convention. We use the letter D for the equivariant pro-linear map $\mathcal{T}A \to \Omega^1(\mathcal{T}A)$ usually denoted by d. This will help us not to confuse this map with the differential d in $\mathcal{T}A = \theta\Omega^{ev}(A)$.

PROPOSITION 3.21. *Let A be any pro-G-algebra. The following maps are equivariant pro-linear isomorphisms.*

$$\begin{aligned}
&\mu_1 : (\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+ \to \Omega^1(\mathcal{T}A), && \mu_1(x\otimes a\otimes y) = xD\sigma_A(a)y\\
&\mu_2 : (\mathcal{T}A)^+\hat{\otimes}A \to \mathcal{T}A, && \mu_2(x\otimes a) = x\circ\sigma_A(a)\\
&\mu_3 : A\hat{\otimes}(\mathcal{T}A)^+ \to \mathcal{T}A, && \mu_3(a\otimes x) = \sigma_A(a)\circ x.
\end{aligned}$$

Hence $\Omega^1(\mathcal{T}A)$ is a free $\mathcal{T}A$-bimodule and $\mathcal{T}A$ is free as a left and right $\mathcal{T}A$-module.

PROOF. First we construct an inverse for μ_1. The derivation rule for D implies $D(dxdy) = D(xy) - xDy - D(x)y$ and consequently

$$\begin{aligned}
D(x_0dx_1\cdots dx_{2n}) &= D(x_0\circ(dx_1dx_2)\circ\cdots\circ(dx_{2n-1}dx_{2n}))\\
&= D(x_0)dx_1\cdots dx_{2n} + \sum_{j=1}^{n} x_0dx_1\cdots dx_{2j-2}D(dx_{2j-1}dx_{2j})dx_{2j+1}\cdots dx_{2n}\\
&= D(x_0)dx_1\cdots dx_{2n} + \sum_{j=1}^{n} x_0dx_1\cdots dx_{2j-2}D(x_{2j-1}x_{2j})dx_{2j+1}\cdots dx_{2n}\\
&\quad - \sum_{j=1}^{n} x_0dx_1\cdots dx_{2j-2}\circ x_{2j-1}D(x_{2j})dx_{2j+1}\cdots dx_{2n}\\
&\quad - \sum_{j=1}^{n} x_0dx_1\cdots dx_{2j-2}D(x_{2j-1})x_{2j}dx_{2j+1}\cdots dx_{2n}.
\end{aligned}$$

This formula gives an explicit preimage for $D(x_0dx_1\cdots dx_{2n})$. We can extend the resulting map $f : D(\mathcal{T}A) \to (\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+$ to a left $\mathcal{T}A$-module homomorphism $\nu_1 : \Omega^1(\mathcal{T}A) \to (\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+$ by setting $\nu_1(\omega D\eta) = \omega f(D\eta)$. By construction we have $\mu_1\nu_1 = \mathrm{id}$ on $D(\mathcal{T}A)$ and hence on $\Omega^1(\mathcal{T}A)$ because μ_1 and ν_1 are left $\mathcal{T}A$-linear. One can check directly that ν_1 is right $\mathcal{T}A$-linear. This implies $\nu_1\mu_1 = \mathrm{id}$.
Since μ_1 is an isomorphism we obtain a natural admissible extension

$$\begin{tikzcd} (\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+ \arrow[r, shift left] & (\mathcal{T}A)^+\hat{\otimes}(\mathcal{T}A)^+ \arrow[l, dashed, shift left] \arrow[r, shift left] & (\mathcal{T}A)^+ \arrow[l, dashed, shift left] \end{tikzcd}$$

of $\mathcal{T}A$-bimodules with right $\mathcal{T}A$-linear splitting. We tensor this extension over $\mathcal{T}A$ with $\mathbb{C}$ viewed as left $\mathcal{T}A$-module with the zero module structure to obtain an admissible extension of pro-G-modules

$$\begin{tikzcd} (\mathcal{T}A)^+\hat{\otimes}A \arrow[r, shift left] & (\mathcal{T}A)^+ \arrow[l, dashed, shift left] \arrow[r, shift left] & \mathbb{C} \arrow[l, dashed, shift left] \end{tikzcd}$$

where the projection $(\mathcal{T}A)^+ \to \mathbb{C}$ is the usual augmentation. This yields the desired isomorphism $(\mathcal{T}A)^+\hat{\otimes} A \cong \mathcal{T}A$. The proof for μ_3 is similar and will be ommitted. $\square$

Using proposition 3.21 we see that the map $\mu_1 : (\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+ \to \Omega^1(\mathcal{T}A)$ induces a covariant isomorphism $\mathcal{O}_G\hat{\otimes}(\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+ \cong \Omega^1_G(\mathcal{T}A)$. Identifying equivariant commutators under this isomorphism and dividing them out yields a covariant isomorphism

$$\Omega^1_G(\mathcal{T}A)/b_G(\Omega^2_G(\mathcal{T}A)) \cong \mathcal{O}_G\hat{\otimes}(\mathcal{T}A)^+\hat{\otimes}A.$$

Using again $\mathcal{T}A = \theta\Omega^{ev}(A)$ we obtain a covariant isomorphism

$$X^1_G(\mathcal{T}A) \cong \theta\Omega^{odd}_G(A)$$

where $\theta\Omega^{odd}_G(A)$ is the odd part of $\theta\Omega_G(A)$.

Having identified $X_G(\mathcal{T}A)$ and $\theta\Omega_G(A)$ as covariant pro-modules we want to compare the differentials on both sides. To this end let $f(s) \otimes xda$ be an element of $\theta\Omega^{odd}_G(A)$ where $x \in \mathcal{T}A \cong \theta\Omega^{ev}_G(A)$ and $a \in A$. The differential $X^1_G(\mathcal{T}A) \to X^0_G(\mathcal{T}A)$ in the equivariant X-complex corresponds to

$$\begin{aligned}\partial_1(f(s)\otimes xda) &= f(s)\otimes(x\circ a - (s^{-1}\cdot a)\circ x)\\ &= f(s)\otimes(xa - (s^{-1}\cdot a)x - dxda + (s^{-1}\cdot da)dx)\\ &= b(f(s)\otimes xda) - (\mathrm{id}+\kappa)d(f(s)\otimes xda).\end{aligned}$$

To compute the other differential we map $f(s)\otimes Dx \in \Omega^1_G(\mathcal{T}A)$ to $(\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+$ using the inverse of the isomorphism μ_1 in proposition 3.21 and compose with the covariant map $(\mathcal{T}A)^+\hat{\otimes}A\hat{\otimes}(\mathcal{T}A)^+ \to \theta\Omega^{odd}_G(A)$ sending $f(s)\otimes x_0\otimes a\otimes x_1$ to $f(s)\otimes(s^{-1}\cdot x_1)\circ x_0da$. The explicit formula in the proof of proposition 3.21 yields

$$\begin{aligned}\partial_0(f(s)\otimes x_0dx_1\cdots dx_{2n}) &= f(s)\otimes D(x_0dx_1\cdots dx_{2n}) \mod [\ ,\ ]_G\\
&= f(s)\otimes s^{-1}\cdot(dx_1\cdots dx_{2n})Dx_0\\
&\quad + \sum_{j=1}^{n} f(s)\otimes s^{-1}\cdot(dx_{2j+1}\cdots dx_{2n})\circ x_0dx_1\cdots dx_{2j-2}D(x_{2j-1}x_{2j})\\
&\quad - \sum_{j=1}^{n} f(s)\otimes s^{-1}\cdot(dx_{2j+1}\cdots dx_{2n})\circ x_0dx_1\cdots dx_{2j-2}\circ x_{2j-1}Dx_{2j}\\
&\quad - \sum_{j=1}^{n} f(s)\otimes s^{-1}\cdot(x_{2j}dx_{2j+1}\cdots dx_{2n})\circ x_0dx_1\cdots dx_{2j-2}Dx_{2j-1} \mod [\ ,\ ]_G\\
&= \sum_{j=0}^{2n} f(s)\otimes s^{-1}\cdot(dx_j\cdots dx_{2n})dx_0dx_1\cdots dx_{j-1}\\
&\quad - \sum_{j=1}^{n} b(f(s)\otimes s^{-1}\cdot(dx_{2j+1}\cdots dx_{2n})x_0dx_1\cdots dx_{2j-1}dx_{2j}\\
&= B(f(s)\otimes x_0dx_1\cdots dx_{2n}) - \sum_{j=0}^{n-1}\kappa^{2j}b(f(s)\otimes x_0dx_1\cdots dx_{2n}).\end{aligned}$$

This can be summarized as follows.

PROPOSITION 3.22. *Under the identification $X_G(\mathcal{T}A) \cong \theta\Omega_G(A)$ as above the differentials of the equivariant X-complex correspond to*

$$\partial_1 = b - (\mathrm{id} + \kappa)d \qquad \text{on } \theta\Omega_G^{odd}(A)$$

$$\partial_0 = -\sum_{j=0}^{n-1} \kappa^{2j} b + B \qquad \text{on } \Omega_G^{2n}(A).$$

We would like to show that the paracomplexes $X_G(\mathcal{T}A)$ and $\theta\Omega_G(A)$ are covariantly homotopy equivalent. However, at this point we cannot directly proceed as in the nonequivariant case.

Let us recall the situation for the ordinary X-complex. The proof of the homotopy equivalence between $X(\mathcal{T}A)$ and $\theta\Omega(A)$ given by Cuntz and Quillen [**28**], [**30**] is based on the spectral decomposition of the Karoubi operator κ. This decomposition is obtained from the polynomial relation

$$(\kappa^{n+1} - \mathrm{id})(\kappa^n - \mathrm{id}) = 0$$

which holds on $\Omega^n(A)$. Remark that this formula is related to the fact that the cyclic permutation operator is of finite order on $\Omega^n(A)$.

In the equivariant theory the situation is different. The equivariant cyclic permutation operator is in general of infinite order, due to lemma 3.16 e) the relevant relation for κ is

$$(\kappa^{n+1} - T)(\kappa^n - T) = 0$$

on $\Omega^n_G(A)$. It is not clear how to obtain a reasonable decomposition of $\Omega^n_G(A)$ using this relation. Hence the argument from [**28**] cannot be carried over directly.

In fact we do not know if $X_G(\mathcal{T}A)$ and $\theta\Omega_G(A)$ are homotopy equivalent for all pro-G-algebras A. However, we can prove a weaker statement which is sufficient for our purposes.

THEOREM 3.23. *Let A be a G-algebra. Then the pro-parasupercomplexes $X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G))$ and $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ are covariantly homotopy equivalent.*

The proof of theorem 3.23 depends essentially on the fact that the spaces $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ of equivariant differential forms over $A\hat{\otimes}\mathcal{K}_G$ are projective covariant modules. We remark that the projective system $X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)) \cong \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ is not a projective object in $\mathrm{pro}(G\text{-}\mathfrak{Mod})$.

Using proposition 3.22 we see that it suffices to prove that the pro-parasupercomplexes $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \partial)$ and $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), B + b)$ are covariantly homotopy equivalent. This will be done in two steps.

In the first step we divide out the action of T. More precisely, let $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T$ be the quotient of $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ by the image of the operator $\mathrm{id} - T$ on $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$. Due to lemma 3.13 the covariant module $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ is isomorphic to a covariant module of the form $\mathcal{O}_G\hat{\otimes}W$ with a free G-module W. Using this description it is easy to see that we obtain an admissible extension

$$\begin{tikzcd}(\mathrm{id} - T)\Omega^n_G(A\hat{\otimes}\mathcal{K}_G) \arrow[r, shift right] & \Omega^n_G(A\hat{\otimes}\mathcal{K}_G) \arrow[l, dashed, shift right] \arrow[r, shift right, "\pi"'] & \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T \arrow[l, dashed, shift right, "\sigma"']\end{tikzcd}$$

of covariant modules for all n where $\pi : \Omega^n_G(A\hat{\otimes}\mathcal{K}_G) \to \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T$ denotes the natural projection. These constructions can be extended to $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ in the obvious way and we obtain an admissible extension

$$(\mathrm{id}-T)\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \rightleftarrows \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \underset{\pi}{\overset{\sigma}{\rightleftarrows}} \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$$

of covariant pro-modules. Remark that on the quotient $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$ the relation $T = \mathrm{id}$ holds. Due to lemma 3.15 all operators constructed on equivariant differential forms in section 3.2 descend to $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$. In particular we see using lemma 3.16 that both ∂ and $B + b$ descend to ordinary differentials on $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$ which will again be denoted by ∂ and $B + b$, respectively.

THEOREM 3.24. *The pro-supercomplexes $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, \partial)$ and $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, B+b)$ are covariantly homotopy equivalent.*

PROOF. The proof from the nonequivariant situation can be carried over easily. We define $c_{2n} = c_{2n+1} = (-1)^n n!$ for $n \in \mathbb{N}$. Consider the isomorphism $c : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ given by $c(\omega) = c_n\,\omega$ for $\omega \in \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ and let $\delta = c^{-1}(B + b)c$ be the boundary corresponding to $B + b$ under this isomorphism. It is easy to check that

$$\delta = B - nb \qquad \text{on } \Omega^{2n}_G(A\hat{\otimes}\mathcal{K}_G)$$

and

$$\delta = -\frac{1}{n+1}B + b \qquad \text{on } \Omega^{2n+1}_G(A\hat{\otimes}\mathcal{K}_G).$$

Hence in order to prove theorem 3.24 it is enough to show that $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, \partial)$ and $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, \delta)$ are covariantly homotopy equivalent.
On $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T$ the polynomial relation $(\kappa^{n+1} - \mathrm{id})(\kappa^n - \mathrm{id}) = 0$ holds. Multiplying both sides with $(\kappa^{n+1} + \mathrm{id})(\kappa^n + \mathrm{id})$ we see that κ^2 also satisfies this relation. Since we are working over the complex numbers $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T$ decomposes into a direct sum $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T = \bigoplus_{\lambda\in N} V_\lambda$ of eigenspaces V_λ where N is the set of roots of the polynomial $(z^{n+1}-1)(z^n-1)$. We denote by P be the projection onto the eigenspace V_1 and let H be the operator specified by the equations $PH = HP = 0$ and $H(\mathrm{id}-\kappa^2) = (\mathrm{id}-\kappa^2)H = \mathrm{id}-P$. Since P and H can be written as polynomials in κ it follows that both operators are covariant. Moreover P and H are chain maps with respect to ∂ and δ. Using lemma 3.16 a) we see that $\kappa^2 d = d$ on the range of P. Part d) of lemma 3.16 yields in the same way $\kappa^2 b = b$ on $P\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$. Hence the restriction of ∂_0 to $P\theta\Omega^{2n}_G(A\hat{\otimes}\mathcal{K}_G)$ is

$$B - \sum_{j=0}^{n-1}\kappa^{2j}b = B - nb = \delta.$$

The restriction of ∂_1 to $P\theta\Omega^{2n+1}_G(A\hat{\otimes}\mathcal{K}_G)$ is

$$b - (\mathrm{id}+\kappa)d = b - \frac{1}{n+1}\sum_{j=0}^{n}(\mathrm{id}+\kappa)\kappa^{2j}d = b - \frac{1}{n+1}B = \delta.$$

This shows that the boundaries ∂ and δ are equal on the range of P. It remains to show that $\mathrm{id}-P$ is homotopic to zero with respect to both boundaries ∂ and δ. We have $\mathrm{id}-P = H(\mathrm{id}-\kappa^2)$ and it suffices to show that $\mathrm{id}-\kappa^2$ is homotopic to zero. Since $Bd = dB = 0$ we compute

$$[\delta, c^{-1}dc] = c^{-1}[B + b, d]c = c^{-1}(bd + db)c = c^{-1}(\mathrm{id}-\kappa)c = \mathrm{id}-\kappa.$$

This means that $\mathrm{id}-\kappa$ is homotopic to zero and hence this is true also for $\mathrm{id}-\kappa^2 = (\mathrm{id}-\kappa)+(\kappa-\kappa^2)$. To treat the differential ∂ we compute

$$(b-(\mathrm{id}+\kappa)d)^2 = -(\mathrm{id}+\kappa)(bd+db) = -(\mathrm{id}+\kappa)(\mathrm{id}-\kappa) = \kappa^2-\mathrm{id}\,.$$

Since we have $\partial^2 = 0$ we deduce $[\partial, b-(\mathrm{id}+\kappa)d] = (b-(\mathrm{id}+\kappa)d)^2 = \kappa^2-\mathrm{id}$. This shows that $\mathrm{id}-\kappa^2$ is homotopic to zero with respect to ∂. We remark that the homotopies constructed in this discussion map $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_T$ to $\Omega^{n-1}_G(A\hat{\otimes}\mathcal{K}_G)_T \oplus \Omega^{n+1}_G(A\hat{\otimes}\mathcal{K}_G)_T$. □

In the second step of the proof of theorem 3.23 we have to lift the homotopy equivalences obtained in theorem 3.24 to $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$. We shall treat the elliptic and hyperbolic components of $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ separately.

Let us first consider the hyperbolic part. From theorem 3.24 we obtain covariant chain maps $f : (\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, B+b) \to (\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, \partial)$ and $g : (\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, \partial) \to (\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T, B+b)$ which are inverse up to homotopy. We have to lift f and g to $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$. Consider the map $p(\kappa) : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ where $p(z)$ is the polynomial

$$p(z) = \Big(\sum_{j=0}^{n-1} z^j\Big)\Big(\sum_{j=0}^{n} z^j\Big)(1-(n-\frac{1}{2})(z-1)).$$

Using the terminology of theorem 3.24 it is straightforward to check that $p(\kappa)$ is a lifting of the projection onto the eigenspace V_1. If we define $F = G = p(\kappa)$ we see that F and G are covariant liftings for the homotopy equivalences f and g in the sense that

$$\pi F = f\pi, \qquad \pi G = g\pi.$$

However, F and G are not compatible with the differentials. In order to obtain appropriate chain maps we have to proceed as follows. First observe that the relation $\partial(f) = f(B+b)-\partial f = 0$ implies $\pi\partial(F) = \pi(F(B+b)-\partial F) = 0$. This means that $\partial(F)$ maps $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ into the kernel $(\mathrm{id}-T)\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ of the natural projection $\pi : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$. Now remark that on the hyperbolic part the map $\mathrm{id}-T : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to (\mathrm{id}-T)\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ is a covariant isomorphism since the spaces $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ are free G-modules. Hence we obtain a covariant map $K_f : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ such that $(\mathrm{id}-T)K_f = \partial(F)$. Due to lemma 3.15 we have $\partial^2(F) = F(\mathrm{id}-T)-(\mathrm{id}-T)F = TF-FT = 0$ and hence we deduce

$$\partial K_f + K_f(B+b) = 0.$$

In the same way we obtain a covariant map $K_g : \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G) \to \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ such that $(\mathrm{id}-T)K_g = \partial(g)$. Let us define

$$L_f = \frac{1}{2}\partial K_f = -\frac{1}{2}K_f(B+b), \qquad L_g = \frac{1}{2}(B+b)K_g = -\frac{1}{2}K_g\partial$$

and

$$\Phi = F + L_f, \qquad \Psi = G + L_g.$$

We compute

$$\begin{aligned}\partial(\Phi) = \partial(F)+\partial(L_f) &= (\mathrm{id}-T)K_f - \frac{1}{2}K_f(B+b)^2 - \frac{1}{2}\partial^2 K_f \\ &= (\mathrm{id}-T)K_f - (\mathrm{id}-T)K_f = 0.\end{aligned}$$

This shows that Φ is a chain map. In the same way we obtain that Ψ is a chain map.
We will now show that $\Phi\Psi$ is homotopic to the identity on $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\partial)$. According to theorem 3.24 there exists a covariant map $h:\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T\to\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T$ such that $\partial h+h\partial=\mathrm{id}-fg$. Due to the fact that $\Omega^n_G(A\hat{\otimes}\mathcal{K}_G)$ is a projective covariant module for all n we see from the last remark in the proof of theorem 3.24 that h can be lifted to a covariant map $H:\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)\to\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ of degree 1 such that $\pi H=h\pi$. Now we compute

$$\begin{aligned}\pi(\mathrm{id}-\Phi\Psi)&=\pi(\mathrm{id}-(F+L_f)(G+L_g))=\pi(\mathrm{id}-FG)-\pi(FL_g+L_fG+L_fL_g)\\&=(\mathrm{id}-fg)\pi-\pi(FL_g+L_fG+L_fL_g)=(\partial h+h\partial)\pi-\pi(FL_g+L_fG+L_fL_g)\\&=\pi(\partial H+H\partial)-\pi(FL_g+L_fG+L_fL_g).\end{aligned}$$

It follows that the difference

$$\lambda=\mathrm{id}-\Phi\Psi-(\partial H+H\partial)+(FL_g+L_fG+L_fL_g)$$

maps $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ into the kernel $(\mathrm{id}-T)\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ of π. As before we obtain a covariant map $\Lambda:\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)\to\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ such that $(\mathrm{id}-T)\Lambda=\lambda$. Moreover we have

$$0=\partial(FG)=\partial(F)G+F\partial(G)$$

since the map $FG=p(\kappa)^2$ is a chain map due to our definitions. We deduce $(\mathrm{id}-T)K_fG+F(\mathrm{id}-T)K_g=0$ and hence

$$K_fG+FK_g=0$$

since $(\mathrm{id}-T)$ is injective. This yields

$$\begin{aligned}FL_g+L_fG+L_fL_g&=\frac{1}{2}(FK_g\partial-\partial K_fG)-\frac{1}{4}K_f(B+b)^2K_g\\&=\frac{1}{2}(FK_g\partial+\partial FK_g)-\frac{1}{4}(\mathrm{id}-T)K_fK_g.\end{aligned}$$

Now observe that the map $\mathrm{id}-T$ is homotopic to zero in $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\partial)$. Since Λ and K_fK_g are chain maps it follows that $(\mathrm{id}-T)(\Lambda-L_fL_g)$ is also homotopic to zero. Hence

$$\mathrm{id}-\Phi\Psi-(\partial H+H\partial)+FL_g+L_fG+L_fL_g$$

is homotopic to zero. This shows that $\Phi\Psi$ is homotopic to the identity as desired. In the same way one can show that $\Psi\Phi$ is homotopic to the identity on the hyperbolic part of $(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),B+b)$. This finishes the proof of theorem 3.23 for the hyperbolic part of $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$.
It remains to treat the elliptic part. Let $s\in G$ be an element of finite order and let A be any pro-G-algebra. Denote by n the order of s and define an operator $e:\theta\Omega_G(A)_{\langle s\rangle}\to\theta\Omega_G(A)_{\langle s\rangle}$ by

$$e=\frac{1}{n}\sum_{j=1}^{n}T^j.$$

Obviously e is covariant and the relation $T^n=\mathrm{id}$ which holds on $\theta\Omega_G(A)_{\langle s\rangle}$ yields $e^2=e$ and $eT=Te=e$. Furthermore T commutes with d and b. It follows that e commutes with all operators made up out of d and b, in particular e commutes with B. From $e^2 = e$ we get a direct sum decomposition

$$\theta\Omega_G(A)_{\langle s\rangle} = e\theta\Omega_G(A)_{\langle s\rangle} \oplus (1-e)\theta\Omega_G(A)_{\langle s\rangle}$$

and this decompositions is compatible with the differentials $B+b$ and ∂.
Since the order of s is finite the canonical projection

$$e\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle} \to (\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_T)_{\langle s\rangle}$$

is an isomorphism. Hence the assertion of theorem 3.23 for the elliptic part is a consequence of the following more general statement.

PROPOSITION 3.25. *Let $s \in G$ be an element of finite order and let A be any pro-G-algebra. Then the localized para-supercomplexes $X_G(\mathcal{T}A)_{\langle s\rangle}$ and $\theta\Omega_G(A)_{\langle s\rangle}$ are covariantly homotopy equivalent.*

PROOF. We need a formula for the projection $\mathrm{id}-e$ constructed above. An easy computation shows

$$\mathrm{id}-e = \sum_{j=0}^{n-1} \frac{n-j}{n}\, T^j(\mathrm{id}-T).$$

Consider the covariant operator $h : \theta\Omega_G(A)_{\langle s\rangle} \to \theta\Omega_G(A)_{\langle s\rangle}$ of degree -1 defined by

$$h = \sum_{j=0}^{n-1} \frac{n-j}{n}\, T^j b.$$

It is immmediate that $h^2 = 0$ since $b^2 = 0$ and we compute

$$[B+b, h] = \sum_{j=0}^{n-1} \frac{n-j}{n}\, T^j[B,b] = \sum_{j=0}^{n-1} \frac{n-j}{n}\, T^j(\mathrm{id}-T) = \mathrm{id}-e.$$

This implies that the para-supercomplex $((\mathrm{id}-e)\theta\Omega_G(A)_{\langle s\rangle}, B+b)$ is covariantly contractible. In the same way one shows that $((\mathrm{id}-e)X_G(\mathcal{T}A)_{\langle s\rangle}, \partial)$ is covariantly contractible. Now let us consider the remaining summands $(e\theta\Omega_G(A)_{\langle s\rangle}, B+b)$ and $(eX_G(\mathcal{T}A)_{\langle s\rangle}, \partial)$. Since we have $e\,\theta\Omega_G(A)_{\langle s\rangle} \cong (\theta\Omega_G(A)_T)_{\langle s\rangle}$ we obtain due to theorem 3.24 a covariant homotopy equivalence between $(eX_G(\mathcal{T}A)_{\langle s\rangle}, \partial)$ and $(e(\theta\Omega_G(A)_{\langle s\rangle}, B+b)$. This finishes the proof of proposition 3.25. □

Using proposition 2.12 we see that proposition 3.25 yields in particular the following result.

PROPOSITION 3.26. *Let G be a finite group and let A be any pro-G-algebra. Then the pro-parasupercomplexes $X_G(\mathcal{T}A)$ and $\theta\Omega_G(A)$ are covariantly homotopy equivalent.*

4. Equivariant periodic cyclic homolgy

Let G be a discrete group. We want to define bivariant equivariant periodic cyclic homology for pro-G-algebras.

DEFINITION 3.27. *Let A and B be pro-G-algebras. The bivariant equivariant periodic cyclic homology of A and B is*

$$HP_*^G(A,B) = H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)), X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G)))).$$

There are some explanations in order. On the right hand side of this definition we take homology with respect to the usual boundary in a Hom-complex given by

$$\partial(\phi) = \phi\partial_A - (-1)^{|\phi|}\partial_B\phi$$

for a homogenous element $\phi \in \mathfrak{Hom}_G(X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)), X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G)))$ where ∂_A and ∂_B denote the boundary operators of $X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G))$ and $X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))$, respectively. However, since the equivariant X-complexes are only a para-supercomplexes we have to check that we indeed obtain a supercomplex in this way!
From the discussion in section 3.3 we know $\partial_A^2 = \mathrm{id} - T$ and $\partial_B^2 = \mathrm{id} - T$. Using these relations we compute

$$\partial^2(\phi) = \phi\,\partial_A^2 + (-1)^{|\phi|}(-1)^{|\phi|-1}\partial_B^2\,\phi = \phi(\mathrm{id} - T) - (\mathrm{id} - T)\phi = T\phi - \phi T$$

and hence $\partial^2(\phi) = 0$ follows from lemma 3.15. Thus the failure of the individual differentials to satisfy $\partial^2 = 0$ is cancelled out in the Hom-complex. This shows that our definition of HP_*^G makes sense. There is a similar result in the situation where the equivariant X-complexes are replaced by Hodge towers.
Let us discuss basic properties of the equivariant homology groups defined above. Of course HP_*^G is a bifunctor, contravariant in the first variable and covariant in the second variable. As usual we define $HP_*^G(A) = HP_*^G(\mathbb{C}, A)$ to be the equivariant periodic cyclic homology of A and $HP^*_G(A) = HP_*^G(A, \mathbb{C})$ to be equivariant periodic cyclic cohomology. There is a natural product

$$HP_i^G(A, B) \times HP_j^G(B, C) \to HP_{i+j}^G(A, C), \qquad (x, y) \mapsto x \cdot y$$

induced by the composition of maps. This product is clearly associative. Every equivariant homomorphism $f : A \to B$ defines an element in $HP_0^G(A, B)$ denoted by $[f]$. The element $[\mathrm{id}] \in HP_0^G(A, A)$ is simply denoted by 1 or 1_A. An element $x \in HP_*^G(A, B)$ is called invertible if there exists an element $y \in HP_*^G(B, A)$ such that $x \cdot y = 1_A$ and $y \cdot x = 1_B$. An invertible element of degree zero will also be called an HP^G-equivalence. Such an element induces isomorphisms $HP_*^G(A, D) \cong HP_*^G(B, D)$ and $HP_*^G(D, A) \cong HP_*^G(D, B)$ for all G-algebras D. An HP^G-equivalence exists if and only if the pro-parasupercomplexes $X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G))$ and $X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))$ are covariantly homotopy equivalent.
For every conjugacy class $\langle s\rangle$ of an element $s \in G$ we define the localisation $HP_*^G(A, B)_{\langle s\rangle}$ of the equivariant periodic theory at $\langle s\rangle$ by

$$HP_*^G(A, B)_{\langle s\rangle} = H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle}, X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle})).$$

Clearly we have

$$HP_*^G(A, B) = \prod_{\langle s\rangle \in \langle G\rangle} HP_*^G(A, B)_{\langle s\rangle}$$

for all pro-G-algebras A and B. In this way HP_*^G can be viewed as a delocalized equivariant homology theory parametrized by the conjugacy classes of elements of the group. The contributions coming from conjugacy classes of elements of finite order will be called elliptic components, those coming from elements of infinite order will be called hyperbolic components of HP_*^G. In particular we may consider the localisation at the identity element $\langle e\rangle$. This localisation will be denoted simply by $HP_*^G(A, B)_e$.

Remark that there is a natural chain map

$$\mathfrak{Hom}_G(X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G))_e, X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))_e) \to \mathrm{Hom}(X(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)), X(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))).$$

This map induces a forgetful map $HP^G_*(A, B) \to HP_*(A, B)$ which is compatible with the composition product.

5. Homotopy invariance

In this section we show that HP^G_* is invariant under smooth equivariant homotopies in both variables.
Let B be a pro-G-algebra and consider the Fréchet algebra $C^\infty[0,1]$ of smooth functions on the interval $[0,1]$. We denote by $B[0,1]$ the pro-G-algebra $B\hat{\otimes}C^\infty[0,1]$ where the action on $C^\infty[0,1]$ is trivial. By definition a (smooth) equivariant homotopy is an equivariant homomorphism $\Phi : A \to B[0,1]$ of pro-G-algebras. Evaluation at a point $t \in [0,1]$ yields an equivariant homomorphism $\Phi_t : A \to B$. Two equivariant homomorphisms from A to B are called equivariantly homotopic if they can be connected by an equivariant homotopy. A homology theory h_* for algebras is called homotopy invariant if the induced maps $h_*(\phi_0)$ and $h_*(\phi_1)$ of homotopic homomorphisms ϕ_0 and ϕ_1 are equal. In our situation we will prove the following assertion.

Theorem 3.28 (Homotopy invariance). *Let A and B be pro-G-algebras and let $\Phi : A \to B[0,1]$ be a smooth equivariant homotopy. Then the elements $[\Phi_0]$ and $[\Phi_1]$ in $HP^G_0(A, B)$ are equal. Hence the functor HP^G_* is homotopy invariant in both variables with respect to smooth equivariant homotopies.*
Moreover the elements $[\Phi_0]$ and $[\Phi_1]$ in $H_0(\mathfrak{Hom}_G(X_G(A), X_G(B)))$ are equal provided A is quasifree.

We recall that $\theta^2\Omega_G(A)$ is the pro-parasupercomplex $\Omega^0_G(A)\oplus\Omega^1_G(A)\oplus\Omega^2_G(A)/b_G(\Omega^3_G(A))$ with the usual differential $B_G + b_G$ and the grading into even and odd forms for any pro-G-algebra A. Clearly there is a natural map of pro-parasupercomplexes $\xi^2 : \theta^2\Omega_G(A) \to X_G(A)$. The first step in the proof of theorem 3.28 is to show that ξ^2 is a covariant homotopy equivalence provided A is quasifree.

Proposition 3.29. *Let A be a quasifree pro-G-algebra. Then the map $\xi^2 : \theta^2\Omega_G(A) \to X_G(A)$ is a covariant homotopy equivalence.*

Proof. Since A is quasifree there exists by theorem 3.5 an equivariant pro-linear map $\nabla : \Omega^1(A) \to \Omega^2(A)$ satisfying

$$\nabla(x\omega) = x\nabla(\omega), \qquad \nabla(\omega x) = \nabla(\omega)x - \omega dx$$

for all $x \in A$ and $\omega \in \Omega^1(A)$. We extend ∇ to forms of higher degree by setting $\nabla(a_0da_1 \cdots da_n) = \nabla(a_0da_1)da_2 \cdots da_n$ Then we have

$$\nabla(a\omega) = a\nabla(\omega), \qquad \nabla(\omega\eta) = \nabla(\omega)\eta + (-1)^{|\omega|}\omega d\eta$$

for $a \in A$ and differential forms ω and η. Moreover we put $\nabla(a) = 0$ for $a \in \Omega^0(A) = A$. Now we construct a covariant map $\nabla_G : \Omega^n_G(A) \to \Omega^{n+1}_G(A)$ using the formula

$$\nabla_G(f(s) \otimes \omega) = f(s) \otimes \nabla(\omega).$$

Let us compute the commutator of b_G and ∇_G. Take $\omega \in \Omega^j_G(A)$ for $j > 0$. For $a \in A$ we obtain

$$\begin{aligned}
[b_G, \nabla_G](f(s) \otimes \omega da) &= b_G(f(s) \otimes \nabla(\omega)da) + \nabla_G(b_G(f(s) \otimes \omega da)) \\
&= (-1)^{j+1}(f(s) \otimes \nabla(\omega)a - f(s) \otimes (s^{-1} \cdot a)\nabla(\omega)) \\
&\quad + (-1)^j(\nabla_G(f(s) \otimes \omega a - f(s) \otimes (s^{-1} \cdot a)\omega)) \\
&= (-1)^j \Big(f(s) \otimes (s^{-1} \cdot a)\nabla(\omega) - f(s) \otimes \nabla(\omega)a \\
&\quad + f(s) \otimes \nabla(\omega a) - f(s) \otimes \nabla((s^{-1} \cdot a)\omega) \Big) \\
&= (-1)^j \Big(f(s) \otimes (s^{-1} \cdot a)\nabla(\omega) - f(s) \otimes \nabla(\omega)a + f(s) \otimes \nabla(\omega)a \\
&\quad + (-1)^j f(s) \otimes \omega da - f(s) \otimes (s^{-1} \cdot a)\nabla(\omega) \Big) \\
&= f(s) \otimes \omega da
\end{aligned}$$

Hence $[b_G, \nabla_G] = \mathrm{id}$ on $\Omega^n(A)$ for $n \geq 2$. Since $[b_G, \nabla_G]$ commutes with b_G this holds also on $b_G(\Omega^2_G(A))$. Let us determine the behaviour of $[b_G, \nabla_G]$ on $\Omega^0_G(A)$ and $\Omega^1_G(A)$. Clearly $[b_G, \nabla_G] = 0$ on $\Omega^0_G(A)$ since ∇_G vanishes on $\Omega^0(A)$. On $\Omega^1_G(A)$ we have $[b_G, \nabla_G] = b_G\nabla_G$ because ∇_G is zero on $\Omega^0_G(A)$. Hence

$$[b_G, \nabla_G][b_G, \nabla_G] = b_G\nabla_G b_G\nabla_G = b_G(\mathrm{id} - b_G\nabla_G)\nabla_G = b_G\nabla_G = [b_G, \nabla_G] \quad \text{on } \Omega^1_G(A)$$

and it follows that $[b_G, \nabla_G]$ is idempotent. The range of the map $[b_G, \nabla_G] = b_G\nabla_G$ restricted to $\Omega^1_G(A)$ is contained in $b_G(\Omega^2_G(A))$. Equality holds because $[b_G, \nabla_G]$ is equal to the identity on $b_G(\Omega^2_G(A))$ as we have seen before.
We will use ∇_G to construct an inverse of ξ^2 up to homotopy. In order to do this consider the commutator of ∇_G with the boundary $B_G + b_G$. Clearly we have $[\nabla_G, B_G + b_G] = [\nabla_G, B_G] + [\nabla_G, b_G]$. Since $[\nabla_G, B_G]$ has degree $+2$ we see from our previous computation that $\mathrm{id} - [\nabla_G, B_G + b_G]$ maps $F_j\Omega_G(A)$ to $F_{j+1}\Omega_G(A)$ for all $j \geq 1$. This implies in particular that $\mathrm{id} - [\nabla_G, B_G + b_G]$ descends to a covariant map $\nu : X_G(A) \to \theta^2\Omega_G(A)$. Using that ∇_G is covariant we see that ν is a chain map. Explicitly we have

$$\begin{aligned}
\nu &= \mathrm{id} - \nabla_G d && \text{on } \Omega^0_G(A) \\
\nu &= \mathrm{id} - [\nabla_G, b_G] = \mathrm{id} - b_G\nabla_G && \text{on } \Omega^1_G(A)/b_G(\Omega^2_G(A))
\end{aligned}$$

and we deduce $\xi^2\nu = \mathrm{id}$. Moreover $\nu\xi^2 = \mathrm{id} - [\nabla_G, B_G + b_G]$ is homotopic to the identity. This completes the proof of proposition 3.29. □
Now let $\Phi : A \to B[0, 1]$ be an equivariant homotopy. The derivative of Φ is an equivariant pro-linear map $\Phi' : A \to B[0, 1]$. If we view $B[0, 1]$ as a bimodule over itself the map Φ' is a derivation with respect to Φ in the sense that $\Phi'(xy) = \Phi'(x)\Phi(y) + \Phi(x)\Phi'(y)$ for $x, y \in A$. We define a covariant map $\eta : \Omega^n_G(A) \to \Omega^{n-1}_G(B)$ for $n > 0$ by

$$\eta(f(s) \otimes x_0dx_1 \dots dx_n) = \int_0^1 f(s) \otimes \Phi_t(x_0)\Phi'_t(x_1)d\Phi_t(x_2) \cdots d\Phi_t(x_n)dt.$$

Since integration is a bounded linear map we see that η is bounded. In addition we set $\eta = 0$ on $\Omega^0_G(A)$. Using the fact that Φ' is a derivation with respect to Φ we compute

$$\begin{aligned}
\eta b_G(f(s)\otimes x_0dx_1\dots dx_n) = \int_0^1 & f(s)\otimes \Phi_t(x_0x_1)\Phi'_t(x_2)d\Phi_t(x_3)\cdots d\Phi_t(x_n)\\
& - f(s)\otimes\Phi_t(x_0)\Phi'_t(x_1x_2)d\Phi_t(x_3)\cdots d\Phi_t(x_n)\\
& + f(s)\otimes\Phi_t(x_0)\Phi'_t(x_1)\Phi_t(x_2)d\Phi_t(x_3)\cdots d\Phi_t(x_n)\\
& - (-1)^n f(s)\otimes\Phi_t(x_0)\Phi'_t(x_1)(d\Phi_t(x_2)\cdots d\Phi_t(x_{n-1}))\Phi_t(x_n)\\
& + (-1)^n f(s)\otimes\Phi_t((s^{-1}\cdot x_n)x_0)\Phi'_t(x_1)d\Phi_t(x_2)\cdots d\Phi_t(x_{n-1})dt\\
= \int_0^1 & (-1)^{n-1}(f(s)\otimes\Phi_t(x_0)\Phi'_t(x_1)(d\Phi_t(x_2)\cdots d\Phi_t(x_{n-1}))\Phi_t(x_n)\\
& - f(s)\otimes\Phi_t((s^{-1}\cdot x_n)x_0)\Phi'_t(x_1)d\Phi_t(x_2)\cdots d\Phi_t(x_{n-1}))dt\\
= -b_G & \eta(f(s)\otimes x_0dx_1\cdots dx_n).
\end{aligned}$$

This implies that η maps $b_G(\Omega^3_G(A))$ into $b_G(\Omega^2_G(B)$ and hence induces a covariant map $\eta : \theta^2\Omega_G(A)\to X_G(B)$.

Lemma 3.30. *We have $X_G(\Phi_1)\xi^2 - X_G(\Phi_0)\xi^2 = \partial\eta + \eta\partial$. Hence the chain maps $X_G(\Phi_t)\xi^2 : \theta^2\Omega_G(A)\to X_G(B)$ for $t = 0, 1$ are covariantly homotopic.*

Proof. We compute both sides on $\Omega^j_G(A)$ for $j = 0, 1, 2$. For $j = 0$ we have

$$[\partial,\eta](f(s)\otimes x) = \eta(f(s)\otimes dx) = \int_0^1 f(s)\otimes\Phi'_t(x)dt = f(s)\otimes\Phi_1(x) - f(s)\otimes\Phi_0(x).$$

For $j = 1$ we get

$$\begin{aligned}
[\partial,\eta](f(s)\otimes x_0dx_1) &= d\eta(f(s)\otimes x_0dx_1) + \eta B(f(s)\otimes x_0dx_1)\\
&= \int_0^1 f(s)\otimes d(\Phi_t(x_0)\Phi'_t(x_1)) + f(s)\otimes\Phi'_t(x_0)d\Phi_t(x_1) -\\
&\quad f(s)\otimes\Phi'_t(s^{-1}\cdot x_1)d\Phi_t(x_0)dt\\
&= \int_0^1 b_G(f(s)\otimes d\Phi_t(x_0)d\Phi'_t(x_1)) + \frac{\partial}{\partial t}\Big(f(s)\otimes\Phi_t(x_0)d\Phi_t(x_1)\Big)dt\\
&= f(s)\otimes\Phi_1(x_0)d\Phi_1(x_1) - f(s)\otimes\Phi_0(x_0)d\Phi_0(x_1).
\end{aligned}$$

Here we can forget about the term

$$\int_0^1 b_G(f(s)\otimes d\Phi_t(x_0)d\Phi'_t(x_1))dt$$

since the range of η is $X_G(B)$. Finally, on $\Omega^3_G(A)/b_G(\Omega^2_G(A))$ we have $\partial\eta+\eta\partial = \eta b_G + b_G\eta = 0$ due to the computation above. □

Now we come back to the proof of theorem 3.28. Let $\Phi : A\to B[0,1]$ be an equivariant homotopy. Tensoring both sides with $\mathcal{K}_G$ we obtain an equivariant homotopy $\Phi\hat{\otimes}\mathcal{K}_G : A\hat{\otimes}\mathcal{K}_G\to(B\hat{\otimes}\mathcal{K}_G)[0,1]$. The map $\Phi\hat{\otimes}\mathcal{K}_G$ induces an equivariant homomorphism $\mathcal{T}(\Phi\hat{\otimes}\mathcal{K}_G) : \mathcal{T}(A\hat{\otimes}\mathcal{K}_G)\to\mathcal{T}((B\hat{\otimes}\mathcal{K}_G)[0,1])$. Now consider the equivariant pro-linear map

$$l : B\hat{\otimes}\mathcal{K}_G\hat{\otimes}C^\infty[0,1]\to\mathcal{T}(B\hat{\otimes}\mathcal{K}_G)\hat{\otimes}C^\infty[0,1],\qquad l(b\otimes T\otimes f) = \sigma_{B\hat{\otimes}\mathcal{K}_G}(b\otimes T)\otimes f.$$

Since $\sigma_{B\hat{\otimes}\mathcal{K}_G}$ is a lonilcur it follows that the same holds true for l. Hence we obtain an equivariant homomorphism $[[l]] : \mathcal{T}((B\hat{\otimes}\mathcal{K}_G)[0,1]) \to \mathcal{T}(B\hat{\otimes}\mathcal{K}_G)[0,1]$ due to proposition 3.3. Composition of $\mathcal{T}(\Phi\hat{\otimes}\mathcal{K}_G)$ with the homomorphism $[[l]]$ yields an equivariant homotopy $\Psi = [[l]]\mathcal{T}(\Phi\hat{\otimes}\mathcal{K}_G) : \mathcal{T}(A\hat{\otimes}\mathcal{K}_G) \to \mathcal{T}(B\hat{\otimes}\mathcal{K}_G)[0,1]$. From the definition of Ψ it follows easily that $\Psi_t = \mathcal{T}(\Phi_t\hat{\otimes}\mathcal{K}_G)$ for all t. Since $\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)$ is quasifree we can apply proposition 3.24 and lemma 3.30 to obtain $[\Phi_0] = [\Phi_1] \in HP_0^G(A,B)$. The second assertion of theorem 3.28 follows directly from proposition 3.24 and lemma 3.30. This finishes the proof of theorem 3.28.
As a first application of homotopy invariance we show that HP_*^G can be computed using arbitrary universal locally nilpotent extensions.

Proposition 3.31. *Let $0 \to I \to R \to A \to 0$ be a universal locally nilpotent extension of the pro-G-algebra A. Then $X_G(R)$ is covariantly homotopy equivalent to $X_G(\mathcal{T}A)$ in a canonical way. More precisely, any morphism of extensions (ξ, ϕ, id) from $0 \to \mathcal{J}A \to \mathcal{T}A \to A \to 0$ to $0 \to I \to R \to A \to 0$ induces a covariant homotopy equivalence $X_G(\phi) : X_G(\mathcal{T}A) \to X_G(R)$. The class of this homotopy equivalence in $H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}A), X_G(R)))$ is independent of the choice of ϕ.*

Proof. From propositions 3.11 and 3.12 it follows that $\phi : \mathcal{T}A \to R$ is an equivariant homotopy equivalence of algebras. Hence $X_G(\phi) : X_G(\mathcal{T}A) \to X_G(R)$ is a covariant homotopy equivalence due to theorem 3.28. Since ϕ is unique up to equivariant homotopy it follows that the class of this homotopy equivalence does not depend on the particular choice of ϕ. □
In particular there is a natural covariant homotopy equivalence between $X_G(\mathcal{T}A)$ and $X_G(A)$ if A itself is quasifree.

6. Stability

In this section we want to investigate stability properties of HP_*^G. We will show that HP_*^G is stable with respect to tensoring with the algebra $l(\mathcal{H})$ of finite rank operators on an arbitrary G-pre-Hilbert space $\mathcal{H}$. Recall from section 2.2 that for every G-pre-Hilbert space $\mathcal{H}$ there is a natural action of G on $l(\mathcal{H})$ given by

$$(s \cdot T)(\xi) = s \cdot T(s^{-1} \cdot \xi)$$

for $T \in l(\mathcal{H})$. Equipped with this action and the fine bornology the algebra $l(\mathcal{H})$ becomes a G-algebra.
First we have to consider a special class of G-pre-Hilbert spaces.

Definition 3.32. *A G-pre-Hilbert space $\mathcal{H}$ is called admissible if there exists a nonzero subspace of $\mathcal{H}$ where the G-action is trivial.*

One should compare this definition with the notion of stability formulated in [**60**].
Now let A be a pro-G-algebra and let $\mathcal{H}$ be an admissible G-pre-Hilbert space. We fix a one-dimensional subspace of $\mathcal{H}$ where the action of G is trivial and let $p \in l(\mathcal{H})$ be the corresponding projection of rank one. Then we have $p = pU_s = U_s p$ for all $s \in G$ where U_s is the unitary operator on $\mathcal{H}$ associated to s in the representation. In particular p is G-invariant. Consider the equivariant homomorphism $\iota_A : A \to A\hat{\otimes}l(\mathcal{H}), \iota_A(a) = a \otimes p$.

THEOREM 3.33. *Let A be a pro-G-algebra and let $\mathcal{H}$ be an admissible G-pre-Hilbert space. Then the class $[\iota_A] \in H_0(\mathfrak{Hom}_G(X_G(\mathcal{T}A), X_G(\mathcal{T}(A\hat{\otimes}l(\mathcal{H})))))$ is invertible.*

PROOF. We have to find an inverse for $[\iota_A]$. Our argument is a generalization of a well-known proof of stability in the nonequivariant case.
First observe that the canonical equivariant pro-linear map $A\hat{\otimes}l(\mathcal{H}) \to \mathcal{T}A\hat{\otimes}l(\mathcal{H})$ is a lonilcur and induces consequently an equivariant homomorphism $\lambda_A : \mathcal{T}(A\hat{\otimes}l(\mathcal{H})) \to \mathcal{T}A\hat{\otimes}l(\mathcal{H})$. Define the map $tr_A : X_G(\mathcal{T}A\hat{\otimes}l(\mathcal{H})) \to X_G(\mathcal{T}A)$ by

$$tr_A(f(s) \otimes x \otimes T) = tr_s(T)f(s) \otimes x$$

and

$$tr_A(f(s) \otimes x_0 \otimes T_0\, d(x_1 \otimes T_1)) = tr_s(T_0T_1)f(s) \otimes x_0dx_1.$$

Here we use the twisted trace tr_s for $s \in G$ defined as follows. As above let U_s be the operator associated to s in the representation. Then we set

$$tr_s(T) = tr(T\,U_s)$$

for each element $T \in l(\mathcal{H})$. Here tr is the usual trace on $l(\mathcal{H})$ which is inherited from the trace on the algebra of finite rank operators on the completion of $\mathcal{H}$.
Now it is easily verified that

$$tr_s(T_0T_1) = tr_s((s^{-1} \cdot T_1)T_0)$$

for all $T_0, T_1 \in l(\mathcal{H})$.
One checks that tr_A is a covariant map of pro-parasupercomplexes. We define $\tau_A = tr_A \circ X_G(\lambda_A)$ and claim that $[\tau_A]$ is an inverse for $[\iota_A]$. Using the relation $p\,U_s = p$ one computes $[\iota_A]\cdot[\tau_A] = 1$. We have to show that $[\tau_A]\cdot[\iota_A] = 1$. Consider the equivariant homomorphisms $i_j : A\hat{\otimes}l(\mathcal{H}) \to A\hat{\otimes}l(\mathcal{H})\hat{\otimes}l(\mathcal{H})$ for $j = 1, 2$ given by

$$i_1(a \otimes T) = a \otimes T \otimes p$$
$$i_2(a \otimes T) = a \otimes p \otimes T$$

As before we see $[i_1] \cdot [\tau_{A\hat{\otimes}l(\mathcal{H})}] = 1$ and we determine $[i_2] \cdot [\tau_{A\hat{\otimes}l(\mathcal{H})}] = [\tau_A] \cdot [\iota_A]$. Let us show that the maps i_1 and i_2 are equivariantly homotopic. We denote by $\mathcal{H} \otimes \mathcal{H}$ the algebraic tensor product of $\mathcal{H}$ with itself. It is again a G-pre-Hilbert space in a natural way. Let σ be the unitary operator on $\mathcal{H} \otimes \mathcal{H}$ defined by the coordinate flip $\sigma(\xi \otimes \eta) = \eta \otimes \xi$. Then σ is equivariant and $\sigma^2 = 1$. Consider for $t \in [0, 1]$ the bounded operator $\sigma_t = \cos(\pi t/2)\,\mathrm{id} + \sin(\pi t/2)\sigma$ on $\mathcal{H} \otimes \mathcal{H}$. Each σ_t is an invertible equivariant operator and $\sigma_t^{-1} = \cos(\pi t/2)\,\mathrm{id} - \sin(\pi t/2)\sigma$. Furthermore the family σ_t depends smoothly on t and we have $\sigma_0 = \mathrm{id}$ and $\sigma_1 = \sigma$. Now the formula $\mathrm{Ad}(\sigma_t)(T) = \sigma_t T \sigma_t^{-1}$ defines equivariant homomorphisms $\mathrm{Ad}(\sigma_t) : l(\mathcal{H} \otimes \mathcal{H}) \to l(\mathcal{H} \otimes \mathcal{H})$ since the operators σ_t are multipliers for $l(\mathcal{H} \otimes \mathcal{H})$. We use $\mathrm{Ad}(\sigma_t)$ to define an equivariant homomorphism $h_t : A\hat{\otimes}l(\mathcal{H}) \to A\hat{\otimes}l(\mathcal{H})\hat{\otimes}l(\mathcal{H})$ by $h_t(a \otimes T) = a \otimes \mathrm{Ad}(\sigma_t)(1 \otimes T)$. One computes $h_0 = i_1$ and $h_1 = i_2$ and the family h_t again depends smoothly on t. Hence we have indeed defined a smooth homotopy between i_1 and i_2. Theorem 3.28 yields $[i_1] = [i_2]$ and hence $[\tau_A] \cdot [\iota_A] = 1$. □
Now we can prove the following stability theorem.

THEOREM 3.34 (Stability). *Let A be a pro-G-algebra and let $\mathcal{H}$ be any G-pre-Hilbert space. Then there exists an invertible element in $HP^G_0(A, A\hat{\otimes} l(\mathcal{H}))$. Hence there are natural isomorphisms*

$$HP^G_*(A\hat{\otimes} l(\mathcal{H}), B) \cong HP^G_*(A, B), \qquad HP^G_*(A, B) \cong HP^G_*(A, B\hat{\otimes} l(\mathcal{H}))$$

for all pro-G-algebras A and B.

PROOF. We have a natural equivariant isomorphism $\mathcal{K}_G\hat{\otimes} l(\mathcal{H}) \cong l(\mathbb{C}G \otimes \mathcal{H})$. Let us show that there is a unitary equivalence of representations $\mathbb{C}G \otimes \mathcal{H} \cong \mathbb{C}G \otimes \mathcal{H}_\tau$ where $\mathcal{H}_\tau$ is the space $\mathcal{H}$ with the trivial G-action. Define the operator $V : \mathbb{C}G \otimes \mathcal{H} \to \mathbb{C}G \otimes \mathcal{H}_\tau$ by

$$V(s \otimes \xi) = s \otimes s^{-1} \cdot \xi$$

for $s \in G \subset \mathbb{C}G$ and $\xi \in \mathcal{H}$. It is easy to check that V is an intertwining operator implementing the desired unitary equivalence. Consequently V induces an equivariant isomorphism of G-algebras

$$\mathcal{K}_G\hat{\otimes} l(\mathcal{H}) \cong \mathcal{K}_G\hat{\otimes} l(\mathcal{H}_\tau).$$

Now we can apply theorem 3.33 with A replaced by $A\hat{\otimes}\mathcal{K}_G$ and $\mathcal{H}$ replaced by $\mathcal{H}_\tau$ to obtain the assertion. ☐

Another application of theorem 3.33 gives a simpler description of HP^G_* if G is a finite group.

PROPOSITION 3.35. *Let G be a finite group. Then we have a natural isomorphism*

$$HP^G_*(A, B) \cong H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}A), X_G(\mathcal{T}B)))$$

for all pro-G-algebras A and B.

PROOF. If G is finite the trivial one-dimensional representation is contained in $\mathbb{C}G$. Hence $\mathbb{C}G$ itself is an admissible G-Hilbert space in this case. ☐

Using stability we can define a restriction map $\mathrm{res}^G_H : HP^G_*(A, B) \to HP^H_*(A, B)$ for every subgroup H of G. First observe that the inclusion $H \to G$ induces an H-equivariant homomorphism $\mathcal{O}_G \to \mathcal{O}_H$. In this way $\mathcal{O}_H$ becomes an $\mathcal{O}_G$-module. In order to define res^G_H take an element $\phi \in \mathfrak{Hom}_G(X_G(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)), X_G(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G)))$. Tensoring with $\mathcal{O}_H$ over $\mathcal{O}_G$ we obtain an element $\mathrm{res}^G_H(\phi) \in \mathfrak{Hom}_H(X_H(\mathcal{T}(A\hat{\otimes}\mathcal{K}_G)), X_H(\mathcal{T}(B\hat{\otimes}\mathcal{K}_G))$. It is easy to see that res^G_H defines a chain map. Since $\mathcal{K}_G$ is H-equivariantly isomorphic to $\mathcal{K}_H\hat{\otimes} l(\mathbb{C}[G/H])$ where $\mathbb{C}[G/H]$ is equipped with the trivial H-action we can apply theorem 3.33 to see that the target of res^G_H identifies naturally with $HP^H_*(A, B)$. More precisely we have the following result.

PROPOSITION 3.36. *Let H be a subgroup of G. The restriction map*

$$\mathrm{res}^G_H : HP^G_*(A, B) \to HP^H_*(A, B)$$

is functorial and compatible with the composition product.

For H equal to the trivial group we reobtain the forgetful map $HP^G_*(A, B) \to HP_*(A, B)$ described in section 3.4.

7. Excision

The goal of this section is the proof of the following theorem.

THEOREM 3.37 (Excision). *Let A be a pro-G-algebra and let $(\iota, \pi) : 0 \to K \to E \to Q \to 0$ be an admissible extension of pro-G-algebras. Then there are two natural exact sequences*

$$\begin{tikzcd}
HP_0^G(A,K) \arrow[r] & HP_0^G(A,E) \arrow[r] & HP_0^G(A,Q) \arrow[d] \\
HP_1^G(A,Q) \arrow[u] & HP_1^G(A,E) \arrow[l] & HP_1^G(A,K) \arrow[l]
\end{tikzcd}$$

and

$$\begin{tikzcd}
HP_0^G(Q,A) \arrow[r] & HP_0^G(E,A) \arrow[r] & HP_0^G(K,A) \arrow[d] \\
HP_1^G(K,A) \arrow[u] & HP_1^G(E,A) \arrow[l] & HP_1^G(Q,A) \arrow[l]
\end{tikzcd}$$

The horizontal maps in these diagrams are induced by the maps in the extension.

Our proof is an adaption of the method used in [**53**] to prove excision in cyclic homology theories. Since the definition of HP_*^G involves tensoring with the algebra $\mathcal{K}_G$ we have to study the admissible extension $(\iota \hat{\otimes} \mathrm{id}, \pi \hat{\otimes} \mathrm{id}) : 0 \to K \hat{\otimes} \mathcal{K}_G \to E \hat{\otimes} \mathcal{K}_G \to Q \hat{\otimes} \mathcal{K}_G \to 0$ in the discussion below. However, to improve legibility we will omit the occurences of $\mathcal{K}_G$ and work with the original extension $(\iota, \pi) : 0 \to K \to E \to Q \to 0$. It can easily be seen that this simplification is irrelevant for the arguments needed in the proof. Throughout we let $\sigma : Q \to E$ be an equivariant pro-linear splitting for the quotient map $\pi : E \to Q$.
Consider $X_G(\mathcal{T}E : \mathcal{T}Q) = \ker(X_G(\mathcal{T}\pi) : X_G(\mathcal{T}E) \to X_G(\mathcal{T}Q))$. The equivariant pro-linear splitting $\sigma : Q \to E$ yields a direct sum decomposition $X_G(\mathcal{T}E) = X_G(\mathcal{T}E : \mathcal{T}Q) \oplus X_G(\mathcal{T}Q)$ of covariant pro-modules. Moreover there is a natural covariant map $\rho : X_G(\mathcal{T}K) \to X_G(\mathcal{T}E : \mathcal{T}Q)$ of pro-parasupercomplexes. The admissible short exact sequence $0 \to X_G(\mathcal{T}E : \mathcal{T}Q) \to X_G(\mathcal{T}E) \to X_G(\mathcal{T}Q) \to 0$ of pro-parasupercomplexes induces long exact sequences in homology in both variables. Hence theorem 3.37 is a consequence of the following theorem.

THEOREM 3.38. *The map $\rho : X_G(\mathcal{T}K) \to X_G(\mathcal{T}E : \mathcal{T}Q)$ is a covariant homotopy equivalence.*

Let $\mathfrak{L} \subset \mathcal{T}E$ be the left ideal generated by $K \subset \mathcal{T}E$. Using proposition 3.21 we see that

$$(\mathcal{T}E)^+ \hat{\otimes} K \to \mathfrak{L}, \qquad x \otimes k \mapsto x \circ k \tag{3.1}$$

is an equivariant pro-linear isomorphism. Moreover we obtain from this description an equivariant pro-linear retraction for the inclusion $\mathfrak{L} \to \mathcal{T}E$. Clearly $\mathfrak{L}$ is a pro-G-algebra since the ideal $K \subset E$ is G-invariant. The natural projection $\tau_E : \mathcal{T}E \to E$ induces an equivariant homomorphism $\tau : \mathfrak{L} \to K$ and σ_E restricted to K is an equivariant pro-linear splitting for τ. Hence we obtain an admissible extension

$$\begin{tikzcd}
N \arrow[r, shift right] & \mathfrak{L} \arrow[l, dashed, shift right] \arrow[r, shift right, "\tau"'] & K \arrow[l, dashed, shift right, "\sigma_E"']
\end{tikzcd}$$

of pro-G-algebras. The inclusion $\mathfrak{L} \to \mathcal{T}E$ induces a morphism of extensions from $0 \to N \to \mathfrak{L} \to K \to 0$ to $0 \to \mathcal{J}E \to \mathcal{T}E \to E \to 0$. In particular we have a natural equivariant homomorphism $i : N \to \mathcal{J}E$ and it is easy to see that there exists an equivariant pro-linear map $r : \mathcal{J}E \to N$ such that $ri = \mathrm{id}$. Using this retraction we want to show that N is locally nilpotent. If $l : N \to C$ is an equivariant pro-linear map with constant range C we compute $lm_N^n = lpim_N^n = lpm_{\mathcal{J}E}^n i^{\hat{\otimes} n}$ where m_N and $m_{\mathcal{J}E}$ denote the multiplication maps in N and $\mathcal{J}E$, respectively. Since $lp : \mathcal{J}E \to C$ is an equivariant pro-linear map with constant range the claim follows from the fact that $\mathcal{J}E$ is locally nilpotent.
We will establish theorem 3.38 by showing

THEOREM 3.39. *With the notations as above we have*
a) The pro-G-algebra $\mathfrak{L}$ is quasifree.
b) The inclusion $\mathfrak{L} \subset \mathcal{T}E$ induces a covariant homotopy equivalence $\psi : X_G(\mathfrak{L}) \to X_G(\mathcal{T}E : \mathcal{T}Q)$.

Let us indicate how theorem 3.39 implies theorem 3.38. The map ρ is the composition of the natural maps $X_G(\mathcal{T}K) \to X_G(\mathfrak{L})$ and $X_G(\mathfrak{L}) \to X_G(\mathcal{T}E : \mathcal{T}Q)$. Since $\mathfrak{L}$ is quasifree by part a) it follows that $0 \to N \to \mathfrak{L} \to K \to 0$ is a universal locally nilpotent extension of K. Hence the first map is a covariant homotopy equivalence due to proposition 3.31. The second map is a covariant homotopy equivalence by part b). It follows that ρ itself is a covariant homotopy equivalence.
We need some notation. The equivariant pro-linear section $\sigma : Q \to E$ induces an equivariant pro-linear map $\sigma_L : \Omega^n(Q) \to \Omega^n(E)$ definded by

$$\sigma_L(q_0 dq_1 \cdots dq_n) = \sigma(q_0) d\sigma(q_1) \dots d\sigma(q_n).$$

Here σ is extended to an equivariant pro-linear map $Q^+ \to E^+$ in the obvious way by requiring $\sigma(1) = 1$.
We also need a right-handed version of the map σ_L. In order to explain this correctly consider first an arbitrary pro-G-algebra A. There is a natural equivariant isomorphism $\Omega^1(A) \cong A \hat{\otimes} A^+$ of right A-modules. This follows easily from the description of $\Omega^1(A)$ as the kernel of the multiplication map $A^+ \hat{\otimes} A^+ \to A^+$. More generally we obtain equivariant pro-linear isomorphisms $\Omega^n(A) \cong A^{\hat{\otimes} n} \hat{\otimes} A^+$ for all n. Using these identifications we define the equivariant pro-linear map $\sigma_R : \Omega(Q) \to \Omega(E)$ by

$$\sigma_R(dq_1 \cdots dq_n q_{n+1}) = d\sigma(q_1) \dots d\sigma(q_n) \sigma(q_{n+1})$$

which is the desired right-handed version of σ_L.

LEMMA 3.40. *The following maps are equivariant pro-linear isomorphisms:*

$$\begin{aligned} \mu_L : & (\mathcal{T}Q)^+ \oplus (\mathcal{T}E)^+ \hat{\otimes} K \hat{\otimes} (\mathcal{T}Q)^+ \to (\mathcal{T}E)^+ \\ & q_1 \oplus (x \otimes k \otimes q_2) \mapsto \sigma_L(q_1) + x \circ k \circ \sigma_L(q_2) \end{aligned}$$

$$\begin{aligned} \mu_R : & (\mathcal{T}Q)^+ \oplus (\mathcal{T}Q)^+ \hat{\otimes} K \hat{\otimes} (\mathcal{T}E)^+ \to (\mathcal{T}E)^+ \\ & q_1 \oplus (q_2 \otimes k \otimes x) \mapsto \sigma_R(q_1) + \sigma_R(q_2) \circ k \circ x \end{aligned}$$

PROOF. Let us show that μ_R is an isomorphism. Using the equivariant pro-linear isomorphism $E \cong K \oplus Q$ induced by σ the spaces $\Omega^{2n}(E) = E^{\hat{\otimes} 2n} \hat{\otimes} E^+$ and $(\mathcal{T}E)^+$ decompose into direct sums. We shall describe the inverse ν_R of μ_R using implicitly these direct sum decompositions. Take an element $\omega = de_1 \cdots de_{2n}e_{2n+1} \in \Omega^{2n}(E)$. If $e_j = \sigma(q_j)$ for all j then $\omega = \mu_R(dq_1 \cdots dq_{2n}q_{2n+1})$. Otherwise pick the first j with $e_j = k_j \in K$. Hence $e_k = \sigma(q_k)$ for $k = 1, \cdots j-1$. If j is even we obtain

$$\begin{aligned} de_1 \cdots de_{2n}e_{2n+1} &= d\sigma(q_1) \cdots d\sigma(q_{j-2}) \circ (\sigma(q_{j-1})k_j - \sigma(q_{j-1}) \circ k_j) \circ de_{j+1} \cdots de_{2n}e_{2n+1} \\ &= \mu_R(dq_1 \cdots dq_{j-2} \otimes \sigma(q_{j-1})k_j \otimes de_{j+1} \cdots de_{2n}e_{2n+1} \\ &\quad - dq_1 \cdots dq_{j-2}q_{j-1} \otimes k_j \otimes de_{j+1} \cdots de_{2n}e_{2n+1}) \end{aligned}$$

If j is odd and $j < 2n+1$ we have

$$\begin{aligned} de_1 \cdots de_{2n}e_{2n+1} &= d\sigma(q_1) \cdots d\sigma(q_{j-1}) \circ (k_je_{j+1} - k_j \circ e_{j+1}) \circ de_{j+2} \cdots de_{2n}e_{2n+1} \\ &= \mu_R(dq_1 \cdots dq_{j-1} \otimes k_je_{j+1} \otimes de_{j+2} \cdots de_{2n}e_{2n+1} \\ &\quad - dq_1 \cdots dq_{j-1} \otimes k_j \otimes e_{j+1}de_{j+1} \cdots de_{2n}e_{2n+1}) \end{aligned}$$

Finally for $j = 2n+1$ we get

$$de_1 \cdots de_{2n}k_{2n+1} = \mu_R(dq_1 \cdots dq_{2n} \otimes k_{2n+1} \otimes 1).$$

From these formulas we obtain the definition of ν_R. Then by construction we have $\nu_R\mu_R = \mathrm{id}$ and it is easy to verify that $\mu_R\nu_R = \mathrm{id}$. This shows that μ_R is an isomorphism. The proof for μ_L is similar and will be omitted. □

Equation (3.1) and lemma 3.40 yield an equivariant pro-linear isomorphism

$$\mathfrak{L}^+ \hat{\otimes} (\mathcal{T}Q)^+ \cong (\mathcal{T}E)^+, \qquad l \otimes q \mapsto l \circ \sigma_L(q). \tag{3.2}$$

This isomorphism is obviously left $\mathfrak{L}$-linear and it follows that $(\mathcal{T}E)^+$ is a free left $\mathfrak{L}$-module. Furthermore we get from lemma 3.40

$$(\mathcal{T}Q)^+ \hat{\otimes} K \hat{\otimes} \mathfrak{L}^+ \cong (\mathcal{T}Q)^+ \hat{\otimes} K \oplus (\mathcal{T}Q)^+ \hat{\otimes} K \hat{\otimes} (\mathcal{T}E)^+ \hat{\otimes} K \cong (\mathcal{T}E)^+ \hat{\otimes} K \cong \mathfrak{L}.$$

It follows that the equivariant pro-linear map

$$(\mathcal{T}Q)^+ \hat{\otimes} K \hat{\otimes} \mathfrak{L}^+ \to \mathfrak{L}, \qquad q \otimes k \otimes l \mapsto \sigma_R(q) \circ k \circ l \tag{3.3}$$

is an isomorphism. This map is right $\mathfrak{L}$-linear and we see that $\mathfrak{L}$ is a free right $\mathfrak{L}$-module. Denote by $\mathfrak{J}$ the kernel of the map $\mathcal{T}\pi : \mathcal{T}E \to \mathcal{T}Q$. Using again lemma 3.40 we see that

$$(\mathcal{T}Q)^+ \hat{\otimes} K \hat{\otimes} (\mathcal{T}E)^+ \cong \mathfrak{J}, \qquad q \otimes k \otimes x \mapsto \sigma_R(q) \circ k \circ x \tag{3.4}$$

is a right $\mathcal{T}E$-linear isomorphism. In a similar way we have a left $\mathcal{T}E$-linear isomorphism

$$(\mathcal{T}E)^+ \hat{\otimes} K \hat{\otimes} (\mathcal{T}Q)^+ \cong \mathfrak{J}, \qquad x \otimes k \otimes q \mapsto x \circ k \circ \sigma_L(q).$$

Together with equation (3.1) this yields

$$\mathfrak{L} \hat{\otimes} (\mathcal{T}Q)^+ \cong \mathfrak{J}, \qquad l \otimes q \mapsto l \circ \sigma_L(q), \tag{3.5}$$

and using equation (3.2) we get

$$\mathfrak{L} \hat{\otimes}_{\mathfrak{L}^+} (\mathcal{T}E)^+ \cong \mathfrak{J}, \qquad l \otimes x \mapsto l \circ x. \tag{3.6}$$

Now we want to construct a free resolution of the $\mathfrak{L}$-bimodule $\mathfrak{L}^+$. First let A be any pro-G-algebra and consider the admissible short exact sequence of A-bimodules in pro(G-Mod) given by

$$B^A_\bullet : \Omega^1(A) \underset{\alpha_1}{\overset{h_1}{\rightleftarrows}} A^+ \hat{\otimes} A^+ \underset{\alpha_0}{\overset{h_0}{\rightleftarrows}} A^+$$

Here the maps are defined as follows:

$$\alpha_1(xDyz) = xy \otimes z - x \otimes yz, \qquad \alpha_0(x \otimes y) = xy$$
$$h_1(x \otimes y) = Dxy, \qquad h_0(x) = 1 \otimes x$$

It is easy to check that $\alpha h + h\alpha = \mathrm{id}$. The complex $B^A_\bullet$ is a projective resolution of the A-bimodule A^+ in pro(G-Mod) iff A is quasifree. Define a subcomplex $P_\bullet \subset B^{\mathcal{T}E}_\bullet$ as follows:

$$P_0 = (\mathcal{T}E)^+ \hat{\otimes} \mathfrak{L} + \mathfrak{L}^+ \hat{\otimes} \mathfrak{L}^+ \subset (\mathcal{T}E)^+ \hat{\otimes} (\mathcal{T}E)^+$$
$$P_1 = (\mathcal{T}E)^+ D\mathfrak{L} \subset \Omega^1(\mathcal{T}E).$$

There exists an equivariant pro-linear retraction $B^{\mathcal{T}E}_\bullet \to P_\bullet$ for the inclusion $P_\bullet \to B^{\mathcal{T}E}_\bullet$. Since $\mathfrak{L}$ is a left ideal in $\mathcal{T}E$ we see that the boundary operators in $B^{\mathcal{T}E}_\bullet$ restrict to $P_\bullet$ and turn $P_1 \to P_0 \to \mathfrak{L}^+$ into a complex. It is clear that P_0 and P_1 inherit a natural $\mathfrak{L}$-bimodule structure from $B^{\mathcal{T}E}_0$ and $B^{\mathcal{T}E}_1$, respectively. Moreover the homotopy h restricts to a contracting homotopy for the complex $P_1 \to P_0 \to \mathfrak{L}^+$. Hence $P_\bullet$ is an admissible resolution of $\mathfrak{L}^+$ by $\mathfrak{L}$-bimodules in pro(G-Mod). Next we show that the $\mathfrak{L}$-bimodules P_0 and P_1 are free. Using equation (3.2) we obtain the isomorphism

$$\mathfrak{L}^+ \hat{\otimes} \mathfrak{L}^+ \oplus \mathfrak{L}^+ \hat{\otimes} \mathcal{T}Q \hat{\otimes} \mathfrak{L} \cong P_0, \qquad (l_1 \otimes l_2) \oplus (l_3 \otimes q \otimes l_4) \mapsto l_1 \otimes l_2 + (l_3 \circ \sigma_L(q)) \otimes l_4. \tag{3.7}$$

Since $\mathfrak{L}$ is a free right $\mathfrak{L}$-module by (3.3) we see that P_0 is a free $\mathfrak{L}$-bimodule. Now consider P_1. We claim that

$$P_1 = \Omega^1(\mathcal{T}E) \circ K + (\mathcal{T}E)^+ DK.$$

The inclusion $(\mathcal{T}E)^+ DK \subset P_1$ is clear and it is easy to see that $\Omega^1(\mathcal{T}E) \circ K \subset P_1$. Conversely, for $x_0 D(x_1 \circ k) \in P_1$ with $x_0, x_1 \in (\mathcal{T}E)^+$ we compute

$$x_0 D(x_1 \circ k) = x_0 (Dx_1) \circ k + x_0 \circ x_1 Dk$$

which is contained in $\Omega^1(\mathcal{T}E) \circ K + (\mathcal{T}E)^+ DK$. This yields the claim. Under the isomorphism $\Omega^1(\mathcal{T}E) \cong (\mathcal{T}E)^+ \hat{\otimes} E \hat{\otimes} (\mathcal{T}E)^+$ from proposition 3.21 the space $\Omega^1(\mathcal{T}E) \circ K$ corresponds to $(\mathcal{T}E)^+ \hat{\otimes} E \hat{\otimes} (\mathcal{T}E)^+ \circ K = (\mathcal{T}E)^+ \hat{\otimes} E \hat{\otimes} \mathfrak{L}$ and $(\mathcal{T}E)^+ DK$ corresponds to $(\mathcal{T}E)^+ \hat{\otimes} K \hat{\otimes} 1$. Hence

$$\begin{aligned} &((\mathcal{T}E)^+ \hat{\otimes} K \hat{\otimes} \mathfrak{L}^+) \oplus ((\mathcal{T}E)^+ \hat{\otimes} Q \hat{\otimes} \mathfrak{L}) \to P_1, \\ &\qquad (x_1 \otimes k \otimes l_1) \oplus (x_2 \otimes q \otimes l_2) \mapsto x_1 Dk l_1 + x_2 D\sigma(q) l_2 \end{aligned} \tag{3.8}$$

is an equivariant pro-linear isomorphism. Since $(\mathcal{T}E)^+$ is a free left $\mathfrak{L}$-module by equation (3.2) and $\mathfrak{L}$ is a free right $\mathfrak{L}$-module by equation (3.3) we deduce that P_1 is a free $\mathfrak{L}$-bimodule. Consequently we have established that $P_\bullet$ is a free $\mathfrak{L}$-bimodule resolution of $\mathfrak{L}^+$ in the category pro(G-Mod). According to theorem 3.5 this finishes the proof of part a) of theorem 3.39.

We need some more notation. Let $X^\beta_G(\mathcal{T}E)$ be the complex obtained from $X_G(\mathcal{T}E)$ by replacing the differential $\partial_1 : X^1_G(\mathcal{T}E) \to X^0_G(\mathcal{T}E)$ by zero. In the same way we proceed for $X_G(\mathcal{T}E : \mathcal{T}Q)$. Moreover let M be an $\mathfrak{L}$-bimodule in pro(G-Mod). We define the covariant module $(\mathcal{O}_G \hat{\otimes} M)/[\ ,\ ]_G$ as the quotient of $\mathcal{O}_G \hat{\otimes} M$ by twisted commutators $f(s) \otimes ml - f(s) \otimes (s^{-1} \cdot l)m$ where $l \in \mathfrak{L}$ and $m \in M$.

Now we continue the proof of theorem 3.39. The inclusion $P_\bullet \to B_\bullet^{\mathcal{T}E}$ is an $\mathfrak{L}$-bimodule homomorphism and induces a chain map

$$\phi : (\mathcal{O}_G \hat{\otimes} P_\bullet)/[\ ,\ ]_G \to (\mathcal{O}_G \hat{\otimes} B_\bullet^{\mathcal{T}E})/[\ ,\ ]_G \cong X_G^\beta(\mathcal{T}E) \oplus \mathcal{O}_G[0].$$

Let us determine the image of ϕ. We use equations (3.7) and (3.5) to obtain

$$\begin{aligned}(\mathcal{O}_G \hat{\otimes} P_0)/[\ ,\ ]_G &\cong \mathcal{O}_G \hat{\otimes} (\mathfrak{L}^+ \oplus \mathfrak{L} \hat{\otimes} \mathcal{T}Q) \\ &\cong \mathcal{O}_G \oplus (\mathcal{O}_G \hat{\otimes} \mathfrak{L} \hat{\otimes} (\mathcal{T}Q)^+) \cong \mathcal{O}_G \oplus (\mathcal{O}_G \hat{\otimes} \mathfrak{J}) \subset \mathcal{O}_G \hat{\otimes} (\mathcal{T}E)^+.\end{aligned}$$

Using equations (3.8) and (3.6) we get

$$\begin{aligned}(\mathcal{O}_G \hat{\otimes} P_1)/[\ ,\ ]_G &\cong \mathcal{O}_G \hat{\otimes} ((\mathcal{T}E)^+ \hat{\otimes} K) \oplus \mathcal{O}_G \hat{\otimes} (\mathfrak{L} \hat{\otimes}_{\mathfrak{L}^+} (\mathcal{T}E)^+ \hat{\otimes} Q) \\ &\cong \mathcal{O}_G \hat{\otimes} ((\mathcal{T}E)^+ \hat{\otimes} K) \oplus \mathcal{O}_G \hat{\otimes} \mathfrak{J} \hat{\otimes} Q \cong \mathcal{O}_G \hat{\otimes} ((\mathcal{T}E)^+ DK + \mathfrak{J} D\sigma(Q)) \subset \Omega_G^1(\mathcal{T}E).\end{aligned}$$

This implies that ϕ induces a covariant isomorphism of chain complexes

$$(\mathcal{O}_G \hat{\otimes} P_\bullet)/[\ ,\ ]_G \cong X_G^\beta(\mathcal{T}E : \mathcal{T}Q) \oplus \mathcal{O}_G[0].$$

With these preparations we can prove part b) of theorem 3.39.

Proposition 3.41. *The natural map $\psi : X_G(\mathfrak{L}) \to X_G(\mathcal{T}E : \mathcal{T}Q)$ is split injective and we have*

$$X_G(\mathcal{T}E : \mathcal{T}Q) = X_G(\mathfrak{L}) \oplus C_\bullet$$

with a covariantly contractible pro-parasupercomplex $C_\bullet$. Hence $X_G(\mathcal{T}E : \mathcal{T}Q)$ and $X_G(\mathfrak{L})$ are covariantly homotopy equivalent.

Proof. The standard resolution $B_\bullet^{\mathfrak{L}}$ of $\mathfrak{L}^+$ is a subcomplex of $P_\bullet$. Since $P_\bullet$ itself is a free $\mathfrak{L}$-bimodule resolution of $\mathfrak{L}^+$ the inclusion map $f_\bullet : B_\bullet^{\mathfrak{L}} \to P_\bullet$ is a homotopy equivalence. Explicitly set $M_0 = \mathfrak{L}^+ \hat{\otimes} \mathcal{T}Q \hat{\otimes} \mathfrak{L}$ and define $g : M_0 \to P_0$ by

$$g(l_1 \otimes q \otimes l_2) = l_1 \circ \sigma_L(q) \otimes l_2 - l_1 \otimes \sigma_L(q) \circ l_2.$$

Using equation (3.7) it is easy to check that $f_0 \oplus g : \mathfrak{L}^+ \hat{\otimes} \mathfrak{L}^+ \oplus M_0 \to P_0$ is an isomorphism. Furthermore we have $\alpha_0 g = 0$. Since the complex $P_\bullet$ is exact this implies $P_1 = \ker \alpha_0 \cong \Omega^1(\mathfrak{L}) \oplus M_0$. Set $M_1 = M_0$ and define the boundary $M_1 \to M_0$ to be the identity map. The complex $M_\bullet$ of $\mathfrak{L}$-bimodules is obviously contractible and $P_\bullet \cong B_\bullet^{\mathfrak{L}} \oplus M_\bullet$. Applying the functor $(\mathcal{O}_G \hat{\otimes} -)/[\ ,\ ]_G$ we obtain covariant isomorphisms

$$\begin{aligned}X_G^\beta(\mathcal{T}E : \mathcal{T}Q) \oplus \mathcal{O}_G[0] &\cong (\mathcal{O}_G \hat{\otimes} P_\bullet)/[\ ,\ ]_G \cong (\mathcal{O}_G \hat{\otimes} B_\bullet^{\mathfrak{L}})/[\ ,\ ]_G \oplus (\mathcal{O}_G \hat{\otimes} M_\bullet)/[\ ,\ ]_G \\ &\cong X_G^\beta(\mathfrak{L}) \oplus \mathcal{O}_G[0] \oplus (\mathcal{O}_G \hat{\otimes} M_\bullet)/[\ ,\ ]_G.\end{aligned}$$

One checks that the two copies of $\mathcal{O}_G$ are identified under this isomorphism. Moreover the map $X_G^\beta(\mathfrak{L}) \to X_G^\beta(\mathcal{T}E : \mathcal{T}Q)$ arising from these identifications is equal to ψ. Hence ψ is split injective. Let $C_\bullet$ be the image of $(\mathcal{O}_G \hat{\otimes} M_\bullet)/[\ ,\ ]_G$ in $X_G^\beta(\mathcal{T}E : \mathcal{T}Q)$. One checks that C_0 is the range of the map

$$\mathcal{O}_G \hat{\otimes} \mathfrak{L} \hat{\otimes} \mathcal{T}Q \to X_G^0(\mathcal{T}E), \qquad f(s) \otimes l \otimes q \mapsto f(s) \otimes l \circ s_L(q) - f(s) \otimes (s^{-1} \cdot s_L(q)) \circ l$$

and that C_1 is the range of the map

$$\mathcal{O}_G \hat{\otimes} \mathfrak{L} \hat{\otimes} \mathcal{T}Q \to X_G^1(\mathcal{T}E), \qquad f \otimes l \otimes q \mapsto f \otimes l D s_L(q).$$

The boundary $C_1 \to C_0$ is the boundary induced from $X_G(\mathcal{T}E : \mathcal{T}Q)$. On the other hand the boundary $\partial_0 : X_G^0(\mathcal{T}E : \mathcal{T}Q) \to X_G^1(\mathcal{T}E : \mathcal{T}Q)$ does not vanishes on C_0. However, we have $\partial^2 = \mathrm{id} - T$ and this implies that $C_\bullet$ is a sub-paracomplex of $X_G(\mathcal{T}E : \mathcal{T}Q)$. Since ψ is compatible with ∂_0 we obtain the desired direct sum decomposition

$$X_G(\mathcal{T}E : \mathcal{T}Q) \cong X_G(\mathfrak{L}) \oplus C_\bullet.$$

It is clear that the paracomplex $C_\bullet$ is covariantly contractible. □

This completes the proof of the excision theorem 3.37. We conclude this section with the following corollary of theorem 3.37.

Corollary 3.42. *Let $(\iota, \pi) : 0 \to K \to E \to Q \to 0$ be an extension of G-algebras with a bounded linear splitting. Then there are natural exact sequences*

$$\begin{tikzcd} HP_0^G(A,K) \arrow[r] & HP_0^G(A,E) \arrow[r] & HP_0^G(A,Q) \arrow[d] \\ HP_1^G(A,Q) \arrow[u] & HP_1^G(A,E) \arrow[l] & HP_1^G(A,K) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} HP_0^G(Q,A) \arrow[r] & HP_0^G(E,A) \arrow[r] & HP_0^G(K,A) \arrow[d] \\ HP_1^G(K,A) \arrow[u] & HP_1^G(E,A) \arrow[l] & HP_1^G(Q,A) \arrow[l] \end{tikzcd}$$

for every G-algebra A.

Proof. By assumption there exists a bounded linear splitting for the quotient map $\pi : E \to Q$. Since $Q \hat{\otimes} \mathcal{K}_G$ is a free G-module according to lemma 2.8 we see that the extension $0 \to K \hat{\otimes} \mathcal{K}_G \to E \hat{\otimes} \mathcal{K}_G \to Q \hat{\otimes} \mathcal{K}_G \to 0$ has an equivariant linear splitting. Hence $0 \to K \hat{\otimes} \mathcal{K}_G \to E \hat{\otimes} \mathcal{K}_G \to Q \hat{\otimes} \mathcal{K}_G \to 0$ is an admissible extension of G-algebras. Using this remark we can proceed in the same way as in the proof of theorem 3.37 to obtain the assertion. □

CHAPTER 4

Finite groups and relations with crossed products

In the first part of this chapter we consider finite groups. We explain how our general definition of HP^G_* is related to previous constructions in the literature. This discussion is based on equivariant Hochschild homology and equivariant cyclic homology. Moreover we prove a universal coefficient theorem which allows in principle to compute HP^G_* for finite groups using suitable SBI-sequences.
In the second part of this chapter we discuss the relation between HP^G_* and the ordinary periodic cyclic homology of crossed products. It turns out that HP^G_* behaves as expected from equivariant KK-theory. The third section contains a version of the Green-Julg theorem for cyclic homology. More precisely, we show $HP^G_*(\mathbb{C}, A) \cong HP_*(A \rtimes G)$ for all finite groups. This result is essentially not new, however, we present a proof which uses the machinery of universal locally nilpotent extensions developed in chapter 3 and thus fits nicely into our general framework. In the last section we prove the dual result $HP^G_*(A, \mathbb{C}) \cong HP^*(A \rtimes G)$ for arbitrary discrete groups. This is the first situation where we have to work with paracomplexes in a very concrete way. We have to develop some tools which will be used again in a slightly different form in chapter 5.

1. Equivariant cyclic homology for finite groups

In this section we discuss equivariant Hochschild homology HH^G_* and equivariant cyclic homology HC^G_* for a finite group G. These theories are essentially already considered by Brylinski in [**17**] although our terminology differs slightly from the one used there.
In order to define cyclic homology one usually one starts with a cyclic object or a mixed complex [**49**]. Both approaches are also available in the equivariant situation and we will explain them briefly.
Let us start with cyclic objects. Assume that A is a unital G-algebra for the finite group G. We define a cyclic object $A^\natural_G$ in the category of complete bornological vector spaces as follows. The space in degree n for this module is

$$A^\natural_G(n) = (\mathcal{O}_G \hat{\otimes} A^{\hat{\otimes} n+1})^G,$$

the G-invariant part of $\mathcal{O}_G \hat{\otimes} A^{\hat{\otimes} n+1}$ with respect to the diagonal action of G. One defines face maps d_j by

$$d_j(f(s) \otimes a_0 \otimes \cdots \otimes a_n) = \begin{cases} f(s) \otimes a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n & 0 \leq j < n, \\ f(s) \otimes (s^{-1} \cdot a_n) a_0 \otimes \cdots \otimes a_{n-1} & j = n. \end{cases}$$

The degeneracy operators s_j are given by

$$s_j(f(s) \otimes a_0 \otimes \cdots \otimes a_n) = f(s) \otimes a_0 \otimes a_j \otimes 1 \otimes a_{j+1} \otimes \cdots \otimes a_n, \quad 0 \leq j \leq n.$$

The cyclic operator t is defined by

$$t(f(s) \otimes a_0 \otimes \cdots \otimes a_n) = f(s) \otimes (s^{-1} \cdot a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}.$$

One checks that these operators satisfy the axioms of a cyclic module. The cyclic relation $t^{n+1} = \mathrm{id}$ holds on $A^\natural_G(n)$ since we are considering only invariant chains.
To each cyclic module one can construct an associated cyclic bicomplex [**49**]. Using this bicomplex one obtains the Hochschild homology and cyclic homology of the cyclic module and other variants of cyclic homology. In the case $A^\natural_G$ we obtain by definition the equivariant Hochschild homology $HH^G_*(A)$ and the equivariant cyclic homology $HC^G_*(A)$ of A.
In order to relate these theories to our definitions in section 3 it is more convenient to use the approach based on mixed complexes. Let again A be a G-algebra for the finite group G. We consider the invariant part $\Omega^n_G(A)^G$ of the equivariant n-forms $\Omega^n_G(A)$ with respect to the action of G. The complete bornological vector space $\Omega^n_G(A)^G$ is no longer a covariant module, we only have a module structure of the algebra $\mathcal{R}(G)$ of class functions on G. By definition the algebra $\mathcal{R}(G)$ is the invariant part of $\mathcal{O}_G$ under the action by conjugation. The covariant operators on equivariant differential forms constructed in section 3.2 restrict to operators on $\Omega^n_G(A)^G$ since they commute with the G-action on $\Omega^n_G(A)$. In particular the operators b_G and B_G still make sense on $\Omega_G(A)^G$. Since we work with invariant forms it follows that the operator T is equal to the identity on $\Omega_G(A)^G$. Hence we see from lemma 3.16 that $\Omega_G(A)^G$ together with the operators b_G and B_G satisfies the axioms of a mixed complex. We refer also to the section 1.3 for the definitions in the more general case of compact Lie groups.
For any mixed complex one can construct the (B, b)-bicomplex which computes the associated cyclic homology. In our situation we obtain the bicomplex

$$\begin{tikzcd}
{} \arrow[d, "b_G"'] & {} \arrow[d, "b_G"'] & {} \arrow[d, "b_G"'] & {} \arrow[d, "b_G"'] \\
\Omega^3_G(A)^G \arrow[d, "b_G"'] & \Omega^2_G(A)^G \arrow[l, "B_G"'] \arrow[d, "b_G"'] & \Omega^1_G(A)^G \arrow[l, "B_G"'] \arrow[d, "b_G"'] & \Omega^0_G(A)^G \arrow[l, "B_G"'] \\
\Omega^2_G(A)^G \arrow[d, "b_G"'] & \Omega^1_G(A)^G \arrow[l, "B_G"'] \arrow[d, "b_G"'] & \Omega^0_G(A)^G \arrow[l, "B_G"'] & \\
\Omega^1_G(A)^G \arrow[d, "b_G"'] & \Omega^0_G(A)^G \arrow[l, "B_G"'] & & \\
\Omega^0_G(A)^G & & &
\end{tikzcd}$$

The equivariant cyclic homology $HC^G_*(A)$ of A is the homology of the total complex of this bicomplex. The equivariant Hochschild homology $HH^G_*(A)$ is the homology of the first column. These definitions are equivalent to the ones given above.
As in the non-equivariant situation we obtain an SBI-sequence connecting $HH^G_*(A)$ and $HC^G_*(A)$

$$\cdots \longrightarrow HC^G_n(A) \xrightarrow{S} HC^G_{n-2}(A) \xrightarrow{B} HH^G_{n-1}(A) \xrightarrow{I} HC^G_{n-1}(A) \longrightarrow \cdots$$

Moreover there is a natural definition of equivariant periodic cyclic homology arising from the mixed complex $\Omega_G(A)^G$. Namely, one has to continue the cyclic bicomplex to the left and consider the homology of the resulting total complex

$$\prod_{j\in\mathbb{N}} \Omega_G^{2j}(A)^G \underset{B_G+b_G}{\overset{B_G+b_G}{\rightleftarrows}} \prod_{j\in\mathbb{N}} \Omega_G^{2j+1}(A)^G$$

obtained by taking direct products over the diagonals.
Let us show that this is precisely the theory that comes out of our general definition in chapter 3 in this case. First recall that for finite groups we have a natural isomorphism

$$HP_*^G(\mathbb{C}, A) \cong H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}\mathbb{C}), X_G(\mathcal{T}A)))$$

for all G-algebras A according to proposition 3.35. The G-algebra $\mathbb{C}$ is quasifree and we get

$$HP_*^G(\mathbb{C}, A) \cong H_*(\mathfrak{Hom}_G(X_G(\mathbb{C}), X_G(\mathcal{T}A))) = H_*(\mathfrak{Hom}_G(\mathcal{O}_G[0], X_G(\mathcal{T}A)))$$

due to the remark after proposition 3.31. Using theorem 3.23 we may switch to differential forms in the second variable to obtain

$$HP_*^G(\mathbb{C}, A) \cong H_*(\mathfrak{Hom}_G(\mathcal{O}_G[0], \theta\Omega_G(A))) = H_*(\mathrm{Hom}_G(\mathbb{C}[0], \theta\Omega_G(A))).$$

We have

$$\mathrm{Hom}_G(\mathbb{C}[0], \theta\Omega_G(A)) = \mathrm{Hom}(\mathbb{C}[0], \theta\Omega_G(A)^G) = \varprojlim_n \Omega_G^{\leq n}(A)^G = \prod_{n\in\mathbb{N}} \Omega_G^n(A)^G$$

with grading into even and odd forms and differential given by $B_G + b_G$. This shows that our general approach is compatible with the definitions given above.
Equivariant periodic cyclic homology is related to equivariant cyclic homology through a short exact sequence

$$0 \longrightarrow \varprojlim{}^1_S HC^G_{2n+*+1}(A) \longrightarrow HP_*^G(A) \longrightarrow \varprojlim{}_S HC^G_{2n+*}(A) \longrightarrow 0$$

as in the non-equivariant case.
We can also define the dual theories HH_G^* and HC_G^* in a straightforward way. The resulting definitions are compatible with the constructions in [**47**]. We shall not go into details here and mention only that one obtains

$$HP_G^*(A, \mathbb{C}) = HP_G^*(A) = \varinjlim_S HC_G^{2n+*}(A)$$

similar to the situation in ordinary cyclic cohomology.
We emphasize that we do *not* define HH_*^G and HC_*^G for infinite groups. It seems to be unclear how a reasonable definition of such theories should look like. Clearly one would like to have SBI-sequences and a close connection to equivariant periodic cyclic homology HP_*^G as above.
It turns out that one can obtain interesting results about HP_*^G without having a definition of HH_*^G and HC_*^G. This is illustrated in the proof of theorem 4.8 in section 4.4 and in theorem 5.12 in chapter 5. In these examples methods originating in Hochschild homology play a crucial role. However, ordinary cyclic homology does not appear at all.

2. A universal coefficient theorem

In this section we prove a universal coefficient theorem for the bivariant equivariant periodic cyclic homology of finite groups. This generalizes the universal coefficient theorem in [**38**]. Throughout this section we assume that G is a finite group.
First recall that for finite groups we have a natural isomorphism

$$HP^G_*(A, B) \cong H_*(\mathfrak{Hom}_G(X_G(\mathcal{T}A), X_G(\mathcal{T}B)))$$

for all G-algebras A and B according to proposition 3.35. Moreover we have seen in proposition 3.26 that $X_G(\mathcal{T}A)$ and $\theta\Omega_G(A)$ are covariantly homotopy equivalent. For the proof of this proposition one constructs a projection e on $\theta\Omega_G(A)$ by averaging over the operator T such that the inclusion $e\theta\Omega_G(A) \to \theta\Omega_G(A)$ is a covariant homotopy equivalence of pro-parasupercomplexes. Consequently there exists a natural isomorphism

$$HP^G_*(A, B) \cong H_*(\mathfrak{Hom}_G(e\theta\Omega_G(A), e\theta\Omega_G(B))).$$

Remark that $e\theta\Omega_G(A)$ and $e\theta\Omega_G(B)$ are pro-supercomplexes. Hence we have already removed the occurence of paracomplexes in this description.
In addition the category of covariant modules for a finite group has special properties. In section 2.3 we have defined the notion of a projective covariant module. Dually one can study injective covariant modules. A covariant module I is injective iff it satisfies the following property. Given an injective covariant map $f : M \to N$ with a bounded linear retraction $r : N \to M$ any covariant map $g : M \to I$ can be extended to a covariant map $h : N \to I$ such that $hf = g$.

PROPOSITION 4.1. *Let G be a finite group. Then every covariant module in G-$\mathfrak{Mod}$ is projective and injective.*

PROOF. From the description of covariant maps in proposition 2.12 we see that it suffices to prove that H-modules for a finite group H satisfy the corresponding lifting and extension properties. This can easily be shown using an averaging argument. □
We will now restrict attention to fine spaces. From proposition 4.1 we see that the category of fine covariant modules is very close to a category of vector spaces over a field in the sense that all objects are projective and injective in a purely algebraic sense. In fact this shows that the machinery of homological algebra for pro-vector spaces developed in [**38**] can be carried over to fine covariant modules over a finite group without change. In particular one can define and study Ext-functors in this setting. Since we do not want to explain the details here we simply define

$$\mathfrak{Ext}^1_G(M, N) = \varprojlim_n{}^1 \varinjlim_m \mathfrak{Hom}_G(M_m, N_n) \tag{4.1}$$

for fine countable covariant pro-modules $M = (M_m)_{m\in\mathbb{N}}$ and $N = (N_n)_{n\in\mathbb{N}}$.
If A is a fine G-algebra we can view $e\theta\Omega_G(A)$ as projective system of the supercomplexes $e\theta^n\Omega_G(A)$. We let

$$H_*(e\theta\Omega_G(A)) = (H_*(e\theta^n\Omega_G(A)))_{n\in\mathbb{N}}$$

be the covariant pro-module obtained by taking the homology of these supercomplexes. Let us describe more explicitly the structure of this covariant pro-module. Since $e\Omega_G(A)$ is a mixed complex we can define its Hochschild homology $HH_n(e\Omega_G(A))$ and its cyclic homology $HC_n(e\Omega_G(A))$ as usual. The natural inclusion $\Omega^n_G(A)^G \to e\Omega^n_G(A)$ for all n

induces a map $HC_*^G(A) \to HC_*(e\Omega_G(A))$. It can be seen as in [**38**] that there is a natural covariant isomorphism

$$H_*(e\theta\Omega_G(A)) \cong (HC_{*+2n}(e\Omega_G(A))_{n\in\mathbb{N}}$$

where the structure maps in the projective system $(HC_{*+2n}(e\Omega_G(A))_{n\in\mathbb{N}}$ are given by the S-operator.
After these preparations we are able to formulate the following universal coefficient theorem for HP_*^G.

THEOREM 4.2 (UCT). *Let G be a finite group and let A and B be fine G-algebras. Then there is a natural short exact sequence*

$$0 \longrightarrow \mathfrak{Ext}^1_G(H_*(e\theta\Omega_G(A)), H_*(e\theta\Omega_G(B))) \longrightarrow HP_*^G(A,B)$$
$$\longrightarrow \mathfrak{Hom}_G(H_*(e\theta\Omega_G(A)), H_*(e\theta\Omega_G(B))) \longrightarrow 0.$$

PROOF. The proof from [**38**] can be copied. We have

$$\mathfrak{Hom}_G(e\theta\Omega_G(A), e\theta\Omega_G(B)) = \varprojlim_n \varinjlim_m \mathfrak{Hom}_G(e\theta^m\Omega_G(A), e\theta^n\Omega_G(B))$$

and put $C_n = \varinjlim_m \mathfrak{Hom}_G(e\theta^m\Omega_G(A), e\theta^n\Omega_G(B))$ for all n. Remark that each C_n is a supercomplex. Using Milnor's description of $\varprojlim^1$ we obtain an exact sequence

$$0 \longrightarrow \varprojlim_n C_n \longrightarrow \prod_{n\in\mathbb{N}} C_n \xrightarrow{\ \mathrm{id}-\sigma\ } \prod_{n\in\mathbb{N}} C_n \longrightarrow \varprojlim{}^1_n C_n \longrightarrow 0$$

where σ denotes the structure maps in $(C_n)_{n\in\mathbb{N}}$. Since all structure maps in $e\theta\Omega_G(B)$ are surjective and $e\theta\Omega_G(A)$ is locally projective the structure maps σ in the inverse system $(C_n)_{n\in\mathbb{N}}$ are again surjective. This implies $\varprojlim^1 C_n = 0$. Therefore the exact sequence above reduces to a short exact sequence

$$0 \longrightarrow \varprojlim_n C_n \longrightarrow \prod_{n\in\mathbb{N}} C_n \xrightarrow{\ \mathrm{id}-\sigma\ } \prod_{n\in\mathbb{N}} C_n \longrightarrow 0$$

of supercomplexes. We obtain an associated long exact sequence in homology and unsplicing this long exact sequence yields the assertion. Here we use the fact that

$$H_*\left(\prod_{n\in\mathbb{N}} C_n\right) = \prod_{n\in\mathbb{N}} H_*(C_n) = \prod_{n\in\mathbb{N}} \varinjlim_m \mathfrak{Hom}_G(H_*(e\theta^m\Omega_G(A)), H_*(e\theta^n\Omega_G(B)))$$

since the functors $\mathfrak{Hom}_G$ and $\varinjlim$ are exact. Moreover one has to insert the definition of $\mathfrak{Ext}^1_G$ in equation (4.1) to identify the $\varprojlim^1$-term. □

The universal coefficient theorem 4.2 can be extended to G-algebras with nontrivial bornologies under additional assumptions.
We point out that in order to formulate theorem 4.2 it is crucial that the paracomplexes $e\theta\Omega_G(A)$ and $e\theta\Omega_G(B)$ are in fact complexes. Moreover proposition 4.1 is used in an essential way in the proof of theorem 4.2. This shows that our universal coefficient theorem is limited to finite groups.

3. The Green-Julg theorem

The Green-Julg theorem [**35**], [**44**] asserts that for a compact group G the equivariant K-theory $K_*^G(A)$ of a G-C^*-algebra A is naturally isomorphic to the ordinary K-theory $K_*(A \rtimes G)$ of the crossed product C^*-algebra $A \rtimes G$. For a comprehensive treatment of K-theory for C^*-algebras including a proof of this theorem we refer to [**11**].
In this section we present an analogue of the Green-Julg theorem in cyclic homology. In its original form this result is due to Brylinski [**17**], [**18**] who studied smooth actions of compact Lie groups. Independently it was obtained by Block [**13**]. We follow the work of Bues [**19**], [**20**] and prove a variant of this theorem for pro-algebras and finite groups.
Our version of the Green-Julg theorem involves crossed products of pro-G-algebras. We remark that the construction of crossed products for G-algebras in section 2.2 can immediately be extended to pro-G-algebras.

Theorem 4.3. *Let G be a finite group and let A be a pro-G-algebra. Then there is a natural isomorphism*

$$HP_*^G(\mathbb{C}, A) \cong HP_*(A \rtimes G).$$

This isomorphism is compatible with the decompositions of $HP_*^G(\mathbb{C}, A)$ and $HP_*(A \rtimes G)$ over the conjugacy classes of G.

For the proof of theorem 4.3 we need some preparations.
First we want to discuss a variant of the construction of the periodic tensor algebra. Let R be an arbitrary pro-G-algebra. The crossed product $B = R \rtimes G$ is a unitary bimodule over the group algebra $C = \mathbb{C}G$ in the obvious way and the multiplication in B induces a C-bimodule map $B \hat{\otimes}_C B \to B$. In other words B can be viewed as an algebra in the category of unitary C-bimodules.
The construction of differential forms can be carried over to this setting as follows. We define the space of relative differential forms $\Omega^n_C(B) = B^{+_C} \hat{\otimes}_C (B)^{\hat{\otimes}_C n}$ for $n > 0$ where the unitarization B^{+_C} now has to be taken in the category of unitary C-bimodules. Hence $B^{+_C} = B \oplus C$ with multiplication given by $(b_1, c_1) \cdot (b_2, c_2) = (b_1 b_2 + c_1 b_2 + b_1 c_2, c_1 c_2)$. Moreover we set $\Omega^0_C(B) = B$. Since B is a free unitary right C-module we have a natural right C-linear isomorphism $\Omega^n_C(B) \cong \Omega^n(R) \hat{\otimes} C$ for all n.
Multiplication of relative differential forms and the differential d can be constructed as usual. We define the relative periodic tensor algebra $\mathcal{T}_C B$ of B simply by replacing differential forms with relative differential forms in the formalism of section 3.1. The pro-linear section $\sigma_B : B \to \mathcal{T}_C B$ for the canonical homomorphism $\tau_B : \mathcal{T}_C B \to B$ is a C-bimodule map. One can easily adapt the proof of proposition 3.3 to see that $\mathcal{T}_C B$ satisfies the following universal property: If D is a pro-algebra in the category of unitary C-bimodules then for any C-bimodule map $l : B \to D$ with locally nilpotent curvature there exists a unique homomorphism $[[l]] : \mathcal{T}_C B \to D$ such that $[[l]]\sigma_B = l$.
Using the description $\Omega^n_C(B) \cong \Omega^n(R) \hat{\otimes} C$ of relative differential forms we check that the pro-algebra $\mathcal{T}_C B$ is isomorphic to the crossed product $\mathcal{T}R \rtimes G$.

Proposition 4.4. *Let G be a finite group and let R be a unital quasifree pro-G-algebra. Then the pro-algebra $R \rtimes G$ is quasifree.*

Proof. We have to construct a lifting homomorphism $w : R \rtimes G \to \mathcal{T}(R \rtimes G)$ for the canoncial projection $\tau_{R \rtimes G}$. Since R is assumed to be quasifree there exists an equivariant lifting homomorphism $v : R \to \mathcal{T}R$ for the homomorphism $\tau_R : \mathcal{T}R \to R$. Taking crossed products yields a homomorphism $v \rtimes G : R \rtimes G \to \mathcal{T}R \rtimes G$ lifting $\tau_R \times G : \mathcal{T}R \rtimes G \to R \rtimes G$. It is easy to see that the natural pro-linear map $\sigma_R \hat{\otimes} \mathrm{id} : R \rtimes G \to \mathcal{T}R \rtimes G$ induced by σ_R is a lonilcur. Hence proposition 3.3 yields a homomorphism $p : \mathcal{T}(R \rtimes G) \to \mathcal{T}R \rtimes G$ such that $p\sigma_R = \sigma_R \hat{\otimes} \mathrm{id}$. Moreover we have $(\tau_R \rtimes G)p = \tau_{R \rtimes G}$. Let us show that p has a pro-linear splitting. We define $s : \Omega^n(R)\hat{\otimes}\mathbb{C}G \to \Omega^n(R \rtimes G)$ by

$$s(x_0dx_1 \cdots dx_{2n}\hat{\otimes} f) = (x_0 \rtimes e)d(x_1 \rtimes e) \cdots d(x_{2n-1} \rtimes e)d(x_{2n} \rtimes f)$$

where $e \in G$ is the unit element. These maps assemble to a pro-linear map $s : \mathcal{T}R \rtimes G \to \mathcal{T}(R \rtimes G)$ satisfying $ps = \mathrm{id}$. Hence we obtain an admissible extension

$$I \underset{i}{\overset{}{\rightleftarrows}} \mathcal{T}(R \rtimes G) \underset{p}{\overset{s}{\rightleftarrows}} \mathcal{T}R \rtimes G$$

of pro-algebras where I is the kernel of p and $i : I \to \mathcal{T}(R \rtimes G)$ is the inclusion. Comparing this extension with the universal locally nilpotent extension $0 \to \mathcal{J}(R \rtimes G) \to \mathcal{T}(R \rtimes G) \to R \rtimes G \to 0$ we see that the ideal I is locally nilpotent.

Consider now the homomorphism $h : \mathbb{C}G \to R \rtimes G, h(f) = 1_R \rtimes f$. We compose h with the map $v \rtimes G : R \rtimes G \to \mathcal{T}R \rtimes G$ to obtain a homomorphism $(v \rtimes G)h : \mathbb{C}G \to \mathcal{T}R \rtimes G$. Since the group G is finite the group algebra $\mathbb{C}G$ is separable in the sense of definition 3.9 and hence quasifree. Due to theorem 3.5 this implies the existence of a lifting homomorphism $k : \mathbb{C}G \to \mathcal{T}(R \rtimes G)$ such that $pk = (v \rtimes G)h$. Using this homomorphism $\mathcal{T}(R \rtimes G)$ becomes a $\mathbb{C}G$-bimodule. We construct a pro-linear map $l : R \rtimes G \to \mathcal{T}(R \rtimes G)$ as follows. If n denotes the order of G we set

$$l(x \rtimes f) = \frac{1}{n} \sum_{s \in G} k(s) \circ \sigma_{R \rtimes G}(s^{-1} \cdot x \rtimes e) \circ k(s^{-1}f).$$

where we view $s \in G$ as an element of $\mathbb{C}G$. It is easy to check that l is a $\mathbb{C}G$-bimodule map. Moreover we have $\tau_{R \rtimes G} l = \mathrm{id}$ where $\tau_{R \rtimes G} : \mathcal{T}(R \rtimes G) \to R \rtimes G$ is the canonical homomorphism. This implies that l has locally nilpotent curvature.

We can apply the universal property of the relative periodic tensor algebra $\mathcal{T}_C B \cong \mathcal{T}R \rtimes G$ from above to obtain a homomorphism $u : \mathcal{T}R \rtimes G \to \mathcal{T}(R \rtimes G)$ such that $u(\sigma_R \hat{\otimes} \mathrm{id}) = l$ where $\sigma_R \hat{\otimes} \mathrm{id} : R \rtimes G \to \mathcal{T}R \rtimes G$ is the canonical splitting. Since we have $(\tau_R \rtimes G)pl = \mathrm{id}$ the universal property of $\mathcal{T}_C B$ yields $(\tau_R \rtimes G)pu = \tau_R \rtimes G$. We put $w = u(v \rtimes G)$ and compute $\tau_{R \rtimes G} w = (\tau_R \rtimes G)pu(v \rtimes G) = (\tau_R \rtimes G)(v \rtimes G) = \mathrm{id}$. Hence w is the desired splitting homomorphism for $\tau_{R \rtimes G} : \mathcal{T}(R \rtimes G) \to R \rtimes G$. □

We keep the notation $B = R \rtimes G$ and $C = \mathbb{C}G$ from above and assume in addition that R is unital. Then the group algebra C is a subalgebra of B using the inclusion $h : C \to B, h(f) = 1_R \rtimes f$. We let $[B, C]$ be the image of the pro-linear map $B\hat{\otimes}C \to B, b \otimes f \mapsto bh(f) - h(f)b$. If we denote the order of G by n we obtain a pro-linear section $\sigma_0 : B/[B, C] \to B$ for the natural quotient map π_0 by setting

$$\sigma_0(x \rtimes f) = \frac{1}{n} \sum_{s \in G} s \cdot x \rtimes sfs^{-1}$$

where we view again $s \in G$ as element of $\mathbb{C}G$. Hence we get an admissible extension

$$[B, C] \underset{\iota_0}{\rightleftarrows} B \underset{\pi_0}{\overset{\sigma_0}{\rightleftarrows}} B/[B, C].$$

Since the group G is finite the algebra $C = \mathbb{C}G$ is separable. This means that there exists a pro-linear C-bimodule splitting for the admissible short exact sequence

$$\Omega^1(C) \rightleftarrows C^+ \hat{\otimes} C^+ \underset{m}{\rightleftarrows} C^+$$

where $m : C^+ \hat{\otimes} C^+ \to C^+$ is multiplication. Tensoring this split exact sequence with B^+ over C^+ on both sides we get an admissible short exact sequence of B-bimodules

$$B^+ \hat{\otimes}_{C^+} \Omega^1(C) \hat{\otimes}_{C^+} B^+ \rightleftarrows B^+ \hat{\otimes} B^+ \rightleftarrows B^+ \hat{\otimes}_{C^+} B^+$$

with B-bimodule splitting. From this one obtains a B-bimodule isomorphism

$$\Omega^1(B) \cong (B^+ \hat{\otimes}_{C^+} \Omega^1(C) \hat{\otimes}_{C^+} B^+) \oplus \Omega^1_C(B)$$

where $\Omega^1(B)_C$ is by definition the kernel of the multiplication map $B^+ \hat{\otimes}_{C^+} B^+ \to B^+$. We set

$$\begin{aligned} K_0 &= [B, C] \\ K_1 &= B^+ \hat{\otimes}_{C^+} \Omega^1(C) \hat{\otimes}_{C^+} B^+ / [B^+ \hat{\otimes}_{C^+} \Omega^1(C) \hat{\otimes}_{C^+} B^+, B] \end{aligned}$$

and check that the differentials of $X(B)$ restrict to K. Our discussion yields an admissible extension of pro-supercomplexes

$$K \underset{\iota}{\rightleftarrows} X(B) \underset{\pi}{\rightleftarrows} X(B)_C \tag{4.2}$$

where $X(B)_C$ is the quotient of $X(B)$ by K. Explicitly we have

$$\begin{aligned} X(B)^0_C &= B/[B, C] \\ X(B)^1_C &= \Omega^1(B)_C / [\Omega^1(B)_C, B]. \end{aligned}$$

Proposition 4.5. *With the notation as above the natural map $\pi : X(B) \to X(B)_C$ is a homotopy equivalence.*

Proof. Let us show that $b : K_1 \to K_0$ is an isomorphism. Consider the map $\alpha : K_0 \to K_1$ given by $\alpha(x) = x \otimes (1 \otimes 1) \otimes 1$ where we identify $\Omega^1(C)$ with the kernel of the multiplication map $m : C^+ \hat{\otimes} C^+ \to C^+$. We compute for $x \in B$ and $f \in C$

$$\begin{aligned} \alpha(xf - fx) &= xf \otimes (1 \otimes 1) \otimes 1 - fx \otimes (1 \otimes 1) \otimes 1 \\ &= x \otimes (f \otimes 1) \otimes 1 - x \otimes (1 \otimes 1) \otimes f \\ &= x \otimes (f \otimes 1 - 1 \otimes f) \otimes 1 = x \otimes df \otimes 1, \end{aligned}$$

hence α is well-defined and $b\alpha = \mathrm{id}$. On the other hand we have

$$(\alpha b)(x \otimes f_0 df_1 \otimes 1) = \alpha(x f_0 f_1 - f_1 x f_0) = x \otimes f_0 df_1 \otimes 1$$

and thus $\alpha b = \mathrm{id}$. Hence b is an isomorphism and it follows that K is contractible. Using the map α and the pro-linear splittings in (4.2) it is not hard to show that the pro-supercomplex $X(B)$ is isomorphic to the direct sum of the pro-supercomplexes K and $X(B)_C$ such that the canonical projection $X(B) \cong K \oplus X(B)_C \to X(B)_C$ is given by π. Together with the fact that K is contractible this yields the claim. □

If R is a pro-G-algebra we denote by $X_G(R)^G$ the invariant part of the equivariant X-complex of R. It is easy to see that $X_G(R)^G$ is in fact a pro-supercomplex. Recall from above the definition of the relative X-complex $X(B)_C = X(R \rtimes G)_{\mathbb{C}G}$.

PROPOSITION 4.6. *Let R be a unital pro-G-algebra. There is a natural isomorphism*

$$X_G(R)^G \cong X(R \rtimes G)_{\mathbb{C}G}$$

of pro-supercomplexes.

PROOF. Since the group G is finite we can identify $X_G(R)^G$ with the G-coinvariants of $X_G(R)$ which we denote by $X_G(R)_G$. The map $c : X_G(R)_G \to X_G(R)^G$ given by

$$c(f \otimes \omega) = \frac{1}{n} \sum_{s \in G} s \cdot (f \otimes \omega)$$

is inverse to the natural map $X_G(R)^G \to X_G(R)_G$.
As before we view $s \in G$ as element of $\mathbb{C}G$. Moreover since elements of both $\mathcal{O}_G$ and $\mathbb{C}G$ are functions on the group G we will identify these spaces. The action of $s \in G$ on $f \in \mathcal{O}_G$ corresponds to the adjoint action of s on $f \in \mathbb{C}G$.
We define a map $\alpha : X_G(R)_G \to X(R \rtimes G)_{\mathbb{C}G}$ by

$$\begin{aligned} \alpha_0(f \otimes x) &= x \rtimes f \\ \alpha_1(f \otimes xdy) &= (x \rtimes e)d(y \rtimes f) \\ \alpha_1(f \otimes dy) &= d(y \rtimes f) \end{aligned}$$

where $e \in G$ is the unit element and a map $\beta : X(R \rtimes G)_{\mathbb{C}G} \to X_G(R)_G$ by

$$\begin{aligned} \beta_0(x \rtimes f) &= f \otimes x \\ \beta_1((x \rtimes s)d(y \rtimes g)) &= sg \otimes xd(s \cdot y) \\ \beta_1(d(y \rtimes g)) &= g \otimes dy \end{aligned}$$

for $f \in \mathbb{C}G$ and $s \in G \subset \mathbb{C}G$. To check that α_0 and β_0 are well-defined we compute

$$\begin{aligned} \alpha_0(s \cdot (f \otimes x)) &= \alpha_0(s \cdot f \otimes s \cdot x) = s \cdot x \rtimes sfs^{-1} \\ &= (1_R \rtimes s)(x \rtimes fs^{-1}) = (x \rtimes fs^{-1})(1_R \rtimes s) = x \rtimes f = \alpha_0(f \otimes x) \end{aligned}$$

where 1_R is the unit element in R and

$$\begin{aligned} \beta_0([x \rtimes f, 1_R \rtimes s]) &= \beta_0(x \rtimes fs - s \cdot x \rtimes sf) \\ &= fs \otimes x - sf \otimes s \cdot x = fs \otimes x - s \cdot (fs \otimes x) = 0. \end{aligned}$$

It follows that α_0 is an isomorphism with inverse β_0. For α_1 and β_1 one has to do similar computations. In order to show that commutators are mapped to commutators we calculate

$$\begin{aligned} \alpha_1(s \otimes xdyz - s \otimes (s^{-1} \cdot z)xdy) &= \alpha_1(s \otimes xd(yz) - s \otimes xydz - s \otimes (s^{-1} \cdot z)xdy) \\ &= (x \rtimes e)d(yz \rtimes s) - (xy \rtimes e)d(z \rtimes s) - ((s^{-1} \cdot z)x \rtimes e)d(y \rtimes s) \\ &= (x \rtimes e)d(y \rtimes e)(z \rtimes s) - ((s^{-1} \cdot z)x \rtimes e)d(y \rtimes s) \\ &= (x \rtimes e)d(y \rtimes s)(s^{-1} \cdot z \rtimes e) - (s^{-1} \cdot z \rtimes e)(x \rtimes e)d(y \rtimes s) \\ &= [(x \rtimes e)d(y \rtimes s), (s^{-1} \cdot z \rtimes e)] \end{aligned}$$

and

$$\begin{aligned}\beta_1([(x \rtimes e)d(y \rtimes s), z \rtimes t]) &= \beta_1((x \rtimes e)d(y(s \cdot z) \rtimes st) - (xy \rtimes s)d(z \rtimes t) \\ &\qquad - (z(t \cdot x) \rtimes t)d(y \rtimes s)) \\ &= st \otimes xd(y(s \cdot z)) - st \otimes xyd(s \cdot z) - ts \otimes z(t \cdot x)d(t \cdot y) \\ &= st \otimes xdy(s \cdot z) - t(st)t^{-1} \otimes (tt^{-1} \cdot z)(t \cdot x)d(t \cdot y) \\ &= st \otimes xdy(s \cdot z) - st \otimes (t^{-1} \cdot z)xdy \\ &= st \otimes xdy(s \cdot z) - st \otimes ((st)^{-1}s \cdot z)xdy\end{aligned}$$

for $s, t \in G \subset \mathbb{C}G$. Moreover we have to check that $\alpha_1 : X^1_G(R) \to X^1(R \rtimes G)_{\mathbb{C}G}$ vanishes on coinvariants. We get

$$\begin{aligned}\alpha_1(s \cdot (f \otimes xdy) - f \otimes xdy) &= (s \cdot x \rtimes e)d(s \cdot y \rtimes sfs^{-1}) - (x \rtimes e)d(y \rtimes f) \\ &= (s \cdot x \rtimes e)d((s \cdot y \rtimes sf)(1_R \rtimes s^{-1})) - (x \rtimes e)d(y \rtimes f) \\ &= (s \cdot x \rtimes e)d(s \cdot y \rtimes sf)(1_R \rtimes s^{-1}) + (s \cdot x \rtimes e)(s \cdot y \rtimes sf)d(1_R \rtimes s^{-1}) \\ &\qquad - (x \rtimes e)d(y \rtimes f) \\ &= (1_R \rtimes s^{-1})(s \cdot x \rtimes e)d(s \cdot y \rtimes sf) + (s \cdot (xy) \rtimes sf)d(1_R \rtimes s^{-1}) \\ &\qquad - (x \rtimes e)d(y \rtimes f) \\ &= (x \rtimes s^{-1})d(s \cdot y \rtimes sf) + (s \cdot (xy) \rtimes sf)d(1_R \rtimes s^{-1}) - (x \rtimes e)d(y \rtimes f) \\ &= (s \cdot (xy) \rtimes sfs^{-1})d(1_R \rtimes e)\end{aligned}$$

and using the fact that $(1_R \rtimes e) = (1_R \rtimes e)^2$ we easily obtain $(s \cdot (xy) \rtimes sfs^{-1})d(1_R \rtimes e) = 0$. Similarly we compute

$$\begin{aligned}\alpha_1(s \cdot (f \otimes dy) - f \otimes dy) &= d(s \cdot y \rtimes sfs^{-1}) - d(y \rtimes f) \\ &= (1_R \rtimes s)d(y \rtimes fs^{-1}) - d(y \rtimes f) \\ &= d(y \rtimes fs^{-1})(1_R \rtimes s) - d(y \rtimes f) \\ &= d(y \rtimes f) + (y \rtimes fs^{-1})d(1_R \rtimes s) - d(y \rtimes f) \\ &= (y \rtimes f)d(1_R \rtimes e)\end{aligned}$$

which is equal to zero. Finally we have to verify that $\beta_1 : X^1(R \rtimes G) \to X^1_G(R)_G$ factors over $X^1(R \rtimes G)_{\mathbb{C}G}$. We obtain

$$\beta_1((x \rtimes st)d(y \rtimes g)) = stg \otimes xd((st) \cdot y) = \beta_1((x \rtimes s)d(t \cdot y \rtimes tg))$$

for $t \in G \subset \mathbb{C}G$. Moreover

$$\beta_1((1_R \rtimes t)d(y \rtimes g)) = tg \otimes 1_R d(t \cdot y) = tg \otimes d(t \cdot y) - tg \otimes d(1_R)(t \cdot y)$$

which is equal to $tg \otimes d(t \cdot y) = \beta_1(d(t \cdot y \times tg))$ since $tg \otimes d(1_R)(t \cdot y)$ is zero.
Once we have checked that α_1 and β_1 are well-defined it is easy to see that they are inverse isomorphisms. Let us now show that α is a chain map. We compute

$$d\alpha_0(f \otimes x) = d(x \rtimes f) = \alpha_1(f \otimes dx) = \alpha_1 d(f \otimes x)$$

and

$$\begin{aligned}
b\alpha_1(s \otimes xdy) &= b((x \rtimes e)d(y \rtimes s)) = xy \rtimes s - y(s \cdot x) \rtimes s \\
&= xy \rtimes s - ((ss^{-1}) \cdot y)(s \cdot x) \rtimes s = xy \rtimes s - (1_R \rtimes s)((s^{-1} \cdot y)x \rtimes e) \\
&= xy \rtimes s - ((s^{-1} \cdot y)x \rtimes e)(1_R \rtimes s) = xy \rtimes s - (s^{-1} \cdot y)x \rtimes s \\
&= \alpha_0(s \otimes xy - s \otimes (s^{-1} \cdot y)x) = \alpha_0 b(s \otimes xdy).
\end{aligned}$$

This finishes the proof of proposition 4.6. □

Now we come back to the proof of theorem 4.3. Using the long exact sequences obtained in theorem 3.37 both for equivariant cyclic homology and ordinary cyclic homology it suffices to prove the assertion for an augmented pro-G-algebra of the form A^+.

On the one hand we have to compute the equivariant periodic cyclic homology of A^+. Due to proposition 3.8 we can use the universal locally nilpotent extension $0 \to \mathcal{J}A \to (\mathcal{T}A)^+ \to A^+ \to 0$ to do this. Since the group G is finite and the G-algebra $\mathbb{C}$ is quasifree the equivariant periodic cyclic homology of A is consequently the homology of

$$\begin{aligned}
\mathfrak{Hom}_G(X_G(\mathbb{C}), X_G((\mathcal{T}A)^+) &= \mathfrak{Hom}_G(\mathcal{O}_G[0], X_G((\mathcal{T}A)^+) \\
&= \mathrm{Hom}_G(\mathbb{C}[0], X_G((\mathcal{T}A)^+) = \mathrm{Hom}(\mathbb{C}[0], X_G((\mathcal{T}A)^+)^G)
\end{aligned}$$

where $X_G((\mathcal{T}A)^+)^G$ is the G-invariant part of $X_G((\mathcal{T}A)^+)$.

On the other hand we have to calculate the cyclic homology of the crossed product $A^+ \rtimes G$. Consider the admissible extension

$$\begin{tikzcd}[column sep=huge]
\mathcal{J}A \rtimes G \arrow[r, "\iota_A \rtimes G"'] & (\mathcal{T}A)^+ \rtimes G \arrow[l, dashed, bend right] \arrow[r, "\tau_A^+ \rtimes G"'] & A^+ \rtimes G \arrow[l, dashed, bend right, "\sigma_A^+ \hat{\otimes}\, \mathrm{id}"']
\end{tikzcd} \tag{4.3}$$

of pro-algebras where $\sigma_A^+ : A^+ \to (\mathcal{T}A)^+$ is the unital extension of σ_A determined by $\sigma_A^+(1) = 1$. It is easy to check that the pro-G-algebra $\mathcal{J}A \rtimes G$ is locally nilpotent. Proposition 4.4 shows that $(\mathcal{T}A)^+ \rtimes G$ is quasifree and hence (4.3) is in fact a universal locally nilpotent extension. This means that $HP_*(A^+ \rtimes G)$ can be computed using $X((\mathcal{T}A)^+ \rtimes G)$. Consider the relative X-complex $X((\mathcal{T}A)^+ \rtimes G)_{\mathbb{C}G}$ described above. Due to proposition 4.5 the pro-supercomplexes $X((\mathcal{T}A)^+ \rtimes G)$ and $X((\mathcal{T}A)^+ \rtimes G)_{\mathbb{C}G}$ are homotopy equivalent. From proposition 4.6 we obtain a natural isomorphism

$$X((\mathcal{T}A)^+ \rtimes G)_{\mathbb{C}G} \cong X_G((\mathcal{T}A)^+)^G.$$

Comparing this with the result from above we see that both theories agree as desired. The assertion regarding the decompositions over the conjugacy classes of G can be checked easily. This finishes the proof of theorem 4.3.

We conclude this section by stating an important particular case of theorem 4.3.

COROLLARY 4.7. *Let G be a finite group. Then $HP_*^G(\mathbb{C}, \mathbb{C}) = \mathcal{R}(G)$ is the complexified representation ring of G.*

Remark that this corollary can also be obtained directly from our computation of $X_G(\mathbb{C})$ in lemma 3.20.

4. The dual Green-Julg theorem

In this paragraph we study equivariant periodic cyclic cohomology. Our main result is contained in the following theorem which is dual to theorem 4.3.

THEOREM 4.8. *Let G be any discrete group and let A be a G-algebra. Then there is a natural isomorphism*

$$HP^G_*(A,\mathbb{C}) \cong HP^*(A \rtimes G).$$

This isomorphism is compatible with the decompositions of $HP^G_*(A,\mathbb{C})$ and $HP^*(A \rtimes G)$ over the conjugacy classes of G.

The proof of theorem 4.8 is divided into two parts. In the first part we obtain a simpler description of $HP^G_*(A,\mathbb{C})$. Recall from theorem 3.23 that we have a natural isomorphism

$$HP^G_*(A,\mathbb{C}) \cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\theta\Omega_G(\mathcal{K}_G)))$$

for all G-algebras A. We show that there exists a natural isomorphism

$$HP^G_*(A,\mathbb{C}) \cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\theta\Omega_G(\mathbb{C}))).$$

It follows that $HP^G_*(A,\mathbb{C})$ is equal to the periodic cyclic cohomology of the mixed complex $\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$ obtained by taking coinvariants in $\Omega_G(A\hat{\otimes}\mathcal{K}_G)$.
In the second part of the proof we construct maps of mixed complexes

$$\phi : \Omega(A \rtimes G) \to \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G, \qquad \tau : \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G \to \Omega(A \rtimes G)$$

that are inverse up to homotopy with respect to the Hochschild boundary. Using the SBI-sequence this implies the desired result.
Let us begin with the first step described above. Consider the trace map $\mathrm{Tr} : \Omega^n_G(\mathcal{K}_G) \to \Omega^n_G(\mathbb{C})$ defined by

$$\mathrm{Tr}(f(s)\otimes |r_0\rangle\langle s_0|\, d(|r_1\rangle\langle s_1|)\cdots d(|r_n\rangle\langle s_n|)) = \delta_{s_0,r_1}\cdots\delta_{s_{n-1},r_n}\delta_{s^{-1}s_n,r_0} f(s)\otimes ede\cdots de$$

and

$$\mathrm{Tr}(f(s)\otimes d(|r_1\rangle\langle s_1|)\cdots d(|r_n\rangle\langle s_n|)) = \delta_{s_1,r_2}\cdots\delta_{s_{n-1},r_n}\delta_{s^{-1}s_n,r_1} f(s)\otimes de\cdots de.$$

One checks that Tr is a covariant map and that it commutes with b and d. It follows that Tr is a map of paramixed complexes. In fact this map is closely related to the trace map that occured already in our proof of the stability theorem 3.33.
The map Tr induces a chain map

$$\mathrm{Tr} : \mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\theta\Omega_G(\mathcal{K}_G)) \to \mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G),\theta\Omega_G(\mathbb{C}))$$

and we want to show that this map becomes an isomorphism in homology. First we need some information about the homology of $\Omega_G(\mathcal{K}_G)$ with respect to the equivariant Hochschild boundary b_G.

PROPOSITION 4.9. *The trace map $Tr : \Omega_G(\mathcal{K}_G) \to \Omega_G(\mathbb{C})$ induces an isomorphism on the homology with respect to the Hochschild boundary. Hence*

$$H_*(\Omega_G(\mathcal{K}_G), b_G) = \begin{cases} \mathcal{O}_G & \text{for } * = 0 \\ 0 & \text{for } * > 0. \end{cases}$$

Proof. For any G-algebra B we view $\Omega_G(B)$ together with the equivariant Hochschild boundary as a double complex in the following way:

$$\begin{tikzcd}
{} \arrow[d, "b_G"'] & {} \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}3} \arrow[d, "b_G"'] & \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}3} \arrow[l, "\mathrm{id}-t_G"'] \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}2} \arrow[d, "b_G"'] & \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}2} \arrow[l, "\mathrm{id}-t_G"'] \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B & \mathcal{O}_G\hat{\otimes}B \arrow[l, "\mathrm{id}-t_G"']
\end{tikzcd}$$

Here $t_G : \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n+1} \to \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n+1}$ is given by

$$t_G(f(s) \otimes b_0 \otimes \cdots \otimes b_n) = (-1)^n f(s) \otimes (s^{-1} \cdot b_n) \otimes b_0 \otimes \cdots \otimes b_{n-1}.$$

The operators $b_G : \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n+1} \to \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n}$ and $b' : \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n+1} \to \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n}$ are defined by

$$\begin{aligned} b_G(f(s) \otimes b_0 \otimes \cdots \otimes b_n) = \sum_{j=0}^{n-1} &(-1)^j f(s) \otimes b_0 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_n \\ &+ (-1)^n f(s) \otimes (s^{-1} \cdot b_n) b_0 \otimes \cdots \otimes b_{n-1} \end{aligned}$$

and

$$b'(f(s) \otimes b_0 \otimes \cdots \otimes b_n) = \sum_{j=0}^{n-1} (-1)^j f(s) \otimes b_0 \otimes \cdots \otimes b_j b_{j+1} \otimes \cdots \otimes b_n.$$

Using the natural identification $\Omega^n_G(B) = (\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n+1}) \oplus (\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}n})$ it is easy to see that the total complex of this bicomplex is precisely the complex $(\Omega_G(B), b_G)$. Observe that we use the symbol b_G here in two different meanings, however, this should not give rise to confusion. We will denote the complex obtained from the first column of the bicomplex and the operator b_G defined above by $C^G_\bullet(B)$.
We apply this description to the G-algebras $\mathcal{K}_G$ and $\mathbb{C}$. Since both algebras have local units the second columns in the corresponding bicomplexes are acyclic (see also the discussion in section 5.1). Thus it suffices to show that the map Tr induces a quasiisomorphism between the first columns. Denote by $\mathcal{K}$ the algebra of finite rank operators on $\mathbb{C}G$ equipped with the trivial G-action. We define a map $\lambda : C^G_\bullet(\mathcal{K}_G) \to C^G_\bullet(\mathcal{K})$ by

$$\lambda(f(s) \otimes T_0 dT_1 \cdots dT_n) = f(s) \otimes U_s T_0 dT_1 \cdots dT_n$$

where U_s is the operator on $\mathbb{C}G$ corresponding to $s \in G$ in the left regular representation. The map λ commutes with the boundary b_G and is clearly an isomorphism. Under this isomorphism Tr corresponds to the usual trace map between the Hochschild complexes $C_\bullet(\mathcal{K})$ and $C_\bullet(\mathbb{C})$ tensored with $\mathcal{O}_G$. Hence the assertion follows from Morita invariance of ordinary Hochschild homology [**49**]. $\square$

We need the concept of a locally projective covariant pro-module and give the following definition.

DEFINITION 4.10. *A covariant pro-module P is called locally projective if for every surjective covariant map $f : M \to N$ of constant covariant modules with bounded linear splitting and each covariant map $g : P \to N$ there exists a covariant map $h : P \to M$ such that $fh = g$.*

It is clear from the definitions that projective covariant modules are locally projective. Conversely, not every locally projective covariant modules is projective. The following lemma gives a simple criterion for local projectivity.

LEMMA 4.11. *Let $P = (P_i)_{i\in I}$ be a covariant pro-module such that all P_i are projective covariant modules. Then P is locally projective.*

PROOF. Let $f : M \to N$ be a surjective covariant map with bounded linear splitting between covariant modules. A morphism $g \in \varinjlim_m \mathrm{Hom}(P_m, N)$ is represented by a covariant map $g_k : P_k \to N$. Since P_k is projective we can find a lifting $h_k : P_k \to M$ such that $fh_k = g_k$. It follows that the class h defined by h_k in $\mathrm{Hom}(P, M) = \varinjlim_m \mathrm{Hom}(P_m, N)$ satisfies $fh = g$. □

Using lemma 4.11 and proposition 3.18 we obtain the following fact.

COROLLARY 4.12. *Let B be any G-algebra. The covariant pro-module $\theta\Omega_G(B\hat{\otimes}\mathcal{K}_G)$ is locally projective.*

In the following discussion we will abbreviate $P = \theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ and set

$$D = \Omega_G(\mathcal{K}_G), \qquad E = \Omega_G(\mathbb{C}).$$

Since both $\mathcal{K}_G$ and $\mathbb{C}$ are fine algebras the mixed complexes D and E consists of fine spaces. The Hodge filtration yields complete spaces $\theta^n D$ and $\theta^n E$ for all n.
Clearly the map $\mathrm{Tr} : D \to E$ is surjective. Hence we obtain an admissible short exact sequence of paramixed complexes $0 \to K \to D \to E \to 0$ where K denotes the kernel of Tr. Since Tr is a quasiisomorphism with respect to b due to proposition 4.9 it follows from the long exact homology sequence that K is acyclic with respect to the Hochschild boundary. Consider the n-th level $\theta^n K = K/F^n K$ of the Hodge tower of K. The Hodge filtration yields a finite decreasing filtration $F^j = F^j K/F^n K$ of $\theta^n K$. In this filtration one has $F^{-1} = \theta^n K$ and $F^n = 0$.

PROPOSITION 4.13. *With the notation as above*

$$H_*(\mathfrak{Hom}_G(P, F^j K/F^{j+1}K)) = 0$$

holds for all j.

PROOF. By definition we have

$$F^j K/F^{j+1}K = b(K_{j+1}) \oplus K_{j+1}/b(K_{j+2}).$$

Since K is acyclic with respect to b it follows that $b : K_{j+1}/b(K_{j+2}) \to b(K_{j+1})$ is an isomorphism. This implies immediately that the para-supercomplex $(F^j K/F^{j+1}K, B + b)$ is covariantly contractible and the claim follows. □

Since P is locally projective the Hodge filtration $F^j K/F^n K$ of $\theta^n K$ induces a finite decreasing filtration of the supercomplex $\mathfrak{Hom}_G(P, \theta^n K)$. We view the supercomplex $\mathfrak{Hom}_G(P, \theta^n K)$ as a $\mathbb{Z}$-graded complex by putting its even part into even degrees and the odd part into odd degrees. Clearly the filtration of $\mathfrak{Hom}_G(P, \theta^n K)$ yields a filtration of this $\mathbb{Z}$-graded complex. Recall that a filtration of a $\mathbb{Z}$-graded chain complex determines a spectral sequence in a natural way [**61**]. Since in our situation the filtration is bounded we obtain convergent spectral sequences

$$E_{pq}(K) \Rightarrow H_*(\mathfrak{Hom}_G(P, \theta^n K))$$

for all n. From proposition 4.13 we see that these spectral sequences degenerate at the E^1-term because

$$E^1_{pq}(K) = H_{p+q}(\mathfrak{Hom}_G(P, F^p K/F^{p+1} K)) = 0$$

for all p and q. Hence we obtain

Proposition 4.14. *With the notation as above we have*

$$H_*(\mathfrak{Hom}_G(P, \theta^n K)) = 0$$

for all n.

Lemma 4.15. *With the notation as above put $C_n = \mathfrak{Hom}_G(P, \theta^n K)$. Then there exists an exact sequence*

$$\begin{tikzcd} H_0(\mathfrak{Hom}_G(P,\theta K)) \arrow[r] & H_0(\prod_{n\in\mathbb{N}} C_n) \arrow[r] & H_0(\prod_{n\in\mathbb{N}} C_n) \arrow[d] \\ H_1(\prod_{n\in\mathbb{N}} C_n) \arrow[u] & H_1(\prod_{n\in\mathbb{N}} C_n) \arrow[l] & H_1(\mathfrak{Hom}_G(P,\theta K)) \arrow[l] \end{tikzcd}$$

Proof. First remark that each C_n is indeed a complex. We let C be the corresponding inverse system of complexes. Using Milnor's description of $\varprojlim^1$ we obtain an exact sequence

$$0 \longrightarrow \varprojlim_n C_n \longrightarrow \prod_{n\in\mathbb{N}} C_n \xrightarrow{\ \mathrm{id}-\sigma\ } \prod_{n\in\mathbb{N}} C_n \longrightarrow \varprojlim^1_n C_n \longrightarrow 0$$

where σ denotes the structure maps in $(C_n)_{n\in\mathbb{N}}$. Since all structure maps in θK are surjective and P is locally projective the structure maps σ in the inverse system $(C_n)_{n\in\mathbb{N}}$ are again surjective. This implies $\varprojlim^1 C_n = 0$. Therefore the exact sequence above reduces to a short exact sequence

$$0 \longrightarrow \varprojlim_n C_n \longrightarrow \prod_{n\in\mathbb{N}} C_n \xrightarrow{\ \mathrm{id}-\sigma\ } \prod_{n\in\mathbb{N}} C_n \longrightarrow 0$$

of supercomplexes. The associated long exact sequence in homology yields the claim. $\square$

From this lemma and proposition 4.14 we obtain

$$H_*(\mathfrak{Hom}_G(P, \theta K)) = 0.$$

Using again the fact that P is locally projective it is not hard to show that the short exact sequence of mixed complexes $0 \to K \to D \to E \to 0$ induces a short exact sequence of supercomplexes $0 \to \mathfrak{Hom}_G(P, \theta K) \to \mathfrak{Hom}_G(P, \theta D) \to \mathfrak{Hom}_G(P, \theta E) \to 0$. The induced six-term exact sequence in homology yields the following theorem.

THEOREM 4.16. *The chain map*

$$Tr : \mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \theta\Omega_G(\mathcal{K}_G)) \to \mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \theta\Omega_G(\mathbb{C}))$$

induces an isomorphism

$$HP^G_*(A, \mathbb{C}) \cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \theta\Omega_G(\mathbb{C})).$$

This description of $HP^G_*(A, \mathbb{C})$ can be simplified further. Using the methods developed in section 3.3 we obtain easily that the supercomplexes $\theta\Omega_G(\mathbb{C})$ and $X_G(\mathcal{T}\mathbb{C})$ are covariantly homotopy equivalent. Hence passing to the X-complex in the second variable yields

$$HP^G_*(A, \mathbb{C}) \cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), X_G(\mathcal{T}\mathbb{C})).$$

Since the G-algebra $\mathbb{C}$ is quasifree the remark after proposition 3.31 shows that $X_G(\mathcal{T}\mathbb{C})$ and $X_G(\mathbb{C})$ are covariantly homotopy equivalent. Using lemma 3.20 we obtain

$$\begin{aligned} HP^G_*(A, \mathbb{C}) &\cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), X_G(\mathbb{C})) \\ &\cong H_*(\mathfrak{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \mathcal{O}_G[0])) \\ &\cong H_*(\mathrm{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \mathbb{C})). \end{aligned}$$

Observe that

$$\mathrm{Hom}_G(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G), \mathbb{C}) \cong \mathrm{Hom}(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G, \mathbb{C})$$

where $\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$ is the mixed complex obtained by taking coinvariants in $\Omega_G(A\hat{\otimes}\mathcal{K}_G)$. Summarizing what we have done so far we obtain the following theorem.

THEOREM 4.17. *Let A be a G-algebra. There is a natural isomorphism*

$$HP^G(A, \mathbb{C}) \cong HP^*(\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G)$$

where $HP^*(\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G)$ denotes the periodic cyclic cohomology of the mixed complex $\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$.

The remaining step of the proof consists in showing

$$HP^*(\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G) \cong HP^*(A \rtimes G).$$

Using excision in equivariant periodic cyclic theory and in ordinary periodic cyclic theory we reduce the general case to the situation where the algebra A is unital. So let us assume now that A has a unit element which will be denoted by 1_A. As before we view $s \in G$ as element of $\mathbb{C}G$. Since elements of both $\mathcal{O}_G$ and $\mathbb{C}G$ are functions on the group G we identify these spaces. Moreover we write $T = \sum_{r,s} T_{rs}[r, s]$ for an element $\sum_{r,s} T_{rs}|r\rangle\langle s|$ in $\mathcal{K}_G$ in the sequel.
Rearranging tensor powers we define the map $\phi : \Omega^n(A \rtimes G) \to \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_G$ by

$$\begin{aligned} \phi(s_0 ds_1 \cdots ds_n \otimes a_0 da_1 \cdots da_n) = s_0 \dots s_n \otimes [e, s_0]d[s_0, s_0 s_1] \cdots d[s_0 \cdots s_{n-1}, s_0 \cdots s_n] \otimes \\ \otimes a_0 d(s_0 \cdot a_1) d((s_0 s_1) \cdot a_2) \cdots d((s_0 \cdots s_{n-1}) \cdot a_n) \end{aligned}$$

for $a_0 \rtimes s_0 \in A \rtimes G$ and

$$\begin{aligned} \phi(ds_1 \cdots ds_n \otimes da_1 \cdots da_n) = s_1 \dots s_n \otimes d[e, s_1]d[s_1, s_1 s_2] \cdots d[s_1 \cdots s_{n-1}, s_1 \cdots s_n] \otimes \\ \otimes da_1 d(s_1 \cdot a_2) \cdots d((s_1 \cdots s_{n-1}) \cdot a_n). \end{aligned}$$

The trace map $\tau : \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_G \to \Omega^n(A \rtimes G)$ is defined by

$$\tau(s \otimes T^0 dT^1 \cdots dT^n \otimes a_0 da_1 \cdots da_n)$$
$$= \sum_{r_0,\dots,r_n \in G} T^0_{r_0 r_1} r_0^{-1} r_1 dT^1_{r_1 r_2} r_1^{-1} r_2 \cdots dT^n_{r_n, s r_0} r_n^{-1} s r_0 \otimes (r_0^{-1} \cdot a_0) d(r_1^{-1} \cdot a_1) \cdots d(r_n^{-1} \cdot a_n)$$

for $a_0 \otimes T^0 \in A\hat{\otimes}\mathcal{K}_G$ and

$$\tau(s \otimes dT^1 \cdots dT^n \otimes da_1 \cdots da_n)$$
$$= \sum_{r_1,\dots,r_n \in G} dT^1_{r_1 r_2} r_1^{-1} r_2 \cdots dT^n_{r_n, s r_1} r_n^{-1} s r_1 \otimes d(r_1^{-1} \cdot a_1) \cdots d(r_n^{-1} \cdot a_n).$$

Observe that the sums occuring here are finite since only finitely many entries in the matrices T^j are nonzero.

PROPOSITION 4.18. *The bounded linear maps $\phi : \Omega(A \rtimes G) \to \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$ and $\tau : \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G \to \Omega(A \rtimes G)$ are maps of mixed complexes and we have $\tau\phi = \mathrm{id}$.*

PROOF. The formulas given above clearly define bounded linear maps. Moreover remark that τ is well-defined since it vanishes on coinvariants. It is immediate from the definitions that ϕ and τ commute with d. A direct calculation shows that both maps also commute with the Hochschild operators. This implies that ϕ and τ are maps of mixed complexes. Furthermore one computes easily that $\tau\phi$ is the identity on $\Omega(A \rtimes G)$. This yields the claim. □

We calculate explicitly

$$(\phi\tau)(s \otimes T^0 dT^1 \cdots dT^n \otimes a_0 da_1 \cdots da_n)$$
$$= \phi\Big(\sum_{r_0,\dots,r_n \in G} T^0_{r_0 r_1} r_0^{-1} r_1 dT^1_{r_1 r_2} r_1^{-1} r_2 \cdots dT^n_{r_n, s r_0} r_n^{-1} s r_0 \otimes$$
$$\otimes (r_0^{-1} \cdot a_0) d(r_1^{-1} \cdot a_1) \cdots d(r_n^{-1} \cdot a_n)\Big)$$
$$= \sum_{r_0,\dots,r_n \in G} r_0^{-1} s r_0 \otimes T^0_{r_0 r_1}[e, r_0^{-1} r_1] d(T^1_{r_1 r_2}[r_0^{-1} r_1, r_0^{-1} r_2]) \cdots d(T^n_{r_n, s r_0}[r_0^{-1} r_n, r_0^{-1} s r_0]) \otimes$$
$$\otimes (r_0^{-1} \cdot a_0) d(r_0^{-1} \cdot a_1) \cdots d(r_0^{-1} \cdot a_n)$$
$$= \sum_{r_0,\dots,r_n \in G} s \otimes T^0_{r_0 r_1}[r_0, r_1] d(T^1_{r_1 r_2}[r_1, r_2]) \cdots d(T^n_{r_n, s r_0}[r_n, s r_0]) \otimes a_0 da_1 \cdots da_n.$$

In the same way one obtains

$$(\phi\tau)(s \otimes dT^1 \cdots dT^n \otimes da_1 \cdots da_n)$$
$$= \sum_{r_1,\dots,r_n \in G} s \otimes d(T^1_{r_1 r_2}[r_1, r_2]) \cdots d(T^n_{r_n, s r_1}[r_n, s r_1]) \otimes da_1 \cdots da_n.$$

PROPOSITION 4.19. *The map $\phi\tau : \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G \to \Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$ is homotopic to the identity with respect to the Hochschild boundary.*

Proof. We have to construct a chain homotopy connecting id and $\phi\tau$ on the Hochschild complex associated to the mixed complex $\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$.
As a first step we associate to an element of the form $s \otimes [r_0, s_0]d[r_1, s_1]\cdots d[r_n, s_n] \otimes a_0da_1\cdots da_n$ a certain number M. If $s_j = r_{j+1}$ for all $j = 0, \dots, n-1$ and $s^{-1}s_n = r_0$ we set $M = \infty$. If at least one of these conditions is not fulfilled, we let M be the smallest number i such that $s_i \neq r_{i+1}$ (or $M = n$ if all $s_j = r_{j+1}$ for $j = 0, \dots, n-1$ and $s^{-1}s_n \neq r_0$).
In a similar way we proceed with elements of the form $s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1\cdots da_n$. Here the first condition disappears and the last condition becomes $s^{-1}s_n = r_1$. The number M is then defined as before.
We construct bounded linear maps $h : \Omega^n_G(A\hat{\otimes}\mathcal{K}_G)_G \to \Omega^{n+1}_G(A\hat{\otimes}\mathcal{K}_G)_G$ for all n as follows. For an element $s \otimes [r_0, s_0]d[r_1, s_1]\cdots d[r_n, s_n] \otimes a_0da_1\cdots da_n$ we set

$$\begin{aligned}
&h(s \otimes [r_0, s_0]d[r_1, s_1]\cdots d[r_n, s_n] \otimes a_0da_1\cdots da_n)\\
&= (-1)^M s \otimes [r_0, s_0]d[r_1, s_1]\cdots d[r_M, s_M]d[s_M, s_M]\cdots d[r_n, s_n]\otimes\\
&\qquad \otimes a_0da_1\cdots da_M d1_A da_{M+1}\cdots da_n
\end{aligned}$$

if $M < \infty$ and

$$h(s \otimes [r_0, s_0]d[r_1, s_1]\cdots d[r_n, s_n] \otimes a_0da_1\cdots da_n) = 0$$

if $M = \infty$.
For elements of the form $s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1\cdots da_n$ we have to distinguish four cases. The first case is $s^{-1}s_n = r_1$ and $M < \infty$. In this case we set

$$\begin{aligned}
&h(s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1 \dots da_n)\\
&= (-1)^M s \otimes d[r_1, s_1]\cdots d[r_M, s_M]d[s_M, s_M]\cdots d[r_n, s_n] \otimes da_1\cdots da_M d1_A da_{M+1}\cdots da_n
\end{aligned}$$

as before. The second case is $s^{-1}s_n \neq r_1$ and $M = n$. We set

$$\begin{aligned}
&h(s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1\cdots da_n)\\
&\quad= (-1)^M s \otimes d[r_1, s_1]\cdots d[r_n, s_n]d[s_n, s_n] \otimes da_1\cdots da_n d1_A\\
&\quad+ (-1)^{M+n} s \otimes d[s^{-1}s_n, s^{-1}s_n]d[r_1, s_1]\cdots d[r_n, s_n] \otimes d1_A da_1\cdots da_n.
\end{aligned}$$

The third case is $s^{-1}s_n \neq r_1$ and $M < n$. We set

$$\begin{aligned}
&h(s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1\cdots da_n)\\
&= (-1)^M s \otimes d[r_1, s_1]\cdots d[r_M, s_M]d[s_M, s_M]\cdots d[r_n, s_n] \otimes da_1\cdots da_M d1_A da_{M+1}\cdots da_n\\
&+ (-1)^{M+n} s \otimes [s^{-1}r_n, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1, s_1]\cdots d[r_M, s_M]d[s_M, s_M]\cdots d[r_{n-1}, s_{n-1}]\otimes\\
&\qquad \otimes (s^{-1}\cdot a_n)d1_A da_1\cdots d1_A\cdots da_{n-1}.
\end{aligned}$$

Finally if $M = \infty$ we set

$$h(s \otimes d[r_1, s_1]\cdots d[r_n, s_n] \otimes da_1\cdots da_n) = 0.$$

Remark that in all cases h maps coinvariants to coinvariants and is therefore well-defined.
We have to check that $bh + hb = \mathrm{id} - \phi\tau$. Let us start with an element of the form

$s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n$ and assume that $M < n$. Then one gets

$$
\begin{aligned}
& bh(s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n) \\
&= b((-1)^M s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_M, s_M]d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots da_M d1_A da_{M+1} \cdots da_n) \\
&= (-1)^M \sum_{j=0}^{M-1} (-1)^j s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_M, s_M]d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_M d1_A da_{M+1} \cdots da_n \\
&+ s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n \\
&+ (-1)^M \sum_{j=M+1}^{n-1} (-1)^{j+1} \\
&\qquad \delta_{s_j, r_{j+1}} s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_M, s_M]d[s_M, s_M] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots da_M d1_A da_{M+1} \ldots d(a_j a_{j+1}) \cdots da_n \\
&+ (-1)^M (-1)^{n+1} \delta_{s^{-1}s_n, r_0} s \otimes [s^{-1}r_n, s_0]d[r_1, s_1] \cdots d[r_M, s_M]d[s_M, s_M] \ldots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) a_0 da_1 \cdots da_M d1_A da_{M+1} \cdots da_{n-1}.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
& hb(s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n) \\
&= h\Big(\sum_{j=0}^{M-1} (-1)^j s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_n \\
&\quad + \sum_{j=M+1}^{n-1} (-1)^j \delta_{s_j, r_{j+1}} s \otimes [r_0, s_0]d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_n \\
&\quad + (-1)^n \delta_{s^{-1}s_n, r_0} s \otimes [s^{-1}r_n, s_0]d[r_1, s_1] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) a_0 da_1 \cdots da_n \Big)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^{M-1} \sum_{j=0}^{M-1} (-1)^j s \otimes [r_0, s_0] d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots d(a_j a_{j+1}) \cdots da_M d1_A da_{M+1} \cdots da_n \\
&+ (-1)^M \sum_{j=M+1}^{n-1} (-1)^j \\
&\qquad \delta_{s_j, r_{j+1}} s \otimes [r_0, s_0] d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_0 da_1 \cdots da_M d1_A da_{M+1} \ldots d(a_j a_{j+1}) \cdots da_n \\
&+ (-1)^M (-1)^n \delta_{s^{-1} s_n, r_0} s \otimes [s^{-1} r_n, s_0] d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) a_0 da_1 \cdots da_M d1_A da_{M+1} \cdots da_n.
\end{aligned}
$$

Hence we get in this case

$$
\begin{aligned}
&(bh + hb)(s \otimes [r_0, s_0] d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n) \\
&\qquad = s \otimes [r_0, s_0] d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n
\end{aligned}
$$

as desired. In fact the same computation holds true also in the case $M = n$ where we have $\delta_{s^{-1} s_n, r_0} = 0$ by construction. For $M = \infty$ we have by definition

$$
(bh + hb)(s \otimes [r_0, s_0] d[r_1, s_1] \cdots d[r_n, s_n] \otimes a_0 da_1 \cdots da_n) = 0.
$$

This is again precisely what is required.
It remains to deal with elements of the form $s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n$. The case $s^{-1} s_n = r_1$ and $M < \infty$ is very similar to the previous computation and will be ommitted. Moreover the case $M = \infty$ is easy. Let us deal with the case $s^{-1} s_n \neq r_1$ and $M = n$. We compute

$$
\begin{aligned}
&bh(s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n) \\
&\quad = b((-1)^M s \otimes d[r_1, s_1] \cdots d[r_n, s_n] d[s_n, s_n] \otimes da_1 \cdots da_n d1_A \\
&\qquad + (-1)^{M+n} s \otimes d[s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_n, s_n] \otimes d1_A da_1 \cdots da_n)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^M s \otimes [r_1, s_1] d[r_2, s_2] \cdots d[r_n, s_n] d[s_n, s_n] \otimes a_1 da_2 \cdots da_n d1_A \\
&\quad + (-1)^M \sum_{j=1}^{n-1} (-1)^j s \otimes d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] d[s_n, s_n] \otimes \\
&\qquad \otimes da_1 \cdots d(a_j a_{j+1}) \cdots da_n d1_A \\
&\quad + s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n \\
&\quad + (-1)^M (-1)^{n+1} s \otimes [s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_n, s_n] \otimes 1_A da_1 \cdots da_n \\
&\quad + (-1)^{M+n} s \otimes [s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_n, s_n] \otimes 1_A da_1 \cdots da_n \\
&\quad + (-1)^{M+n} \sum_{j=1}^{n-1} (-1)^{j+1} s \otimes d[s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes d1_A da_1 \cdots d(a_j a_{j+1}) \cdots da_n \\
&\quad + (-1)^{M+n} (-1)^{n+1} s \otimes [s^{-1} r_n, s^{-1} s_n] d[s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots da_{n-1}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&hb(s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n) \\
&= h\Bigg(s \otimes [r_1, s_1] d[r_2, s_2] \cdots d[r_n, s_n] \otimes a_1 da_2 \cdots da_n \\
&\quad + \sum_{j=1}^{n-1} (-1)^j s \otimes d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes da_1 \cdots d(a_j a_{j+1}) \cdots da_n \\
&\quad + (-1)^n s \otimes [s^{-1} r_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_{n-1}, s_{n-1}] \otimes (s^{-1} \cdot a_n) da_1 \cdots da_{n-1} \Bigg) \\
&= (-1)^{M-1} s \otimes [r_1, s_1] d[r_2, s_2] \cdots d[r_n, s_n] d[s_n, s_n] \otimes a_1 da_2 \cdots da_n d1_A \\
&\quad + (-1)^{M-1} \sum_{j=1}^{n-1} (-1)^j s \otimes d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] d[s_n, s_n] \otimes \\
&\qquad \otimes da_1 \cdots d(a_j a_{j+1}) \cdots da_n d1_A \\
&\quad + (-1)^{(M-1)+(n-1)} \sum_{j=1}^{n-1} (-1)^j s \otimes d[s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes d1_A da_1 \cdots d(a_j a_{j+1}) \cdots da_n \\
&\quad + (-1)^n s \otimes [s^{-1} r_n, s^{-1} s_n] d[s^{-1} s_n, s^{-1} s_n] d[r_1, s_1] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots da_{n-1}.
\end{aligned}
$$

Comparing both expressions one checks that the answer is correct. Finally we have to deal with the case $s^{-1}s_n \neq r_1$ and $M < n$. We compute

$$
\begin{aligned}
&bh(s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n) \\
&= b((-1)^M s \otimes d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes da_1 \cdots da_M d1_A da_{M+1} \cdots da_n \\
&+ (-1)^{M+n} s \otimes [s^{-1}r_n, s^{-1}s_n] d[s^{-1}s_n, s^{-1}s_n] d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots da_M d1_A da_{M+1} \cdots da_{n-1}) \\
&= (-1)^M s \otimes [r_1, s_1] d[r_2, s_2] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes a_1 da_2 \cdots da_M d1_A da_{M+1} \cdots da_n \\
&+ (-1)^M \sum_{j=1}^{M-1} (-1)^j s \otimes d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes da_1 \cdots d(a_j a_{j+1}) \cdots da_M d1_A da_{M+1} \cdots da_n \\
&+ s \otimes d[r_1, s_1] \cdots d[r_n, s_n] \otimes da_1 \cdots da_n \\
&+ (-1)^M \sum_{j=M+1}^{n-1} (-1)^{j+1} \delta_{s_j, r_{j+1}} s \otimes d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_j, s_{j+1}] \cdots d[r_n, s_n] \otimes \\
&\qquad \otimes da_1 \cdots da_M d1_A da_{M+1} \cdots d(a_j a_{j+1}) \cdots da_n \\
&+ (-1)^M (-1)^{n+1} s \otimes [s^{-1}r_n, s^{-1}s_n] d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) da_1 \cdots da_M d1_A da_{M+1} \cdots da_{n-1} \\
&+ (-1)^{M+n} s \otimes [s^{-1}r_n, s^{-1}s_n][r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) da_1 \cdots da_M d1_A da_{M+1} \cdots da_{n-1} \\
&+ (-1)^{M+n} \sum_{j=1}^{M-1} (-1)^{j+1} \\
&s \otimes [s^{-1}r_n, s^{-1}s_n] d[s^{-1}s_n, s^{-1}s_n] d[r_1, s_1] \cdots d[r_j, s_{j+1}] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots d(a_j a_{j+1}) \cdots da_M d1_A da_{M+1} \cdots da_{n-1} \\
&+ (-1)^{M+n} (-1)^{M+1} s \otimes [s^{-1}r_n, s^{-1}s_n] d[s^{-1}s_n, s^{-1}s_n] d[r_1, s_1] \cdots d[r_M, s_M] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots da_M da_{M+1} \cdots da_{n-1} \\
&+ (-1)^{M+n} \sum_{j=M+1}^{n-2} (-1)^{j+2} \delta_{s_j, r_{j+1}} \\
&s \otimes [s^{-1}r_n, s^{-1}s_n] d[s^{-1}s_n, s^{-1}s_n] d[r_1, s_1] \cdots d[r_M, s_M] d[s_M, s_M] \cdots d[r_j, s_{j+1}] \cdots d[r_{n-1}, s_{n-1}] \otimes \\
&\qquad \otimes (s^{-1} \cdot a_n) d1_A da_1 \cdots da_M d1_A da_{M+1} \cdots d(a_j a_{j+1}) \cdots da_{n-1}
\end{aligned}
$$

$$+ (-1)^{M+n}(-1)^{n+1}$$
$$\delta_{s_{n-1},r_n} s \otimes [s^{-1}r_{n-1}, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1,s_1]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_{n-2},s_{n-2}]\otimes$$
$$\otimes s^{-1}\cdot(a_{n-1}a_n)da_1\cdots da_M d1_A da_{M+1}\cdots da_{n-2}.$$

On the other hand

$$hb(s\otimes d[r_1,s_1]\cdots d[r_n,s_n]\otimes da_1\cdots da_n)$$
$$= h\Bigg(s\otimes [r_1,s_1]d[r_2,s_2]\cdots d[r_n,s_n]\otimes a_1da_2\cdots da_n$$
$$+ \sum_{j=1}^{M-1}(-1)^j s\otimes d[r_1,s_1]\cdots d[r_j,s_{j+1}]\cdots d[r_n,s_n]\otimes da_1\cdots d(a_ja_{j+1})\cdots da_n$$
$$+ \sum_{j=M+1}^{n-1}(-1)^j\delta_{s_j,r_{j+1}} s\otimes d[r_1,s_1]\cdots d[r_j,s_{j+1}]\cdots d[r_n,s_n]\otimes$$
$$\otimes da_1\cdots d(a_ja_{j+1})\cdots da_n$$
$$+ (-1)^n s\otimes [s^{-1}r_n, s^{-1}s_n]d[r_1,s_1]\cdots d[r_{n-1},s_{n-1}]\otimes (s^{-1}\cdot a_n)da_1\cdots\cdots da_{n-1}\Bigg)$$
$$= (-1)^{M-1}s\otimes [r_1,s_1]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_n,s_n]$$
$$\otimes a_1da_2\cdots da_M d1_A da_{M+1}\cdots da_n$$
$$+ (-1)^{M-1}\sum_{j=1}^{M-1}(-1)^j s\otimes d[r_1,s_1]\cdots d[r_j,s_{j+1}]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_n,s_n]\otimes$$
$$\otimes da_1\cdots d(a_ja_{j+1})\cdots da_M d1_A da_{M+1}\cdots da_n$$
$$+ (-1)^{(M-1)+(n-1)}\sum_{j=1}^{M-1}(-1)^j$$
$$s\otimes [s^{-1}r_n, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1,s_1]\cdots d[r_j,s_{j+1}]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_{n-1},s_{n-1}]\otimes$$
$$\otimes (s^{-1}\cdot a_n)d1_A da_1\cdots d(a_ja_{j+1})\cdots da_M d1_A da_{M+1}\cdots da_{n-1}$$
$$+ (-1)^M\sum_{j=M+1}^{n-1}(-1)^j\delta_{s_j,r_{j+1}} s\otimes d[r_1,s_1]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_j,s_{j+1}]\cdots d[r_n,s_n]\otimes$$
$$\otimes da_1\cdots da_M d1_A da_{M+1}\cdots d(a_ja_{j+1})\cdots da_n$$
$$+ (-1)^{M+n-1}\sum_{j=M+1}^{n-2}(-1)^j\delta_{s_j,r_{j+1}}$$
$$s\otimes [s^{-1}r_n, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1,s_1]\cdots d[r_M,s_M]d[s_M,s_M]\cdots d[r_j,s_{j+1}]\cdots d[r_{n-1},s_{n-1}]\otimes$$
$$\otimes (s^{-1}\cdot a_n)d1_A da_1\cdots da_M d1_A da_{M+1}\cdots d(a_ja_{j+1})\cdots da_{n-1}$$

$$+ (-1)^{M+n}(-1)^n \delta_{s_{n-1},r_n}$$
$$s \otimes [s^{-1}r_{n-1}, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1, s_1]\cdots d[r_M, s_M]d[s_M, s_M]\cdots d[r_{n-2}, s_{n-2}]\otimes$$
$$\otimes\, s^{-1}\cdot (a_{n-1}a_n)da_1\cdots da_M d1_A da_{M+1}\cdots da_{n-2}$$
$$+ (-1)^n s \otimes [s^{-1}r_n, s^{-1}s_n]d[s^{-1}s_n, s^{-1}s_n]d[r_1, s_1]\cdots d[r_{n-1}, s_{n-1}]$$
$$\otimes\, (s^{-1}\cdot a_n)d1_A da_1\cdots da_{n-1}.$$

This finishes the computation showing $bh + hb = \mathrm{id} - \phi\tau$. □

COROLLARY 4.20. *The periodic cyclic cohomologies of* $\Omega(A \rtimes G)$ *and* $\Omega_G(A\hat{\otimes}\mathcal{K}_G)_G$ *are isomorphic. Inverse isomorphisms are induced by the maps* ϕ *and* τ*.*

PROOF. This follows after dualizing from proposition 4.19, the SBI-sequence and the fact that periodic cyclic cohomology is the direct limit of the cyclic cohomology groups. □

We remark that all identifications that have been made so far are compatible with the decompositions over the conjugacy classes. This observation finishes the proof of theorem 4.8.

As a special case of theorem 4.8 we obtain

COROLLARY 4.21. *Let* G *be a discrete group. Localisation at the identity element gives an isomorphism*

$$HP^G_*(\mathbb{C},\mathbb{C})_e = \bigoplus_j H^{*+2j}(G;\mathbb{C})$$

where $H^n(G;\mathbb{C})$ *is the* n*-th group cohomology of* G *with coefficients in* $\mathbb{C}$*.*

CHAPTER 5

Comparison with the theory of Baum and Schneider

In this chapter we study the relation between equivariant periodic cyclic homology and the bivariant equivariant cohomology theory introduced by Baum and Schneider [**7**]. The latter is defined using methods of algebraic topology and can be computed to some extent using standard machinery from homological algebra. Moreover the theory of Baum and Schneider generalizes and unifies a number of constructions which appeared earlier in the literature. We will review the definition of this theory in section 5.3 below.
In our discussion we focus on simplicial actions of groups on simplicial complexes. Simplicial complexes are an appropriate class of spaces in our situation for two reasons. On the one hand they are special enough to have a nice de Rham-theoretic description of their cohomology. On the other hand they are general enough to cover important examples, in particular in connection with the Baum-Connes conjecture.
In an abstract sense our main theorem 5.12 is a computation of HP^G_* for an interesting class of commutative G-algebras. We remark that theorem 5.12 contains as a special case a simplicial version of Connes' theorem computing the cyclic cohomology of the algebra $C^\infty(M)$ of smooth functions on a compact smooth manifold M. In this sense our discussion yields an equivariant generalization of this important result.

1. Smooth functions on simplicial complexes

In this section we study smooth functions and smooth differential forms on simplicial complexes. Definitions and results presented here will be used in the following sections.
First we have to fix some notation. We denote by Δ^k the k-dimensional standard simplex

$$\Delta^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} |\, 0 \leq x_j \leq 1, \sum_{j=0}^{k} x_j = 1\}$$

in $\mathbb{R}^{k+1}$. By construction Δ^k is contained in a unique k-dimensional affine subspace of $\mathbb{R}^{k+1}$ which will be denoted by A^k. A function $f : \Delta^k \to \mathbb{C}$ is called smooth if it is the restriction of a smooth function on the affine space A^k.
To obtain an appropriate class of functions for our purposes we have to require conditions on the behaviour of such smooth functions near the boundary $\partial\Delta^k$ of the simplex Δ^k. Roughly speaking we shall consider only those functions which are constant in the direction orthogonal to the boundary in a neigborhood of $\partial\Delta^k$.
Let us explain this precisely. We denote by $\partial^i\Delta^k$ the i-th face of the standard simplex consisting of all points $(x_0, \cdots, x_k) \in \Delta^k$ satisfying $x_i = 0$. Then $\partial^i\Delta^k$ defines a hyperplane $A^k_i \subset A^k$ in a natural way. To this hyperplane we associate the vector space V_i which contains all vectors in $\mathbb{R}^k$ that are orthogonal to A^k_i. For $v \in \mathbb{R}^k$ denote by $\partial_v(f)$ the partial derivative of a smooth function f on A^k in direction v. We say that a smooth function

$f : \Delta^k \to \mathbb{C}$ is i-regular if there exists a neighborhood U_i of $\partial^i\Delta^k$ such that $\partial_v(f)(x) = 0$ for all $x \in U_i$ and all $v \in V_i$. If we want to emphasize the particular neighborhood U_i we also say that f is i-regular on U_i. The function f is called regular if it is i-regular for all $i = 0, \ldots, k$. We denote by $C^\infty(\Delta^k)$ the algebra of regular smooth functions on Δ^k.
The idea behind these definitions is simple. Let us denote by $C^\infty(\Delta^k, \partial\Delta^k) \subset C^\infty(\Delta^k)$ the subalgebra consisting of those functions that vanish on the boundary $\partial\Delta^k$ of Δ^k. It is not hard to check that $C^\infty(\Delta^k, \partial\Delta^k)$ can be identified with the algebra $C^\infty_c(\Delta^k \setminus \partial\Delta^k)$ of smooth functions with compact support on the open set $\Delta^k \setminus \partial\Delta^k$. Moreover the inclusion $\partial^i : \partial^i\Delta^k \to \Delta^k$ of a face induces a well-defined homomorphism $C^\infty(\Delta^k) \to C^\infty(\partial^i\Delta^k)$. If one tries to find a natural class of smooth functions satisfying these properties one is led to the definitions given above.
Let us explicitly consider the case $k = 1$. If we identify Δ^1 with the unit interval $[0, 1]$ the algebra $C^\infty(\Delta^1)$ corresponds to the algebra of smooth functions on $[0, 1]$ which are constant around the endpoints. The algebra $C^\infty(\Delta^1, \partial\Delta^1)$ can be identified with the algebra $C^\infty_c(0, 1)$ of smooth functions with compact support on the open interval $(0, 1)$.
We want to extend the definition of regular smooth functions to arbitrary simplicial complexes. A regular smooth function on a simplicial complex X is given by a family $(f_\sigma)_{\sigma\subset X}$ of regular smooth functions on the simplices of X which is compatible with restriction to faces in the obvious way. The function f is said to have compact support if only finitely many f_σ in the corresponding family are different from zero. We denote by $C^\infty_c(X)$ the algebra of regular smooth functions with compact support on X. If the simplicial complex X is finite we simply write $C^\infty(X)$ instead of $C^\infty_c(X)$.
Let us now describe the natural locally convex topology on the algebra $C^\infty_c(X)$ of regular smooth functions on the simplicial complex X. Again we first consider the case $X = \Delta^k$. Let $\mathcal{U} = (U_0, \cdots, U_k)$ be a family of open subsets of Δ^k where each U_i is a neighborhood of $\partial^i\Delta^k$. The collection of all such families is partially ordered where $\mathcal{U} \prec \mathcal{V}$ iff $V_j \subset U_j$ for all j in the corresponding families. For a family $\mathcal{U} = (U_0, \cdots, U_k)$ we let $C^\infty(\Delta^k, \mathcal{U}) \subset C^\infty(\Delta^k)$ be the subalgebra of smooth functions which are i-regular on U_i for all i. We equip $C^\infty(\Delta^k, \mathcal{U})$ with the natural Fréchet topology of uniform convergence of all derivatives on Δ^k. For $\mathcal{U} \prec \mathcal{V}$ we have an obvious inclusion $C^\infty(\Delta^k, \mathcal{U}) \subset C^\infty(\Delta^k, \mathcal{V})$ which is compatible with the topologies. Moreover $C^\infty(\Delta^k)$ is obtained as the union of the algebras $C^\infty(\Delta^k, \mathcal{U})$. We equip $C^\infty(\Delta^k)$ with the resulting inductive limit topology. In this way the algebra $C^\infty(\Delta^k)$ becomes a nuclear LF-algebra. Remark that the natural restriction homomorphism $C^\infty(\Delta^k) \to C^\infty(\partial^i\Delta^k)$ associated to the inclusion of a face is continuous.
In order to introduce a topology on $C^\infty_c(X)$ for arbitrary X let $K \subset X$ be a finite subcomplex. A function $f = (f_\sigma)_{\sigma\subset X} \in C^\infty_c(X)$ is said to have support in K if $f_\sigma = 0$ for all $\sigma \subset X$ not contained in K. We let $C^\infty_K(X) \subset C^\infty_c(X)$ be the algebra of regular smooth functions with support in K. The algebra $C^\infty_K(X)$ is equipped with the subspace topology from the finite direct sum of algebras $C^\infty(\sigma)$ for $\sigma \subset K$. If $K \subset L$ are finite subcomplexes the obvious inclusion $C^\infty_K(X) \to C^\infty_L(X)$ is compatible with the topologies. Moreover $C^\infty_c(X)$ is the union over all finite subcomplexes K of the algebras $C^\infty_K(X)$. Hence we obtain a natural inductive limit topology on $C^\infty_c(X)$. We equip the algebra $C^\infty_c(X)$ with the associated precompact bornology which equals the bounded bornology. In this way $C^\infty_c(X)$ becomes a complete bornological algebra.

Let us have a closer look at the natural continuous restriction homomorphism $C^\infty(\Delta^k) \to C^\infty(\partial\Delta^k)$. As above we denote by $C^\infty(\Delta^k, \partial\Delta^k) \subset C^\infty(\Delta^k)$ the kernel of this homomorphism which consists of all regular smooth functions on Δ^k that have compact support in the interior $\Delta^k \setminus \partial\Delta^k$.

PROPOSITION 5.1. *For all k the restriction homomorphism $C^\infty(\Delta^k) \to C^\infty(\partial\Delta^k)$ has a continuous linear splitting. Hence we obtain an admissible extension*

$$C^\infty(\Delta^k, \partial\Delta^k) \rightleftarrows C^\infty(\Delta^k) \rightleftarrows C^\infty(\partial\Delta^k)$$

of complete bornological algebras.

PROOF. By definition we have $C^\infty(\partial\Delta^0) = 0$ and hence the case $k = 0$ is trivial.
For $k = 1$ we identify Δ^1 with the unit interval $[0, 1]$. Choose a smooth function $h : [0, 1] \to [0, 1]$ such that $h = 1$ on $[0, 1/3]$ and $h = 0$ on $[2/3, 1]$. We set $e_0 = h, e_1 = 1 - h$ and define $\sigma_1 : C^\infty(\partial\Delta^1) = \mathbb{C} \oplus \mathbb{C} \to C^\infty(\Delta^1)$ by $\sigma_1(f_0, f_1) = f_0e_0 + f_1e_1$. It is clear that σ_1 is a continuous linear splitting for the restriction map.
In order to treat the case $k = 2$ we first consider the corresponding lifting problem for a corner of Δ^2. Let us formulate precisely what we mean by this. We write $\mathbb{R}^+$ for the set of nonnegative real numbers. A corner of Δ^2 can be viewed as a neighborhood of the point $(0, 0)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Given smooth functions $f_1, f_2 : \mathbb{R}^+ \to \mathbb{C}$ that are both constant in a neighborhood of 0 and satisfy $f_1(0) = f_2(0)$ we want to construct a smooth function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ such that

a) we have $f(x_1, 0) = f_1(x_1)$ and $f(0, x_2) = f_2(x_2)$ for all $x_1, x_2 \in \mathbb{R}^+$,
b) the function f is constant in the transversal direction in a neighborhood of the boundary $(\mathbb{R}^+ \times \{0\}) \cup (\{0\} \times \mathbb{R}^+)$,
c) f depends linearly and continuously on f_1 and f_2.

In order to construct such a function we first extend f_1 and f_2 to smooth functions F_1 and F_2 on $\mathbb{R}^+ \times \mathbb{R}^+$ by setting

$$F_1(x_1, x_2) = f_1(x_1), \qquad F_2(x_1, x_2) = f_2(x_2).$$

Then we use polar coordinates (r, θ) in $(\mathbb{R}^+ \times \mathbb{R}^+) \setminus \{(0, 0)\}$ to define a smooth function g_1 by

$$g_1(r, \theta) = h\left(\frac{2\theta}{\pi}\right)$$

where h is the function from above. We extend g_1 to $\mathbb{R}^+ \times \mathbb{R}^+$ by setting $g_1(0, 0) = 0$. Moreover we define g_2 by $g_2 = 1 - g_1$ on $(\mathbb{R}^+ \times \mathbb{R}^+) \setminus \{(0, 0)\}$ and $g_2(0, 0) = 0$. Remark that g_1 and g_2 are not continuous in $(0, 0)$. With these preparations we can define the desired function $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{C}$ by

$$f = f_1(0)\delta + F_1g_1 + F_2g_2 = f_2(0)\delta + F_1g_1 + F_2g_2$$

where δ is the characteristic function of the point $(0, 0)$. The function f is smooth in $(0, 0)$ since the assumptions on f_1 and f_2 imply that f is constant in a neighborhood of this point. Moreover it is easy to verify that f satisfies the conditions a), b) and c) above. Hence this construction solves the lifting problem for a corner of Δ^2.
Now we want to show that the restriction map $C^\infty(\Delta^2) \to C^\infty(\partial\Delta^2)$ has a continuous linear

splitting. One can combine the functions g_0 and g_1 constructed above for the corners of Δ^2 to obtain functions e_j on Δ^2 for $j = 0, 1, 2$ such that

a) each e_j is a regular smooth function on Δ^2 except in the vertices where e_j is zero,
b) $e_j = 1$ in the interior of $\partial^j\Delta^2$ and $e_j = 0$ in the interior of $\partial^i\Delta^2$ for $i \neq j$,
c) for each j there exists a neighborhood U_j of the j-th vertex v_j of Δ^2 such that
$$\sum_{i \neq j} e_i = 1$$
holds on U_j except in v_j.

Now assume that a regular smooth function $f = (f_0, f_1, f_2)$ on $\partial\Delta^2$ is given where f_i is defined on the face $\partial^i\Delta^2$. The functions f_i can be extended to i-regular smooth functions F_i on Δ^2 by
$$\begin{aligned} F_0(x_0, x_1, x_2) &= f_0\left(x_1 + \frac{x_0}{2}, x_2 + \frac{x_0}{2}\right) \\ F_1(x_0, x_1, x_2) &= f_1\left(x_0 + \frac{x_1}{2}, x_2 + \frac{x_1}{2}\right) \\ F_2(x_0, x_1, x_2) &= f_2\left(x_0 + \frac{x_2}{2}, x_1 + \frac{x_2}{2}\right). \end{aligned}$$
Moreover let $\chi : \Delta^2 \to \mathbb{C}$ be the characteristic function of the set $\{v_0, v_1, v_2\}$ consisting of the three vertices of Δ^2. Using these functions we define $\sigma_2(f) : \Delta^2 \to \mathbb{C}$ by
$$\sigma_2(f) = f\chi + F_0e_0 + F_1e_1 + F_2e_2.$$
To avoid confusion we point out that $f\chi$ is the function which is equal to f in the vertices of Δ^2 and extended by zero to the whole simplex. It is easy to see that the restriction of $\sigma_2(f)$ to the boundary of Δ^2 is equal to f. Using the fact that F_i is i-regular one checks that $\sigma_2(f)$ is a regular smooth function on Δ^2. Our construction yields a continuous linear map $\sigma_2 : C^\infty(\partial\Delta^2) \to C^\infty(\Delta^2)$ which splits the natural restriction homomorphism.

To prove the assertion for $k > 2$ one proceeds in a similar way as in the case $k = 2$. Essentially one only has to combine the functions constructed above in an appropriate way. First we consider again the lifting problem for a corner of Δ^k. Such a corner can be viewed as a neighborhood of $(0, \dots, 0)$ in $(\mathbb{R}^+)^k$. We are given smooth functions $f_1, \dots, f_k :$ $(\mathbb{R}^+)^{k-1}$ which are transversally constant in a neighborhood of the boundary and satisfy certain compatibility conditions. The function f_j is extended to a smooth function $F_j :$ $(\mathbb{R}^+)^k \to \mathbb{C}$ by setting
$$F_j(x_1, \dots, x_k) = f_j(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_k).$$
For $1 \leq i < j \leq k$ we define
$$g_{ij}(x_1, \dots, x_k) = g_1(x_i, x_j), \qquad g_{ji}(x_1, \dots, x_k) = g_2(x_i, x_j)$$
where g_1 and g_2 are the functions from above. Each function g_{ij} is smooth except in some $k-2$-dimensional subspace inside the boundary and transversally constant in a neighborhood of the boundary. If we expand the product
$$\prod_{0 \leq i < j \leq k} (g_{ij} + g_{ji})$$

we obtain a sum of functions which are smooth except in the boundary and transversally constant in a neighborhood of the boundary. Moreover these functions vanish in the interior of all faces except possibly one. Using the functions F_j constructed before we can proceed as in the case $k = 2$ to solve the lifting problem for the k-dimensional corner. Since this is a straightforward but lengthy verification we omit the details.
To treat the simplex Δ^k we construct functions e_j for $j = 0, \dots, k$ such that

a) e_j is regular smooth except in the $(k-2)$-skeleton of Δ^k where $e_j = 0$,
b) $e_j = 1$ in the interior of $\partial^j \Delta^k$ and $e_j = 0$ in the interior of $\partial^i \Delta^k$ for $i \neq j$,
c) for each point $x \in \partial \Delta^k$ there exists a neighborhood U_x of x such that
$$\sum_{j \in S(x)} e_j = 1$$
in U_x except the $(k-2)$-skeleton of Δ^k where $S(x)$ is the collection of all i such that $x \in \partial^i \Delta^k$.

Assume that a regular smooth function $f = (f_0, \dots, f_k)$ on $\partial \Delta^k$ is given where f_i is defined on the face $\partial^i \Delta^k$. The function f_i can be extended to an i-regular smooth function F_i on Δ^k by
$$F_i(x_0, \dots, x_k) = f_i\left(x_0 + \frac{x_i}{k}, \dots, x_{i-1} + \frac{x_i}{k}, x_{i+1} + \frac{x_i}{k}, \dots, x_k + \frac{x_i}{k} \right).$$
Moreover let $\chi : \Delta^k \to \mathbb{C}$ be the characteristic function of the $k-2$-skeleton of Δ^k. We define $\sigma_k(f) : \Delta^k \to \mathbb{C}$ by
$$\sigma_k(f) = f\chi + \sum_{j=0}^{k} F_j \, e_j.$$
Using the properties of the functions e_j and the fact that F_j is j-regular one checks that $\sigma_k(f)$ is a regular smooth function. The restriction of $\sigma_k(f)$ to $\partial \Delta^k$ is equal to f. In this way we obtain a continuous linear splitting $\sigma_k : C^\infty(\partial \Delta^k) \to C^\infty(\Delta^k)$ for the natural restriction homomorphism. □

For a simplicial complex X let X^k denote its k-skeleton. Consider the natural continuous restriction homomorphism $C_c^\infty(X^k) \to C_c^\infty(X^{k-1})$. The kernel of this homomorphism will be denoted by $C_c^\infty(X^k, X^{k-1})$.

Proposition 5.2. *Let X be a simplicial complex. For all k the restriction homomorphism $C_c^\infty(X^k) \to C_c^\infty(X^{k-1})$ has a continuous linear splitting. Hence we obtain an admissible extension*
$$C_c^\infty(X^k, X^{k-1}) \leftarrow\!\!\dashrightarrow C_c^\infty(X^k) \leftarrow\!\!\dashrightarrow C_c^\infty(X^{k-1})$$
of complete bornological algebras.

Proof. We construct a retraction $\rho : C_c^\infty(X^k) \to C_c^\infty(X^k, X^{k-1})$ for the natural inclusion. The algebra $C_c^\infty(X^k, X^{k-1})$ can be identified with a direct sum $\bigoplus_{i \in I} C^\infty(\Delta^k, \partial \Delta^k)$. Recall that the elements $f \in C_c^\infty(X^k)$ are families $(f_\sigma)_{\sigma \subset X^k}$. For each k-simplex $\sigma \in X^k$ we define a map
$$\rho_\sigma : C_c^\infty(X^k) \to C^\infty(\Delta^k, \partial \Delta^k), \qquad \rho_\sigma((f_\sigma)) = \rho_k(f_\sigma)$$

where $\rho_k : C^\infty(\Delta^k) \to C^\infty(\Delta^k, \partial\Delta^k)$ is the retraction obtained in proposition 5.1. It is easy to check that ρ_σ is continuous. The maps ρ_σ assemble to yield a map

$$\rho : C_c^\infty(X^k) \to \bigoplus_{i\in I} C^\infty(\Delta^k, \partial\Delta^k) = C^\infty(X^k, X^{k-1})$$

which is again continuous. Moreover by construction ρ is a retraction for the inclusion $C^\infty(X^k, X^{k-1}) \to C^\infty(X^k)$. $\square$

For the proof of theorem 5.12 we will need that certain algebras have local units in the following sense. We say that a complete bornological algebra K has local units if for every small subset $S \subset K$ there exists an element $e \in K$ such that $es = se = s$ for all $s \in S$. Clearly every unital complete bornological algebra has local units. In the bornological framework complete bornological algebras with local units behave like H-unital algebras [**62**], [**63**] in the algebraic context. Remark that a complete bornological algebra having local units is in particular H-unital in the purely algebraic sense. The proofs in [**39**] can easily be adapted to show that every admissible extension

$$\begin{tikzcd} K \arrow[r, shift right] & E \arrow[l, dashed, shift right] \arrow[r, shift right] & Q \arrow[l, dashed, shift right] \end{tikzcd}$$

of complete bornological algebras induces a long exact sequence in (bornological) Hochschild homology provided K has local units. A similar assertion holds also for the homology with respect to the twisted Hochschild boundary in the equivariant context. This is the reason why we are interested in the existence of local units in some situations. In particular we will need the following result.

Proposition 5.3. *Let X be a locally finite simplicial complex. For every finite subcomplex $K \subset X$ there exists a positive function $e \in C_c^\infty(X)$ such that $e = 1$ on K. In particular $C_c^\infty(X)$ has local units.*

Proof. First recall that a simplicial complex X is called locally finite if every vertex of X is contained in only finitely many simplices of X. A simplicial complex is locally finite iff it is a locally compact space in the weak topology [**58**].

The desired function e will be constructed inductively. On X^0 we define $e(x) = 1$ if $x \in K^0$ and $e(x) = 0$ otherwise. Assuming that e is constructed on X^{k-1} we essentially have to extend functions which are defined on the boundary of k-dimensional simplices to the whole simplices. If e is constant on the boundary we extend it to the whole simplex as a constant function. In general the extension can be done using the liftings for the restriction map $C^\infty(\Delta^k) \to C^\infty(\partial\Delta^k)$ constructed in proposition 5.1. It is clear that the resulting regular smooth function e is equal to 1 on K. The fact that X is locally finite guarantees that e has compact support. Since every small subset of $C_c^\infty(X)$ is contained in $C_K^\infty(X)$ for some finite subcomplex $K \subset X$ the previous discussion shows that $C_c^\infty(X)$ has local units. $\square$

Apart from smooth functions we also have to consider differential forms on simplicial complexes in the following sections. A smooth differential form on the standard simplex Δ^k is defined as the restriction of a smooth differential form on the k-dimensional affine space A^k to Δ^k. Again we have to impose some conditions on the behaviour near the boundary. Let us consider forms of a fixed degree p. For $v \in \mathbb{R}^k$ we denote by $\mathcal{L}_v$ the Lie derivative in direction v and by ι_v the interior product with the vector field associated to v. Using the notation established in the beginning of this section we say that a smooth p-form ω on Δ^k is i-regular if there exist a neighborhood U_i of $\partial^i\Delta^k$ such that $\mathcal{L}_v(\omega)(x) = 0$

and $\iota_v(\omega)(x) = 0$ for all $x \in U_i$ and all $v \in V_i$. The form ω is called regular if it is i-regular for all $i = 0, \ldots, k$.
Given a simplicial complex X a regular smooth p-form ω on X is a family $(\omega_\sigma)_{\sigma\subset X}$ of regular smooth p-forms on the simplices of X which is compatible with the natural restriction maps. A form $\omega = (\omega_\sigma)_{\sigma\subset X}$ is said to have compact support if only finitely many ω_σ in the corresponding family are nonzero. We denote by $\mathcal{A}^p_c(X)$ the space of regular smooth p-forms on X with compact support. The exterior differential d can be defined on $\mathcal{A}_c(X)$ in the obvious way and turns it into a complex. Also the exterior product of differential forms extends naturally. Note that $\mathcal{A}^0_c(X)$ is isomorphic to the algebra $C^\infty_c(X)$ of regular smooth functions defined above.
As in the case of functions there is a natural topology on the space $\mathcal{A}^p_c(X)$ of regular smooth p-forms. Let us start with $X = \Delta^k$ and consider a family $\mathcal{U} = (U_0, \cdots, U_k)$ of open subsets of Δ^k where each U_i is a neighborhood of $\partial^i\Delta^k$. We let $\mathcal{A}^p(\Delta^k,\mathcal{U}) \subset C^\infty(\Delta^k)$ be the space of smooth p-forms which are i-regular on U_i for all i and equip this space with the natural Fréchet topology. We obtain a corresponding inductive limit topology on $\mathcal{A}^p(\Delta^k)$. Since one proceeds for an arbitrary simplicial complex X as in the case of functions we shall not work out the details. Most of the time we will not take into account the resulting bornology on $\mathcal{A}^p_c(X)$ in our considerations anyway.
We will have to consider differential forms not only as globally defined objects but also from the point of view of sheaf theory. The regularity conditions for smooth differential forms on a simplicial complex X obviously make sense also for an open subset U of X. Hence we obtain in a natural way sheaves $\mathcal{A}^p_X$ on X by letting $\Gamma(U, \mathcal{A}^p_X)$ be the space of regular smooth p-forms on the open set $U \subset X$. We also write C^∞_X for the sheaf $\mathcal{A}^0_X$. The sheaf C^∞_X is a sheaf of rings and the sheaves $\mathcal{A}^p_X$ are sheaves of modules for C^∞_X. Clearly the space $\Gamma_c(X, \mathcal{A}^p_X)$ of global sections with compact support of $\mathcal{A}^p_X$ can be identified with $\mathcal{A}^p_c(X)$.

Proposition 5.4. *Let X be a locally finite simplicial complex. The sheaves $\mathcal{A}^p_X$ are c-soft for all p and*

$$\mathbb{C}_X \longrightarrow \mathcal{A}^0_X \xrightarrow{d} \mathcal{A}^1_X \xrightarrow{d} \mathcal{A}^2_X \xrightarrow{d} \cdots$$

is a resolution of the constant sheaf $\mathbb{C}_X$ on X.

Proof. In this proof we will tacitly use some standard results in sheaf theory which can be found for instance in [**16**]. Let us first show that the sheaves $\mathcal{A}^p_X$ are c-soft. Since the sheaves $\mathcal{A}^p_X$ are sheaves of modules for the sheaf of rings C^∞_X it suffices to show that C^∞_X is c-soft. Moreover we may assume without loss of generality that X is a finite complex. We have to show that the restriction map $\Gamma(X, C^\infty_X) \to \Gamma(K, C^\infty_X)$ is surjective for all closed subsets $K \subset X$. Given a regular smooth function f on K we shall construct a regular smooth function $F : X \to \mathbb{C}$ which extends f. For $x \in X^0$ we put $F(x) = f(x)$ if $x \in K$ and $F(x) = 0$ otherwise. Now assume that F has been constructed on X^{k-1}. In order to extend F to X^k we can consider each k-simplex of X separately. If σ is a k-simplex then F is already given on $\partial\sigma$ by induction hypothesis and on the closed subset $\sigma \cap K$ by assumption. The resulting function can be extended to a smooth regular function in a small neigborhood U of $\partial\sigma \cup (\sigma \cap K)$. We find a regular smooth function h on σ such that the support of h is contained in U and $h = 1$ on $\partial\sigma \cup (\sigma \cap K)$. Using the function h we

can extend F to the whole simplex σ.
To show that the complex of sheaves $\mathcal{A}^\bullet_X$ is a resolution of the constant sheaf on X we have to prove that the stalks $(\mathcal{A}^\bullet_X)_x$ of this complex are resolutions of $\mathbb{C}$ for all $x \in X$. Each point $x \in X$ is contained in $X^k \setminus X^{k-1}$ for some k and we find a k-dimensional simplex σ in X such that x is an element in the interior $\sigma \setminus \partial\sigma$ of σ. From the definition of regular smooth differential forms we see that the stalks $(\mathcal{A}^\bullet_X)_x$ depend only on the coordinates of σ. Hence we can identify these stalks in a natural way with stalks of the sheaves $\mathcal{A}^\bullet_{\mathbb{R}^k}$ of smooth differential forms on k-dimensional Euclidean space. Using the Poincaré lemma we easily obtain the assertion. □

2. G-simplicial complexes

In this section we collect some material concerning group actions on simplicial complexes.
Recall that a simplicial map between simplicial complexes X and Y is a continuous map $f : X \to Y$ such that the restriction of f to any simplex of X is an affine map into a simplex of Y. We say that the discrete group G acts simplicially on X if every $s \in G$ acts as a simplicial map.
An interesting example of a simplicial action is the model for the universal space for proper actions $\underline{E}G$ constructed as follows [**6**]. By definition

$$\underline{E}G = \{f \in C_c(G, [0, 1])| \sum_{s\in G} f(s) = 1\}$$

is the geometric realization of the simplicial complex whose n-simplices are all $(n + 1)$-element subsets of G and the action of G is given by translation. The Baum-Connes conjecture asserts that the K-theory of the reduced group C^*-algebra $C^*_r(G)$ of G is isomorphic to the equivariant K-homology with compact supports of $\underline{E}G$. This is the reason why the universal space for proper actions plays an important role in this context. We refer to [**4**], [**6**] for more information.
Assume that G acts simplicially on a simplicial complex X. The action is called type-preserving if for each simplex σ of X the stabilizer G_σ fixes the vertices of σ. In other words, an element of G which fixes a simplex actually acts trivially on this simplex. Passing to the barycentric subdivision one may always achieve that G acts type-preserving.
Let us now specify the class of G-spaces we are interested in.

Definition 5.5. *Let G be a discrete group. A G-simplicial complex is a locally finite and finite dimensional simplicial complex X with a type-preserving simplicial action of the group G.*

The algebra $C^\infty_c(X)$ of regular smooth functions on a G-simplicial complex is equipped with a natural G-action. Our goal is to describe the equivariant periodic cyclic homology of the G-algebras arising in this way.
If X is a G-simplicial complex the space X^H of invariants with respect to a subgroup $H \subset G$ is a subcomplex of X. The action of G on X is proper iff the stabilizer of every point is a finite subgroup of G. Equivalently, X is proper iff the fixed point set X^H is empty whenever H is an infinite subgroup of G.

3. Bivariant equivariant cohomology

In this section we review the definition of bivariant equivariant cohomology in the sense of Baum and Schneider [**7**].
Let G be a discrete group and let X be a locally compact G-space. Consider the space

$$\hat{X} = \{(s,x) \in G \times X |\, s \text{ is of finite order and } s \cdot x = x\} \subset G \times X.$$

Since G is discrete we may also view $\hat{X}$ as the disjoint union of the fixed point sets $X^s = \{x \in X |\, s \cdot x = x\}$ for elements $s \in G$ of finite order. There is a G-action on $\hat{X}$ given by

$$t \cdot (s,x) = (tst^{-1}, t \cdot x)$$

for $t \in G$, $(s,x) \in \hat{X}$.
Recall from section 2.3 the definition of an equivariant sheaf. Since the category of equivariant sheaves on any G-space has enough injectives [**37**] we can choose an injective resolution

$$\mathbb{C}_{\hat{X}} \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

of the constant sheaf $\mathbb{C}_{\hat{X}}$ in the category of equivariant sheaves on $\hat{X}$. Consider the complex $C_c^\bullet(\hat{X})$ obtained by taking global sections with compact support of the sheaves $I^\bullet$. Since the sheaves I^j are equivariant there is a natural G-action on $C_c^j(\hat{X})$ for all j. Moreover we have an $\mathcal{O}_G$-module structure on $C_c^j(\hat{X})$ given by

$$(f\sigma)(s,x) = f(s)\sigma(s,x)$$

for $f \in \mathcal{O}_G$ and $\sigma \in C_c^j(\hat{X})$. Observe that only the elliptic part of $\mathcal{O}_G$ acts nontrivially on $C_c^j(\hat{X})$. It is easy to check that this $\mathcal{O}_G$-module structure and the natural G-action combine to give each $C_c^j(\hat{X})$ the structure of a fine covariant module.
With these preparations the definition of bivariant equivariant cohomology given by Baum and Schneider can be formulated as follows.

Definition 5.6. *Let G be a discrete group and let X and Y be G-spaces. The (delocalized) bivariant equivariant cohomology of X and Y is*

$$H^n_G(X,Y) = \mathfrak{Ext}^n_G(C_c^\bullet(\hat{X}), C_c^\bullet(\hat{Y}))$$

where $\mathfrak{Ext}_G$ denotes the hyperext functor in the category of fine covariant modules.

The functor $\mathfrak{Ext}_G$ can be viewed as the Hom-functor in the derived category of fine covariant modules. For general information on derived categories we refer to [**33**], [**61**]. Here we shall only explain how the right-hand side in definition 5.6 can be computed. Choose a complex $I^\bullet(\hat{Y})$ consisting of injective fine covariant modules together with a quasiisomorphism $C_c^\bullet(\hat{Y}) \to I^\bullet(\hat{Y})$. Then

$$\mathfrak{Ext}^n_G(C_c^\bullet(\hat{X}), C_c^\bullet(\hat{Y})) = H_n(\mathfrak{Hom}_G(C_c^\bullet(\hat{X}), I^\bullet(\hat{Y})),$$

hence in order to calculate $\mathfrak{Ext}_G$ we have to compute the homology of a certain Hom-complex.

Now let X be a G-simplicial complex. Since $\hat{X}$ is again a simplicial complex we may consider the resolution

$$\mathbb{C}_{\hat{X}} \longrightarrow \mathcal{A}^0_{\hat{X}} \xrightarrow{d} \mathcal{A}^1_{\hat{X}} \xrightarrow{d} \mathcal{A}^2_{\hat{X}} \xrightarrow{d} \cdots$$

of the constant sheaf $\mathbb{C}_{\hat{X}}$ by regular smooth differential forms which was constructed in section 5.1. The sheaves $\mathcal{A}^p_{\hat{X}}$ are G-equivariant and we obtain in fact a resolution in the category $\mathrm{Sh}_G(\hat{X})$ of equivariant sheaves on $\hat{X}$. The spaces $\mathcal{A}^p_c(\hat{X})$ become G-modules in a natural way. Moreover there is an $\mathcal{O}_G$-module structure on $\mathcal{A}^p_c(\hat{X})$ induced by the projection $\hat{X} \to G$. In the same way as above one checks that these actions combine to give $\mathcal{A}^p_c(\hat{X})$ the structure of a covariant module for every p .
Although there is a natural bornology on $\mathcal{A}^\bullet_c(\hat{X})$ this structure will not be taken into account for the the remaining part of this section. In order to make this explicit let us denote by $\mathfrak{Fine}$ the functor on covariant modules which changes the bornology to the fine bornology.

PROPOSITION 5.7. *Let X and Y be G-simplicial complexes. Then we have an isomorphism*

$$H^n_G(X,Y) = \mathfrak{Ext}^n_G(\mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{X})), \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{Y})))$$

which is natural with respect to equivariant proper simplicial maps in both variables.

PROOF. This isomorphism follows from proposition 5.4 and the fact that $\mathfrak{Ext}_G$ does not distinguish between quasiisomorphic complexes. The assertion concerning naturality is clear. ☐
Since X is finite dimensional the complex $\mathcal{A}^\bullet_c(\hat{X})$ is not only bounded below but also bounded above. This means that in order to compute $\mathfrak{Ext}^n_G(\mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{X})), \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{Y})))$ we may use a complex $P^\bullet(\hat{X})$ consisting of projective fine covariant modules together with a quasiisomorphism $p : P^\bullet(\hat{X}) \to \mathcal{A}^\bullet_c(\hat{X})$ and obtain

$$\mathfrak{Ext}^n_G(\mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{X})), \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{Y}))) = H^n(\mathfrak{Hom}_G(P^\bullet(\hat{X}), \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{Y})))).$$

If $D = \dim(\hat{X})$ is the dimension of $\hat{X}$ we can construct a Cartan-Eilenberg resolution $P^{\bullet\bullet}(\hat{X})$ of $\mathcal{A}^\bullet_c(\hat{X})$ in such a way that its total complex $P^\bullet(\hat{X})$ fits into a commutative diagram of the form

$$\begin{tikzcd}
0 & P^D(\hat{X}) \arrow[l] \arrow[d, "p^D"] & P^{D-1}(\hat{X}) \arrow[l, "\delta"'] \arrow[d, "p^{D-1}"] & P^{D-2}(\hat{X}) \arrow[l, "\delta"'] \arrow[d, "p^{D-2}"] & \cdots \arrow[l, "\delta"'] \\
0 & \mathcal{A}^D_c(\hat{X}) \arrow[l] & \mathcal{A}^{D-1}_c(\hat{X}) \arrow[l, "d"] & \mathcal{A}^{D-2}_c(\hat{X}) \arrow[l, "d"] & \cdots \arrow[l, "d"]
\end{tikzcd}$$

Moreover we shall require that the covariant modules $P^j(\hat{X})$ have an additional property. Recall from section 2.3 that $G_s \subset G$ denotes the centralizer of an element $s \in G$. Proposition 2.12 shows that $P^j(\hat{X})$ is projective iff all localisations $P^j(\hat{X})_s$ for $s \in G$ are projective G_s-modules. Using an averaging argument we may assume without loss of generality that for all elements s of finite order the action of s on the projective G_s-module $P^j(\hat{X})_s$ is trivial. Moreover we may require that $P^j(\hat{X})_s = 0$ if s is of infinite order since $\mathcal{A}^\bullet_c(\hat{X})_s = 0$ in this case. If all $P^j(\hat{X})$ satisfy these conditions we call the projective resolution $P^\bullet(\hat{X})$

regular. Hence we may assume without loss of generality that the resolution $P^\bullet(\hat{X})$ is regular. This fact will be used in the proof of theorem 5.12 below.
Our goal is to obtain a description of $H^n_G(X,Y)$ which is closer to the definition of equivariant cyclic homology. In order to achieve this we view $\mathcal{A}^\bullet_c(\hat{X})$ as a mixed complex by setting the b-boundary equal to zero and letting $B = d$ be the exterior differential. We associate to this mixed complex a tower of supercomplexes $\mathcal{A}_c(\hat{X}) = (\mathcal{A}_c(\hat{X})_k)$ as follows. We define

$$\mathcal{A}_c(\hat{X})_k = \bigoplus_{j=0}^{k} \mathcal{A}^j_c(\hat{X})$$

and equip this space with the ordinary grading into even and odd forms and differential $B+b = d$. Observe that $\mathcal{A}_c(\hat{X})_k = \mathcal{A}_c(\hat{X})_D$ for $k \geq D$. Hence the tower of supercomplexes $\mathcal{A}_c(\hat{X})$ is isomorphic to the constant supercomplex

$$\mathcal{A}_c(\hat{X}) \cong \bigoplus_{j=0}^{D} \mathcal{A}^j_c(\hat{X}).$$

In a similar way the projective resolution $P^\bullet(\hat{X})$ satisfies the axioms of a mixed complex except that it it is not bounded below. Let us define a tower of supercomplexes $P(\hat{X}) = (P(\hat{X})_k)$ as follows. We set

$$P(\hat{X})_k = P^{-(k+1)}(\hat{X})/\delta(P^{-(k+2)}(\hat{X})) \oplus \bigoplus_{j=-k}^{k} P^j(\hat{X}) \oplus \delta(P^k(\hat{X}))$$

Remark that for $k \geq D$ this becomes

$$P(\hat{X})_k = P^{-(k+1)}(\hat{X})/\delta(P^{-(k+2)}(\hat{X})) \oplus \bigoplus_{j=-k}^{D} P^j(\hat{X}).$$

Clearly we consider the grading into even and odd components on $P(\hat{X})_k$ and equip these spaces with the differential δ. Since the covariant modules $P^j(\hat{X})$ are projective for all j it is easy to see that the inverse system $P(\hat{X})$ is locally projective in the sense of definition 4.10. The chain map $p : P^\bullet(\hat{X}) \to \mathcal{A}^\bullet_c(\hat{X})$ induces a covariant chain map of pro-supercomplexes $p : P(\hat{X}) \to \mathcal{A}_c(\hat{X})$.

PROPOSITION 5.8. *Let X and Y be G-simplicial complexes. Then*

$$\begin{aligned}\bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X,Y) &= H_*(\varinjlim_k \mathfrak{Hom}_G(P(\hat{X})_k, \mathfrak{Fine}(\mathcal{A}_c(\hat{Y})))) \\ &= H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}))))\end{aligned}$$

where in the last expression we take homomorphisms in $\mathrm{pro}(G\text{-}\mathfrak{Mod})$.

PROOF. The degree n component of $\mathfrak{Hom}_G(P^\bullet(\hat{X}), \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{Y})))$ is

$$\bigoplus_{i\in\mathbb{Z}} \mathfrak{Hom}_G(P^i(\hat{X}), \mathfrak{Fine}(\mathcal{A}^{i+n}_c(\hat{Y}))),$$

here a direct sum occurs because $\mathcal{A}_c^\bullet(\hat{Y})$ is a bounded complex. We deduce

$$\bigoplus_{j\in\mathbb{Z}} \mathfrak{Hom}_G^{*+2j}(P^\bullet(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{Y}))) = \bigoplus_{j\in\mathbb{Z}} \bigoplus_{i\in\mathbb{Z}} \mathfrak{Hom}_G(P^i(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c^{i+2j+*}(\hat{Y})))$$

and obtain natural maps

$$\lambda_k : \mathfrak{Hom}_G(P(\hat{X})_k, \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}))) \to \bigoplus_{j\in\mathbb{Z}} \mathfrak{Hom}_G^{*+2j}(P^\bullet(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{Y})))$$

for all $k \geq \dim(\hat{X})$. It is easy to check that each λ_k is a chain map of supercomplexes. Moreover the maps λ_k are compatible with the projections in the first variable. The resulting map

$$\lambda : \varinjlim_k \mathfrak{Hom}_G(P(\hat{X})_k, \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}))) \to \bigoplus_{j\in\mathbb{Z}} \mathfrak{Hom}_G^{*+2j}(P^\bullet(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{Y})))$$

is an isomorphism of complexes. □

4. The equivariant Hochschild-Kostant-Rosenberg theorem

The goal of this section is to identify the homology of $\Omega_G(C_c^\infty(X))$ with respect to the equivariant Hochschild boundary. This will be an important ingredient in the proof of our main theorem 5.12 below.
For each element $s \in G$ we view $\mathcal{A}_c(X^s)$ as a (para-) mixed complex with b-boundary equal to zero and B-boundary equal to the exterior differential d. We define the (localized) equivariant Hochschild-Kostant-Rosenberg map $\alpha_s : \Omega_G(C_c^\infty(X))_s \to \mathcal{A}_c(X^s)$ by

$$\alpha_s(f \otimes a_0 da_1 \cdots da_n) = \frac{1}{n!}\, f(s)\, a_0 da_1 \wedge \cdots \wedge da_{n|X^s}$$

where we recall that X^s denotes the set of fixed points under the action of s. The main result of this section is

THEOREM 5.9. *Let X be a G-simplicial complex. For all elements $s \in G$ the equivariant Hochschild-Kostant-Rosenberg map $\alpha_s : \Omega_G(C_c^\infty(X))_s \to \mathcal{A}_c(X^s)$ is a map of paramixed complexes and induces an isomorphism*

$$H_*(\Omega_G(C_c^\infty(X))_s, b) = \mathcal{A}_c(X^s).$$

PROOF. Let us first remark that Block and Getzler have obtained a similar result in the context of smooth actions of compact Lie groups on compact manifolds [**14**]. In our case the proof is actually simpler. This is in particular due to the fact that the G-spaces we are working with are already triangulated in a way which is compatible with the action.

Let us first show that α_s is a map of paramixed complexes. We compute

$$\begin{aligned}\alpha_s b(f\otimes a_0da_1\cdots da_n) &= \sum_{j=0}^{n-1}(-1)^j\alpha_s(f(s)\otimes a_0da_1\cdots d(a_ja_{j+1})\cdots da_n)\\ &\quad + (-1)^n\alpha_s(f(s)\otimes(s^{-1}\cdot a_n)a_0da_1\cdots da_{n-1})\\ &= \frac{1}{(n-1)!}\Big(\sum_{j=0}^{n-1}(-1)^j f(s)\; a_0da_1\wedge\cdots\wedge d(a_ja_{j+1})\cdots\wedge da_{n|X^s}\\ &\quad + (-1)^n f(s)a_na_0da_1\wedge\cdots\wedge da_{n-1|X^s}\Big) = 0\end{aligned}$$

where we use $(s^{-1}\cdot a_n)(x) = a_n(s\cdot x) = a_n(x)$ for all $x\in X^s$. Moreover we have

$$\begin{aligned}\alpha_s B(f\otimes a_0da_1\cdots da_n) &= \sum_{j=0}^{n}(-1)^{nj}\alpha_s(f(s)\otimes s^{-1}\cdot(da_{n-j+1}\cdots da_n)da_0\cdots da_{n-j})\\ &= \frac{1}{(n+1)!}\sum_{j=0}^{n}(-1)^{nj}f(s)\,da_{n-j+1}\wedge\cdots\wedge da_n\wedge da_0\wedge\cdots\wedge da_{n-j|X^s}\\ &= \frac{1}{n!}\,f(s)\,da_0\wedge\cdots\wedge da_{n|X^s} = d\alpha_s(f\otimes a_0da_1\cdots da_n)\end{aligned}$$

and hence α_s commutes with the boundary operators as desired.
In order to show that α_s is a quasiisomorphism with respect to the Hochschild boundary we first consider the case that X is an equivariant simplex. By definition an equivariant simplex is a space of the form $X = G/H\times\Delta^k$ where H is a subgroup of G and the action on G/H is given by translation. Remark that the space of fixed points X^s for the action of s is given by $X^s = (G/H)^s\times\Delta^k$ in this situation where $(G/H)^s$ is the set of fixed points in the homogenous space G/H. We define

$$C^\infty_c(X,\partial X) = C_c(G/H)\hat\otimes C^\infty(\Delta^k,\partial\Delta^k)$$

where we recall that $C^\infty(\Delta^k,\partial\Delta^k)$ is the kernel of the restriction homomorphism $C^\infty(\Delta^k)\to C^\infty(\partial\Delta^k)$. Moreover we set

$$\mathcal{A}_c(X^s,\partial X^s) = C_c((G/H)^s)\hat\otimes\mathcal{A}(\Delta^k,\partial\Delta^k)$$

where $\mathcal{A}(\Delta^k,\partial\Delta^k)$ denotes the space of regular differential forms on Δ^k which vanish on the boundary $\partial\Delta^k$. It is easy to check that the equivariant Hochschild-Kostant-Rosenberg map for X restricts to a chain map

$$\alpha_s:\Omega_G(C^\infty_c(X,\partial X))_s\to\mathcal{A}_c(X^s,\partial X^s).$$

Our first goal is to show that this map induces an isomorphism in homology.
In order to do this we shall reduce the question to the case $G=\mathbb{Z}$. This can be done as follows. Consider the cyclic subgroup S of G generated by s. There is a canonical group homomorphism $\mathbb{Z}\to S$ which maps 1 to s and for convenience we will also denote the generator 1 of $\mathbb{Z}$ by s in the sequel. We may view $X = G/H\times\Delta^k$ as a $\mathbb{Z}$-space using the group homomorphism $\mathbb{Z}\to G$ induced by the inclusion $S\subset G$. Clearly the complexes $\Omega_G(C^\infty_c(X,\partial X))_s$ and $\Omega_{\mathbb{Z}}(C^\infty_c(X,\partial X))_s$ are isomorphic since the equivariant Hochschild

boundary in $\Omega_G(C_c^\infty(X, \partial X))_s$ only depends on s. Now observe that as a $\mathbb{Z}$-space X can be written as disjoint union

$$X = \bigcup_{j \in J} X_{n_j}$$

for some index set J where X_n is the $\mathbb{Z}$-equivariant simplex

$$X_n = \mathbb{Z}/n\mathbb{Z} \times \Delta^k$$

for $n \geq 0$. In this decomposition the spaces X_n may appear with multiplicity. Observe that if the order of $s \in G$ is finite only those X_n can appear for which n divides the order of s. Let us determine how the map $\alpha_s : \Omega_{\mathbb{Z}}(C_c^\infty(X, \partial X))_s \to \mathcal{A}_c(X^s, \partial X^s)$ can be described in terms of the spaces X_{n_j}. On the right hand side the decomposition of X induces a direct sum decomposition $\mathcal{A}_c(X^s, \partial X^s) = \bigoplus_{j \in J} \mathcal{A}(X^s_{n_j}, \partial X^s_{n_j})$. On the left hand side we obtain an isomorphism $C_c^\infty(X) = \bigoplus_{j \in J} C_c^\infty(X_{n_j})$ of $\mathbb{Z}$-algebras and a natural inclusion of complexes

$$\iota : \bigoplus_{j \in J} \Omega_{\mathbb{Z}}(C_c^\infty(X_{n_j}, \partial X_{n_j}))_s \to \Omega_{\mathbb{Z}}(\bigoplus_{j \in J} C_c^\infty(X_{n_j}, \partial X_{n_j}))_s.$$

The map ι is a quasiisomorphism on homology. If J is a finite set this follows directly from the general discussion preceeding proposition 5.3 since the algebras $C_c^\infty(X_{n_j}, \partial X_{n_j})$ have local units. The general case is obtained from this by an inductive limit argument.
It follows that $\alpha_s : \Omega_G(C_c^\infty(X, \partial X))_s \to \mathcal{A}_c(X^s, \partial X^s)$ is a quasiisomorphism iff $\alpha_s \iota$ is a quasiisomorphism. We see easily that it suffices to show that the maps

$$\alpha_s : \Omega_{\mathbb{Z}}(C_c^\infty(X_n, \partial X_n))_s \to \mathcal{A}(X^s_n, \partial X^s_n)$$

for the $\mathbb{Z}$-equivariant simplices X_n are quasiisomorphisms for all $n \geq 0$. Remark that the fixed point set X_n^s is empty for $n = 0$ or $n > 1$. In particular we have $\mathcal{A}(X^s_n, \partial X^s_n) = 0$ for $n = 0$ or $n > 1$.

PROPOSITION 5.10. *With the notation as above we have:*

a) The equivariant Hochschild-Kostant-Rosenberg map

$$\alpha_s : \Omega_{\mathbb{Z}}(C_c^\infty(X_1, \partial X_1))_s \to \mathcal{A}(X^s_1, \partial X^s_1)$$

is a quasiisomorphism.

b) For $n = 0$ or $n > 1$ the homology of $\Omega_{\mathbb{Z}}(C_c^\infty(X_n, \partial X_n))_s$ with respect to the equivariant Hochschild boundary is trivial.

PROOF. *a*) By definition we have $X_1 \cong \Delta^k$ and the action is trivial. The algebra $C_c^\infty(X_1, \partial X_1)$ can be identified with the algebra $C_c^\infty(\Delta^k \setminus \partial \Delta^k)$ of smooth functions with compact support on $\Delta^k \setminus \partial \Delta^k$. The bornology on this nuclear LF-algebra is the bounded bornology which equals the precompact bornology. We recall from section 1.1 that the completed bornological tensor product is given by the inductive tensor product in this situation. Moreover the space $\mathcal{A}(X^s_1, \partial X^s_1)$ consist of smooth differential forms with compact support in $\Delta^k \setminus \partial \Delta^k$. With these observations in mind the assertion follows from the classical Hochschild-Kostant-Rosenberg theorem [**59**].
b) First let A be any unital complete bornological algebra. We equip A with the trivial $\mathbb{Z}$-action and consider the $\mathbb{Z}$-algebra $C_c(\mathbb{Z}/n\mathbb{Z}) \hat{\otimes} A$. An element of this algebra can be written as a linear combination of elements $[i] \otimes a$ where $[i] \in C_c(\mathbb{Z}/n\mathbb{Z})$ for $i \in \mathbb{Z}/n\mathbb{Z}$

denotes the characteristic function located in i and a is an element in A. Observe that since we assume A to be unital the algebras $C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A$ have local units. Using the notation introduced in the proof of proposition 4.9 it follows that the natural inclusion $C^{\mathbb{Z}}_\bullet(C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)_s \to \Omega_{\mathbb{Z}}(C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)_s$ is a quasiisomorphism for all n.
We want to construct a contracting homotopy h for the complex $C^{\mathbb{Z}}_\bullet(C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)_s$. Using the natural identification $C^{\mathbb{Z}}_n(C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)_s \cong (C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)^{\hat{\otimes}n+1}$ we define

$$
\begin{aligned}
&h(([i_0]\otimes a_0)\otimes\cdots\otimes([i_m]\otimes a_m))\\
&\qquad\qquad = (-1)^{l+1}([i_0]\otimes a_0)\otimes\cdots\otimes([i_{l+1}]\otimes 1)\otimes([i_{l+1}]\otimes a_{l+1})\otimes\cdots\otimes([i_m]\otimes a_m)
\end{aligned}
$$

if $0\le l\le m-1$ is the smallest number such that $i_l \neq i_{l+1}$ or

$$h(([i_0]\otimes a_0)\otimes\cdots\otimes([i_m]\otimes a_m)) = ([i_0]\otimes 1)\otimes([i_0]\otimes a_0)\otimes\cdots\otimes([i_m]\otimes a_m)$$

if $i_j = i_{j+1}$ for $0\le j\le m-1$. Easy calculations show that h is indeed a contracting homotopy.
It follows from this discussion that the complexes $\Omega_{\mathbb{Z}}(C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A)_s$ are acyclic for $n=0$ or $n>1$. We will now use this result to show that the homology of $\Omega_{\mathbb{Z}}(C^\infty_c(X_n,\partial X_n))_s$ is trivial for these n. Due to proposition 5.2 we have an extension of $\mathbb{Z}$-algebras with bounded linear splitting

$$
\begin{tikzcd}
C^\infty_c(X_n,\partial X_n) \arrow[r] & C^\infty_c(X_n) \arrow[l, dashed, bend right] \arrow[r] & C^\infty_c(\partial X_n). \arrow[l, dashed, bend right]
\end{tikzcd}
$$

The algebras $C^\infty_c(X_n)$ and $C^\infty_c(\partial X_n)$ are obviously of the form $C_c(\mathbb{Z}/n\mathbb{Z})\hat{\otimes}A$ described above. Hence the homology of $\Omega_{\mathbb{Z}}(C^\infty_c(X_n))_s$ and $\Omega_{\mathbb{Z}}(C^\infty_c(\partial X_n))_s$ is trivial. Moreover the algebra $C^\infty_c(X_n,\partial X_n)$ has local units. We obtain a long exact sequence in homology showing that the complex $(\Omega_{\mathbb{Z}}(C^\infty_c(X_n,\partial X_n))_s, b)$ is acyclic. This finishes the proof of proposition 5.10. □
Together with the discussion preceeding proposition 5.10 we deduce

PROPOSITION 5.11. *Let $X = G/H\times\Delta^k$ be an equivariant simplex. For all $s\in G$ the equivariant Hochschild-Kostant-Rosenberg map*

$$\alpha_s : \Omega_G(C^\infty_c(X,\partial X))_s \to \mathcal{A}_c(X^s,\partial X^s)$$

is a quasiisomorphism.

Now let us finish the proof of theorem 5.9. We use induction on the dimension of X. If $\dim(X)=0$ the space X is a disjoint union of homogenous spaces G/H. As above we see that it suffices to consider an equivariant simplex $X=G/H$. Since $C^\infty_c(X) = C^\infty_c(X,\partial X)$ in this case the assertion follows from proposition 5.11. Now consider the diagram

$$
\begin{tikzcd}
\Omega_G(C^\infty_c(X^k,X^{k-1}))_s \arrow[r] \arrow[d] & \Omega_G(C^\infty_c(X^k))_s \arrow[r] \arrow[d, "\alpha_s"] & \Omega_G(C^\infty_c(X^{k-1}))_s \arrow[d, "\alpha_s"] \\
\mathcal{A}_c((X^k)^s,(X^{k-1})^s) \arrow[r] & \mathcal{A}_c((X^k)^s) \arrow[r] & \mathcal{A}_c((X^{k-1})^s)
\end{tikzcd}
$$

where $\mathcal{A}_c((X^k)^s,(X^{k-1})^s)$ is the kernel of the natural map $\mathcal{A}_c((X^k)^s)\to\mathcal{A}_c((X^{k-1})^s)$. The algebra $C^\infty_c(X^k,X^{k-1})$ is a direct sum of algebras of the form $C^\infty_c(\sigma,\partial\sigma)$ where $\sigma = G/H\times\Delta^k$ is an equivariant simplex. In particular $C^\infty_c(X^k,X^{k-1})$ has local units.

Hence the upper horizontal sequence induces a long exact sequence in homology. Moreover proposition 5.11 implies that the left vertical map is a quasiisomorphism. The right vertical map is a quasiisomorphism by induction hypothesis. Hence the same holds true for $\alpha_s : \Omega_G(C^\infty_c(X^k))_s \to \mathcal{A}_c((X^k)^s)$.
This completes the proof of theorem 5.9.

5. The comparison theorem

In this section we prove the following theorem which explains the relation between equivariant periodic cyclic homology and bivariant equivariant cohomology in the sense of Baum and Schneider.

THEOREM 5.12. *Let G be a discrete group and let X and Y be G-simplicial complexes. If the action of G on X is proper there exists a natural isomorphism*

$$HP^G_*(C^\infty_c(X), C^\infty_c(Y)) \cong \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X,Y).$$

The elliptic part of $HP^G_*(C^\infty_c(X), C^\infty_c(Y))$ is naturally isomorphic to $\bigoplus_{j\in\mathbb{Z}} H^{*+2j}_G(X,Y)$ even it the action on X is not proper. These isomorphisms are natural with respect to equivariant proper simplicial maps in both variables.

In section 3.4 we have seen that equivariant periodic cyclic homology admits a canonical decomposition over the conjugacy classes

$$HP^G_*(A,B) = \prod_{\langle s\rangle\in\langle G\rangle} HP^G_*(A,B)_{\langle s\rangle}$$

for arbitrary G-algebras A and B. The elliptic part of equivariant periodic cyclic homology is by definition the contribution coming from conjugacy classes of elements $s \in G$ of finite order. The contribution from conjugacy classes of elements of infinite order is called the hyperbolic part.
Using this terminology it is immediate from the definitions that the theory H^*_G defined by Baum and Schneider a priori only has an elliptic part. Hence theorem 5.12 states in particular that the hyperbolic part of $HP^G_*(C^\infty_c(X), C^\infty_c(Y))$ is zero provided the action of G on X is proper. In the general case the hyperbolic part of $HP^G_*(C^\infty_c(X), C^\infty_c(Y))$ might be different from zero, however, this cannot be detected using the theory introduced by Baum and Schneider. In this sense theorem 5.12 is the most general statement one should expect in the context of discrete groups and simplicial complexes.
The proof of theorem 5.12 is divided into several steps. First we shall identify HP^G_* with an auxiliary bivariant theory h^G_* under the assumptions of the theorem. As in section 5.3 we denote by $\mathfrak{Fine}$ the natural forgetful functor on covariant modules which changes the bornology to the fine bornology. The functor $\mathfrak{Fine}$ is extended to the category pro(G-$\mathfrak{Mod}$) in the obvious way. With this notation we define the bivariant theory $h^G_*(A,B)$ for G-algebras A and B by

$$h^G_*(A,B) = H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)), \mathfrak{Fine}(\theta\Omega_G(B\hat{\otimes}\mathcal{K}_G)))).$$

Using theorem 3.23 we see that this definition is identical to the definition of HP^G_* except that we do *not* require the covariant maps in the Hom-complex to be bounded. It should be clear that h_*^G shares many properties with HP_*^G. For our purposes it is important that h_*^G satisfies excision in both variables. This follows immediately from the proof of theorem 3.37. Moreover we have a natural composition product for h_*^G and there exists a natural transformation

$$\phi : HP_*^G(A, B) \to h_*^G(A, B)$$

which is obtained by forgetting the bornology. It is clear that ϕ is compatible with the product.

Proposition 5.13. *Let X be a G-simplicial complex and let B be an arbitrary G-algebra. Then the natural map*

$$\phi : HP_*^G(C_c^\infty(X), B) \to h_*^G(C_c^\infty(X), B)$$

is an isomorphism.

Proof. We use induction on the dimension of X. For $\dim(X) = 0$ the algebra $C_c^\infty(X)$ is equipped with the fine bornology and $\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)$ is a projective system of fine spaces. It follows that the complexes used in the definition of HP_*^G and h_*^G are equal. Hence ϕ is an isomorphism in this case. Now assume that the assertion is true for all G-simplicial complexes of dimension smaller than k. Due to proposition 5.2 we have a linearly split extension of G-algebras of the form

$$\bigoplus_{i\in I} C_c(G/H_i)\hat{\otimes}C^\infty(\Delta^k, \partial\Delta^k) \rightleftarrows C_c^\infty(X^k) \rightleftarrows C_c^\infty(X^{k-1}).$$

Using the six-term exact sequences for HP_*^G and h_*^G obtained from the excision theorem 3.37 it suffices to show that

$$\phi : HP_*^G(\bigoplus_{i\in I} C_c(G/H_i)\hat{\otimes}C^\infty(\Delta^k, \partial\Delta^k), B) \to h_*^G(\bigoplus_{i\in I} C_c(G/H_i)\hat{\otimes}C^\infty(\Delta^k, \partial\Delta^k), B)$$

is an isomorphism. Applying excision again it we see that in both theories HP_*^G and h_*^G the G-algebras $\bigoplus_{i\in I} C_c(G/H_i)\hat{\otimes}C^\infty(\Delta^k, \partial\Delta^k)$ and $\bigoplus_{i\in I} C_c(G/H_i)$ are equivalent. Since ϕ is compatible with products the assertion follows now from the case $\dim(X) = 0$ which we have already proved. □

Corollary 5.14. *For all G-simplicial complexes X and Y we have a natural isomorphism*

$$HP_*^G(C_c^\infty(X), C_c^\infty(Y)) \cong h_*^G(C_c^\infty(X), C_c^\infty(Y)).$$

This isomorphism is natural with respect to equivariant proper simplicial maps in both variables.

Hence we can work with the theory h_*^G instead of HP_*^G from now on.
In order to formulate the next ingredient in the proof of theorem 5.12 consider the following situation. Let B be a G-algebra and define the trace map $\mathrm{Tr} : \Omega_G(B\hat{\otimes}\mathcal{K}_G) \to \Omega_G(B)$ by

$$\mathrm{Tr}(f(s) \otimes (x_0 \otimes T_0)\, d(x_1 \otimes T_1) \cdots d(x_x \otimes T_n)) = tr_s(T_0 \cdots T_n)\, f(s) \otimes x_0 dx_1 \cdots dx_n.$$

where as in section 3.6 the map tr_s is the twisted trace defined by

$$tr_s(T) = tr(T\, U_s)$$

for $T \in \mathcal{K}_G$ and $s \in G$. It is easy to check that Tr commutes with the equivariant Hochschild boundary b and the operator d. This implies that Tr is a map of paramixed complexes. We remark that in section 4.4 we have already considered a special case of this map.

PROPOSITION 5.15. *Let X be a G-simplicial complex. The map $Tr : \Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G) \to \Omega_G(C^\infty_c(X))$ induces an isomorphism on the homology with respect to the equivariant Hochschild boundary b.*

PROOF. For a G-algebra B we view $\Omega_G(B)$ together with the equivariant Hochschild boundary as a double complex

$$\begin{tikzcd}
{} \arrow[d, "b_G"'] & {} \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}3} \arrow[d, "b_G"'] & \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}3} \arrow[l, "\mathrm{id}-t_G"'] \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}2} \arrow[d, "b_G"'] & \mathcal{O}_G\hat{\otimes}B^{\hat{\otimes}2} \arrow[l, "\mathrm{id}-t_G"'] \arrow[d, "-b'"] \\
\mathcal{O}_G\hat{\otimes}B & \mathcal{O}_G\hat{\otimes}B \arrow[l, "\mathrm{id}-t_G"']
\end{tikzcd}$$

and refer to the proof of proposition 4.9 for the definition of the operators occuring here. The complex obtained from the first column of this bicomplex and the operator b_G is denoted by $C^G_\bullet(B)$.
Let us apply this description to the G-algebras $C^\infty_c(X)\hat{\otimes}\mathcal{K}_G$ and $C^\infty_c(X)$. Using proposition 5.3 we see that both algebras have local units. It follows that the second columns in the corresponding bicomplexes are acyclic. Thus it suffices to show that the map Tr induces a quasiisomorphism between the first columns. Denote by $\mathcal{K}$ the algebra of finite rank operators on $\mathbb{C}G$ equipped with the trivial G-action. We define a map $\lambda : C^G_\bullet(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G) \to C^G_\bullet(C^\infty_c(X)\hat{\otimes}\mathcal{K})$ by

$$\lambda(f(t) \otimes (a_0 \otimes T_0)d(a_1 \otimes T_1)\cdots d(a_n \otimes T_n)) = f(t) \otimes (a_0 \otimes U_tT_0)d(a_1 \otimes T_1)\cdots d(a_n \otimes T_n)$$

The map λ commutes with the boundary b_G and is clearly an isomorphism. Hence it suffices to show that the map $\tau : C^G_\bullet(C^\infty_c(X)\hat{\otimes}\mathcal{K}) \to C^G_\bullet(C^\infty_c(X))$ defined by

$$\tau(f(s) \otimes (a_0 \otimes T_0) \otimes \cdots \otimes (a_n \otimes T_n)) = f(s) \otimes tr(T_0 \cdots T_n)a_0 \otimes \cdots \otimes a_n$$

is a quasiisomorphism. Using an inductive limit argument it is in fact enough to consider the case where $\mathcal{K}$ is replaced by a finite matrix algebra $M_n(\mathbb{C})$.
For a unital G-algebra the proof of Morita invariance in Hochschild homology in [**49**] can easily be adapted to show that the map $\tau : C^G_\bullet(A\hat{\otimes}M_n(\mathbb{C})) \to C^G_\bullet(A)$ is a quasiisomorphism. Using the fact that $C^\infty_c(X)$ has local units we obtain long exact sequences in homology associated to the admissible short exact sequences

$$\begin{tikzcd}
C^\infty_c(X) \arrow[r, shift right] & C^\infty_c(X)^+ \arrow[l, dashed, shift right] \arrow[r, shift right] & \mathbb{C} \arrow[l, dashed, shift right]
\end{tikzcd}$$

and

$$\begin{tikzcd}
C^\infty_c(X)\hat{\otimes}M_n(\mathbb{C}) \arrow[r, shift right] & C^\infty_c(X)^+\hat{\otimes}M_n(\mathbb{C}) \arrow[l, dashed, shift right] \arrow[r, shift right] & M_n(\mathbb{C}) \arrow[l, dashed, shift right]
\end{tikzcd}$$

of G-algebras. Comparing these long exact sequences using the map τ we see that $\tau : C^G_\bullet(C^\infty_c(X)\hat{\otimes}M_n(\mathbb{C})) \to C^G_\bullet(C^\infty_c(X))$ is a quasiisomorphism with respect to the equivariant Hochschild boundary. $\square$

In section 5.4 we have studied the equivariant Hochschild-Kostant-Rosenberg maps $\alpha_s : \Omega_G(C^\infty_c(X))_s \to \mathcal{A}_c(X^s)$ for $s \in G$. We assemble these maps for all elements $s \in G$ of finite order to obtain a covariant map

$$\alpha : \Omega_G(C^\infty_c(X)) \to \mathcal{A}_c(\hat{X}).$$

Remark that the hyperbolic part of $\Omega_G(C^\infty_c(X))$ is mapped to zero under α. We define a covariant map $q : \mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)) \to \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))$ by composing $\mathrm{Tr} : \Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G) \to \Omega_G(C^\infty_c(X))$ with the map α. Moreover we choose a regular projective resolution $P^\bullet(\hat{X})$ of $\mathcal{A}^\bullet_c(\hat{X})$ as in section 5.3 and let $P(\hat{X})$ be the associated pro-supercomplex. Composition with q yields two maps

$$f_1 : H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)))) \to H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))))$$

and

$$\begin{aligned} f_2 : H_*(\mathfrak{Hom}_G(&\mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)), \mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)))) \\ &\to H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)), \mathfrak{Fine}(\mathcal{A}_c(\hat{X})))). \end{aligned}$$

Recall that in section 5.3 we also obtained a covariant morphism $p : P(\hat{X}) \to \mathcal{A}_c(\hat{X})$ of pro-supercomplexes. Composition with p yields two maps

$$\begin{aligned} g_1 : H_*(\mathfrak{Hom}_G(&\mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)), P(\hat{X}))) \\ &\to H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)), \mathfrak{Fine}(\mathcal{A}_c(\hat{X})))) \end{aligned}$$

and

$$g_2 : H_*(\mathfrak{Hom}_G(P(\hat{X}), P(\hat{X}))) \to H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c(\hat{X})))).$$

It is important to remark that the Hom-complexes occuring in the definition of f_1, f_2 and g_1, g_2 really are complexes. This has to be checked in those situations where the pro-parasupercomplex $\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)$ is involved. One has to perform computations similar to the one in section 3.4 showing that HP^G_* is well-defined. We remark that this is the point where the assumption that $P^\bullet(\hat{X})$ is regular is needed.

Let us have a closer look at the maps f_1 and f_2.

PROPOSITION 5.16. *Let X be a G-simplicial complex. If the action of G on X is proper the maps f_1 and f_2 are isomorphisms. For an arbitrary action the restrictions to the elliptic part of f_1 and f_2 are isomorphisms.*

PROOF. We shall treat the elliptic and the hyperbolic parts separately.

First we consider the elliptic part for arbitrary X. Let $\langle s\rangle$ be the conjugacy class of an element $s \in G$ of finite order. We consider the quotient $\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_T$ of the localisation $\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle}$ by the space of coinvariants of the action of the operator T. Observe that the action of $s \in G$ on $\mathcal{A}_c(X^s)$ is trivial for all s. Using this fact we see that the localized map $q : \Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle} \to \mathcal{A}_c(\hat{X})_{\langle s\rangle}$ factors over $\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_T$. Since s is of finite order the order of the operator T restricted to $\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle}$ is finite. Since taking coinvariants by a finite group is an exact functor it follows from proposition 5.15 and the equivariant Hochschild-Kostant-Rosenberg theorem 5.9 that the map $\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G)_T \to \mathcal{A}_c(\hat{X})_{\langle s\rangle}$ induced by q is a quasiisomorphism with respect to the Hochschild boundary. Let us show that the resulting map

$$H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G))_T)) \to H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))_{\langle s\rangle}))$$

is an isomorphism. We abbreviate $M = \mathfrak{Fine}(\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G))_T$ and $N = \mathfrak{Fine}(\mathcal{A}^\bullet_c(\hat{X}))_{\langle s\rangle}$. Observe that both M and N are mixed complexes. First we consider the map $q_j : F^jM/F^{j+1}M \to F^jN/F^{j+1}N$ induced by q on the Hodge filtrations associated to M and N. Since q is a quasiisomorphism with respect to the Hochschild boundary it follows that q_j is a quasiisomorphism of supercomplexes for all j. This implies that the mapping cone C^{q_j} of q_j is acyclic.
Recall that the mapping cone C^f associated to a chain map $f : D \to E$ of supercomplexes is defined by

$$C^f_0 = D_1 \oplus E_0, \qquad C^f_1 = D_0 \oplus E_1$$

with differential given by the matrix

$$\partial = \begin{pmatrix} -\partial_D & 0 \\ -f & \partial_E \end{pmatrix}$$

There exists an admissible short exact sequence $0 \to E \to C^f \to D[1] \to 0$ of supercomplexes. Moreover the boundary map $H_*(D[1]) = H_{*-1}(D) \to H_{*-1}(E)$ in the associated long exact homology sequence is the map induced by $f : D \to E$.
We need two abstract results.

Lemma 5.17. *Let $f : D \to E$ be a morphism of supercomplexes of covariant modules. Assume that $\partial_0 : D_0 \to D_1$ is zero and that $\partial_1 : E_1 \to E_0$ is surjective. Then in the mapping cone C^f we have $\mathrm{im}(\partial_1) = E_0$. Consequently the image of ∂_1 is a direct summand in $C^f_0 = D_1 \oplus E_0$.*

Proof. By assumption the differential ∂_1 in C^f has the form

$$\begin{pmatrix} 0 & 0 \\ -f & \partial_1 \end{pmatrix}$$

Thus since $\partial_1 : E_1 \to E_0$ is surjective the image of ∂_1 in C^f is precisely E_0. □

Lemma 5.18. *Let $g : P \to C$ be a morphism of pro-supercomplexes of covariant modules where P is locally projective and C is a constant and acyclic supercomplex. Moreover assume that C_0 admits a direct sum decomposition $C_0 = K \oplus R$ where $K = \mathrm{im}(\partial_1) = \ker(\partial_0)$. Then g is homotopic to zero. Consequently we have $H_*(\mathfrak{Hom}_G(P, C)) = 0$.*

Proof. Using the direct sum decomposition the map $g_0 : P_0 \to C_0 = K \oplus R$ may be written as $g_0 = k \oplus r$. Since $\partial_1 : C_1 \to K$ is a surjection we find a map $s : P_0 \to C_1$ such that $\partial_1 s = k$. This means that we may assume without loss of generality that $k = 0$. Now since g is a chain map and the image of $\partial_1 g_1$ is contained in K we deduce $\partial_1 g_1 = 0$. Hence since C is exact we have $\mathrm{im}(g_1) \subset \ker(\partial_1) = \mathrm{im}(\partial_0)$. This means that we may construct a map $h : P_1 \to C_0$ such that $\partial_0 h = g_1$. Furthermore we can of course require that h factorizes over R, that is $h : P_1 \to R \to C_0$. Using this we obtain a homotopy and can change g in such a way that we get $g_1 = 0$ and still $k = 0$ in $g_0 = k \oplus r$. Since the new g is again a chain map we now have $0 = \partial_0 g_0 = \partial_0 r$. But ∂_0 restricted to R is an injection since $\ker(\partial_0) = K$. This implies $g_0 = 0$ and hence our original g is homotopic to zero. Since we have explicitly shown that any chain map $g : P \to C$ is homotopic to zero we obtain $H_0(\mathfrak{Hom}_G(P, C)) = 0$. By reindexing P we deduce in the same way that $H_1(\mathfrak{Hom}_G(P, C)) = 0$. This finishes the proof. $\square$

We shall apply these results to $D = F^jM/F^{j+1}M$, $E = F^jN/F^{j+1}N$ and $P = P(\hat{X})$. Observe that after possibly reindexing $F^jM/F^{j+1}M$ and $F^jN/F^{j+1}N$ the map $q_j : D \to E$ satisfy the assumptions of lemma 5.17. It follows that the mapping cone $C = C^{q_j}$ satisfies the assumptions of lemma 5.18. Since the short exact sequence $0 \to E \to C \to D[1] \to 0$ has a covariant splitting we obtain a short exact sequence $0 \to \mathfrak{Hom}_G(P, E) \to \mathfrak{Hom}_G(P, C) \to \mathfrak{Hom}_G(P, D[1]) \to 0$ of supercomplexes. Consider the associated long exact sequence in homology. From lemma 5.18 we deduce $H_*(\mathfrak{Hom}_G(P, C)) = 0$ and it is easy to check that the boundary map in this long exact sequence is the map induced by q. Hence we obtain the following result.

Proposition 5.19. *With the notation as above the map q induces an isomorphism*

$$H_*(\mathfrak{Hom}_G(P, F^jM/F^{j+1}M)) \to H_*(\mathfrak{Hom}_G(P, F^jN/F^{j+1}N))$$

for all j.

We can now proceed as in the proof of theorem 4.8. Since P is locally projective the Hodge filtration filtrations of $\theta^n M$ and $\theta^n N$ induce bounded filtrations on $\mathfrak{Hom}_G(P, \theta^n M)$ and $\mathfrak{Hom}_G(P, \theta^n N)$, respectively. We obtain convergent spectral sequences

$$E_{pq}(M) \Rightarrow H_{p+q}(\mathfrak{Hom}_G(P, \theta^n M)), \qquad E_{pq}(N) \Rightarrow H_{p+q}(\mathfrak{Hom}_G(P, \theta^n N)).$$

The map q induces a map of spectral sequences from $E^0_{pq}(M)$ to $E^0_{pq}(N)$. From proposition 5.19 we deduce that the induced map $E^1_{pq}(M) \to E^1_{pq}(N)$ is an isomorphism. This implies that the E^∞-terms of these spectral sequences are isomorphic. Hence we obtain

Proposition 5.20. *With the notation as above the map q induces an isomorphism*

$$H_*(\mathfrak{Hom}_G(P, \theta^n M)) \cong H_*(\mathfrak{Hom}_G(P, \theta^n N))$$

for all n.

Let us define $M(n) = \mathfrak{Hom}_G(P, \theta^n M)$. As in lemma 4.15 we obtain a short exact sequence

$$\begin{tikzcd}
H_0(\mathfrak{Hom}_G(P, \theta M)) \arrow[r] & H_0(\prod_{n\in\mathbb{N}} M(n)) \arrow[r] & H_0(\prod_{n\in\mathbb{N}} M(n)) \arrow[d] \\
H_1(\prod_{n\in\mathbb{N}} M(n)) \arrow[u] & H_1(\prod_{n\in\mathbb{N}} M(n)) \arrow[l] & H_1(\mathfrak{Hom}_G(P, \theta M)) \arrow[l]
\end{tikzcd}$$

and a similar short exact sequence with M replaced by N. The map q induces a morphism between these two exact sequences and together with proposition 5.20 we see that the map $\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G))_T \to \mathcal{A}_c(\hat{X})_{\langle s\rangle}$ induces an isomorphism

$$H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C^\infty_c(X)\hat{\otimes}\mathcal{K}_G))_T)) \cong H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))_{\langle s\rangle})).$$

The proof of proposition 3.25 shows that the natural quotient map $\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle} \to \theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)_T$ is a covariant homotopy equivalence. Hence the localized map

$$\begin{aligned} f_1 : H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle})) \\ \to H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))_{\langle s\rangle})) \end{aligned}$$

is an isomorphism. Since s was an arbitrary element of finite order this shows that f_1 is an isomorphisms on the elliptic components. Moreover the proof can easily be adapted to see that f_2 is an isomorphism on the elliptic components.
It remains to treat the hyperbolic part. Assume that X is proper and let $s \in G$ be an element of infinite order. In this case it follows from the equivariant Hochschild-Kostant-Rosenberg theorem 5.9 and proposition 5.15 that the localisation $\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)_{\langle s\rangle}$ is acyclic with respect to the Hochschild boundary. In the same way as in the proof of theorem 4.8 we obtain

$$H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle})) = 0.$$

It follows trivially that

$$\begin{aligned} f_2 : H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle})) \\ \to H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle}, \mathfrak{Fine}(\mathcal{A}_c(\hat{X}))_{\langle s\rangle})) \end{aligned}$$

is an isomorphism because both sides are zero. This finishes the proof that f_2 is an isomorphism on the hyperbolic component if the action of G on X is proper. The correponding assertion for f_1 is clear since $P(\hat{X})_{\langle s\rangle} = 0$ for elements $s \in G$ of infinite order. □
Later we will need the following statement which can be easily obtained by adapting the proof of proposition 5.16.

COROLLARY 5.21. *Let X and Y be G-simplicial complexes. Composition with the chain map $q : \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y)\hat{\otimes}\mathcal{K}_G)) \to \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}))$ induces an isomorphism*

$$H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y)\hat{\otimes}\mathcal{K}_G))_{\langle s\rangle})) \cong H_*(\mathfrak{Hom}_G(P(\hat{X})_{\langle s\rangle}, \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}))_{\langle s\rangle}))$$

for all elements $s \in G$ of finite order.

Now we consider the maps g_1 and g_2 from above.

PROPOSITION 5.22. *The maps g_1 and g_2 are isomorphisms for an arbitrary G-simplicial complex X.*

PROOF. We only prove the assertion for g_1 since g_2 is handled in the same way. Let us abbreviate $M = P^\bullet(\hat{X})$, $N = \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{X}))$ and $P = \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))$. Recall from section 5.3 that M can be viewed as an unbounded mixed complex with b-boundary equal to zero and B-boundary equal to the differential δ. By definition the pro-supercomplex $P(\hat{X})$ is the projective system of supercomplexes $\xi M = (\xi^n M)$ given by

$$\xi^n M = M_{-(n+1)}/BM_{-(n+2)} \oplus \bigoplus_{i=-n}^{n} M_i \oplus B(M_i).$$

Since M is bounded above in the sense that $M_n = 0$ for $n > D = \dim(\hat{X})$ we obtain

$$\xi^n M = M_{-(n+1)}/BM_{-(n+2)} \oplus \bigoplus_{i=-n}^{D} M_i$$

for $n > D$. We define a filtration F^j of $\xi^n M$ for $n > D$ by

$$F^j(\xi^n M) = M_{-(n+1)}/B(M_{-(n+2)}) \oplus \bigoplus_{i=-n}^{j} M_i \oplus B(M_j).$$

Hence $F^j(\xi^n M)$ is a finite increasing filtration with $F^{-(n+2)}(\xi^n M) = 0$ and $F^n(\xi^n M) = \xi^n M$. If we proceed in the same way for N we see that p induces chain maps $\xi^n M \to \xi^n N$ which are compatible with the filtrations. By construction the map $p : P^\bullet(\hat{X}) \to \mathcal{A}_c^\bullet(\hat{X})$ is a quasiisomorphism with respect to the boundary B. It follows easily that $p : F^{j+1}(\xi^n M)/F^j(\xi^n M) \to F^{j+1}(\xi^n N)/F^j(\xi^n N)$ is a quasiisomorphism for each j and $n > D$. Hence the mapping cone C^p of this map is acyclic. Since P is locally projective we see in the same way as in the proof of proposition 5.16 that the map

$$H_*(\mathfrak{Hom}_G(P, \xi^n M)) \to H_*(\mathfrak{Hom}_G(P, \xi^n N))$$

is an isomorphism for $n > D$.
Let us define $M(n) = \mathfrak{Hom}_G(P, \xi^n M)$. Since the projective system $\xi M = (\xi^n M)_{n\in\mathbb{N}}$ is isomorphic to the projective systems $(\xi^n M)_{n>D}$ we obtain as in lemma 4.15 a short exact sequence

$$\begin{array}{ccccc}
H_0(\mathfrak{Hom}_G(P,\xi M)) & \longrightarrow & H_0(\prod_{n>d} M(n)) & \longrightarrow & H_0(\prod_{n>d} M(n)) \\
\uparrow & & & & \downarrow \\
H_1(\prod_{n>d} M(n)) & \longleftarrow & H_1(\prod_{n>d} M(n)) & \longleftarrow & H_1(\mathfrak{Hom}_G(P,\xi M))
\end{array}$$

and a similar short exact sequence with M replaced by N. Comparing these two exact sequences yields the claim. □
From propositions 5.16 and 5.22 we deduce formally the following theorem.

THEOREM 5.23. *Let X be a G-simplicial complex. If the action of G on X is proper the pro-parasupercomplexes $\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))$ and $P(\hat{X})$ are covariantly homotopy equivalent. The elliptic parts of $\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))$ and $P(\hat{X})$ are covariantly homotopy equivalent even if the action on X is not proper.*

PROOF. We shall only consider the case that X is proper since the second assertion is proved in the same way. Denote by x the preimage of $[p]$ under the isomorphism f_1 and by y the preimage of $[q]$ under the isomorphism g_1. Then we have $f_1(x) = x \cdot [q] = [p]$ and $g_1(y) = y \cdot [p] = [q]$. Hence $g_2(x \cdot y) = x \cdot y \cdot [p] = [p]$ and $f_2(y \cdot x) = y \cdot x \cdot [q] = [q]$. Since g_2 and f_2 are isomorphisms we obtain $x \cdot y = \mathrm{id}$ and $y \cdot x = \mathrm{id}$. This implies that $\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))$ and $P(\hat{X})$ are covariantly homotopy equivalent. □
Now we can finish the proof of theorem 5.12. Again we consider only the case that X is a proper G-simplicial complex.

Using proposition 5.8 and corollary 5.21 we obtain an isomorphism

$$\bigoplus_{j\in\mathbb{Z}} H_G^{*+2j}(X,Y) \cong H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y)\hat{\otimes}\mathcal{K}_G)))).$$

Remark that the hyperbolic part of the Hom-complex vanishes independent of the fact that the action of G on Y may not be proper. We apply theorem 5.23 to deduce

$$\begin{aligned} &H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y)\hat{\otimes}\mathcal{K}_G)))) \\ &\qquad \cong H_*(\mathfrak{Hom}_G(\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y)\hat{\otimes}\mathcal{K}_G)))). \end{aligned}$$

Consequently we have an isomorphism

$$\bigoplus_{j\in\mathbb{Z}} H_G^{*+2j}(X,Y) \cong h_*^G(C_c^\infty(X), C_c^\infty(Y)).$$

Combining this with corollary 5.14 we obtain the desired identification of equivariant periodic cyclic homology with the theory of Baum and Schneider.
It remains to check naturality. Using corollary 5.14 it suffices to show that the isomorphism $\bigoplus_{j\in\mathbb{Z}} H_G^{*+2j}(X,Y) \cong h_*^G(C_c^\infty(X), C_c^\infty(Y))$ is natural in both variables. First we consider the second variable. Let $f: Y_1 \to Y_2$ be an equivariant proper simplicial map. This map induces an equivariant homomorphism $F: C_c^\infty(Y_2) \to C_c^\infty(Y_1)$ and a chain map $\mathcal{A}_c(\hat{Y}_2) \to \mathcal{A}_c(\hat{Y}_1)$ which will also be denoted by F. Using this notation we obtain a commutative diagram

$$\begin{tikzcd} H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y_2)\hat{\otimes}\mathcal{K}_G)))) \arrow[r, "q_*"] \arrow[d, "F_*"'] & H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}_2)))) \arrow[d, "F_*"] \\ H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(Y_1)\hat{\otimes}\mathcal{K}_G)))) \arrow[r, "q_*"'] & H_*(\mathfrak{Hom}_G(P(\hat{X}), \mathfrak{Fine}(\mathcal{A}_c(\hat{Y}_1)))) \end{tikzcd}$$

Since $P(\hat{X})$ and $\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G)))$ are covariantly homotopy equivalent we deduce naturality in the second variable. Now let $f: X_1 \to X_2$ be an equivariant proper simplicial map. As before this map induces an equivariant homomorphism $F: C_c^\infty(X_2) \to C_c^\infty(X_1)$ and a chain map $F: \mathcal{A}_c^\bullet(\hat{X}_2) \to \mathcal{A}_c^\bullet(\hat{X}_1)$. Moreover there exists a chain map $P(F):$ $P^\bullet(\hat{X}_2) \to P^\bullet(\hat{X}_1)$ such that the diagram

$$\begin{tikzcd} P^\bullet(\hat{X}_2) \arrow[r, "P(F)"] \arrow[d, "p"'] & P^\bullet(\hat{X}_1) \arrow[d, "p"] \\ \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{X}_2)) \arrow[r, "F"'] & \mathfrak{Fine}(\mathcal{A}_c^\bullet(\hat{X}_1)) \end{tikzcd}$$

is commutative. Consider the commutative diagram

$$\begin{tikzcd}
H_*(\mathfrak{Hom}_G(P(\hat{X}_2), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X_2)\hat{\otimes}\mathcal{K}_G)))) \arrow[r, "\cong"] \arrow[d, "F_*"'] & H_*(\mathfrak{Hom}_G(P(\hat{X}_2), \mathfrak{Fine}(\mathcal{A}_c(\hat{X}_2)))) \arrow[d, "F_*"] \\
H_*(\mathfrak{Hom}_G(P(\hat{X}_2), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X_1)\hat{\otimes}\mathcal{K}_G)))) \arrow[r, "\cong"] & H_*(\mathfrak{Hom}_G(P(\hat{X}_2), \mathfrak{Fine}(\mathcal{A}_c(\hat{X}_1)))) \\
H_*(\mathfrak{Hom}_G(P(\hat{X}_1), \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X_1)\hat{\otimes}\mathcal{K}_G)))) \arrow[r, "\cong"] \arrow[u, "P(F)^*"] & H_*(\mathfrak{Hom}_G(P(\hat{X}_1), \mathfrak{Fine}(\mathcal{A}_c(\hat{X}_1)))) \arrow[u, "P(F)^*"']
\end{tikzcd}$$

An easy diagram chase shows that the diagram

$$\begin{tikzcd}
P(\hat{X}_2) \arrow[r, "x_2"] \arrow[d, "P(F)"'] & \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X_2)\hat{\otimes}\mathcal{K}_G)) \arrow[d, "\theta\Omega_G(F\hat{\otimes}\,\mathrm{id})"] \\
P(\hat{X}_1) \arrow[r, "x_1"'] & \mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X_1)\hat{\otimes}\mathcal{K}_G))
\end{tikzcd}$$

is commutative where x_1 and x_2 are the chain maps obtained in theorem 5.23 for X_1 and X_2, respectively. This implies that the homotopy equivalence between $P(\hat{X})$ and $\mathfrak{Fine}(\theta\Omega_G(C_c^\infty(X)\hat{\otimes}\mathcal{K}_G))$ is natural in X. It follows that the isomorphism in theorem 5.12 is natural in the first variable.

This finishes the proof of theorem 5.12.

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